A MULTIDIMENSIONAL MARKER-IN-CELL HYDRAULIC AND SEDIMENT TRANSPORT MODEL FOR BRAIDED RIVER FLOW

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INTRODUCTION

Recently, fluvial geomorphologists have expended a great deal of effort in developing numerical models to simulate braided rivers. Contingent upon limitations imposed by computing power (Nicholas (2003)) and a lack of integrated field and laboratory datasets at a suitable resolution (Lane and Richards (2001)), approaches have ranged in complexity in relation to the spatial and temporal scales to which they have been applied. For example, at the scale of individual braid bars or confluence units, two-dimensional depth-averaged and fully three-dimensional hydraulic models have been used to simulate flow patterns (e.g. Lane and Richards (1998), Lane et al. (1999)). At the braidplain scale, one-dimensional models of section-averaged flow and bedload transport (e.g. Paola (1996)), simplified two-dimensional cellular automata models (e.g. Murray and Paola (1994)) and, recently, two-dimensional models have been applied (e.g. Nicholas (2003)). However, only with the use of a fully three-dimensional model can the important secondary circulation processes that lead to the scour and deposition patterns seen in braided rivers be accurately represented.

Given the apparent dichotomy between the need for three-dimensional models and the associated computational and data limitations, this paper presents the formulation of a modified 2+ dimensional Marker-in-Cell model (Tetzlaff and Harbaugh (1989)). Lagrangian momentum tracers that have neither mass nor volume are tracked through an Eulerian mesh that characterizes the field variables (e.g. depth and bed elevation). For each tracer, the momentum equation is solved in its Lagrangian form to reduce numerical dispersion, whilst the continuity equation is solved in its Eulerian form to achieve improved mass conservation (e.g. Chin (1997)). As tracers accelerate and decelerate due to interactions with each other and topography, erosion and deposition are induced. The model presented here simulates fluid flow and sediment transport in the horizontal plane and depth and deposits in the vertical plane. Non-uniform total-load sediment transport is simulated by using a non-equilibrium approach, dividing the sediment mixture into size fractions and accounting for the effects of hiding and exposure. Historical erosion and deposition is represented by a mixing layer approach.

EQUATIONS OF TWO-DIMENSIONAL OPEN-CHANNEL FLOW

It is possible to obtain two key equation sets for a single fluid unit from Newton’s laws of motion: (i) the law of conservation of mass for an incompressible fluid in Eulerian form and (ii) the Navier-Stokes momentum equations for an incompressible fluid (Lane (1998)). In two dimensions, and assuming that the fluid is homogeneous, incompressible and at constant temperature, the continuity or mass conservation equation can be expressed as:

$$\frac{\partial h}{\partial t}+\frac{\partial hU}{\partial x}+\frac{\partial hV}{\partial y}=0 \tag{1}$$

where h = flow depth, t = time, and U and V are depth-averaged velocities in the x and y directions, respectively.

In Cartesian coordinates, the simplified depth-averaged shallow water momentum equations may be written as:

$$\frac{\partial U}{\partial t}+\frac{\partial U^2}{\partial x}+\frac{\partial UV}{\partial y}=\frac{DU}{Dt}=-g\frac{\partial H}{\partial x}+\left(\frac{\mu}{\rho}+\nu\right)\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\right)-\frac{c_f UQ}{8h} \tag{2}$$

$$\frac{\partial V}{\partial t}+\frac{\partial UV}{\partial x}+\frac{\partial V^2}{\partial y}=\frac{DV}{Dt}=-g\frac{\partial H}{\partial y}+\left(\frac{\mu}{\rho}+\nu\right)\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right)+-\frac{c_f VQ}{8h} \tag{3}$$

where $\frac{D(\;)}{Dt}$ = material or total derivative, g = gravitational acceleration, H = water surface elevation, μ = molecular viscosity, ρ = density of water, ν = turbulent eddy viscosity = $\frac{(e-1)}{e^2}\frac{\kappa hQ}{\left(8/c_f\right)^{1/2}}$, derived from the Kàrmàn-Prandtl law-of-the-wall, assuming the depth-averaged velocity occurs at h/e within the water column (where e is the base of natural logarithms), κ is the von Kàrmàn constant which has a value of $\approx$ 0.33 in suspended sediment-laden flows (Bennett et al. (1998)), c_f = Darcy-Weisbach friction coefficient, and $Q = \left(U^2+V^2\right)^{1/2}$.

In arriving at equations (2) and (3), a number of simplifying assumptions have been made. The Coriolis force, horizontal variations in atmospheric pressure and water surface wind stress terms have been neglected (e.g. Lane (1998)), and it has been assumed that the hydrostatic pressure distribution (e.g. Vreugdenhill (1994)) and constant density hold over the flow depth. In addition the shear stress and dispersion terms have been modeled very simply, assuming that dispersion dominates turbulent momentum diffusion, and terms involving cross-derivatives have been neglected (Westerink, 2003). In these equations, all velocities are time-averaged and uppercase letters denote depth-averaged velocities. Through the use of the material derivative, equations (2) and (3) express both the Eulerian and Lagrangian forms of the equations.

SEDIMENT EROSION, TRANSPORT AND DEPOSITION

Commonly, three main modes of sediment transport are distinguished: wash load, suspended load and bed load. This research is concerned with the latter two modes. The suspended load fraction is made up of particles that are held in suspension by turbulent eddies, whilst the bedload fraction is made up of particles that roll, slide, or saltate along the bed. In this research, the water column is treated as a single bed-material load layer, since computational efficiency is of key importance and because this technique is less reliant upon empiricism (Langendoen (2000)).

Non-equilibrium bed-material load transport model: Many early sediment transport models assumed that local equilibrium conditions applied when simulating bed-material transport. This sets the actual bed-material transport rate to be equal to the sediment transport capacity, the transported sediment load under equilibrium conditions (i.e. uniform flow and no net erosion or deposition) (Langendoen (2000)). However, this may lead to unrealistic predictions of bed deformation, especially in cases where flows are above (e.g. after urbanization, forest exploitation, channelization; Simon (1992), Stott et al. (2001), Trimble (1997)) or below (e.g. downstream of a reservoir; Sherrard and Erskine (1991)) capacity for much of the time. To overcome this limitation, a non-equilibrium transport model is implemented in this work.

For the determination of sediment transport in a non-uniform sediment mixture, it is convenient to divide the mixture into several size classes. For each size class *k*, the two-dimensional advection-diffusion equation of total load sediment transport can be derived (Wu et al. (2000a)):

$$\frac{\partial (hC_{tk})}{\partial t} + \frac{\partial (hUC_{tk})}{\partial x} + \frac{\partial (hVC_{tk})}{\partial y} = \frac{\partial}{\partial x}\left(\varepsilon_s h \frac{\partial C_{tk}}{\partial x}\right) + \frac{\partial}{\partial y}\left(\varepsilon_s h \frac{\partial C_{tk}}{\partial y}\right) + E_{bk} - D_{bk} \tag{4}$$

where C_{tk} = total load sediment concentration of the k^{th} size class, $\varepsilon_s = \nu / \sigma_s$ = eddy diffusivity of sediment (ν = eddy viscosity and σ_s = turbulent Prandtl-Schmidt number), E_b = entrainment rate of particles from the bed, and D_b = deposition rate of particles onto the bed. The net flux at the bed, $E_{bk} - D_{bk}$, equals the change in bed elevation due to erosion and deposition of each size fraction. The bed elevation change by size fraction can be formulated as:

$$E_{bk} - D_{bk} = (1-\lambda)\frac{\partial Z_k}{\partial t} = \alpha \omega_k \left(C_{tk} - C_{t*k}\right) \tag{5}$$

where λ = bed porosity, Z_k = bed deformation of the k^{th} size class, α = non-equilibrium adaptation coefficient (Armanini and Di Silvio (1988)), ω = fall velocity (available from lookup tables published by, for example, USICWR (1957)), and C_{t*} = the total load transport capacity.

Sediment transport capacity: Solution of equation (5) requires the determination of the sediment transport capacity. Sediment transport capacity is determined utilizing the formulations of Wu et al (2000b), which determine the fractional transport capacities of bed load and suspended load, and take into account hiding and exposure effects among different size classes. The total sediment transport capacity, C_{t*}, is then defined as:

$$C_{t*} = \sum_k p_k C_{*k} \tag{6}$$

where p_k = fraction of sediment in the k^{th} size class available for transport; and C_{*k} = sediment transport capacity of the k^{th} size class. The fraction p_k depends on the fractional content by volume of size class *k* in the surface layer and the fraction of sediment in the k^{th} size class entering the reach from upstream.

Representation of erosion and deposition- sediment layers: To represent historical erosion and deposition, the bed can be divided into a surface or active layer and a subsurface layer. These layers constitute the mixing layer (e.g. Hirano (1971)). Sediment particles are continuously exchanged between the water column and the surface layer. In contrast, sediment particles only exchange between surface layer and substrate when the bed scours or fills (Langendoen (2000). Variation in the bed material composition in the surface layer is determined from mass conservation:

$$\frac{\partial\left(\beta_k^s a\right)}{\partial t} = \frac{\partial z_{bk}}{\partial t} + \beta_k^* \left(\frac{\partial a}{\partial t} - \frac{\partial z_b}{\partial t} \right) \tag{7}$$

where β_k^s = bed material composition in the surface layer, a = thickness of the surface layer, $\partial z_b / \partial t$ = total bed deformation rate, β_k^* = β_k^s when $\partial a/\partial t - \partial z_b/\partial t \le 0$, and β_k^* = the bed material composition in the subsurface layer when $\partial a/\partial t - \partial z_b/\partial t > 0$.

NUMERICAL SOLUTION OF FLOW AND SEDIMENT TRANSPORT EQUATIONS

In this research, a 'segregated' (Tannehill et al. (1997), Roache (1998)) or decoupled approach is utilized to solve the governing equations at each timestep. Two different solution methods for the equations are required- one for the Lagrangian momentum equations and one to solve the Eulerian continuity equation and sediment transport equation and to couple the solutions. This coupling is essential in order to ensure conservation of both mass and momentum (Tannehill et al. (1997), Roache (1998)).

Solution of the Lagrangian momentum equations: The simplified momentum equations in Lagrangian form are solved utilizing the second order in time, fourth order in space Runge Kutta method (e.g. Press et al. (1992)). Press et al. (1992) note that this method provides the maximum accuracy for the minimum of effort (in their words, "it gives the most bang for the buck" (p 716)). Since tracer positions do not necessarily coincide with grid nodes (which contain the values of flow depth and velocities), interpolation is required to determine h and $\partial H/\partial x$ and $\partial H/\partial y$. It is assumed that the water surface is made up of a series of hyperbolic paraboloids, whereby the water surface elevation is fitted exactly at each grid node and so avoids discontinuities at cell edges. A hyperbolic paraboloid degenerates to a plane when its four vertices are coplanar, which is justified by the fact that water surfaces tend to obtain minimum potential energy in the form of a plane (Tetzlaff and Harbaugh (1989)). The interpolation function is:

$$h = h_{x,y}(1-i)(1-j) + h_{x,y+1}(1-i)(j) + h_{x+1,y}(i)(1-j) + h_{x+1,y+1}(i)(j) \tag{8}$$

where i and j are the x- and y-coordinates of the tracer within the cell, respectively ($0 \le i,j \le 1$). It follows by differentiation that:

$$\frac{\partial H}{\partial x} = \frac{\left[\left(H_{x+1,y+1} - H_{x,y+1}\right)(j) + \left(H_{x+1,y} - H_{x,y}\right)(1-j)\right]}{\Delta x} \tag{9}$$

and:

$$\frac{\partial H}{\partial y} = \frac{\left[\left(H_{x+1,y+1} - H_{x+1,y}\right)(i) + \left(H_{x,y+1} - H_{x,y}\right)(1-i)\right]}{\Delta x} \tag{10}$$

Once the positions and velocities of all tracers have been updated, velocities at each grid node are updated by averaging the velocities of all the tracers within a grid cell. The accuracy of the flow representation is hence directly related to the number of tracers in each grid cell.

Solution of the Eulerian continuity equation and solution coupling: Comparison of the Eulerian two-dimensional shallow-water equations and the Euler equations of compressible isentropic gas dynamics indicate that they have the same mathematical structure (Majda (2002)). This suggests that numerical methods designed to solve compressible flows can be tailored to solve the shallow water equations. Several schemes have been devised to solve the decoupled continuity and momentum equations, such as the SIMPLE family (SIMPLE (Patankar and Spalding (1972)), SIMPLER (Patankar (1980))), PISO (Issa (1985)) and fractional step methods (e.g. Chorin (1968)). The SIMPLE method has been applied to compressible flows by, for example, Demirdžić and Perić (1990) and Ferziger and Perić (2002). PISO has been applied to compressible flows by Issa et al. (1986). Since the solution of the Lagrangian momentum equations should yield correct velocity fields, application of the SIMPLE method would lead to initial divergence and hence increase the required number of iterations for convergence (Patankar (1980)). To overcome this difficulty, Patankar (1980) developed the SIMPLER method, and hence it is adopted as the second solution method here.

Demirdžić and Perić (1990) detail the method to assemble the components of the algebraic equations in order to implicitly solve the transport equation for a scalar, ϕ. During solution, the velocities, U and V and the concentration C_{tk} can be considered as scalars. The discretized transport equation for a scalar quantity in integral form and a colocated finite volume grid is:

$$\Delta x^2 \frac{(h\phi)^{n+1} - (h\phi)^n}{\Delta t} + \Delta x \sum_l (h\phi)_l U_i \cdot n - \sum_l (\Gamma_\phi)_l \left(\phi_{l-1/2} - \phi_{l+1/2}\right) = \Delta x^2 q_\phi \tag{11}$$

where $U_i \cdot n$ is the velocity perpendicular to the cell face, l = 1,2,3,4 (east, south, west and north faces), Γ_ϕ is the relevant diffusion coefficient, and q_ϕ is the source term.

The method of Rhie and Chow (1983) to handle ‘checkerboard’ pressure distributions is implemented. By inserting the relevant scalar into the equation, a matrix equation results which can be solved iteratively for ϕ. In this research, the suggestion of Demirdžić and Perić (1990) is followed and Stone’s (1968) SIP solver is implemented. The rate of convergence of this already efficient solver may, at a later date, be improved by utilizing it as a smoother within a multigrid framework (e.g. Ferziger and Perić (2002)).

The solution algorithm can be summarized as follows:

1. Employ the 4th order Runge Kutta solver to update the velocities and positions of the momentum tracers. Average all tracer velocities within each grid cell to determine nodal velocity values.
2. Assemble the coefficients for the discretized equation (11) and hence calculate pseudovelocities (Patankar (1980)). Assemble and solve the SIMPLER pressure equation resulting from the discretized equation (11) (Patankar (1980)).
3. Using this new pressure field, solve the discretized equation (11) with $\phi = U$ and then V to update the grid node velocities.
4. Calculate new mass fluxes using the updated grid node velocities, and determine the mass imbalance in each grid cell.
5. Assemble and solve the pressure correction equation resulting from the discretization. Apply SIP until the sum of absolute residuals is reduced by a factor of 5.
6. Correct the nodal velocity components utilizing the calculated pressure correction. Use the hydrostatic pressure equation to determine the flow depth.
7. Assemble and solve the sediment concentration equation resulting from the discretization, equation (11). Apply SIP until the sum of absolute residuals is reduced by a factor of 5.
8. Return to step 2 and repeat until the sum of the absolute residuals in the momentum and continuity equations has fallen by two orders of magnitude.
9. Advance the time by an increment Δt.

CONCLUSION

In this paper, the formulation of a modified Marker-in-Cell model has been presented. This technique entails tracking Lagrangian momentum tracers that have neither mass nor volume through an Eulerian mesh that characterizes the field variables. As tracers accelerate and decelerate due to interactions with each other and topography, erosion and deposition are induced. For each tracer, the momentum equation is solved by the fourth order in space Runge Kutta method, whilst the Eulerian continuity equation is discretized implicitly and solved by the SIMPLER method (Patankar (1980)) to achieve improved mass conservation and velocity-pressure coupling. The model operates in 2+ dimensions, whereby fluid flow and sediment transport are simulated in the horizontal plane and depth and deposits are simulated in the vertical plane. Non-uniform total-load sediment transport is simulated using a non-equilibrium approach, dividing the sediment mixture into size fractions and accounting for the effects of hiding and exposure. Historical erosion and deposition is represented by a mixing layer approach. It is believed that the use of the Marker-in-Cell method formulated in this paper may permit the simulation of some of the more stochastic features observed in braided rivers (e.g. Paola, 1996).

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