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In a previous paper (1956) I attempted to treat the problem of the movement of granular solids in or by fluids as broadly as possible for the sake of general scientific interest and incidentally for the sake of embracing as wide a range of factual evidence as possible. The section on sediment transport by water streams was therefore treated merely as a special case. So no attempt was made to adapt the reasoning in this special case to the needs of practice.

The treatment was deliberately confined to the simplest conditions of uniform grain size and steady flow. As regards uniformity of grain size it appeared, however, on the basis of the evidence given by Paper No. 17 of the U. S. Waterways Experiment Station (1935) that the results might be extended tentatively to natural materials. The width of the size distribution in the Waterways Station experiments was narrow and happened not to include any appreciable proportion of grains small enough to be within the Stokes' law range of size.

As regards steadiness of the flow, no attempt was made to extend the argument to unsteady flow such as occurs when the Froude number approaches unity and the water surface becomes very much disturbed. On the evidence of data from the Gilbert (1914) experiments unsteadiness of the flow seemed to have no effect on the transport rate. But again, Gilbert's sands being uniform, they did not include any proportion of Stokes' law sizes.

The general case included windblown sand which provided a good deal of useful evidence. This meant that the only relevant flow quantity available for use in the general treatment as an independent variable was the boundary stress. In the stream case, where the gravity slopes are very small, the boundary stress can be written as the fluid stress τ . Further, I was led naturally to think in terms of boundary stress τ because the threshold of grain movement, which must clearly be introduced, has for a long time been assumed to be definable as a critical value of the boundary stress. This definition <u>seemed</u> axiomatic.

The key idea underlying the theory was as follows:

When any real substance (water) impels any other real substance (sediment) to move, all experience shows that energy must be expended by the first substance in maintaining the motion of the second against some kind of dynamic opposition. And power--that is, a time rate of energy expenditure--is necessary to maintain the motion at a given time rate. Thus a stream can be regarded as a transporting machine; and we have the dynamic relation

$$\text{rate of work done} = \text{efficiency} \times \text{available power} \qquad (1)$$

The work rate is a certain proportion of the available power, and both work rate and power are the same thing and measurable in the same units. Both have the nature of force times velocity with which something is pushed along in the direction of the force.

Since the sediment is immersed, the work rate is to be measured in terms of $\frac{\sigma-\rho}{\sigma}gMU$ where M is the dry mass of the sediment, U its velocity of transport, and MU is its transport rate G.

Hence, the sediment transport rate is essentially a measure of the rate of work done in transporting the sediment.

The available power of an open turbulent stream is the time rate of conversion of potential gravity energy into kinetic turbulent eddy energy (and thence into heat energy). Hence the available power of an open stream flowing steadily down a constant slope s is ρgQs, or $\rho gRs \times \bar{u} = \bar{\tau}\,\bar{u}$ per unit boundary area where Q is water discharge, R is hydraulic mean depth, s is energy slope, $\bar{u}$ is mean velocity, and $\bar{\tau}$ is mean distributed stress.

But in the general approach neither Q nor $\bar{u}$ could be used, because in the case of atmospheric wind neither had any definable meaning. So I expressed the available power in terms of $\bar{\tau}$ times the drag-velocity $u_* = \sqrt{\frac{\bar{\tau}}{\rho}}$ times an experimental constant 'A'. Thus

$$\frac{\sigma-\rho}{\sigma} g \frac{G}{P} = e\,\bar{\tau}\,'A'\sqrt{\frac{\bar{\tau}}{\rho}} = 'A'\,e\,\bar{\tau}^{3/2}\frac{1}{\sqrt{\rho}} \qquad (2)$$

where G is the whole transport rate over the perimeter P, and e is an efficiency factor.

When only the bedload was considered, the efficiency e appeared as

$B_b \dfrac{\tau - \tau_t}{\tau}$ whence work rate $i_b = \dfrac{\sigma-\rho}{\sigma} g \dfrac{G_b}{P} = A B_b (\tau - \tau_t)\sqrt{\dfrac{\tau}{\rho}}$ (3)

where B_b depended on grain size via the drag coefficient.

The relation was expressed in non-dimensional form by dividing both sides of the equation by a standard power, $(\sigma-\rho) g D \sqrt{\frac{\sigma-\rho}{\sigma} g D}$. The purpose was to reduce the terms to the same scale in order that different sets of experimental results could be compared. This procedure also avoids the continual repetition of the term $(\sigma-\rho)g$. The idea had previously been used by Shields (1936) and **was** further developed by Einstein (1950).

Owing to the fact that the threshold of bed movement was assumed definable as a critical value of the stress τ, it was necessary to plot experimental transport rate values against τ. The natural method of plotting would have been to plot transport rate against available power direct, according to equation (1), in order to plot like quantities against each other.

Now several considerations, including the inconsistency of experimental measurements of the threshold in terms of stress, point to the probability that the threshold of bed movement should be defined as a threshold power and not as a threshold stress. If we assume a power threshold we have

$$i_b = \frac{\sigma-\rho}{\sigma} g \frac{G_b}{P} = e'_b (\omega - \omega_t) \tag{4}$$

in which ω is power per unit boundary area, equal to total power $\frac{\Omega}{P}$; e'_b should have a value of the same order as A_bB_b in the earlier notation of Bagnold (1956); e'_b would be equal to A_bB_b were it not for the change in definition of the threshold.

We can now plot like against like. This has the great advantage that if the parameter e' is constant over any part of the experimental range, its constancy will appear at once as a straight line if we plot on ordinary graph paper instead of on log-log paper.

Further, confining attention to water streams, we can evaluate the available power ω directly as $\rho gQs/P$--both Q and s being directly measured quantities, instead of $\tau\sqrt{\frac{\tau}{\rho}}$ times some inferred numerical factor. Incidentally, this avoids altogether the often difficult and unreliable measurement of flow depth in order to evaluate τ .

Transport of the bedload uses up a portion of the available kinetic power ω . This portion is converted directly into heat in the process of intergranular friction.

The remainder ω - i_b being still in kinetic eddy-form is available to maintain a suspended load. And if every turbulent eddy could play its part we should have

$$i = i_b + i_s = \omega = \rho gQs/P \qquad (5)$$

That is, regardless of grain size the transport rate expressed in the dynamic form $\left[\frac{\sigma-\rho}{\sigma} g \times \text{measured rate per unit width} \right]$ would become equal to the directly measured stream power. The efficiency of the machine would then be unity.

Under what conditions might this ideal state be possible?

The eddies may be assumed to have a random set of eddy-velocities, ranging from zero to some upper limit determined by the size of the bed grains and by the velocity of flow. In order for any grain suspension to occur, some eddies at least must have an eddy velocity equal to or exceeding the fall velocity of some grains. With sand of uniform size, we therefore find a definite threshold state at which suspension begins. But with uniform grains of finite size it appears that the ideal "equality" state of perfect efficiency could never be reached however great we make the power, because there would always be some eddies having insufficient eddy velocity.

Plotting Gilbert's data for uniform sands as $i = \frac{\sigma-\rho}{\sigma} g$ x sediment discharge per unit flume width, against $\omega = \rho gQs$/width, on log-log scales for comparison with figures 13-17 of my 1956 paper, one gets a pattern of which figure 1A is representative.

The plot appears to be asymptotic to the equality line. i fails to reach equality with ω. On the basis of the previous method of plotting, the work rate i_b for bedload alone would be as sketched by the broken curve. And the threshold of suspension would be indicated by the point at which the plot breaks away from this curve. (The suspension threshold is better indicated in figure 5A.)

But if the sediment contains a range of smaller grain sizes, the plot would approach the equality line much more steeply, and the threshold of suspension would occur earlier and might coincide with the threshold of bed movement.

It would however not be expected to reach equality with ω unless the sediment contains a sufficient proportion of grains so small as to lie within the true Stokes' law range of size; that is, of grains whose fall through the water dissipates energy directly into heat without creating "wasted" wake turbulence. The required proportion appears to be such that the low-grade wake turbulence created by larger grains is all utilized in suspending smaller ones.

These ideas seem to be confirmed both by existing laboratory data and by a small amount of river data so far examined. Provided sufficient proportion of Stokes' law grains are present in the sediment, the sediment discharge in terms of the work rate rises abruptly until it does, in fact, reach equality with the stream power ω. At that point there is a sharp discontinuity, so that for all higher values of the stream power, the sediment discharge remains equal to the stream power. Comparative plots are

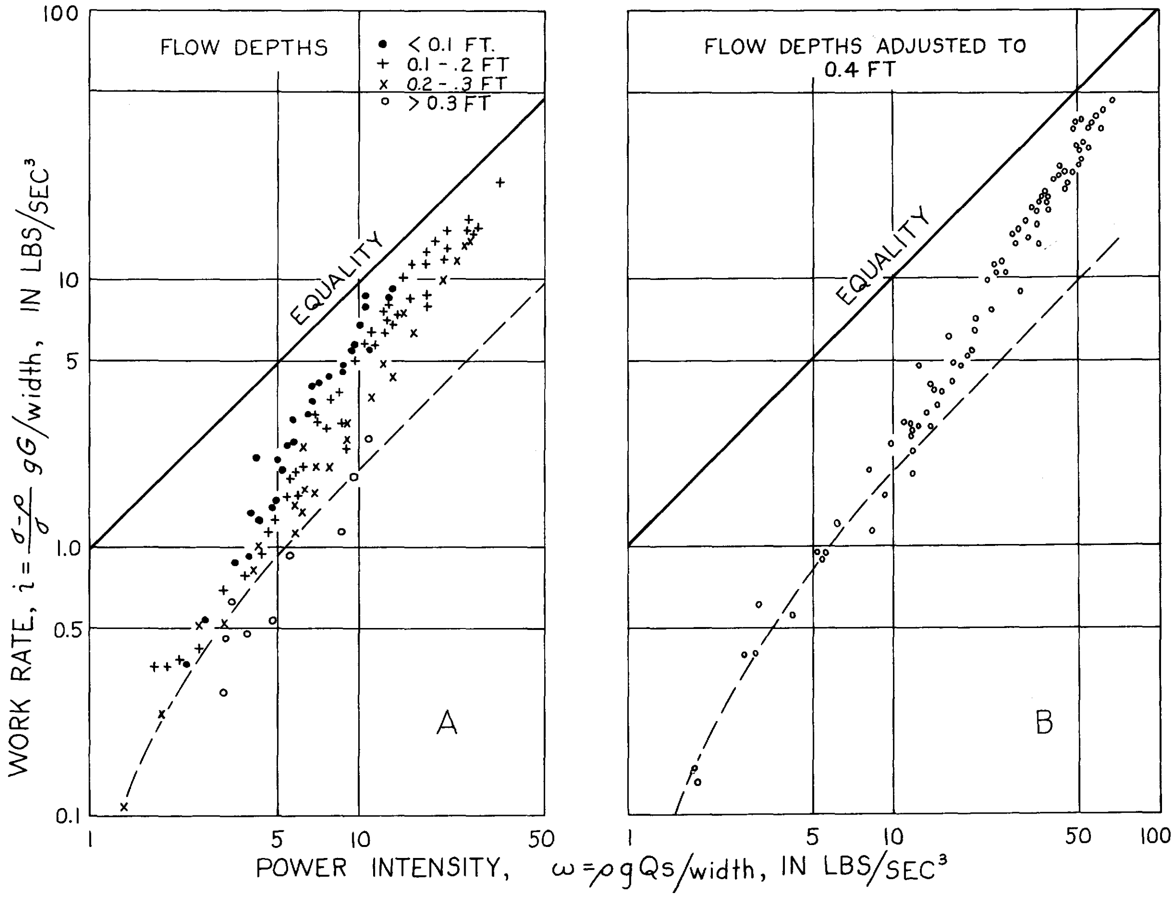

Figure 1.--PLOTS USING DATA FOR GILBERT'S EXPERIMENTS, SAND D, 0.786 MM (UNIFORM)

A. Sediment discharge G lbs/sec expressed as the work rate per unit width $i = \frac{\sigma - \rho}{\sigma} \cdot gG/\text{width}$, lbs/sec^{-3}, plotted against stream power $\omega = \rho gQs/\text{width}$, lbs/sec^{-3} showing gradual approach to equality; and showing correlation of scatter with flow depth.

B. Same data reduced to constant flow depth of 0.4 ft by multiplying each value of i and ω by 0.4 / experimental flow depth. No corrections have been made for wall effects, which appear from figure 1B to be negligible, since this plot shows no significant correlation between sediment discharge and either flow depth or depth-to-width ratio. Residual scatter attributable to errors in the estimation of the true energy slope.

shown in figure 2.

The new modified approach seems entirely consistent with the reasoning in my earlier paper (1956) but is more direct and practical. Moreover, the previous restrictions to uniform grain size are removed, and attention can be directed to the effects of varying the degree of heterogeneity, especially as regards the proportion of very fine sizes present. The only real innovation is the assumption that the threshold of bed movement should be regarded as a threshold of power instead of a threshold of bed stress.

In figure 2 the Brooks-Nomicos data (tabulated in Brooks, 1957) plotted as work rate against stream power show but little scatter. Whereas Brooks (1955 and 1957) pointed out that a very large and inexplicable scatter resulted when the same data were plotted against bed stress. He rightly insisted that velocity of flow must play an important part, but was not able to explain why. Since the power is the product of stress and velocity, the part played by the velocity is now clear.

At the other end of the scale, not far above the threshold of bed movement, interesting ideas emerge from plotting sediment discharge against power on an ordinary linear scale. A plot of data from Waterways Station (1935) Sand No. 1, shows that equation (4) is indeed a linear relation (see figure 3A) except very near the threshold where one finds a "tail" which results from the heterogeneity of the sand. The bed movement in the tail was selective (reported as not "general movement"), whereas all the points on the linear part of the plot correspond to movement reported as "general."

The intercept of the straight line plot with the axis gives the effective

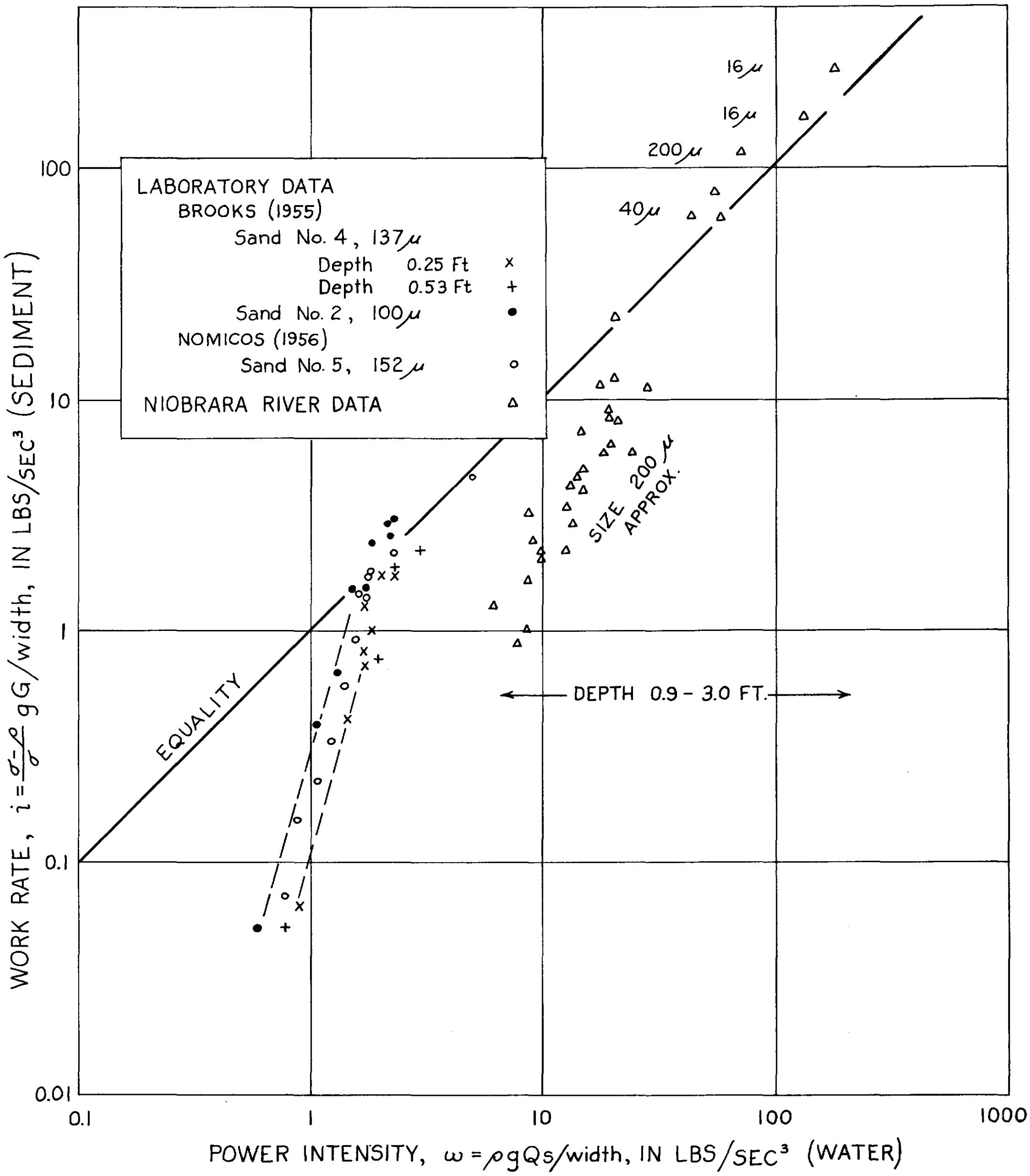

Figure 2.--Comparison of discharge of fine sediment in laboratory experiments and in rivers. Sediment work rate reaches equality with stream power.

threshold power ω_t.

Plotting the same sediment discharge data against the power expressed as $\tau\sqrt{\frac{\tau}{\rho}}$ instead of as ρgQs/width (figure 3B) one obtains another linear plot on a different scale because the constant $C = \bar{u}/\sqrt{\frac{\tau}{\rho}}$ is omitted, but the detailed pattern of the scatter is nearly identical. This shows that such scatter cannot be due to errors in either the measurement of discharge or the measurement of flow depth. The slope being fixed and unadjusted, the scatter seems to be due to the assumption that the energy slope was equivalent to the slope of the flume[a/]. The same data are plotted in figure 4 on log-log scale for comparison of the relative scatter. It should be noticed that threshold conditions are better seen from the linear plottings.

When one makes the same comparison in the case of the finer sand No. 8 (0.20 mm), the scatter in a plot of $\tau\sqrt{\frac{\tau}{\rho}}$ is hopelessly large, but if those same data are plotted on the basis of ρgQs/width, it is reduced in a quite startling way. (See figures 5A and B.) This finding suggests that large dune features on the bed and great difficulty in measuring the mean flow depth prevent a good measurement of τ.

By comparing these linear plots for sand No. 1 (diameter, 0.586 mm) and for sand No. 9 (4.0 mm), both sizes being such that there was no appreciable form drag, we see that the threshold powers are approximately in the ratio $D^{3/2}$, which is consistent with the theory; that is, that threshold stress is proportional to D, and threshold power varies as $D^{3/2}$. Hence, it appears correct to compare experimental results relating to different scales

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a/ The bed surface having been initially made parallel to the flume floor, appreciable changes in the bed slope would not occur within the limited experimental flow period until general grain movement took place. This accounts for the relative lack of scatter in the tail.

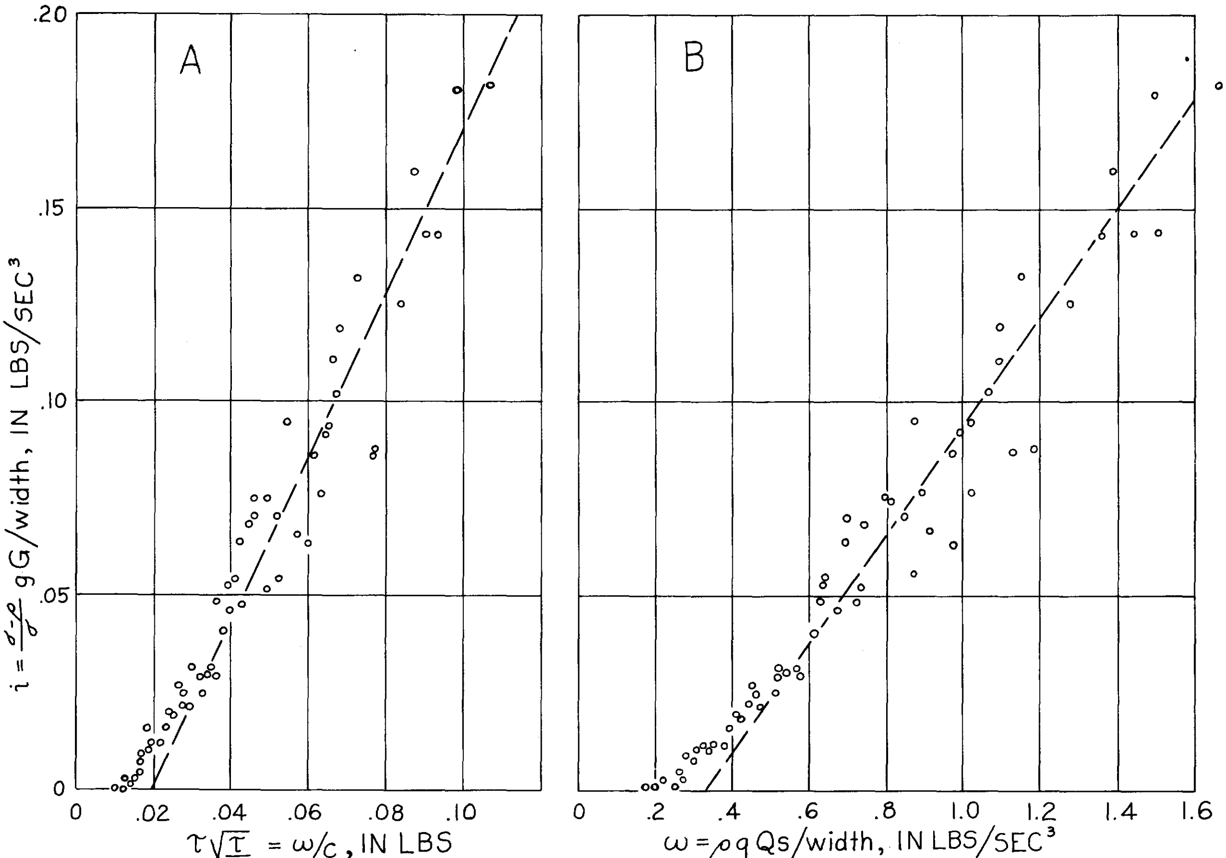

Figure 3.--PLOTS OF U. S. WATERWAYS EXPERIMENT STATION DATA FOR SAND NO. 1, 0.586 MM

A. Experimental bedload work rate i plotted against stream power $\omega = \rho gQs/\text{width}$ on a linear scale to show the linear relation $i \propto \omega - \omega_t$ (Threshold power ω_t substituted for conventional threshold tractive stress τ_t .)

B. Same data as A above, plotted as i against the power function $\tau\sqrt{\frac{\tau}{\rho}} = \frac{\omega}{C}$, the tractive stress τ being obtained from measurement of slope and flow depth, whereas the power in A was obtained from measurement of slope and discharge. The velocity coefficient C is given by the ratio of the gradients in the two plots, $C = \bar{u}/\sqrt{\frac{\tau}{\rho}}$

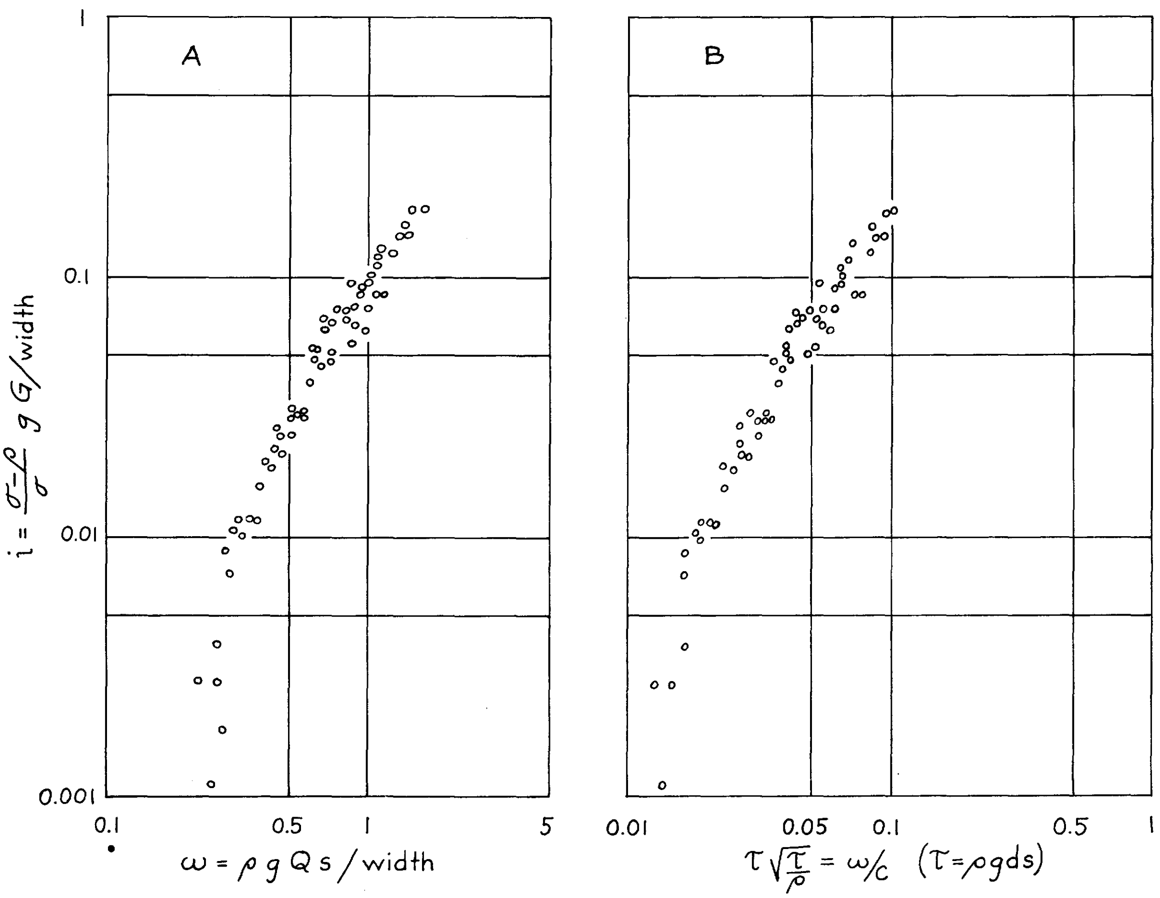

Figure 4.--Values in figure 3A (left) and 3B (right) replotted on log-log scales to compare relative scatter.

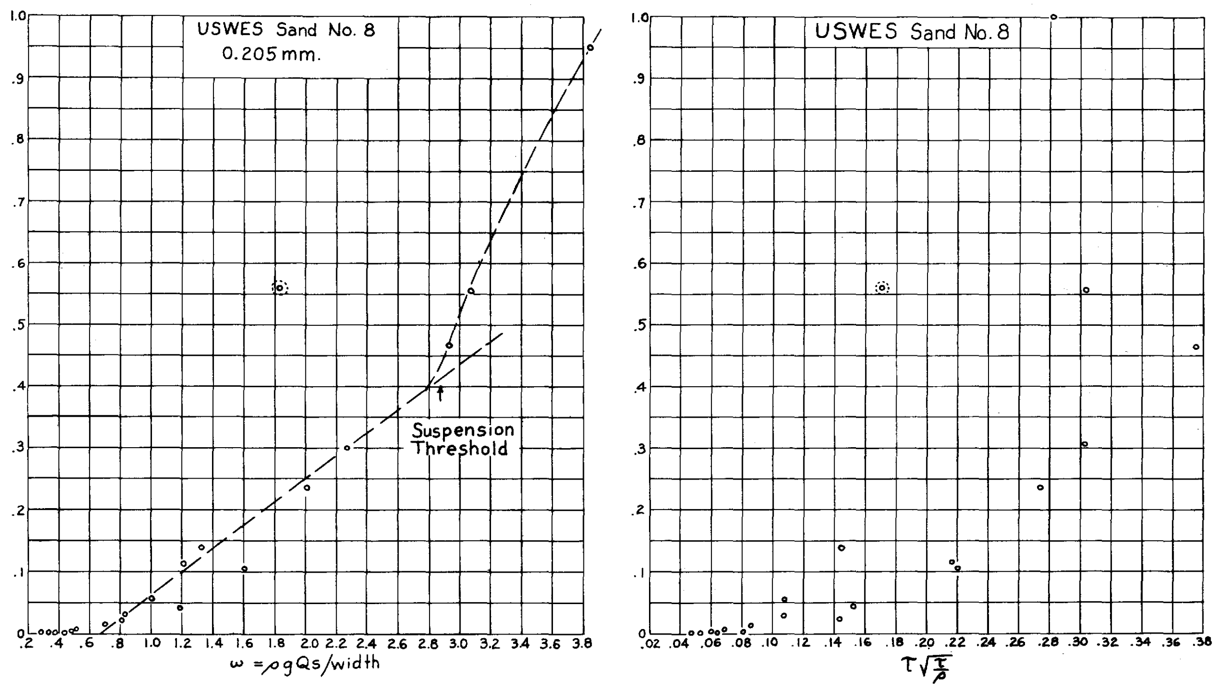

Figure 5.--PLOTS OF U. S. WATERWAYS EXPERIMENT STATION DATA FOR SAND NO. 8, FINE GRAINED, 0.205 MM

A. Experimental discharges of finer grains, plotted as in figure 3A, i against $\omega = \rho gQs/\text{width}$, showing the departure from the linear bedload relation at a definite suspension threshold. Note that this threshold had not been reached in the case of the coarser sediment of figure 3.

B. Same data as figure 5A plotted against the power function showing the gross scatter introduced by difficulties in measuring the true flow depth.

of mean grain size and mean density by dividing both i and ω by the appropriate unit power function

$$\tau\sqrt{\frac{\tau}{\rho}} = (\sigma-\rho)\, g\, D \sqrt{\frac{\sigma-\rho}{\rho}\, g D}$$

So that $i = \frac{\sigma-\rho}{\sigma} g \frac{G}{\rho}$ becomes

$$\frac{\sigma-\rho}{\sigma} g \frac{G}{\rho} \Big/ (\sigma-\rho)\, g\, D \sqrt{\frac{\sigma-\rho}{\rho}\, g D} = \Phi'$$ of Bagnold (1956)

and ω becomes $\frac{\omega}{\tau\sqrt{\tau/\rho}} = C\,\theta^{3/2}$ when $C = \frac{\bar{u}}{\sqrt{\tau/\rho}}$

A number of queries arise about the effects of channel shape and scale. The most direct relation is that between the whole sediment discharge in terms of I = i x width, and the whole stream power $\Omega = \omega$ x width. But the absolute values of I and Ω will then vary with the size of the stream, those for rivers being enormously greater than those for flumes. And if one compares values for the same river at different stages, one is really comparing conditions in effectively different rivers, the cross sections being different.

The scale effect is reduced by dealing in quantities per unit width, but it is not removed, because the unit cross sections still vary in flow depth.

If at low stage the sediment were to move as bedload only, as in the case of Waterways Station sand No. 1, there appears from these experimental data to be no independent correlation between transport rate and flow depth. Flow depth seems to be involved only as determining the discharge Q. In this "low-stage," or bedload, region no evidence seems to exist that scatter is improved by making a systematic allowance for wall effect as some function of the width-depth ratio of a flume. Width-to-depth ratios have generally been too small.

It is very clear, however, when plotting Gilbert's data, most of which cover the "high-stage" or suspension region, that the sediment transport rate figures have a pronounced correlation with flow depth. Gilbert was the only investigator who extended his experimental conditions to include large width-to-depth ratios by varying width as well as depth. Unfortunately the available water discharge did not permit him to combine large flow depth with large width. So large depth had to be associated with narrow width.

The correlation between sediment discharge and flow depth has been known for a long time, but it has always been attributed to wall effect. The correlation is so large, however, that in correcting for it as a wall effect one has to assume the smooth walls to exert the same drag as the grain bed; one has to make the full "hydraulic mean depth allowance $b/(b + 2d)$." This has always seemed to me to be irrational, especially when the bed is rippled and must have a very large form drag.

It seems more reasonable to assume the drag effect of smooth walls to be negligible compared with the drag of a very rough bed, and to assume Gilbert's wide scatter of transport rate values to be due to the variation in cross-sectional shape as the depth was increased.

Taking grain suspension to be a volume effect, as distinct from bedload movement which is a boundary effect, one may reduce Gilbert's data--both transport rate and power--from a "per unit width" basis to a "per unit volume" or unit depth basis by dividing both i and ω by the flow depth of the run concerned. The correlation with depth then disappears without any recourse to an arbitrary correction for wall effect. In figure 1B the data of figure 1A have been reduced to the common basis of a 0.4-ft depth. The residual scatter is no greater than can be attributed to errors in slope estimation.

It seems significant that the Brooks-Nomicos results, which cover the same suspension region up to the approach to "equality," have no appreciable scatter when plotted as i against $\omega = \rho gQs/\text{width}$, whereas Gilbert's unreduced data have a wide scatter. The apparent reason is that Nomicos ran his experiments at constant depth--that is, constant cross section--and Brooks' variation of depth was comparatively small.

This suggests that consistent results from sediment transport experiments require constant cross section, just as do consistent results from water flow experiments without sediment. This aspect seems to have been overlooked previously, presumably because it was difficult to adjust the energy slopes of old-fashioned fixed flumes, experiments could not be made at constant depth.

Sediment Discharge as a Function of Stream Power Measured by Qs Compared With That Measured by Tractive Stress

The stream power $\omega = \rho gQs/\text{width}$ can be expressed as

$$= \rho gRs \times \bar{u} = \bar{\tau}\bar{u}$$

And by introducing a conduction coefficient C of the Chezy type so that

$$C = \bar{u}/\sqrt{\frac{\tau}{\rho}}$$

we have $\omega = C\,\tau\sqrt{\frac{\tau}{\rho}}$

For clear water flowing in the same channel, provided the depth remains reasonably constant, C should remain constant as u is varied (square law). When figures 3A and 3B are compared for flow with bedload sediment transport only, it is evident that C does remain constant, for both plots give straight lines, and C is given by the ratio of the two constant gradients. The transport of bedload, therefore, does not appear to invalidate the square law for turbulent flow.

If, however, we plot Nomicos' sediment discharge data (sand No. 5, Brooks 1957), obtained at constant flow depth and therefore most likely to be consistent, against $\tau\sqrt{\frac{\tau}{\rho}}$ as in figure 3B, the results are highly anomalous. (See figure 6.) Assuming the effective values of $\bar{d}$ in $\tau = \rho gRs$ were correctly estimated; then, as Brooks has pointed out, the sediment discharge may for a given value of τ have two values, the second being as much as 10 times the first.

The reason for this apparent anomaly is now clear. It fits in with much other evidence. Water flow containing varying concentrations of fine solids

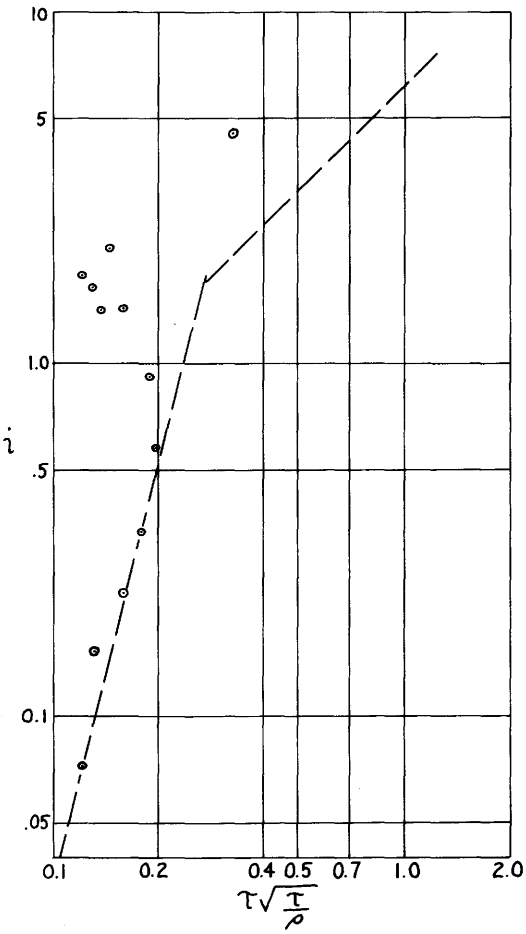

Figure 6.--Data for Nomicos' Sand No. 5, plotted as work rate, i, as a function of tangential stress, τ. These data were obtained at a nearly constant depth of flow.

in <u>suspension</u> does not obey the square law. The velocity coefficient C is not at all constant.

This may be seen as follows: If C were constant, plots of sediment discharges on log-log scales against $\omega = \rho gQs/\text{width}$ and against $\omega/C = \tau\sqrt{\frac{\tau}{\rho}}$ should have identical gradients, as in figure 4. The gradients of the Nomicos plot in figure 2 have therefore been transferred to figure 6 (the two broken lines). They have been positioned with reference to the four lowest of the plotted points; and the agreement here is very good. But as the real power is increased, the apparent power as given by $\tau\sqrt{\frac{\tau}{\rho}}$ becomes relatively smaller, because a real velocity increase has not been taken into account.

The values of the velocity coefficient $C = \frac{\bar{u}}{\sqrt{\frac{\tau}{\rho}}}$ may be calculated directly from the velocity and tractive force data. These values are shown in figure 7 plotted first against the ratio i/ω and also against the Froude number $\bar{u}/\sqrt{gd}$. The similarity between the two curves is remarkable. It seems important to discover how far the variation in C depends on Froude number, sediment size and composition, and other possible factors. It may be significant that the apparent peak in figure 7B occurs at $\bar{F} = 0.8$ which would be very close to unity for a Froude number basis of water surface velocity instead of mean velocity.

Comparing figure 6 with figure 5B for 0.2 mm sand of narrow size distribution, it now appears that the latter is not so hopelessly inconsistent as it seemed at first. The points curl backwards in the same way. The Froude number of the top point was 0.64.

On the other hand, there is no definite indication of any such effect

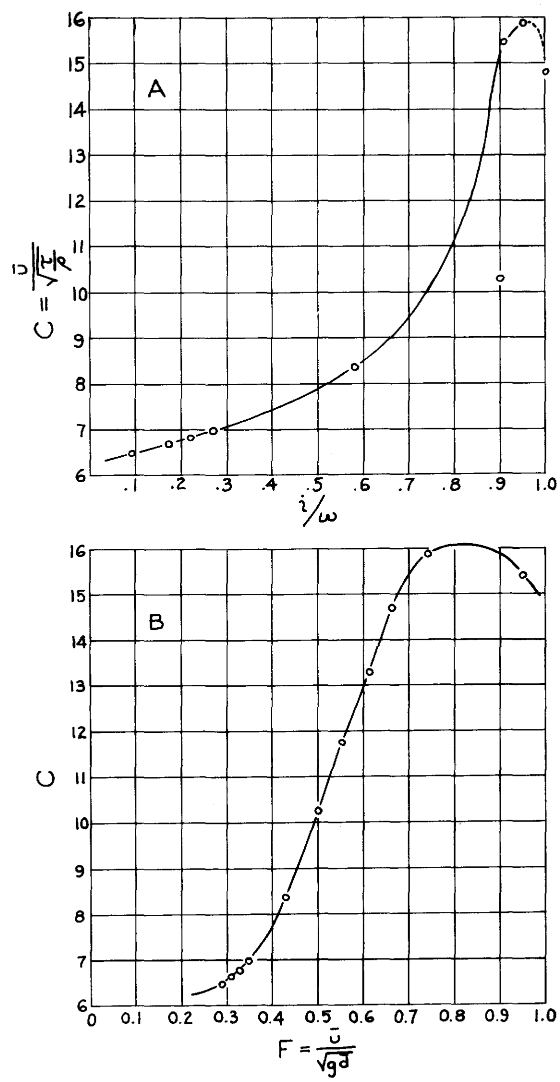

Figure 7.--Data from Nomicos' experiments, sand No. 5. Upper graph, A, shows velocity coefficient as a function of the ratio of work rate to stream power. Lower graph, B, shows same coefficient as a function of Froude number.

from Gilbert's data for uniform sands. And it may well be that the effect is in some way due to the presence of a proportion of fine grains within the Stokes' law range of size. In support of this notion it may be pointed out that in the Brooks-Vanoni-Nomicos experiments four different sands were used, each having approximately the same mean size, but differing widely in the proportion of fine sand present. All four sands exhibited the effect of a progressive increase in the velocity coefficient C. The relative magnitude of the increase is in one for one correlation with the proportion of fine sand.

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LIST OF SYMBOLS

A Experimental constant

B Factor dependent on grain size

b Refers to bedload

C Velocity coefficient

D Diameter of grain

d Depth

e An efficiency factor

$\bar{F}$ Froude number for whole channel

g Gravitational acceleration

G Transport rate

I Work rate for whole width (sediment)

i_b Work rate per unit width

M Dry mass of sediment

P Perimeter

Q Water discharge

R Hydraulic mean depth

s Slope of water surface

t Subscript signifying threshold of movement

U Velocity of transport

$\bar{u}$ Mean velocity of water

u_* Drag velocity

γ Unit stress

σ Grain density

ρ Fluid density

τ Tangential boundary stress

$\bar{\tau}$ Mean distributed tangential boundary stress

ω Power intensity per unit width

Ω Whole stream power for full width

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