Techniques of Water-Resources Investigations of the United States Geological Survey

Chapter B2

INTRODUCTION TO GROUND-WATER HYDRAULICS

A Programed Text for Self-Instruction

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Book 3

APPLICATIONS OF HYDRAULICS

Part VIII. Analog Techniques

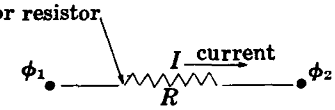
Introduction

In Part VIII we consider another technique of obtaining solutions to the differential equation of ground-water flow. This is the method of the electric analog. It is a powerful technique which has been widely used. The technique depends upon the mathematical similarity between Darcy's law, describing flow in a porous medium, and Ohm's law, describing flow of charge in a conductor. In the case of nonequilibrium modeling, it depends also upon the similarity between the ground-water storage–head relation and the equation describing storage of charge in a capacitor; and upon the similarity between the electrical continuity principle, involving the conservation of electric charge, and the equation of continuity describing the conservation of matter.

1

Ohm's law states that the electrical current through a conducting element is directly proportional to the voltage difference, or potential difference across its terminals. The sketch represents a conducting element, or resistor, across which the voltage difference is $\phi_1-\phi_2$. That is, the voltage at one terminal of the resistor is ϕ_1, while that at the other end is ϕ_2. The current through the resistor is defined as the net rate of movement of positive charge across a cross-sectional plane within the resistor, taken normal to the direction of charge flow. The standard unit of charge is the coulomb, and current is normally measured as the number of coulombs per second crossing the plane under consideration. A charge flow of 1 coulomb per second is designated 1 ampere. The symbol I is frequently used to represent current.

Symbol representing a conducting element, or resistor



R represents value of resistance (ohms)

For the resistor shown in the diagram, Ohm's law may be stated as follows

$$I=\frac{1}{R}(\phi_1-\phi_2)$$

where I is the current through the resistor, and $\phi_1-\phi_2$, as noted above, is the voltage difference across its terminals. The term $1/R$ is the constant of proportionality relating current to voltage; R is termed the resistance of the element. It depends both upon the dimensions of the element and the electrical properties of the conductive material used. The unit of resistance is the ohm. A resistance of 1 ohm will carry 1 ampere of current under a potential difference of 1 volt.

QUESTION

Suppose the voltage at one terminal of a 500-ohm resistor is 17 volts, and the voltage at the other terminal is 12 volts. What would the current through the resistor be?

	Turn to Section:
10 amperes	19
0.10 ampere, or 100 milliamperes	8
0.01 ampere, or 10 milliamperes	6

2 •

Your answer in Section 22 is not correct. The finite-difference form of the equation for two-dimensional nonequilibrium ground-water flow is

$$h_1+h_2+h_3+h_4-4h_0=\frac{Sa^2}{T}\frac{\Delta h_0}{\Delta t},$$

while the equation for our resistance- capacitance network is

$$\phi_1+\phi_2+\phi_3+\phi_4-4\phi_0=RC\frac{d\phi_0}{dt}.$$

Comparison of these equations illustrates that resistance, R, may be considered to be analogous to the term $1/T$; voltage, ϕ, is analogous to head, h; and capacitance, C, may be considered analogous to the term Sa^2.

In the answer which you selected, voltage is treated as analogous to transmissivity, in that the procedure calls for increasing voltage in areas of high transmissivity.

Return to Section 22 and choose another answer.

3 •

Your answer in Section 6,

$$I=\frac{A}{RL}(\phi_1-\phi_2),$$

is not correct. The idea here is to obtain an expression for the current which involves the resistivity, ρ_e, of the material composing the resistance. Your answer involves the resistance, R, rather than the resistivity. It is not a valid statement of Ohm's law in any case, for Ohm's law in terms of resistance was given in Section 1 as

$$I=\frac{1}{R}(\phi_1-\phi_2).$$

Return to Section 6 and choose another answer.

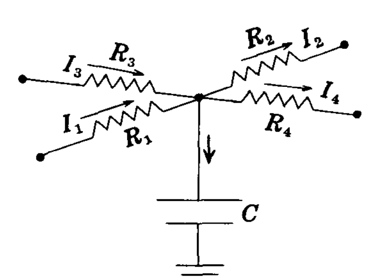
4 •

Your answer in Section 9,

$$\frac{1}{R}(\phi_1-\phi_c)=C\frac{d\phi_c}{dt},$$

is correct. The quantity C, as we have seen, is actually the derivative $d\epsilon/d\phi_c$; thus $C\,(d\phi_c/dt)$ is equivalent to $(d\epsilon/d\phi_c)\cdot(d\phi_c/dt)$, or simply $d\epsilon/dt$.

Without referring to it explicitly, we made use in Section 9 of an electrical equivalent to the hydraulic equation of continuity. In an electric circuit, charge is conserved in the same way that fluid mass is conserved in a hydraulic system. Kirchoff's current law, which is familiar to students of elementary physics, is a statement of this principle. In the circuit of Section 9, we required that the rate of accumulation of charge in the capacitor be equal to the time rate at which charge was transported to the capacitor plate through the resistor—that is, to the current through the resistor. In the circuit



shown in the figure, in which four resistors are connected to a single capacitor, the net

Con.—4

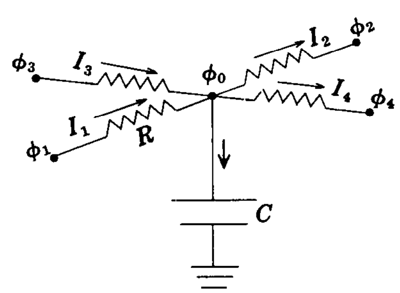
inflow minus outflow of charge, through all four resistors, must equal the rate of accumulation of charge on the capacitor. Let I_1 and I_3 represent currents toward the capacitor, through resistors R_1 and R_3; and let I_2 and I_4 represent currents away from the capacitor, through resistors R_2 and R_4. Then the time rate of inflow of charge, toward the capacitor, will be I_1+I_3; the rate of outflow charge, away from the capacitor, will be I_2+I_4. The net inflow minus outflow of charge will be $I_1-I_2+I_3-I_4$; and this must equal the rate of accumulation of charge on the capacitor. That is, we must have

$$I_1-I_2+I_3-I_4=\frac{d\epsilon}{dt}.$$

QUESTION

The diagram again shows the circuit described above, but we now assume that the four resistances are equal—that is, we assume

$$R_1=R_2=R_3=R_4=R.$$



Let ϕ_0 represent the voltage on the capacitor plate—this is essentially equal to the voltage at the junction point of the four resistors (the resistance of the wire connecting the capacitor to the resistor junction point is assumed negligible). The voltages at the extremities of the four resistors are designated ϕ_1, ϕ_2, ϕ_3, and ϕ_4, as shown in the diagram. If Ohm's law is applied to obtain an expression for the current through each resistor and the capacitor equation is applied to obtain an expression for the rate of accumulation of charge on the capacitor, which of the following equations will be obtained from our circuit equation

$$I_1-I_2+I_3-I_4=\frac{d\epsilon}{dt}?$$

	Turn to Section:
$\dfrac{\phi_1-\phi_2+\phi_3-\phi_4}{R}=C\dfrac{d\phi_0}{dt}$	15
$\dfrac{1}{C}(\phi_1-\phi_2+\phi_3-\phi_4+\phi_0)=R\dfrac{d\phi_0}{dt}$	27
$\phi_1+\phi_2+\phi_3+\phi_4-4\phi_0=RC\dfrac{d\phi_0}{dt}$	22

5

Your answer in Section 22 is correct. This is of course one indication of the power of the analog method, in that problems involving heterogeneous aquifers are handled as easily as those involving a uniform aquifer. Complex boundary conditions can also be accommodated, and three-dimensional problems may be approached by constructing networks of several layers. The method is applicable to water-table aquifers as well as to confined aquifers, provided dewatering is small in relation to total saturated thickness. Some successful simulation has been done even for cases in which this condition is not satisfied, using special electrical components which vary in resistance as voltage changes.

Steady-state problems are sometimes handled by network models constructed solely of resistors—that is, not involving capacitors—rather than by analogs constructed of a continuous conductive mate-

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rial. Such steady-state networks are particularly useful when heterogeneity is involved.

In some cases, the symmetry of a ground-water system may be such that a two-dimensional analog in a vertical plane—that is, representing a vertical cross section through an aquifer, or series of aquifers—may be more useful than a two-dimensional analog representing a map view. In this type of model, anisotropy is frequently a factor; that is, permeability in the vertical direction is frequently much smaller than that in the lateral direction. This is easily accommodated in a network by using higher resistances in the vertical direction; or, equivalently, by using a uniform resistance value but distorting the scales of the model, so that this resistance value is used to simulate different distances and cross-sectional areas of flow in the two directions.

An important special type of network analog is that used to simulate conditions in a vertical plane around a single discharging well. The cylindrical symmetry of the discharging well problem is in effect built into the network; the resistances and scales of the model are chosen in such a way as to simulate the increasing cross-sectional areas of flow, both vertically and radially, which occur in the aquifer with increasing radial distance from the well.

This concludes our discussion of the electric-analog approach. We have given here only a brief outline of some of the more important principles that are involved. The technique is capable of providing insight into the operation of highly complex ground-water systems. Further discussion of the principles of simulation may be found in the text by Karplus (1958). The book "Concepts and Models in Ground-Water Hydrology" by Domenico (1972) contains a discussion of the application of analog techniques to ground water, as does the text "Ground-Water Resource Evaluation" by Walton (1970). Additional discussions may be found in papers by Skibitzke (1960), Brown (1962) Stallman (1963b) Patten (1965), Bedinger, Reed, and Swafford (1970), and many others.

This concludes the studies presented in this text.

6 •

Your answer in Section 1 is correct.

The resistance of an electrical element is given by the formula

$$R=\rho_e\cdot\frac{L}{A}$$

where L is the length of the element in the direction of the current, A is its cross-sectional area normal to that direction, and ρ_e is the electrical *resistivity* of the material of which the resistor is composed. The inverse of the resistivity is termed the conductivity of the material; it is often designated σ; that is, $\sigma=1/\rho_e$. Resistivity and conductivity are normally taken as constants characteristic of a particular material,; however, these properties vary with temperature, and the linear relationships usually break down at extremes of voltage. Moreover, a small change in the composition of some materials can produce a large change in electrical properties. Resistivity is commonly expressed in units of ohm·metre²/metre, or ohm-metres. With this unit of resistivity, the formula,

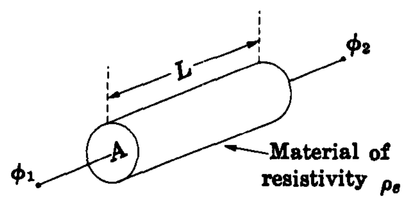
$$R=\rho_e\cdot\frac{L}{A},$$

Con.—6

will yield resistance in ohms if length is expressed in metres and area in square metres.

QUESTION

The sketch shows a resistor of cross-sectional area A and length L, composed of a material of resistivity ρ_e. The potential difference across the resistor is $\phi_1 - \phi_2$. Which of the following expressions is a valid expression of Ohm's law, giving the current through the resistor?



	Turn to Section:
$I = \dfrac{A}{\rho_e L}(\phi_1 - \phi_2)$	28
$I = \dfrac{\rho_e A}{L}(\phi_1 - \phi_2)$	24
$I = \dfrac{A}{RL}(\phi_1 - \phi_2)$	3

7

Your answer in Section 28,

$$Q = \frac{K}{A_p} \cdot \frac{L_p}{h_1 - h_2}$$

is not correct. Darcy's law states that flow is directly proportional to cross-sectional area and to the (negative) gradient of head. In the answer which you chose, flow is given as inversely proportional to cross-sectional area, and proportional to the term $L_p/h_1 - h_2$, which is actually the inverse of the negative head gradient.

Return to Section 28 and choose another answer.

8

Your answer in Section 1 is not correct. Ohm's law was given as

$$I = \frac{1}{R}(\phi_1 - \phi_2),$$

and the discussion pointed out that a resistance of 1 ohm would carry a current of 1 ampere under a potential difference of 1 volt. Thus when the voltage difference is expressed in volts and the resistance ohms, the quotient

$$\frac{\phi_1 - \phi_2}{R}$$

will give the correct current in amperes.

Return to Section 1 and choose another answer.

9

Your answer in Section 21 is correct.

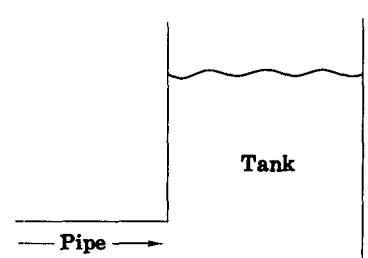
If we monitor the voltage on a capacitor plate in a given circuit and observe that it is changing with time, we know from the relations given in Section 21 that charge is accumulating on the capacitor plate with time. An expression for the rate at which charge is accumulating can be obtained by dividing the capacitor equation by a time increment, Δt. This gives

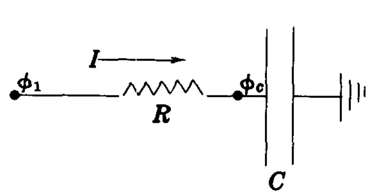
$$\frac{\Delta\epsilon}{\Delta t}=C\frac{\Delta\phi}{\Delta t}$$

or, in terms of derivatives,

$$\frac{d\epsilon}{dt}=C\frac{d\phi}{dt}.$$

The figure shows a hydraulic system and an analogous electrical system. The rate of accumulation of fluid in the tank is equal to the rate of flow of water through the pipe supplying it. Similarly, the rate of accumulation of charge on the capacitor plate is equal to the rate of flow of charge through the resistor connected to the plate. This rate of flow of charge is by definition the current through the resistor. (Recall that the units of current are charge/time—for example, coulombs/second.) We thus have





$$I=\frac{d\epsilon}{dt}$$

where I is the current through the resistor, and $d\epsilon/dt$ is the rate at which charge accumulates on the capacitor.

QUESTION

Suppose the voltage at the left terminal of the resistor is ϕ_1, while the voltage at the right terminal, which is essentially the voltage on the capacitor plate, is ϕ_c. If we use Ohm's law to obtain an expression for I, in terms of the voltages, and the capacitor equation to obtain an expression for $d\epsilon/dt$, which of the following relations will we obtain. (R denotes the resistance of the resistor, and C the capacitance of the capacitor.)

	Turn to Section:
$\frac{1}{R}(\phi_1-\phi_c)=C\frac{d\phi_c}{dt}$	4
$R(\phi_c-\phi_1)=C\frac{d\phi_c}{dt}$	20
$RC(\phi_c-\phi_1)=\frac{d\phi_c}{dt}$	18

10

Your answer in Section 21 is not correct. The equation which we developed for the capacitor was

$$C=\frac{\Delta\epsilon}{\Delta\phi}$$

where C was the capacitance, $\Delta\epsilon$ the quantity of charge placed in storage in the capacitor, and $\Delta\phi$ the increase in the voltage difference across the capacitor plates, observed as the charge $\Delta\epsilon$ is accumulated. For the prism of aquifer used in developing the ground-water equations in Part V, we had

$$\Delta V=SA\Delta h$$

where ΔV was the volume of water taken into storage in the prism, Δh the increase in head associated with this accumulation in storage, S the storage coefficient, and A the base area of the prism. This equation can be rewritten

$$SA=\frac{\Delta V}{\Delta h}$$

to faciliate comparison with the capacitor equation.

Return to Section 21 and choose another answer.

11

Your answer in Section 26 is correct. Note that this equation,

$$\frac{I}{w\cdot b}=-\sigma\frac{\partial\phi}{\partial x},$$

is analogous to the equation we would write for the component of specific discharge in the x direction, through a section of aquifer of width w and thickness b; that is,

$$\frac{Q}{w\cdot b}=-K\frac{\partial h}{\partial x}.$$

In practice, steady-state electric-analog work may be carried out by constructing a scale model of an aquifer from a conductive material and applying electrical boundary conditions similar to the hydraulic boundary conditions prevailing in the ground-water system. The voltage is controlled at certain points or along certain boundaries of the model, in proportion to known values of head at corresponding points in the aquifer; and current may be introduced or withdrawn in proportion to known values of inflow and outflow for the aquifer. When the boundary conditions are applied in this manner, voltages at various points of the model are proportional to heads at corresponding points in the aquifer, and the current density vector in various sections of the model is proportional to the specific-discharge vector in the corresponding sections of the aquifer.

QUESTION

Suppose an analog experiment of this type is set up, and the experimenter traces a line in the model along which voltage has some constant value. To which of the following hydrologic features would this line correspond?

	Turn to Section:
a flowline	16
a line of constant head	21
a line of uniform recharge	17

12 •

Your answer in Section 28,

$$Q=-K\frac{\partial^2 h}{\partial x^2}A_p,$$

is not correct. Darcy's law states that flow is equal to the product of hydraulic conductivity, cross-sectional area, and (negative) head gradient. The gradient of head is by definition a first derivative—the derivative of head with respect to distance. The answer which you chose involves a second derivative. The correct answer must either include a first derivative, or an expression equivalent to or approximating a first derivative.

Return to Section 28 and choose another answer.

13 •

Your answer in Section 21 is not correct. We have seen in dealing with the analogy between steady-state electrical flow and steady-state ground-water flow that voltage is analogous to hydraulic head, whereas current, or rate of flow of charge, is analogous to the volumetric rate of flow of fluid. In the analogy between the capacitor equation and the storage–head relation, voltage must still be analogous to head, or capacitors could not be used to represent storage in a model incorporating the flow analogy between Darcy's law and Ohm's law. Similarly, charge must represent fluid volume, so that rate of flow of charge (current) can represent volumetric fluid discharge. Otherwise the storage-capacitance analogy would be incompatible with the flow analogy.

Return to Section 21 and choose another answer.

14 •

Your answer in Section 22 is not correct. Increasing both R and C, as suggested in the answer which you chose, has the effect of increasing the factor RC in the equation

$$\phi_1+\phi_2+\phi_3+\phi_4-4\phi_0=RC\frac{d\phi_0}{dt}.$$

On the other hand, an increase in T in the aquifer causes the factor Sa^2/T to decrease, in the equation

$$h_1+h_2+h_3+h_4-4h_0=\frac{Sa^2}{T}\frac{\Delta h}{\Delta t}.$$

Thus the proposed technique fails to simulate the hydrologic system.

Notice that head and voltage are analogous and that increases in T can be simulated by decreases in R.

Return to Section 22 and choose another answer.

15

Your answer in Section 4 is not correct. The rate of accumulation of charge on the capacitor plate must equal the net rate at which charge is being transported to the capacitor through the four resistors. To set up the problem, we assume that the current is toward the capacitor in resistors 1 and 3, and away from the capacitor in resistors 2 and 4 in the diagram. The current toward the capacitor in resistor 1 is given by Ohm's law as

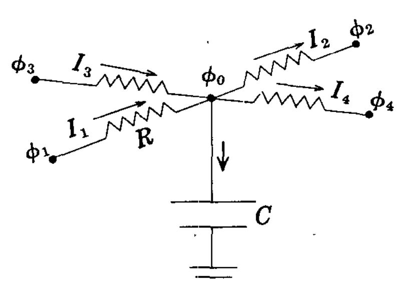
$$I_1=\frac{1}{R}(\phi_1-\phi_0),$$

while that in resistor 3 is given by

$$I_3=\frac{1}{R}(\phi_3-\phi_0).$$

The current away from the capacitor in resistor 2 is given by

$$I_2=\frac{1}{R}(\phi_0-\phi_2),$$



while that in resistor 4 is given by

$$I_4=\frac{1}{R}(\phi_0-\phi_4).$$

If it turns out that any of these currents are not actually in the direction initially assumed, the current value as computed above will be negative; thus the use of these expressions remains algebraically correct whether or not the assumptions regarding current direction are correct.

The net rate of transport of charge toward the capacitor will be the sum of the inflow currents minus the sum of the outflow currents, or

$$I_1+I_3-I_2-I_4.$$

This term must equal the rate of accumulation of charge on the capacitor plate, $d\epsilon/dt$,

$$\frac{d\epsilon}{dt}=C\frac{d\phi_0}{dt}.$$

That is we must have

$$I_1+I_3-I_2-I_4=C\frac{d\phi_0}{dt}.$$

The correct answer to the question of Section 4 can be obtained by substituting our expressions for I_1, I_2, I_3, and I_4 into this equation and rearranging the result.

Return to Section 4 and choose another answer.

16

Your answer in Section 11 is not correct. In steady-state two-dimensional flows, one can specify a function which is constant along a flowline. However, this function—which is termed a stream function—is not analogous to voltage (potential) in electrical theory; thus a flowline, or line along which stream function is constant, cannot correspond to an equipotential, or line along which voltage is constant. In developing the analogy between flow of electricity and flow of fluid through a porous medium, we stressed that voltage is analogous to head; current is analogous to fluid discharge; and electrical conductivity is analogous to hydraulic conductivity.

Return to Section 11 and choose another answer.

17 •

Your answer in Section 11 is not correct. The forms of Darcy's law and Ohm's law which we have used for comparison are repeated below:

Darcy's law:

$$\frac{Q}{w \cdot b} = -K\frac{\partial h}{\partial x}$$

where Q is the volumetric fluid discharge through a cross-sectional area of width w and thickness b, taken at right angles to the x direction; K is the hydraulic conductivity; and $\partial h/\partial x$ is the derivative of head in the x direction.

Ohm's law:

$$\frac{I}{w \cdot b} = -\sigma\frac{\partial \phi}{\partial x}$$

where I is the current through a cross-sectional area of width w and thickness b, taken at right angles to the x direction; σ is the electrical conductivity; and $\partial \phi/\partial x$ is the derivative of voltage, or potential, in the x direction.

A comparison of these equations shows that voltage, or potential, ϕ, occupies a position in electrical theory exactly parallel to head, h, in the theory of ground-water flow. Current, I, is analogous to discharge, Q; while σ, the electrical conductivity, is analogous to the hydraulic conductivity, K. These parallels should be kept in mind in answering the question of Section 11.

Return to Section 11 and choose another answer.

18 •

Your answer in Section 9 is not correct. The question concerns a capacitor which is connected to a resistor. The idea is to equate the rate of accumulation of charge on the capacitor plate to the rate at which charge is carried to the capacitor through the resistor—that is, to the current through the resistor. The rate at which charge accumulates on the capacitor plate is given by the capacitor equation as

$$\frac{d\epsilon}{dt} = C\frac{d\phi_c}{dt}.$$

The current through the resistor, or rate at which charge flows through the resistor, is given by Ohm's law as

$$I = \frac{1}{R}(\phi_1 - \phi_c).$$

Return to Section 9 and choose another answer.

19 •

Your answer in Section 1 is not correct. Ohm's law was given in the form

$$I = \frac{1}{R}(\phi_1 - \phi_2).$$

If R is in ohms and the difference $\phi_1 - \phi_2$ is in volts, current, I will be in amperes. In the example given, $\phi_1 - \phi_2$ was 5 volts and R was 500 ohms. Substitute these values in the equation to obtain the amount of current through the resistor.

Return to Section 1 and choose another answer.

20

Your answer, in Section 9,

$$R(\phi_c-\phi_1)=C\frac{d\phi_c}{dt},$$

is not correct. The rate of accumulation of charge on the capacitor, $d\epsilon/dt$, is equal to $C(d\phi_c/dt)$, and this part of your answer is correct. However, the idea is to equate this rate of accumulation of charge on the capacitor to the rate of transport of the charge toward the capacitor, through the resistor—that is, to the current through the resistor. This current is to be expressed in terms of resistance and voltage, using Ohm's law; and this has not been done correctly in the answer which you chose. Ohm's law states that the current through a resistance is equal to the voltage drop across the resistance divided by the value of the resistance in ohms.

Return to Section 9 and choose another answer.

21

Your answer in Section 11 is correct. The line of constant voltage, or equipotential line, is analogous to the line of constant head in ground-water hydraulics.

The analogy between Darcy's law and Ohm's law forms the basis of steady-state electric-analog modeling. In recent years, the modeling of nonequilibrium flow has become increasingly important; and just as Darcy's law alone is inadequate to describe nonequilibrium ground-water flow, its analogy with Ohm's law is in itself an inadequate basis for nonequilibrium modeling. The theory of nonequilibrium flow is based upon a combination of Darcy's law with the storage equation, through the equation of continuity. To extend analog modeling to nonequilibrium flow, we require electrical equations analogous to the storage and continuity equations.

The analog of ground-water storage is provided by an electrical element known as a capacitor. The capacitor is essentially a storage tank for electric charge; in circuit diagrams it is denoted by the symbol shown in figure A. As the symbol itself suggests, capacitors can be constructed by inserting two parallel plates of conductive material into a circuit, as shown in figure B. When the switch is closed, positive charge flows from the battery to the upper plate and accumulates on the plate in a manner analogous to the accumulation of water in a tank. At the same time, positive charge is drawn from the lower plate, leaving it with a net negative charge. Figure C shows a hydraulic circuit analogous to this simple capacitor circuit; when the valve is opened, the pump delivers water to the left-hand tank, draining the right-hand tank. If the right-hand tank is connected in turn to an effectively limitless water supply, as shown in figure D, both the volume of water and the water level in the right-hand tank will remain essentially constant, while water will still accumulate in the left-hand tank as the pump operates. The analogous electrical arrangement is shown in figure E; here the additional symbol shown adjacent to the lower plate indicates that this plate has been grounded—that is, connected to a large mass of metal buried in the earth, which in effect constitutes a limitless reserve of charge. In this situation, the quantity of charge on the lower plate remains essentially constant, as does the voltage on this plate, but the battery still causes positive charge to accumulate on the upper plate. The voltage on the lower plate is analogous to the water level in the right-hand tank, which is held constant by connection to the unlimited water supply.

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FIGURE *A*

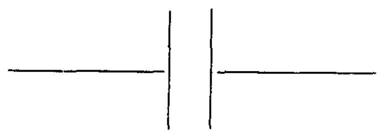


FIGURE *B*

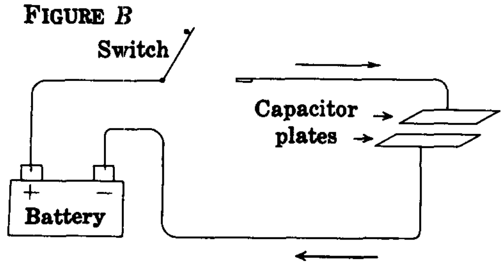


FIGURE *C*

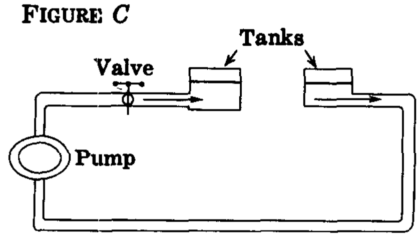


FIGURE *D*

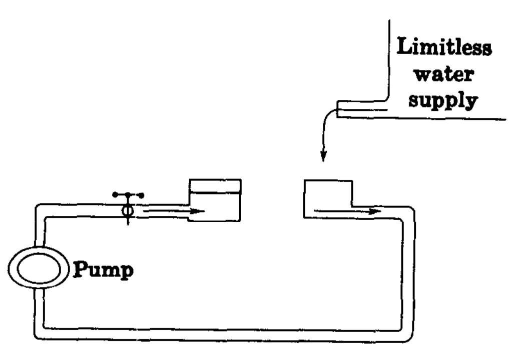
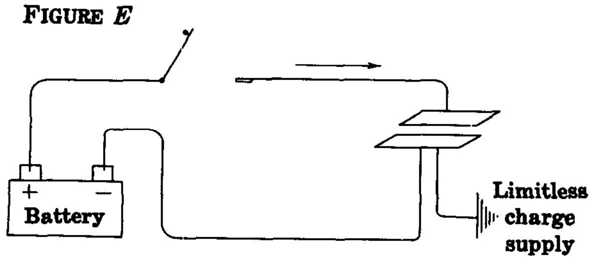


FIGURE *E*



In a circuit such as that shown in figure E, it is customary to designate the constant voltage of the ground plate as zero. This is done arbitrarily—it is equivalent for example, to setting head equal to zero at the constant water level of the right-hand tank of figure D. With the voltage of the grounded plate taken as zero, the voltage difference between the plates becomes simply the voltage, ϕ, measured on the upper plate. In the circuit of figure E, this voltage is equal to the voltage produced by the battery.

Now suppose an experiment is run in which the battery in the circuit of figure E is replaced in turn by batteries of successively higher voltage. At each step the charge on the positive plate is measured in some way, after the circuit has reached equilibrium. The results will show that as the applied voltage is increased, the charge which accumulates on the positive plate increases in direct proportion. If a graph is constructed from the experimental results in which the charge, ϵ, which has accumulated on the positive plate is plotted versus the voltage in each step, the result will be a straight line, as shown in the figure. The slope of this line, $\Delta\epsilon/\Delta\phi$, is termed the capacitance of the capacitor, and is designated C; that is,

$$C=\frac{\Delta\epsilon}{\Delta\phi},\quad C=\frac{d\epsilon}{d\phi},$$

or simply

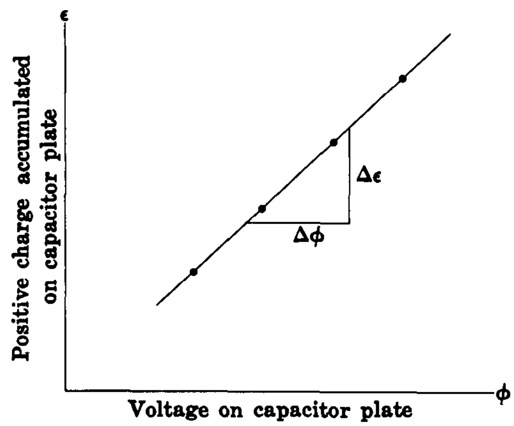
$$C=\frac{\epsilon}{\phi}.$$

Capacitance is measured in farads, or more commonly in microfarads; a farad is equal to 1 coulomb per volt.

These equations serve to define the operation of a capacitor and provide the analog we require for the equation of ground-water storage. It will be recalled that the relation between volume in storage and head can be written

$$\Delta V=S\cdot A\cdot\Delta h$$

Con.—21



where ΔV is the volume of water taken into or released from storage in a prism of aquifer of base area A, as the head changes by an amount Δh.

QUESTION

Which of the following statements correctly describes the analogy between the capacitor equation and the ground-water-storage—head relation?

	Turn to Section:
Charge is analogous to head, voltage is analogous to volume of water, and capacitance, C, is analogous to the factor SA.	13
Charge is analogous to volume of water, voltage is analogous to head, and capacitance, C, is analogous to the factor SA.	9
Charge is analogous to volume of water, voltage is analogous to head, and capacitance, C, is analogous to the factor $\frac{1}{SA}$	10

22

Your answer in Section 4,

$$\phi_1+\phi_2+\phi_3+\phi_4-4\phi_0=RC\frac{d\phi_0}{dt},$$

is correct. In Part VII, we obtained a finite-difference approximation to the differential equation for two-dimensional non-steady-state ground-water flow,

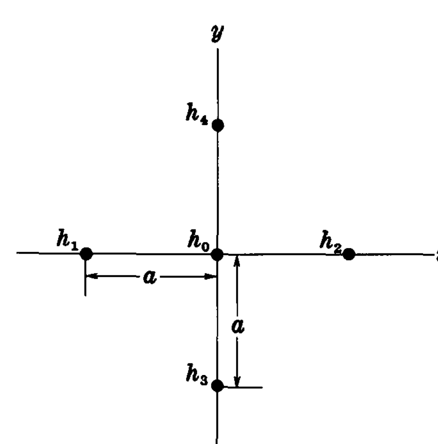
$$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}=\frac{S}{T}\frac{\partial h}{\partial t}.$$

This approximation can be written

$$\frac{h_1+h_2+h_3+h_4-4h_0}{a^2}=\frac{S}{T}\frac{\Delta h_0}{\Delta t}.$$

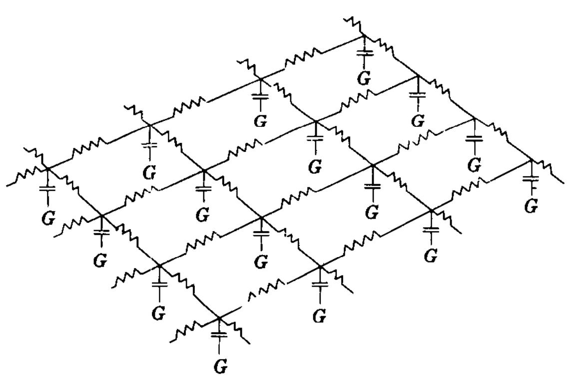
or

$$h_1+h_2+h_3+h_4-4h_0=\frac{Sa^2}{T}\frac{\Delta h_0}{\Delta t}.$$



22 • —Con.

where h_0, h_1, h_2, h_3, and h_4 represent the head values at the nodes of an array such as that shown in the sketch; a is the node spacing; S is storage coefficient; T is transmissivity; and $\Delta h_0/\Delta t$ represents the rate of change of head at the central node. The circuit equation which we have just obtained is directly analogous to this finite-difference form of the ground-water equation, except for the use of the time derivative notation $d\phi_0/dt$ as opposed to the finite-difference form, $\Delta h_0/\Delta t$. In other words, the circuit element composed of the four resistors and the capacitor behaves in approximately the same way as the prism of confined aquifer which was postulated in developing the ground-water equations. It follows that a *network* composed of circuit elements of this type, such as that shown in the figure, should behave



G **indicates grounded terminal**

in the same way as a two-dimensional confined aquifer of similar geometry. The nonequilibrium behavior of such an aquifer may be studied by constructing a model of the aquifer, consisting of a network of this type; electrical boundary conditions similar to the observed hydraulic boundary conditions are imposed on the model, and voltage is monitored at various points in the network as a function of time. The voltage readings constitute, in effect, a finite-difference solution to the differential equation describing head in the aquifer. The time scale of model experiments is of course much different from that of the hydrologic regime. A common practice is to use a very short time scale, in which milliseconds of model time may represent months in the hydrologic system. When time scales in this range are employed, the electrical excitations, or boundary conditions, are applied repeatedly at a given frequency, and the response of the system is monitored using oscilloscopes. The sweep frequency of each recording oscilloscope is synchronized with the frequency of repetition of the boundary-condition inputs, so that the oscilloscope trace represents a curve of voltage, or head, versus time, at the network point to which the instrument is connected.

QUESTION

Suppose we wish to model an aquifer in which transmissivity varies from one area to another, while storage coefficient remains essentially constant throughout the aquifer. Which of the following procedures would you consider an acceptable method of simulating this condition in a resistance–capacitance network analog?

	Turn to Section:
Construct a network using uniform values of resistance and capacitance, but apply proportionally higher voltages in areas having a high transmissivity.	2
Construct a network in which resistance and capacitance are both increased in proportion to local increases in transmissivity.	14
Construct a network in which resistance is varied inversely with the transmissivity to be simulated, while capacitance is maintained at a uniform value throughout the network.	5

23

Your answer in Section 26,

$$\frac{I}{w \cdot l} = -\sigma \frac{\partial \phi}{\partial z},$$

is not correct. The answer which you chose actually expresses the component of current density in the z direction. $w \cdot l$ is an area taken normal to the z direction. If I represents the current through this area, $I/w \cdot l$ will give the component of current density in the z direction; and this should equal $-\sigma$ times the directional derivative of voltage in the z direction, $\partial\phi/\partial z$. However, the question asked for the current density component in the x direction; and in fact, the problem stated that the current flow was two dimensional confined to the x, y plane. This implies that the current component in the vertical direction is zero, and thus that $\partial\phi/\partial z$ is zero as well.

Return to Section 26 and choose another answer.

24

Your answer in Section 6 is not correct. Ohm's law was given in Section 1 as

$$I = \frac{1}{R}(\phi_1 - \phi_2)$$

where $\phi_1 - \phi_2$ is the voltage difference across a resistance, R, and I is the current through the resistance. In Section 6 the expression

$$R = \rho_e \cdot \frac{L}{A}$$

was given for the resistance, where ρ_e is the electrical resistivity of the material of which the resistance is composed; L is the length of the resistance, and A is its cross-sectional area. This expression for resistance should be substituted into the form of Ohm's law given above to obtain the correct answer.

Return to Section 6 and choose another answer.

25

Your answer in Section 26,

$$\frac{I}{w \cdot l} = -\sigma \frac{\partial \phi}{\partial y},$$

is not correct. The component of current density in a given direction is defined as the charge crossing a unit area taken normal to that direction, in a unit time. Here we are concerned with the current density component in the x direction; we must accordingly use an area at right angles to the x direction. In your answer, the area is $w \cdot l$, which is normal to the z direction. Again, the component of current density in a given direction is proportional to the directional derivative of voltage in that direction. Since we are dealing with the component of current density in the x direction, we require the derivative of voltage in the x direction. The answer which you chose, however, uses the derivative of voltage with respect to y.

Return to Section 26 and choose another answer.

26 •

Your answer in Section 28 is correct. The term

$$\frac{h_1 - h_2}{L_p}$$

is equivalent to the negative of the head gradient, $-\partial h/\partial x$, so that this formulation of Darcy's law is equivalent to those we have studied previously. Now let us compare this form of Darcy's law with Ohm's law.

Our expression for Darcy's law was

$$Q = K \cdot \frac{h_1 - h_2}{L_p} \cdot A_p.$$

Our expression for Ohm's law in terms of electrical conductivity was

$$I = \sigma \cdot \frac{\phi_1 - \phi_2}{L} \cdot A.$$

In terms of electrical resistivity, we obtained

$$I = \frac{1}{\rho} \cdot \frac{\phi_1 - \phi_2}{L} \cdot A.$$

In these forms, the analogous quantities are easily identified. Voltage takes the place of head, current takes the place of fluid discharge and as noted in the preceding section σ, or $1/\rho$, takes the place of hydraulic conductivity. We note further that since current is defined as the rate of movement of electric charge across a given plane, while fluid discharge is the rate of transport of fluid volume across a given plane, electric charge may be considered analogous to fluid volume.

In Part II, we noted that Darcy's law could be written in slightly more general form as

$$q_x = \frac{Q_x}{A} = -K \frac{\partial h}{\partial x}$$

$$q_y = \frac{Q_y}{A} = -K \frac{\partial h}{\partial y}$$

and

$$q_z = \frac{Q_z}{A} = -K \frac{\partial h}{\partial z}$$

where q_x is the component of the specific-discharge vector in the x direction, or the discharge through a unit area at right angles to the x axis; q_y is the component of the specific-discharge vector in the y direction, and q_z is the component in the z direction. The three components are added vectorially to obtain the resultant specific discharge. $\partial h/\partial x$, $\partial h/\partial y$, and $\partial h/\partial z$ are the directional derivatives of head in the x, y, and z directions; and K is the hydraulic conductivity, which is here assumed to be the same in any direction. We may similarly write a more general form of Ohm's law, replacing the term $\phi_1 - \phi_2/L$ by derivatives of voltage with respect to distance, and considering components of the current density, or current per unit cross-sectional area, in the three space directions. This gives

$$\left(\frac{I}{A}\right)_x = -\sigma \frac{\partial \phi}{\partial x} = -\frac{1}{\rho_e} \frac{\partial \phi}{\partial x}$$

$$\left(\frac{I}{A}\right)_y = -\sigma \frac{\partial \phi}{\partial y} = -\frac{1}{\rho_e} \frac{\partial \phi}{\partial y}$$

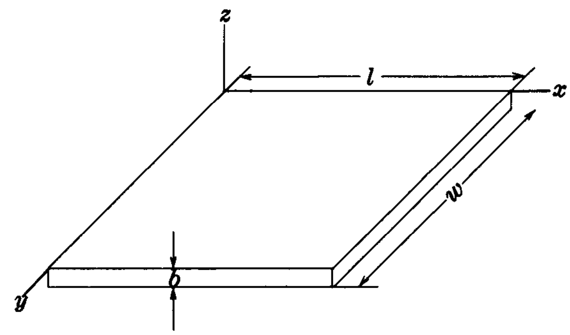
$$\left(\frac{I}{A}\right)_z = -\sigma \frac{\partial \phi}{\partial z} = -\frac{1}{\rho_e} \frac{\partial \phi}{\partial z}.$$

Here $(I/A)_x$ is the current through a unit area oriented at right angles to the x axis, $(I/A)_y$ is current through a unit area perpendicular to the y axis, and $(I/A)_z$ is the current through a unit area perpendicular to the z axis. These terms form the components of the current density vector. $\partial \phi/\partial x$, $\partial \phi/\partial y$, and $\partial \phi/\partial z$ are the voltage gradients, in units of volts/distance, in the three directions. These three expressions simply represent a generalization to three dimensions of the equation given in Section 1 as Ohm's law.

Con.—26

QUESTION

The picture shows a rectangle in a conductive sheet, in which there is a two-dimensional flow of electricity. The flow is in the plane of the sheet, that is, the x, y plane; the thickness of the sheet is b, and the dimensions of the rectangle are l and w. Which of the following expressions gives the magnitude of the component of current density in the x direction?

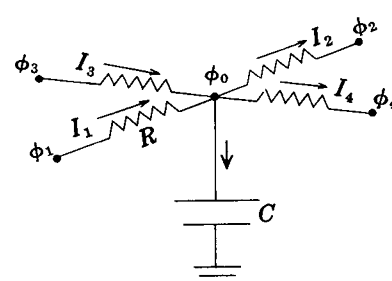


	Turn to Section:
$\dfrac{I}{w \cdot b} = -\sigma \dfrac{\partial \phi}{\partial x}$	11
$\dfrac{I}{w \cdot l} = -\sigma \dfrac{\partial \phi}{\partial y}$	25
$\dfrac{I}{w \cdot l} = -\sigma \dfrac{\partial \phi}{\partial z}$	23

(I represents the current through the area utilized in the equation, $w \cdot b$ or $w \cdot l$.)

27

Your answer in Section 4 is not correct. The essential idea here is that the rate of accumulation of charge on the capacitor must equal the net inflow minus outflow of charge through the four resistors. The inflow of charge through resistor 1 is the current through that resistor, and is given by Ohm's law as



$$I_1 = \frac{1}{R}(\phi_1 - \phi_0).$$

The outflow through resistor 2 is similarly given by

$$I_2 = \frac{1}{R}(\phi_0 - \phi_2).$$

The inflow through resistor 3 is

$$I_3 = \frac{1}{R}(\phi_3 - \phi_0),$$

27 • —Con.

while the outflow through resistor 4 is

$$I_4=\frac{1}{R}(\phi_0-\phi_4).$$

The net inflow minus outflow of charge to the capacitor is

$$I_1+I_3-I_2-I_4,$$

and this must equal the rate of accumulation of charge on the capacitor, $d\epsilon/dt$, that is

$$I_1+I_3-I_2-I_4=\frac{d\epsilon}{dt}.$$

According to the capacitor equation, $d\epsilon/dt$ is given by

$$\frac{d\epsilon}{dt}=C\frac{d\phi_0}{dt}.$$

The answer to the question of Section 4 can be obtained by substituting the appropriate expressions for I_1, I_2, I_3 I_4 and $d\epsilon/dt$ into the relation

$$I_1+I_3-I_2-I_4=\frac{d\epsilon}{dt},$$

and rearranging the result.

Return to Section 4 and choose another answer.

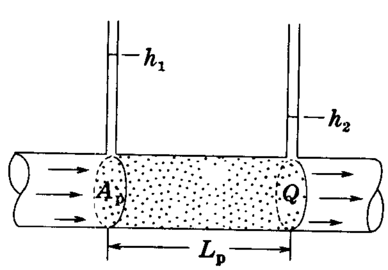
28 •

Your answer in Section 6 is correct.

Electrical conductivity, or 1/resistivity, is the electrical equivalent of hydraulic conductivity. In terms of electrical conductivity, Ohm's law for the problem of Section 6 becomes

$$I=\frac{\sigma A}{L}(\phi_1-\phi_2)$$

where σ is electrical conductivity.



The analogy between Darcy's law and Ohm's law is easily visualized if we consider the flow of water through a sand-filled pipe, of length L_p and cross-sectional area A_p, as shown in the diagram. The head at the inflow end of the pipe is h_1, while that at the outflow end is h_2. The hydraulic conductivity of the sand is K.

QUESTION

Which of the following expressions is obtained by applying Darcy's law to this flow? (Q represents the discharge through the pipe.)

	Turn to Section:
$Q=-K\cdot\frac{\partial^2 h}{\partial x^2}\cdot A_p$	12
$Q=K\cdot\frac{h_1-h_2}{L_p}\cdot A_p$	26
$Q=\frac{K}{A_p}\cdot\frac{L_p}{h_1-h_2}$	7

References

Bedinger, M. S., Reed, J. E., Wells, C. J., and Swafford, B. F., 1970, Methods and applications of electrical simulation in ground-water studies in the lower Arkansas and Verdigris River valleys, Arkansas and Oklahoma: U.S. Geol. Survey Water-Supply Paper 1971, 71 p.

Boulton, N. S., 1954, The drawdown of the water table under nonsteady conditions near a pumped well in an unconfined formation: Proc. Civil Engineers (British), p. 564–579.

Bredehoeft, J. D., and Pinder, G. F., 1970, Digital analysis of areal flow in multi-aquifer ground water systems: A quasi three-dimensional model: Water Resources Research, v. 6, no. 3, p. 883–888.

Brown, R. H., 1962, Progress in ground water studies with the electric analog model: Am. Waterworks Assoc. Jour., v. 54, no. 8, p. 943.

Cooper, H. H., 1966, The equation of ground water flow in fixed and deforming coordinates: Jour. Geophys. Research, v. 71, no. 20, p. 4785–4790.

Cooper, H. H., Bredehoeft, J. D., and Papadopulos, S. S., 1967, Response of a finite diameter well to an instantaneous charge of water: Water Resources Research, v. 3, no. 1.

Cooper, H. H., and Jacob, C. E., 1946, A generalized graphical method of evaluating formation constants and summarizing well-field history: Am. Geophys. Union Trans., v. 27, no. 4, Aug. 1946, p. 526–534.

Darcy, Henri, 1856, Les fontaines publiques de la ville de Dijon (The water supply of Dijon): Paris, Victor Dalmont, 647 p.

Domenico, P. A., 1972, Concepts and models in ground water hydrology: New York, McGraw-Hill Book Co., 405 p.

Dupuit, Jules, 1863, Etudes théoriques et pratiques sur le mouvement des eaux courantes (Theoretical and practical studies on the movement of running water): Paris, Carilion-Goery et V. Dalmont, 275 p.

Ferris, J. G., 1959, Ground water, Chapter 6 *in* Wisler, C. O., and Brater, E. F., Hydrology (2d edition): New York, John Wiley and Sons.

Ferris, J. G., and Knowles, D. B., 1963, The slug-injection test for estimating the transmissibility of an aquifer *in* Methods of determining permeability, transmissibility, and drawdown: U.S. Geol. Survey Water-Supply Paper 1536–I, 299 p.

Ferris, J. G. Knowles, D. B., Brown, R. H., and Stallman, R. W., 1962, Theory of aquifer tests: U.S. Geol. Survey Water-Supply Paper 1536–E, 174 p.

Forsythe, G. E., and Wasow, W. R., 1960, Finite difference methods for partial differential equations: New York, John Wiley and Sons.

Freeze, R. A., 1971, Three-dimensional, transient, saturated-unsaturated flow in a ground water basin: Water Resources Research, v. 7, no. 2, p. 347–366.

Hantush, M. S., 1959, Nonsteady flow to flowing wells in leaky aquifers: Jour. Geophys. Research, v. 64, no. 8, p. 1043.

——— 1960, Modification of the theory of leaky aquifers: Jour. Geophys. Research, v. 65, no. 11, p. 3713–3725.

——— 1967a, Flow of ground water in relatively thick leaky aquifers: Water Resources Research, v. 3, no. 2, p. 583.

——— 1967b, Flow to wells in aquifers separated by a semipervious layer: Jour. Geophys. Research, v. 72, no. 6, p. 1709.

Hantush, M. S., and Jacob, C. E., 1955, Nonsteady radial flow in an infinite leaky aquifer: Am. Geophys. Union Trans., v. 36, no. 1, p. 95.

Hildebrand, F. B., 1968, Finite-difference equations and simulations: Englewood Cliffs, N.J., Prentice-Hall Inc., 338 p.

Hubbert, M. K., 1940, The theory of ground-water motion: Jour. Geology, v. 48, no. 8, pt. 1, p. 785–944.

Jacob, C. E., 1946, Radial flow in a leaky artesian aquifer: Am. Geophys. Union Trans., v. 27, no. 2, p. 198.

——— 1950, Flow of ground water, Chapter 5 *in* Rouse, Hunter, Engineering Hydraulics: New York, John Wiley and Sons, 1039 p.

——— 1963, Correction of drawdown caused by a pumped well tapping less than the full thickness of an aquifer *in* Methods of determining permeability, transmissibility, and drawdown: U.S. Geol. Survey Water-Supply Paper 1536–I, p. 272–282.

Jacob, C. E., and Lohman, S. W., 1952, Nonsteady flow to a well of constant drawdown in an extensive aquifer: Am. Geophys. Union Trans., v. 33, no. 4, p. 559–569.

Karplus, W. J., 1958, Analog simulation: New York, McGraw-Hill Book Co., 434 p.

Lang, S. M., 1963, Drawdown patterns in aquifers having a straight line boundary *in* Shortcuts and special problems in aquifer tests: U.S. Geol. Survey Water-Supply Paper 1545–C.

Muskat, M., 1937, The flow of homogeneous fluids through porous media: Ann Arbor, Mich., J. W. Edwards, Inc. [repr. 1946].

Papadopulos, S. S., and Cooper, H. H., 1967, Drawdown in a well of large diameter: Water Resources Research, v. 3, no. 1.

Patten, E. P., 1965, Design, construction and use of electric analog models, *in* Wood, L. A., and Gabrysch, R. K., Analog model study of ground water in the Houston District, Texas: Texas Water Comm. Bull. 6508.

Peaceman, D. W., and Rachford, H. H., 1955, The numerical solution of parabolic and elliptic differential equations: Jour. Soc. Indust. Appl. Math., v. 3, no. 1, March, 1955.

Pinder, G. F., and Bredehoeft, J. D., 1968, Applications of digital computer for aquifer evaluation: Water Resources Research, v. 4, no. 5, p. 1069–1093.

Prickett, T. A., and Lonnquist, G. C., 1971, Selected digital computer techniques for groundwater resource evaluation: Illinois State Water Survey Bull. 55, 62 p.

——— 1973, Aquifer simulation model for use on disk supported small computer systems: Illinois State Water Survey Circ. 114, 21 p.

Remson, Irwin, Appel, C. A., and Webster, R. A., 1965, Ground water models solved by digital computer: Jour. Hydr. Div., Am. Soc. Civil Engineers, v. 91, no. HY3, p. 133–147.

Remson, Irwin, Hornberger, G. M., and Molz, F. J., 1971, Numerical methods in ground water hydrology: New York, Wiley Interscience Division, John Wiley and Sons, 389 p.

Rubin, Jacob, 1968, Theoretical analysis of two-dimensional transient flow of water in unsaturated and partly unsaturated soils: Soil Sci. Soc. America Proc., v. 32, no. 5, p. 607–615.

Skibitzke, H. E., 1960, Electronic computers as an aid to the analysis of hydrologic problems: Internat. Assoc. Scientific Hydrol. Pub. 52, Comm. Subterranean waters, Gentbrugge, Belgium, p. 347.

——— 1963, Determination of the coefficient of transmissibility from measurements of residual drawdown in a bailed well *in* Methods of determining permeability, transmissibility, and drawdown: U.S. Geol. Survey Water-Supply Paper 1536–I, p. 293.

Slichter, C. S., 1899, Theoretical investigation of the motion of ground waters: U.S. Geol. Survey, 19th Ann. Rept., pt. 2, p. 295.

Stallman, R. W., 1956, Numerical analysis of regional water levels to define aquifer hydrology: Am. Geophys. Union Trans., v. 37, no. 4, p. 451–460.

——— 1963a, Type curves for the solution of single boundary problems *in* Shortcuts and special problems in aquifer tests: U.S. Geol. Survey Water-Supply Paper 1545–C, p. 45–47.

——— 1963b, Electric analog of three dimensional flow to wells and its application to unconfined aquifers: U.S. Geol. Survey Water-Supply Paper 1536–H, 38 p.

Theis, C. V., 1935, The relation between the lowering of the piezometric surface and the rate and duration of discharge of a well using ground water storage: Am. Geophys. Union Trans., pt. 2, Aug. 1935, p. 519–524.

——— 1938, The significance and nature of the cone of depression in ground water bodies: Econ. Geology, v. 33, no. 8, p. 889–902.

Thiem, Gunther, 1906, Hydrologische methoden (Hydrologic methods): Leipzig, J. M. Gebhardt, 56 p.

Thomas, L. H., 1949, Elliptic problems in linear differential equations over a network: Watson Scientific Computing Laboratory, Columbia Univ., N.Y.

Trescott, P. C., 1973, Iterative digital model for aquifer evaluation: U.S. Geol. Survey open-file rept., June 1973, 63 p.

Trescott, P. C., Pinder, G. F., and Jones, J. F., 1970, Digital model of alluvial aquifer: Jour. Hydr. Div., Proc. Am. Soc. Civil Engineers, HY5, p. 1115–1128.

Walton, W. C., 1970, Ground water resource evaluation: New York, McGraw-Hill Book Co., 664 p.

Weeks, E. P., 1969, Determining the ratio of horizontal to vertical permeability by aquifer-test analysis: Water Resources Research, v. 5, no. 1, p. 196–214.

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