Techniques of Water-Resources Investigations of the United States Geological Survey

Chapter B2

INTRODUCTION TO GROUND-WATER HYDRAULICS

A Programed Text for Self-Instruction

By Gordon D. Bennett

Book 3

APPLICATIONS OF HYDRAULICS

Part III. Application of Darcy's Law to Field Problems

Darcy's law, as mentioned in the discussion at the close of Part II, may be generalized to deal with three-dimensional flows; and it may be combined with other laws or concepts to develop equations for relatively complex problems of ground-water hydraulics. Even in the simple form developed in the program of Part II, however, Darcy's law has direct application to many field problems. In Part III we shall consider a few examples of such direct application. Later, in Part V and VI, we will consider the combination of Darcy's law with other concepts to yield equations for more complex problems.

1 □

In Part II we pointed out that Darcy's law is a differential equation—that is, an equation containing a derivative. It gives us some information about the rate at which head changes with distance, under given conditions of flow. In general, in dealing with ground-water problems, we will require expressions that relate *values* of head, rather than the rate of change of head, to flow conditions. To proceed from a differential equation, describing the rate of change of head, to an algebraic equation giving values of head, is to obtain a solution to the differential equation. There are various techniques for doing this. We need not go into these techniques of solution here. For our purposes, it will be sufficient if we can recognize a solution when we are given one—that is, if we can test an algebraic equation to determine whether it is a solution to a given differential equation. This is just a matter of differentiation. When we wish to know whether an algebraic equation is a solution to a differential equation, we may simply differentiate the algebraic equation. If we obtain a result which is equivalent to the given differential equation, then the algebraic equation is a solution to the differential equation. Should we fail to obtain an equivalent result, the algebraic equation is not a solution. Thus, for our present purposes at least, we may consider a solution to a differential equation to be an algebraic equation which, when differentiated, will yield the given differential equation.

QUESTION

Which of the following algebraic equations is a solution to the differential equation

$$\frac{dy}{dx}=K\ ?$$

	Turn to Section:
$y=Kx^2$	15
$x=2y+K$	23
$y=Kx+5$	7

2

Your answer in Section 35,

$$\frac{dh}{d(\ln r)}=\frac{Q}{2\pi Kb},$$

is correct. This equation is equivalent to the original differential equation for the problem and states that the rate of change of hydraulic head, with respect to change in the natural logarithm of radial distance, is constant and equal to

$$\frac{Q}{2\pi Kb}.$$

QUESTION

Suppose we were to plot a graph of hydraulic head versus the natural log of radial distance from the well, in our discharging well problem. Which of the following statements would apply to this graph?

	Turn to Section:
(a) The plot would become progressively steeper with decreasing values of ln r—that is, as the well is approached.	18
(b) Equal changes in head would be observed over intervals representing equal changes in r.	31
(c) The plot would be a straight line.	38

3

Your answer in Section 19 is correct. If the head in the well (and throughout the aquifer) prior to pumping is equal to h_e, the term h_e-h_w is actually the drawdown in the pumping well (assuming that there are no additional losses in head associated with flow through the well screen, or within the well itself). Thus the equation in your answer allows us to predict the drawdown associated with any discharge, Q. Alternatively, the equation can be viewed as a method of calculating the hydraulic conductivity, K, of the aquifer on the basis of field measurements of Q and h_e-h_w, or on the basis of head measurements at any arbitrary radii, r_1 and r_2, using observation wells. The theory of steady-state flow to a well as developed here is often referred to as the Thiem theory, after G. Thiem, who contributed to its development (Thiem, 1906).

While it would not be common, in practice, to find a well conveniently located at the center of a circular island, the example is a very useful one. The hydraulic operation of any well is similar, in many important respects, to that of the well on the island. In particular, the decrease in cross-sectional area of flow as the well is approached, leading to the logarithmic "cone of depression" in the potentiometric surface, is a feature of every discharging well problem. It is in fact the dominant feature of such problems, since the head losses close to the well, within this "cone of depression" are normally the largest head losses associated with the operation of a well. The radial symmetry assumed in the Thiem analysis usually prevails, at least in the area close to the well, in most discharging well problems.

Readers familiar with differential equations will note that the equations of radial flow developed here can be obtained more directly by separating variables in the differential equation

$$\frac{Q}{2\pi br}=K\frac{dh}{dr},$$

and integrating between the limits r_1 and r_2, or r_w and r_e. That is, these radial-flow equations, which state that head will vary with the logarithm of radial distance, are actually solutions to this differential equa-

3 □ —Con.

tion; if they are differentiated with respect to r, the differential equation is obtained. Again readers familiar with the general concepts of potential theory will recognize the pattern of head loss around the well as an example of the "logarithmic potential" associated with potential-flow problems involving cylindrical symmetry in other branches of physics.

You have completed Part III. You may go on to Part IV.

4 □

Your answer in Section 9,

$$h = h_0 - \frac{2Q}{Kw}x$$

is not correct. If we differentiate this equation, treating h_0 as a constant, we obtain the result

$$\frac{dh}{dx} = -\frac{2Q}{Kw}$$

which is not the differential equation we developed for the problem. Keep in mind that in order to find a solution to the differential equation

$$\frac{d(h^2)}{dx} = -\frac{2Q}{Kw}$$

we must find an expression which will yield this equation upon differentiation.

Return to Section 9 and choose another answer.

5 □

Your answer in Section 8 is not correct. The differential equation tells us that any solution we obtain, giving h as a function of x, must be such that the derivative of h with respect to x, dh/dx is a constant, $-(Q/KA)$. Thus we know that (1) since the derivative is a constant (does not involve x), the plot of h versus x for any solution must have a constant slope—that is, the plot must be a straight line; and (2) since the constant has the same value for any solution, the graphs of different or distinct solutions must all have the same slope—that is, these plots must be parallel straight lines. A family of curves all intersecting the x axis at a common point, as in the answer which you chose, could not have these characteristics.

Return to Section 8 and choose another answer.

6 □

Your answer in Section 41 is not correct. The direction of flow in this problem is radial, toward the well as an axis. The cross-sectional area of flow must be taken at right angles to this radial flow direction; that is, it must be a cylindrical surface within the aquifer having the centerline of the well as its axis. At a radial distance r from the well, the cross-sectional area of flow will be the area of a cylindrical surface of radius r and of height equal to the thickness of the aquifer.

Return to Section 41 and select another answer.

7

Your answer, $y=Kx+5$, in Section 1 is correct; of the three expressions given, it is the only one which yields $dy/dx=K$ upon differentiation. However, $y=Kx+5$ is obviously not the only equation which will give this result upon differentiation. For example, differentiation of the equations $y=Kx+7$, $y=Kx-3$, or $y=Kx$ will also yield $dy/dx=K$. The constant term which is added or subtracted on the right does not affect the differentiation; regardless of the value of the constant, the derivative of y with respect to x always turns out to be K. Since we have an infinite choice of constants to add or subtract, there are an infinite number of algebraic equations which qualify as solutions to our differential equation. This is a general characteristic of differential equations—the solutions to a differential equation are always infinite in number.

QUESTION

Given the following three algebraic equations relating head, h, to distance, x.

(a) $h=-\dfrac{Q}{KA}x$

(b) $h=h_0-\dfrac{Q}{KA}x$

(c) $h=h_0-\dfrac{Q}{KA}x^2+7$

where h_0, Q, K, and A are constants; which of the equations are solutions to the differential equation

$$\frac{Q}{A}=-K\frac{dh}{dx}\ ?$$

	Turn to Section:
all three	29
only (a)	14
(a) and (b) but not (c)	8

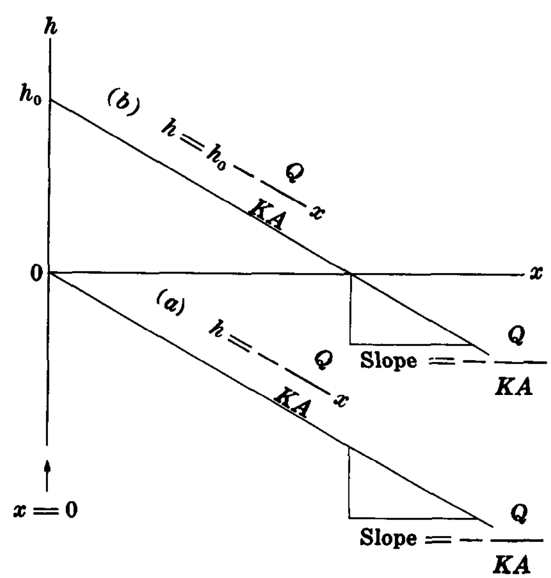
8

Your answer in Section 7 is correct. Either (a) or (b), when differentiated and rearranged, will yield the equation

$$\frac{Q}{A}=-K\frac{dh}{dx}.$$

Differentiation of (c) leads to an entirely different equation.

In the preceding example, the algebraic equations deal with values of hydraulic head, h, at various distances from some reference point; while the differential equation deals with the rate of change of head with distance. The differential equation is, of course, Darcy's law and states that if head is plotted versus distance, the slope of the plot will be constant—that is, the graph will be a straight



8 □ —Con.

line. The graphs of equations (a) and (b) of Section 7 are shown in the diagram. Each is a straight line having a slope equal to

$$-\frac{Q}{KA};$$

the intercept of equation (a) on the h axis is $h=0$, while the intercept of equation (b) on the h axis is $h=h_0$. These intercepts give the values of h at $x=0$; they provide the reference points from which changes in h are measured.

QUESTION

If we were to graph all possible solutions to the differential equation

$$\frac{dh}{dx}=-\frac{Q}{KA},$$

the result would be:

Turn to Section:

A family of curves, infinite in number, each intersecting the x axis at

$$x=-\frac{Q}{KA}$$ 5

An infinite number of parallel straight lines, all having a slope

$$-\frac{Q}{KA},$$

and distinguished by different intercepts on the $x=0$ axis. 10

A finite number of parallel straight lines, all having a slope

$$-\frac{Q}{KA},$$

which intersect the $x=0$ axis at various positive values of h. 20

9 □

Your answer in Section 25,

$$Q=-Kwh\frac{dh}{dx},$$

is correct. From the rules of differentiation, the derivative of h^2 with respect to x is given by

$$\frac{d(h^2)}{dx}=2h\frac{dh}{dx}.$$

Therefore, substituting

$$\frac{1}{2}\frac{d(h^2)}{dx}$$

for $h(dh/dx)$ in the equation

$$Q=-Kwh\frac{dh}{dx}$$

and rearranging, we have

$$\frac{d(h^2)}{dx}=\frac{-2Q}{Kw}.$$

In this rearranged form, the differential equation states that the derivative of h^2 with respect to x must equal the constant term

$$\frac{-2Q}{Kw}.$$

QUESTION

Which of the following expressions, when differentiated, yields the above form of the differential equation—that is, which of the following expressions constitutes a solution to the differential equation? (h_0 is a constant, representing the value of h at $x=0$.)

Turn to Section:

$$h^2=h_0^2-\frac{2Q}{Kw}x^2$$ 16

$$h^2=h_0^2-\frac{2Q}{Kw}x$$ 41

$$h=h_0-\frac{2Q}{Kw}x$$ 4

□ *10*

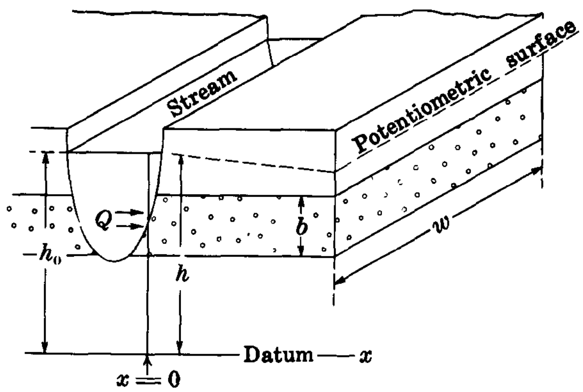
Your answer in Section 8 is correct. Any straight line having the slope

$$-\frac{Q}{KA}$$

will be the graph of a solution to the differential equation

$$\frac{dh}{dx}=-\frac{Q}{KA}.$$

There are an infinite number of lines which may have this slope, corresponding to the infinite number of solutions to the differential equation.



The figure shows a confined aquifer of thickness b. The aquifer is completely cut by a stream, and seepage occurs from the stream into the aquifer. The stream level stands at an elevation h_0 above the head datum, which is an arbitrarily chosen level surface. The direction at right angles to the stream is denoted the x direction, and we take x as 0 at the edge of the stream. We assume that the system is in steady state, so that no changes occur with time. Along a reach of the stream having length w, the total rate of seepage loss from the stream (in, say, cubic feet per second) is denoted $2Q$. We assume that half of this seepage occurs through the right bank of the stream, and thus enters the part of the aquifer shown in our sketch. This seepage then moves away from the stream in a steady flow along the x direction. The resulting distribution of hydraulic head within the aquifer is indicated by the dashed line marked "potentiometric surface" in the sketch. This surface, sometimes referred to as the "piezometric surface," actually traces the static water levels in wells or pipes tapping the aquifer at various points. The differential equation applicable to this problem is obtained by applying Darcy's law to the flow, Q, across the cross-sectional area, bw, and may be written

$$\frac{dh}{dx}=-\frac{-Q}{Kbw},$$

where K is the hydraulic conductivity of the aquifer. The head distribution—that is, the potentiometric surface—is described by one of the solutions to this differential equation. In addition to satisfying the differential equation, the required solution must yield the correct value of h at the edge of the stream—that is, at $x=0$.

QUESTION

Which of the following expressions gives the particular solution (to the above differential equation) which applies to the problem described in this section?

	Turn to Section:
$h=-\frac{Q}{Kbw}x$	22
$h=2Q-\frac{Q}{Kbw}x$	36
$h=h_0-\frac{Q}{Kwb}x$	24

11 □

Your answer in Section 27 is not correct. The decrease in radius does not compensate for the decrease in cross-sectional area; it is, rather, the cause of this decrease in cross-sectional area. The decreasing cross-sectional area, along the path of flow, is a fundamental characteristic of the problem we are considering. It has a major—in fact, dominant—effect upon the solution to the problem.

Return to Section 27 and choose another answer.

12 □

Your answer in Section 41 is not correct. The flow of water is directed radially inward toward the well. Any cross-sectional area of flow, taken normal to this radial direction of movement, would be a cylindrical surface in the aquifer, having the centerline of the well as its axis. The area of flow at a radial distance r from the well would thus be the area of a cylindrical surface of radius r, having a height equal to the thickness of the aquifer.

Return to Section 41 and choose another answer.

13 □

Your answer in Section 35,

$$(\ln r)\frac{dh}{dr}=\frac{Q}{2\pi Kb},$$

is not correct. The differential equation as given in Section 35 was

$$r\frac{dh}{dr}=\frac{Q}{2\pi Kb}.$$

In your answer, $\ln r$ has simply been substituted for r. This is obviously not what we want; $\ln r$ is not equal to r. The relations given in Section 35 can be used to obtain an expression which is equivalent to dh/dr. This expression can then be substituted for dh/dr in the above differential equation to obtain the required result.

Return to Section 35 and choose another answer.

14 □

Your answer in Section 7 is not correct. It is true that expression (a),

$$h=-\frac{Q}{KA}x,$$

yields the result

$$\frac{dh}{dx}=-\frac{Q}{KA}$$

upon differentiation and is thus a solution to the given equation. However, it is not the only one of the given expressions which yields the required result upon differentiation.

Return to Section 7 and test the remaining expressions, by differentiation, in order to find the correct answer.

15

Your answer, $y=Kx^2$, in Section 1 is not correct. If we differentiate the equation $y=Kx^2$, we obtain

$$\frac{dy}{dx}=2Kx,$$

which is not the differential equation with which we started. Our differential equation was

$$\frac{dy}{dx}=K,$$

and we are looking for a solution to this differential equation—that is, we are looking for an algebraic expression which, when differentiated, will produce the differential equation $(dy/dx)=K$.

Return to Section 1 and test the remaining choices, by differentiating them, to see which will yield the given differential equation.

16

Your answer in Section 9,

$$h^2=h_0^2-\frac{2Q}{Kw}x^2,$$

is not correct. If we differentiate this answer, treating h_0^2 as a constant, we obtain

$$\frac{d(h^2)}{dx}=-\frac{2Q}{Kw}\cdot 2x,$$

since the derivative of x^2 with respect to x is $2x$. This result is not the differential equation with which we started, so the equation of your answer is not the solution we require.

Return to Section 9 and choose another answer. Keep in mind that the equation you select must yield the result

$$\frac{d(h^2)}{dx}=-\frac{2Q}{Kw}$$

when it is differentiated.

17

Your answer in Section 40,

$$\frac{Q}{2\pi rb}=K\frac{d(h^2)}{dr},$$

is not correct. Darcy's law states that flow, divided by cross-sectional area, must be proportional to the head gradient. Your answer states that flow, divided by cross-sectional area, is proportional to the gradient of the square of head. Thus it cannot be a valid application of Darcy's law to the problem.

Return to Section 40 and choose another answer.

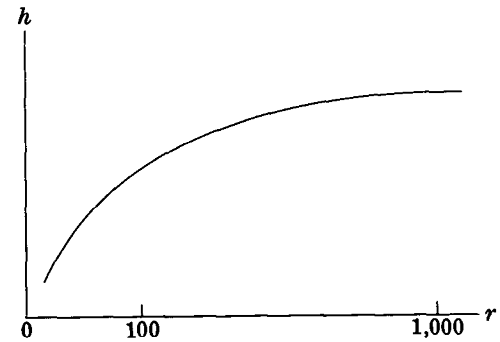
18

Your answer in Section 2 is not correct. The equation in Section 2 states that the derivative of head with respect to ln r is a constant. This derivative is simply the slope of a plot of h versus ln r. If such a plot changes slope, as in the answer you chose, the derivative cannot be constant.

Return to Section 2 and choose another answer.

19 □

Your answer in Section 38 is correct; inasmuch as log r changes by the same amount between 10 and 1 as it does between 1,000 and 100, the head changes by the same amount in these two intervals. If we were to replot head directly versus radius, r, rather than versus log r, we would no longer have a straight line, but rather a "logarithmic" curve, as shown in the sketch. The gradient becomes progressively steeper as we approach the well, to compensate for the decreasing cross-sectional area of flow. This logarithmic pattern of head decline is sometimes referred to as the "cone of depression" in the potentiometric surface around the well.



Arithmetic scale

QUESTION

The equation obtained in Section 38 can be applied between the radius of the island, r_e, and the radius of the well, r_w, to obtain an expression for the head difference between the well and edge of the island. If h_e represents the head at the edge of the island (that is, the level of the open water surrounding the island) and h_w represents the head in the well which of the following expressions would result from this procedure?

	Turn to Section:
$h_e - h_w = \frac{2.3Q}{2\pi Kb} \log \frac{r_w}{r_e}$	28
$h_e - h_w = \frac{2.3Q}{2\pi Kb} \log \frac{r_e}{r_w}$	3
$h_e - h_w = \frac{2.3Q}{2\pi Kb} (\log r_w - \log r_e)$	30

20 □

Your answer in Section 8 is not correct. If we were to write the solution to the equation

$$\frac{Q}{A} = -K\frac{dh}{dx}$$

in the most general form, we would write

$$h = -\frac{Q}{KA}x + c$$

where c could represent any constant term we wish. No matter what value we assign c, so long as it is constant (not dependent on x) its derivative with respect to x will be zero. Thus regardless of the value of c, differentiation will yield the result

$$\frac{dh}{dx} = -\frac{Q}{KA}$$

which is equivalent to our given differential equation. Clearly we can assign an infinite number of values to the term c, and obtain an infinite number of distinct equations (solutions) which we can differentiate to obtain our differential equation. Each of these solutions is the equation of a straight line; that is, each has a slope, dh/dx, equal to $-(Q/KA)$, and each has a distinct intercept on the h axis, where $x=0$. This intercept is simply the value of the constant c, since if we set $x=0$ in the solution we obtain $h=c$.

Return to Section 8 and choose another answer.

□ 21

Your answer in Section 24 is not correct. According to Darcy's law, the specific discharge, Q/A, is given by

$$\frac{Q}{A} = -K\frac{dh}{dx}.$$

If the specific discharge increases as the stream is approached, the head gradient dh/dx must also increase—that is, become steeper—as the stream is approached. A plot of h versus distance would thus be some sort of curve. In the statement of the problem in Section 24, however, head was described as increasing *linearly* with distance away from the stream. Since head increases in a linear fashion, dh/dx is constant.

Return to Section 24 and choose another answer.

□ 22

Your answer in Section 10,

$$h = -\frac{Q}{Kbw}x,$$

is not correct. It is true that differentiation of this equation yields the result

$$\frac{dh}{dx} = -\frac{Q}{Kbw}$$

which is our given differential equation; but this in itself is not enough to make it the answer to our problem. If we set x equal to zero in the expression

$$h = -\frac{Q}{Kwb}x,$$

we obtain the result $h=0$. That is, this equation says that where x is zero, at the edge of the stream, hydraulic head is also zero. According to the statement of our problem, however, head is equal to h_0, the elevation of the stream surface above datum, at $x=0$. The solution which we require must not only have the property of yielding the given differential equation

$$\frac{dh}{dx} = -\frac{Q}{Kbw}$$

when it is differentiated; it must also have the property that when x is set equal to zero in the solution, hydraulic head will be h_0. This is an example of what is meant by a *boundary condition;* the solution must satisfy a certain condition ($h=h_0$) along a certain boundary ($x=0$) of the problem.

Return to Section 10 and choose another answer.

□ 23

Your answer, $x=2y+K$, in Section 1 is not correct. We can rearrange the equation you selected as follows

$$y = \tfrac{1}{2}x - \frac{K}{2}.$$

Now if we differentiate this equation, we obtain

$$\frac{dy}{dx} = \tfrac{1}{2},$$

which is not the differential equation with which we started. We were asked to find a solution to the differential equation

$$\frac{dy}{dx} = K;$$

that is, we were asked to find an algebraic equation which, when differentiated, would yield the result $dy/dx=K$.

Return to Section 1 and test the remaining answers by differentiation, to see which one satisfies this condition.

24 □

Your answer in Section 10,

$$h = h_0 - \frac{Q}{Kbw}x$$

is correct. The differential equation tells us that a plot of h versus x will be a straight line with slope

$$-\frac{Q}{Kbw};$$

while from the other information given, we know that at $x=0$, h is equal to h_0. Thus, to describe h as a function of x we require the equation of a straight line, with h_0 as the intercept and $-(Q/Kbw)$ as the slope. We can make two tests to verify that we have obtained the correct solution; first, we may differentiate the solution with respect to x, to see whether we obtain the differential equation; second, we may let x equal 0 in the solution to see whether the condition that h is h_0 at $x=0$ is satisfied. Only if our equation meets both of these tests is it the solution we require. The condition that h must be h_0 at $x=0$ is an example of what is commonly termed a *boundary condition;* it is a condition which states that h must have a certain value along one or another of the boundaries of our problem. The differential equation,

$$\frac{dh}{dx} = -\frac{Q}{Kbw},$$

is in itself insufficient to define head as a function of x. It establishes that the graph of h versus x will be a straight line with slope

$$-\frac{Q}{Kbw},$$

but there are an infinite number of such straight lines which we might draw. The additional information given by the boundary condition—that h must be h_0 at $x=0$—permits us to pick out the particular straight line we require, by giving us its intercept. A boundary condition is thus a bit of information on the value of head at a known point; it provides a reference from which the changes in head indicated by a differential equation may be measured. The processes of (1) differentiation to establish that a given equation is a solution to a differential equation and (2) application of boundary conditions to establish that it is the particular solution that we require may be applied to problems much more complex than the one we have considered here.

QUESTION

Suppose that, in measuring observation wells tapping a confined aquifer, we observe a linear increase in head with distance away from a stream or channel which cuts completely through the aquifer; and suppose this pattern remains unchanged through a considerable period of time. Which of the following conclusions could we logically draw on the basis of this evidence?

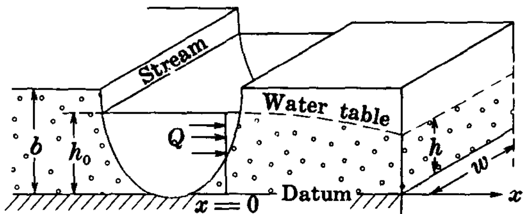
	Turn to Section:
There is no flow within the aquifer.	42
There is a steady flow through the aquifer into the stream.	25
A flow which increases in specific discharge as one approaches the stream occurs in the aquifer.	21

25 □

Your answer in Section 24 is correct. This serves to illustrate the dual utility of flow equations in ground-water hydraulics—they enable us to predict the head distributions associated with various conditions of flow and they enable us to draw conclusions regarding ground-water flow on the basis of head distributions observed in the field.

Suppose we now consider an aquifer in which the flow is unconfined, so that the upper limit of the flow system at any point is the water surface, or water table, itself. Again we consider uniform flow away from a stream, as shown in the diagram. It is convenient in this case to take the base of the unconfined aquifer as our head datum. We

Con.— □ 25



assume that vertical components of flow are negligible. This assumption is never wholly satisfied, as movement cannot be entirely lateral in and near the free surface, owing to the slope of the surface itself. Frequently, however, the vertical velocity component is slight compared to the lateral and therefore can be neglected, as we are doing here. An important difference between this problem and the confined-flow problem is that here the cross-sectional area of flow diminishes along the path of flow, as h decreases, whereas in the confined problem it remains constant.

Along a reach of the stream having a length w, seepage into the aquifer occurs at a rate $2Q$; and we assume that half of this seepage moves to the right, into the part of the aquifer shown in the sketch.

QUESTION

According to the assumptions outlined above, which of the following relations is obtained by applying Darcy's law to this problem?

	Turn to Section:
$Q = -Kxw\dfrac{dh}{dx}$	26
$\dfrac{Q}{bw} = -K\dfrac{dh}{dx}$	43
$Q = -Kwh\dfrac{dh}{dx}$	9

□ 26

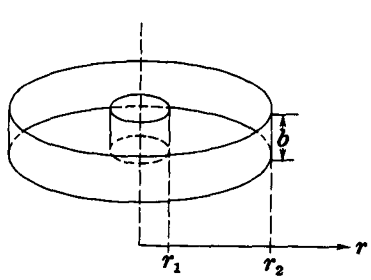
Your answer in Section 25,

$$Q = -Kxw\frac{dh}{dx},$$

is not correct. Darcy's law states that the flow is the product of the hydraulic conductivity, the cross-sectional area of flow, and the (negative) head gradient. Referring to the diagram of Section 25, the cross-sectional area of the flow—that is, the cross-sectional area taken at right angles to the direction of movement—can be seen to be equal to wh. In the answer which you chose, the term xw appears as the area of flow.

Return to Section 25 and choose another answer.

□ 27



Your answer, $2\pi rb$, in Section 41 is correct. The flow is radially inward in the (negative) r direction—that is, parallel to the r axis of polar coordinates. The cross-sectional area of flow is a surface which is everywhere normal to this direction of flow; hence it is a cylindrical surface, and its area is given by the expression for the area of a cylinder.

As we proceed inward along the path of flow in this problem, the cylindrical area of flow becomes smaller and smaller, as illustrated in the sketch. This is also evident from our expression for the cross-sectional area, which tells us that as r decreases, the area must decrease.

27 □ —Con.

QUESTION

Which of the following statements is correct?

Turn to Section:

(a) Although cross-sectional area is decreasing, radius is also decreasing. These factors combine in such a way that the hydraulic gradient remains constant. 11

(b) Cross-sectional area decreases along the path of flow, while discharge remains constant; therefore, the hydraulic gradient must increase along the path of flow. 40

(c) Cross-sectional area of flow decreases along the path of flow, but this is offset by convergence of the flowlines toward the well, and no increase in the hydraulic gradient occurs. 32

28 □

Your answer in Section 19,

$$h_e - h_w = \frac{2.3Q}{2\pi Kb} \log \frac{r_w}{r_e},$$

is not correct. If we let h_e and r_e be represented by h_2 and r_2, and if we let h_w and r_w be represented by h_1 and r_1, your answer can be restated in the form

$$h_2 - h_1 = \frac{2.3Q}{2\pi Kb} \log \frac{r_1}{r_2}.$$

Comparison with the equations in Section 38 will show that this is not the form which we require.

Return to Section 19 and choose another answer.

29 □

Your answer in Section 7 is not correct. The given differential equation

$$\frac{Q}{A} = -K\frac{dh}{dx}$$

can be rearranged to

$$\frac{dh}{dx} = -\frac{Q}{KA}.$$

In order for all three of the given expressions to be solutions to this equation, all three would have to yield $-(Q/KA)$ as the derivative of h with respect to x. But if we differentiate expression (c), for example, which was

$$h = h_0 - \frac{Q}{KA}x^2 + 7,$$

we obtain

$$\frac{dh}{dx} = \frac{-2Q}{KA}x,$$

which is not the given differential equation. Thus we can see that at least expression (c) does not satisfy the given equation.

Return to Section 7 and test the remaining expressions, by differentiation, in order to find the correct answer.

30

Your answer in Section 19,

$$h_e - h_w = \frac{2.3Q}{2\pi Kb}(\log r_w - \log r_e),$$

is not correct. The term log r_e will obviously be greater than log r_w, since r_e is much greater than r_w. Thus the expression on the right in your answer will be negative, implying that h_w is greater than h_e. This does not make sense; the head in a discharging well cannot be greater than the head at the radius of influence of the well.

Return to Section 19 and choose another answer.

31

Your answer in Section 2 is not correct. If equal changes in head were observed over intervals representing equal changes in r, we could write

$$\frac{\Delta h}{\Delta r} = \text{constant}$$

where Δh is the change in head which is always observed over any interval of radial width Δr. In derivative form this would be

$$\frac{dh}{dr} = \text{constant},$$

and this is not the condition which has been shown to apply to this problem. The condition our plot must satisfy, rather, is

$$\frac{dh}{d(\ln r)} = \text{constant}.$$

Return to Section 2 and choose another answer.

32

Your answer in Section 27 is not correct. The convergence of flowlines toward the well does not compensate for the decrease in flow area; it is, rather, caused by this decrease in flow area. The decrease in flow area as the well is approached is a fundamental characteristic of the discharging well problem; in effect the decreasing flow area has a dominant influence on the form of the head distribution around the well.

Return to Section 27 and select another answer.

33

Your answer in Section 40,

$$\frac{Q}{A} = K\frac{dh}{dx},$$

is not correct. The x coordinate was not used in our analysis of this problem; we did not set up an x axis along which head could vary. The answer which you selected involves a derivative of head with respect to x and thus cannot apply to our problem.

Return to Section 40 and choose another answer.

34 □

Your answer in Section 38 is not correct. The equation

$$h_2-h_1=\frac{2.3Q}{2\pi Kb}\cdot\log\frac{r_2}{r_1}$$

indicates that if the ratio r_2/r_1—that is, the ratio of the outer radius to the inner radius—is the same for two different intervals, then the head drops across those intervals must be equal. For the two intervals mentioned in the answer which you chose, these ratios are 10/1 and 1000/100.

Return to Section 38 and choose another answer.

35 □

Your answer in Section 40 is correct. The hydraulic gradient here is dh/dr, since flow is in the r direction. We assume radial symmetry around the well, so that the angular polar coordinate, θ, need not appear at all. We now rewrite the equation which you selected in the form:

$$r\frac{dh}{dr}=\frac{Q}{2\pi Kb},$$

and we focus our attention for a moment on the left-hand member. According to the rules of differentiation we may write:

$$\frac{dh}{dr}=\frac{dh}{d(\ln r)}\cdot\frac{d(\ln r)}{dr}$$

where ln r denotes the natural logarithm of r; and we may recall from introductory calculus that the derivative of ln r with respect to r is given by

$$\frac{d(\ln r)}{dr}=\frac{1}{r}$$

QUESTION

Using these expressions, which of the following may be obtained as a correct restatement of the differential equation for the problem?

	Turn to Section:
$\frac{dh}{dr}=\frac{Q(\ln r)}{2\pi Kb}$	39
$\frac{dh}{d(\ln r)}=\frac{Q}{2\pi Kb}$	2
$(\ln r)\frac{dh}{dr}=\frac{Q}{2\pi Kb}$	13

36 □

Your answer in Section 10,

$$h=2Q-\frac{Q}{Kwb}x,$$

is not correct. This answer is indeed a solution to our differential equation, for when we differentiate it we obtain the differential equation

$$\frac{dh}{dx}=-\frac{Q}{Kbw}.$$

However, if we set x equal to zero in the answer which you chose, we find that hydraulic head, h, is equal to $2Q$ at the point where x is zero—that is, at the edge of the stream. In the discussion of Section 10, however, it was stated that hydraulic head was equal to h_0 at the edge of the stream—h_0 being the elevation of the stream surface above datum. This problem illustrates what is meant by the term *boundary condition;* the solution must satisfy a condition along one boundary ($h=h_0$ at $x=0$) in addition to satisfying the given differential equation. There are an infinite number of possible solutions to the above differential equation, but only one which satisfies this required boundary condition.

Return to Section 10 and choose another answer.

37

Your answer in Section 38 is not correct. If the equation

$$h_2-h_1=\frac{2.3Q}{2\pi Kb}\log\left(\frac{r_2}{r_1}\right)$$

is applied to the two intervals in question, we have

$$h_{10}-h_1=\frac{2.3Q}{2\pi Kb}\log\left(\frac{10}{1}\right)=\frac{2.3Q}{2\pi Kb}\cdot 1$$

and

$$h_{100}-h_{10}=\frac{2.3Q}{2\pi Kb}\log\left(\frac{100}{10}\right)=\frac{2.3Q}{2\pi Kb}\cdot 1.$$

Return to Section 38 and choose another answer.

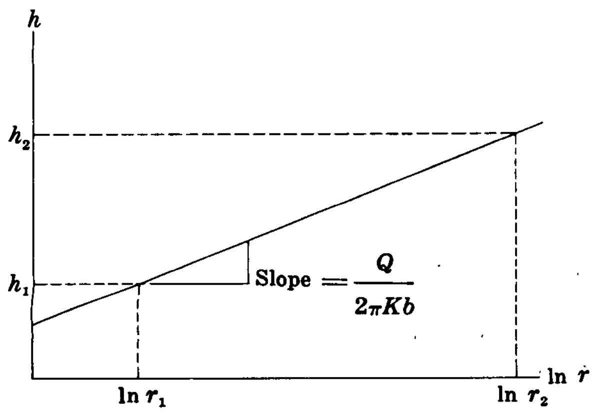
38

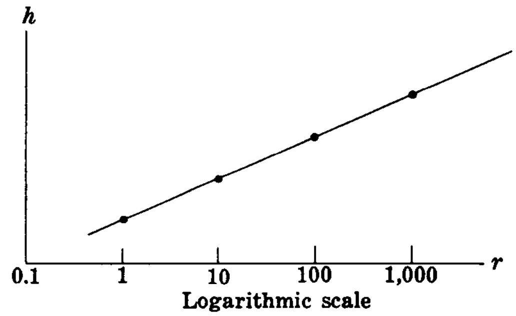
Your answer in Section 2 is correct. The equation states that the derivative of h with respect to $\ln r$ is a constant. Thus a graph of h versus $\ln r$ will be a straight line, which will have a slope equal to

$$\frac{Q}{2\pi Kb}.$$

The sketch shows such a graph. As $\ln r$ changes from $\ln r_2$ to $\ln r_1$, head decreases from h_2 to h_1; and as with any straight line function, the change in head can be obtained by multiplying the change in the independent variable by the slope of the line; that is,

$$h_2-h_1=\frac{Q}{2\pi Kb}(\ln r_2-\ln r_1).$$





This can be written in the equivalent form

$$h_2-h_1=\frac{Q}{2\pi Kb}\ \ln\frac{r_2}{r_1}$$

inasmuch as the difference between $\ln r_2$ and $\ln r_1$ is simply the log of the quotient $\ln\ (r_2/r_1)$. At this point it is convenient to change from natural logs to common logs. This involves only multiplication by a constant—that is $\ln r=2.3 \log r$, where $\log r$ denotes the common logarithm, or log to the base 10. Making this change, our equation takes the form

$$h_2-h_1=\frac{2.3Q}{2\pi Kb}\log\left(\frac{r_2}{r_1}\right)$$

or

$$h_2-h_1=\frac{2.3Q}{2\pi Kb}(\log r_2-\log r_1).$$

Again a graph can be plotted of h versus $\log r$—or, to do the same thing more con-

38 □ —Con.

veniently, a graph can be plotted of h versus r on semilog paper, as shown in the sketch. Since we have only multiplied by a constant, the graph remains a straight line.

QUESTION

On the basis of the graph shown in the figure and the equations given above, which of the following statements is correct?

Turn to Section:

(a) The head drop between $r=10$ and $r=1$ is equal to that between $r=1{,}000$ and $r=100$. 19

(b) The head drop between $r=10$ and $r=1$ is less than that between $r=1{,}000$ and $r=100$. 34

(c) The head drop between $r=10$ and $r=1$ is much greater than that between $r=100$ and $r=10$. 37

39 □

Your answer in Section 35,

$$\frac{dh}{dr}=\frac{Q(\ln r)}{2\pi Kb},$$

is not correct. The following relations were given in Section 35:

$$\frac{dh}{dr}=\frac{dh}{d(\ln r)}\cdot\frac{d(\ln r)}{dr}$$

and

$$\frac{d(\ln r)}{dr}=\frac{1}{r}.$$

Combining these,

$$\frac{dh}{dr}=\frac{1}{r}\cdot\frac{dh}{d(\ln r)}.$$

In the question of Section 35, the idea is to substitute the term

$$\frac{1}{r}\cdot\frac{dh}{d(\ln r)}$$

for the term

$$\frac{dh}{dr}$$

in the differential equation for our problem.

Return to Section 35 and choose another answer.

40 □

Your answer in Section 27 is correct. The decrease in cross-sectional area must, according to Darcy's law, be accompanied by a steepening of the hydraulic gradient. When we apply Darcy's law to this problem, we will omit the customary negative sign. This is done because Q, the well discharge, must itself carry a negative sign in this problem, since it is oriented toward the well, in the direction of decreasing values of r. The negative sign on Q combines with the negative sign used by convention in Darcy's law to yield an equation in positive terms.

QUESTION

Which of the following expressions is a valid application of Darcy's law to this problem, and hence a valid differential equation for the problem?

Turn to Section:

$$\frac{Q}{A}=K\frac{dh}{dx}$$ 33

$$\frac{Q}{2\pi rb}=K\frac{dh}{dr}$$ 35

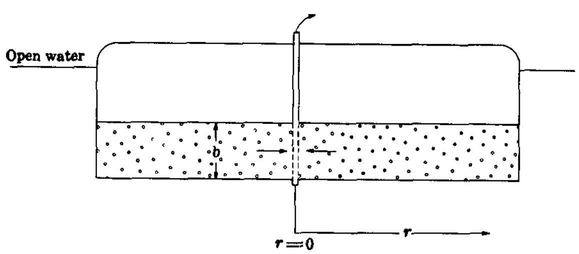
$$\frac{Q}{2\pi rb}=K\frac{d(h^2)}{dr}$$ 17

□ 41

Your answer in Section 9,

$$h^2 = h_0^2 - \frac{2Q}{Kw}x,$$

is correct. The solution indicates that h will have the form of a parabola when plotted versus x in this case. The parabolic steepening of the hydraulic gradient compensates for the progressive decrease in flow area, in such a way that Darcy's law is always satisfied. This approximate theory of unconfined flow was introduced by Dupuit (1863) and the assumptions involved in it are frequently referred to as the Dupuit assumptions. If the method is used in cases where these assumptions do not apply, serious errors can be introduced.



We next consider another problem in which the cross-sectional area of flow diminishes along the path of flow, leading to a progressive steepening of the hydraulic gradient. In this case, however, the decrease in area is generated by cylindrical geometry rather than by the slope of a free surface.

The figure shows a well located at the center of a circular island. The well taps a confined aquifer which is recharged by the open water around the perimeter of the island. During pumping, water flows radially inward toward the well. We assume that the open water around the island maintains the head at a constant level along the periphery of the aquifer and that the recharge along this periphery equals the well discharge. Since the well is at the center of the island and the island is circular, we can assume that cylindrical symmetry will prevail; we can therefore introduce polar coordinates to simplify the problem.

QUESTION

If b represents the thickness of the aquifer, which of the following expressions represents the cross-sectional area of flow at a radial distance r from the axis of the well?

	Turn to Section:
$2\pi rb$	27
$\pi r^2 b$	12
$2\pi r^2$	6

□ 42

Your answer in Section 24 is not correct. The statement that there is a linear increase in head with distance away from the stream implies that there is a *non-zero* slope, dh/dx, in the potentiometric surface, and this in turn implies that flow exists in the aquifer. Darcy's law states that

$$Q = -KA\frac{dh}{dx}.$$

Hydraulic conductivity, K, may be very low, but cannot be considered equal to zero as long as we are dealing with an aquifer in the normal sense of the word. Thus in order for Q to be zero, through a given area A, the head gradient dh/dx normal to A must be zero. In this case we have observed a head gradient which is not zero in the aquifer, so we know that flow of some magnitude must exist in the aquifer.

Return to Section 24 and choose another answer.

43 □

Your answer in Section 25,

$$\frac{Q}{bw} = -K\frac{dh}{dx},$$

is not correct. You have taken the cross-sectional area of flow to be bw—that is, the product of aquifer thickness and width of section. An examination of the figure in Section 25 will show that this does not represent the actual area of flow. The aquifer is not saturated through its full thickness, but rather to a distance h above the base of the aquifer. Thus, the cross-sectional area of flow is wh, rather than bw.

Return to Section 25 and choose another answer.