Three-Dimensional Heat Conduction in Permafrost Beneath Heated Buildings

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By ARTHUR H. LACHENBRUCH

EXPERIMENTAL AND THEORETICAL GEOPHYSICS

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A discussion of three-dimensional heat conduction in permafrost with special reference to thermal disturbance caused by heated structures

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[Plates in pocket]

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SYMBOLS

Θ or T	temperature (degrees C).
t	time since initiation of temperature disturbance in S (sec)
t_o	time of initiation of temperature disturbance in S (sec)
t'	$=t+t_o$
α	thermal diffusivity (cm^2 per sec)
ω	angular frequency ($\omega=1.99\times10^{-7}$ radians per sec for a period equal to 1 year)
Ω	solid angle (steradians)
S	finite region of plane $z=0$
ρ, λ	polar coordinates in plane $z=0$
r	$=[(x-x')^2+(y-y')^2+z^2]^{1/2}=[\rho^2+z^2]^{1/2}$
A	amplitude of temperature variation in undisturbed portion of plane $z=0$ (degrees C)
B	mean value of surface temperature in disturbed region, S, of plane $z=0$ (degrees C)
C	amplitude of temperature variation in disturbed region, S, of plane $z=0$ (degrees C)
D	$=C-A$, amplitude of seasonal temperature disturbance in disturbed region of plane $z=0$ (degrees C)
erf x	$=\frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du$
erfc x	$=\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-u^2}\,du$
Φ	weighting factor for principal-disturbance calculation $=\frac{2}{\sqrt{\pi}}\frac{r}{2\sqrt{\alpha t}}e^{-\frac{r^2}{4\alpha t}}+\text{erfc}\,\frac{r}{2\sqrt{\alpha t}}$
Ψ	weighting factor for seasonal-disturbance calculation $=e^{-r\sqrt{\frac{\omega}{2\alpha}}}\left\{\sin\left(\omega t-r\sqrt{\frac{\omega}{2\alpha}}\right)+r\sqrt{\frac{\omega}{2\alpha}}\left[\cos\left(\omega t-r\sqrt{\frac{\omega}{2\alpha}}\right)+\sin\left(\omega t-r\sqrt{\frac{\omega}{2\alpha}}\right)\right]\right\}$

THREE-DIMENSIONAL HEAT CONDUCTION IN PERMAFROST BENEATH HEATED BUILDINGS

By Arthur H. Lachenbruch

ABSTRACT

The general Green's function solution has been integrated for the case of heat conduction in a homogeneous semi-infinite medium in which the temperature at the surface varies sinusoidally with time but the mean temperature and amplitude of the variation are different within and outside an arbitrarily shaped region at the surface. The amplitude and mean temperature can be treated as functions of position within the arbitrary surface region. For certain simple surface regions the results can be expressed in terms of tabulated functions. Numerical results for the general case can be obtained by simple graphical procedures.

The results can be applied to the study of disturbances in ground temperature caused by the presence of bodies of water or by engineering surface modifications such as those produced by erecting a heated building. The primary application of such studies is in high-latitude regions where much of the undisturbed ground is perennially frozen. In such areas, a method of predicting the extent of thawing induced by various modifications of the temperature of the ground surface is important in problems of engineering design and logistics.

INTRODUCTION

The temperature distribution in permanently frozen ground is of considerable importance in Arctic and Antarctic regions. Inasmuch as engineering operations in these regions are carried out on normally permanently frozen mediums (snow, ice, and permafrost) a knowledge of the temperature distribution in these mediums under various disturbing influences is important to the engineer so that he can, for example, design in such a way as to keep critical parts of structural foundations frozen, or water-supply sources and conduits unfrozen. Several problems of more scientific than practical nature, such as the relation between the age of a lake and the transient disturbance of ground temperature it produces, can also be approached through the study of idealized models such as those considered in this paper.

One of the important general problems is to determine the disturbance of subsurface temperature that results when the temperature at the ground surface within a finite region, S, is different from the surface temperature characteristic of the area outside S. The conditions correspond to the presence of natural features such as a lake or river, or a region in which the thermal properties of the surface cover are appreciably different from those characteristic of the area in general, and to modifications of the surface such as those resulting from erecting a building, stripping the vegetation, or emplacing a gravel fill.

Inasmuch as probably the most important application is to study of the temperature distribution beneath a heated building, this general problem will be referred to for convenience as the heated-building problem.

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Francis Birch and D. C. Hilton offered valuable comments on the manuscript from the physical and engineering points of view respectively. S. H. Lachenbruch provided a thorough check of the paper's mathematical content.

OUTLINE OF THE HEATED-BUILDING PROBLEM

For study of the general aspects of ground temperatures the simple two-parameter sinusoidal approximation for the undisturbed surface temperature, $A \sin \omega t$, usually serves quite well. The temperature origin is taken as the mean annual temperature just beneath the ground surface in the undisturbed area. The angular frequency, ω, corresponds to a period of 1 year. In the finite region, S, of the surface in which the surface temperature is disturbed, the significant parts of the problem can usually be treated by representing the temperature by $B(x,y)+C(x,y) \sin \omega t$. Here $B(x,y)$ is the difference in mean annual temperature between the undisturbed area and the surface point (x,y) in the disturbed region S. $C(x,y)$ is the amplitude of temperature variation at (x,y) in S.

It is generally convenient to assume that the surface temperature presently characteristic of the undisturbed region (outside S) formerly obtained over the entire surface, and that any initial transient effects had died out by the time, t_0, that the disturbing condition was introduced in the region, S.

Thus the heated-building problem can be stated formally as follows: Find the temperature, $\Theta(x,y,z,t)$, in the semi-infinite solid $z>0$, subject to the following conditions:

$$\frac{\partial^2\Theta}{\partial x^2}+\frac{\partial^2\Theta}{\partial y^2}+\frac{\partial^2\Theta}{\partial z^2}=\frac{1}{\alpha}\frac{\partial\Theta}{\partial t'},\ z>0,\ t'>0 \tag{1}$$

$$\Theta(x,y,z,0)=f\ (z),\ z>0,\ t'=0 \tag{2}$$

$$\Theta(x,y,0,t')=A\sin\omega t',\ z=0\begin{cases}\text{a) } 0<t'<t_0,\ \text{all } (x,y)\\ \text{b) } t_0<t',\ (x,y) \text{ not in } S\end{cases} \tag{3}$$

$$\Theta(x,y,0,t')=B\ (x,y)+C(x,y)\ \sin\omega t',\ z=0,\ t_0<t',\ (x,y) \text{ in } S. \tag{4}$$

Here t_0 is taken sufficiently large that transient initial effects have died out by the time $(t'=t_0)$ of initiation of the disturbance in S. $f\ (z)$ represents the normal geothermal profile assumed independent of time.

It has been implicitly assumed above that the ground surface has no appreciable relief, and thus can be represented by the plane $z=0$. In addition, we shall assume that the ground is homogeneous and neglect the effects of latent heat resulting from the change of state of interstitial water or ice. The physical implications of these limiting assumptions will be discussed briefly later.

With latent heat neglected the relations become linear and it is generally possible to consider B and C to be constant over finite subregions of S and solve the problem for each of these subregions. These solutions are later superposed to give the collective effect of the entire region, S. Thus the necessity of considering B and C as functions of position is eliminated. A further simplification resulting from neglecting latent heat arises from the fact that it is then possible to disregard the normal static geothermal profile, $f(z)$, and consider the surface effects separately. To get the complete solution, $f(z)$ can be added to the final result. It can be neglected, however, for most problems involving small heated structures.

With the assumption of homogeneity and no latent heat, the problem is reduced to

$$\frac{\partial^2\Theta}{\partial x^2}+\frac{\partial^2\Theta}{\partial y^2}+\frac{\partial^2\Theta}{\partial z^2}=\frac{1}{\alpha}\frac{\partial\Theta}{\partial t},\ z>0,\ t>0 \tag{5}$$

$$\Theta(x,y,0,t)=A\ \sin\omega t,\ z=0,\ t>0,\ (x,y) \text{ not in } S \tag{6}$$

$$\Theta(x,y,0,t)=B+C\ \sin\omega t,\ z=0,\ t>0,\ (x,y) \text{ in } S \tag{7}$$

and

$$\Theta(x,y,z,0)=\Theta_0(z,o),\ z>0,\ t=0 \tag{8}$$

where $t=t'-t_0$.
This solution can be expressed as the sum of three independent solutions as follows:

$$\Theta=\Theta_0+\Theta_1+\Theta_2 \tag{9}$$

where Θ_0 is the limit, as t' becomes infinite, of the solution to the problem

$$\frac{\partial^2 T}{\partial z^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t'},\ z>0,\ t'>0 \tag{10}$$

$$T(0,t)=A\sin \omega t',\ z=0,\ t'>0 \tag{11}$$

$$T(z,\ 0)=0,\ z>0,\ t'=0. \tag{12}$$

Θ_1 is the solution to

$$\frac{\partial^2\Theta_1}{\partial x^2}+\frac{\partial^2\Theta_1}{\partial y^2}+\frac{\partial^2\Theta_1}{\partial z^2}=\frac{1}{\alpha}\frac{\partial\Theta_1}{\partial t},\ z>0,\ t>0 \tag{13}$$

$$\Theta_1(x,\ y,\ 0,\ t)=B,\ t>0,\ z=0,\ (x,\ y)\text{ in }S \tag{14}$$

$$\Theta_1\ (x,\ y,\ 0,\ t)=0,\ t>0,\ z=0,\ (x,\ y)\text{ not in }S \tag{15}$$

$$\Theta_1\ (x,\ y,\ z,\ 0)=0,\ t=0,\ z>0. \tag{16}$$

and Θ_2 is the solution to

$$\frac{\partial^2\Theta_2}{\partial x^2}+\frac{\partial^2\Theta_2}{\partial y^2}+\frac{\partial^2\Theta_2}{\partial z^2}=\frac{1}{\alpha}\frac{\partial\Theta_2}{\partial t},\ z>0,\ t>0 \tag{17}$$

$$\Theta_2\ (x,\ y,\ 0,\ t)=D\sin \omega t,\ z=0,\ t>0,\ (x,\ y)\text{ in }S \tag{18}$$

$$\Theta_2\ (x,\ y,\ 0,\ t)=0,\ z=0,\ t>0,\ (x,\ y)\text{ not in }S \tag{19}$$

$$\Theta_2\ (x,\ y,\ z,\ 0)=0,\ z>0,\ t=0 \tag{20}$$

where $D=C-A$. (21)

The solution for Θ_0 is well-known (see, for example, Carslaw and Jaeger, 1947, p. 47, eq 3) and is given by

$$\Theta_0(z,\ t)=Ae^{-z\sqrt{\frac{\omega}{2\alpha}}}\sin\left(\omega t-z\sqrt{\frac{\omega}{2\alpha}}\right). \tag{22}$$

The solutions for Θ_1 and Θ_2 can be approached through the use of Green's functions in heat conduction. The following method has been used by Birch (1950) for a similar problem.

The temperature T (x, y, z, t) in a semi-infinite medium initially at zero, due to a surface temperature ϕ (x, y, t) in S and zero outside S, is given by Carslaw and Jaeger (1947, p. 308, eq 3) as

$$T(x,y,z,t)=\frac{z}{8(\pi\alpha)^{3/2}}\int_0^t\left[\iint\frac{\phi(x',y',t')}{(t-t')^{5/2}}e^{-\frac{[(x-x')^2+(y-y')^2+z^2]}{4\alpha(t-t')}}dx'dy'\right]dt'. \quad (23)$$

By making the substitution

$$\beta=\left(\frac{(x-x')^2+(y-y')^2+z^2}{4\alpha(t-t')}\right)^{1/2}$$

and reversing the order of integration, equation 23 is reduced to

$$T(x,y,z,t)=\frac{2}{\pi^{3/2}}\iint\left\{\int_{\frac{r}{2\sqrt{\alpha t}}}^{\infty}\phi\left(x',y',t-\frac{r^2}{4\alpha\beta^2}\right)e^{-\beta^2}\beta^2 d\beta\right\}\frac{z}{r^3}dx'dy' \quad (24)$$

where $r^2=(x\text{-}x')^2+(y\text{-}y')^2+z^2$ (that is, r is the distance between the field point (x, y, z) and the element of surface $dx'dy'$).

Equation 24 can be rewritten

$$T(x,y,z)=\frac{2}{\pi^{3/2}}\iint\left\{\int_{\frac{r}{2\sqrt{\alpha t}}}^{\infty}\phi\left(x',y',t-\frac{r^2}{4\alpha\beta^2}\right)e^{-\beta^2}\beta^2 d\beta\right\}d\Omega \quad (25)$$

where $d\Omega$ is the element of solid angle subtended by $dx'dy'$ at the field point (x, y, z). By setting $\phi\left(x',y',t-\frac{r^2}{4\alpha\beta^2}\right)=B$, and integrating with respect to β, an expression for Θ_1 is obtained:

$$\Theta_1(x,y,z,t)=\frac{B}{2\pi}\iint\left\{\frac{r}{\sqrt{\pi\alpha t}}e^{-\frac{r^2}{4\alpha t}}+\text{erfc}\frac{r}{2\sqrt{\alpha t}}\right\}d\Omega. \quad (26)$$

By setting

$$\phi\left(x',y',t-\frac{r^2}{4\alpha\beta^2}\right)=D\sin\omega\left(t-\frac{r^2}{4\alpha\beta^2}\right)$$

an expression for Θ_2 is obtained:

$$\Theta_2(x,y,z,t)=\frac{2D}{\pi^{3/2}}\iint\left\{\int_{\frac{r}{2\sqrt{\alpha t}}}^{\infty}\sin\omega\left(t-\frac{r^2}{4\alpha\beta^2}\right)e^{-\beta^2}\beta^2 d\beta\right\}d\Omega. \quad (27)$$

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The sum of these two temperature fields gives the subsurface temperature disturbance due to a heated building (or other surface disturbance) in the region S. Θ_1 (equation 26) will be referred to as the principal disturbance inasmuch as it gives the temperature disturbance due to a building (or other feature) except for the effects of seasonal fluctuations. It can be thought of as giving a mean annual value of temperature disturbance at any subsurface position at any time after the surface region, S, was disturbed (for example, after the building was erected). The sum of Θ_0 and Θ_2 (equations 22 and 27) will be referred to as the seasonal effect as it gives the seasonal oscillation about the principal disturbance. Θ_2 will be called the seasonal disturbance. These three equations (22, 26, and 27) provide a complete solution to the heated-building problem for a homogeneous medium in which latent heat is neglected. The next two sections are discussions of equations 26 and 27 respectively.

THE PRINCIPAL DISTURBANCE, Θ_1

The principal disturbance as defined above is the temperature resulting in the ground ($z>0$) from a constant temperature difference, B, between a point inside the disturbed region, S, and the mean annual temperature of the undisturbed region outside S.

In equation 26 the disturbed region may have an arbitrary configuration (for example, the shape of the foundation of a heated building). It is evident, however, that this equation cannot be integrated directly for arbitrary S. The integration can be performed for certain simple situations, for example under the vertex of a circular sector of radius R and central angle λ in the plane $z=0$. In this case

$$\Theta_1(0,0,z,t)=\frac{Bz}{\pi^{3/2}}\int_0^{\lambda}d\lambda\int_z^{\sqrt{z^2+R^2}}\left[\frac{r}{2\sqrt{\alpha t}}\,e^{-\frac{r^2}{4\alpha t}}+\frac{\sqrt{\pi}}{2}\,\mathrm{erfc}\,\frac{r}{2\sqrt{\alpha t}}\right]\frac{dr}{r^2} \tag{28}$$

$$=\frac{B\lambda}{2\pi}\left[\mathrm{erfc}\,\frac{z}{2\sqrt{\alpha t}}-\frac{z}{\sqrt{z^2+R^2}}\,\mathrm{erfc}\,\frac{\sqrt{z^2+R^2}}{2\sqrt{\alpha t}}\right]. \tag{29}$$

For a circular region this reduces to

$$\Theta_1(0,0,z,t)=B\left[\mathrm{erfc}\,\frac{z}{2\sqrt{\alpha t}}-\frac{z}{\sqrt{z^2+R^2}}\,\mathrm{erfc}\,\frac{\sqrt{z^2+R^2}}{2\sqrt{\alpha t}}\right]. \tag{30}$$

The first term in equation 30 corresponds to the effect that would result if S covered the entire plane $z=0$, and the second term may be thought of as a correction for the finite radius, R, of S. By setting $\lambda=\pi$ and $\pi/2$ in equation 29, results are obtained for a semicircle and a quarter circle respectively. These results may be used to estimate the principal disturbance under the edges and corners of a building for depths that are small with respect to building dimensions. Under these conditions the correction term is negligible and the disturbance is respectively one-half and one-quarter of that produced by the "infinite building" (that is, S=entire plane $z=0$).

From equation 26 we obtain the relation

$$\lim_{t\to\infty} \Theta_1(x, y, z, t)=\frac{B}{2\pi}\,\Omega(x,y,z;S) \tag{31}$$

where Ω is the solid angle subtended by S at the field point (x, y, z). This special case is in agreement with results from potential theory. A useful result for computing the steady-state principal disturbance at a point (x, y, z) beneath or to one side of the rectangular region $-a\leq x\leq +a$, $-b\leq y\leq +b$ is given by

$$\Theta_1(x, y, z)=\frac{B}{2\pi}\left\{\tan^{-1}\left[\frac{(x+a)(y+b)}{z\sqrt{z^2+(x+a)^2+(y+b)^2}}\right]-\right.$$
$$\tan^{-1}\left[\frac{(x-a)(y+b)}{z\sqrt{z^2+(x-a)^2+(y+b)^2}}\right]-\tan^{-1}\left[\frac{(x+a)(y-b)}{z\sqrt{z^2+(x+a)^2+(y-b)^2}}\right]+$$
$$\left.\tan^{-1}\left[\frac{(x-a)(y-b)}{z\sqrt{z^2+(x-a)^2+(y-b)^2}}\right]\right\}. \tag{32}$$

This result is obtained by substituting the appropriate expression for Ω in equation 31.

From equation 29 we obtain

$$\Delta\Theta_1=\frac{B\lambda}{2\pi}\left\{\frac{z}{\sqrt{z^2+R_1^2}}\,\mathrm{erfc}\left(\frac{\sqrt{z^2+R_1^2}}{2\sqrt{\alpha t}}\right)-\frac{z}{\sqrt{z^2+R_2^2}}\,\mathrm{erfc}\left(\frac{\sqrt{z^2+R_2^2}}{2\sqrt{\alpha t}}\right)\right\}, \tag{33}$$

which represents the principal disturbance at a depth z beneath the vertex of the sector of a circular annulus with central angle λ, inner radius R_1, and outer radius R_2. The principal disturbance for a region of arbitrary shape may thus be evaluated by dividing the region into concentric annular sectors and applying equation 33. This method can be used to give any desired degree of accuracy by choosing λ and R_2-R_1 sufficiently small near the region's periphery. It can also be used to give rapid approximations in many cases. Generally, however, the calculation is simpler if the integration with respect to equation 26 is performed graphically. The graphical methods described below have been used previously by Birch (1950).

To carry out the graphical integration, a grid is constructed by gnomonic projection of meridians and parallels of a hemisphere onto its north polar tangent plane. The meridians and parallels are so selected that the plane of projection is subdivided into concentric annular sectors each subtending equal elements of solid angle at the sphere's center. The field point (x, y, z) is identified with the point at the center of the sphere, the projection plane is identified with the ground surface. Thus the radius of the sphere corresponds to the depth, z, of the field point beneath the surface.

A grid of this type is shown in plate 1. It is divided into twelve concentric zones designated by letters A through L. The region in zone A subtends 2 percent of the total solid angle (2π steradians) subtended by the plane $z=0$. Zone B subtends 4 percent; zone C, 6 percent; zone D, 8 percent; and zones E through J each subtend 10 percent of that subtended by the entire plane. Zones K and L each subtend 5 percent. Thus, according to equation 31, the steady-state effect of a circular building whose periphery coincided with the outer limit of zone L would be 90 percent of the effect of a building of infinite dimensions covering the entire plane $z=0$. The zones are cut by radial lines in such a way that each element of area of the grid subtends 0.1 percent of the solid angle subtended by the entire plane $z=0$. The depth, z, of the field point corresponds to 1 inch on the grid (pl. 1). Columns 2 and 3 of the table in plate 1 give the inner and outer radius of each zone in units of z. This grid can be used to calculate the effects of surface disturbances at horizontal distances from the field point as much as about 10 times the depth. Grids with other specifications are easily constructed.

The integrand in equation 26 depends only upon the ratio $r/\sqrt{\alpha t}$ (where α is the thermal diffusivity of the medium). In figure 22, the value, Φ, of the integrand in equation 26 is plotted against $r^2/\alpha t$. For any assumed αt, the value of the integrand corresponding to any r is read directly from this graph. For purposes of the graphical integration it will be assumed that all points in a given zone are at the same distance, $\bar{r}$, from the field point and that this distance is given by

$$\bar{r}=\sqrt{z^2+\bar{\rho}^2};\text{ that is, }\bar{r}^2=z^2\,[1+(\bar{\rho}/z)^2]$$

where $\bar{\rho}$ is the mean radius of that zone. The error introduced by this simplification should be taken into account in designing grids for special purposes.

The method of integration is as follows. First a depth, z, is selected at which temperature information is desired. Next a plan drawing of the outline of the building foundation is made on tracing paper. The drawing is in units of z; that is, if the depth under study is 10 feet, then, as z corresponds to 1 inch on the grid (pl. 1), the building will be drawn on the scale of 1 foot=0.1 inch. Next, the tracing paper

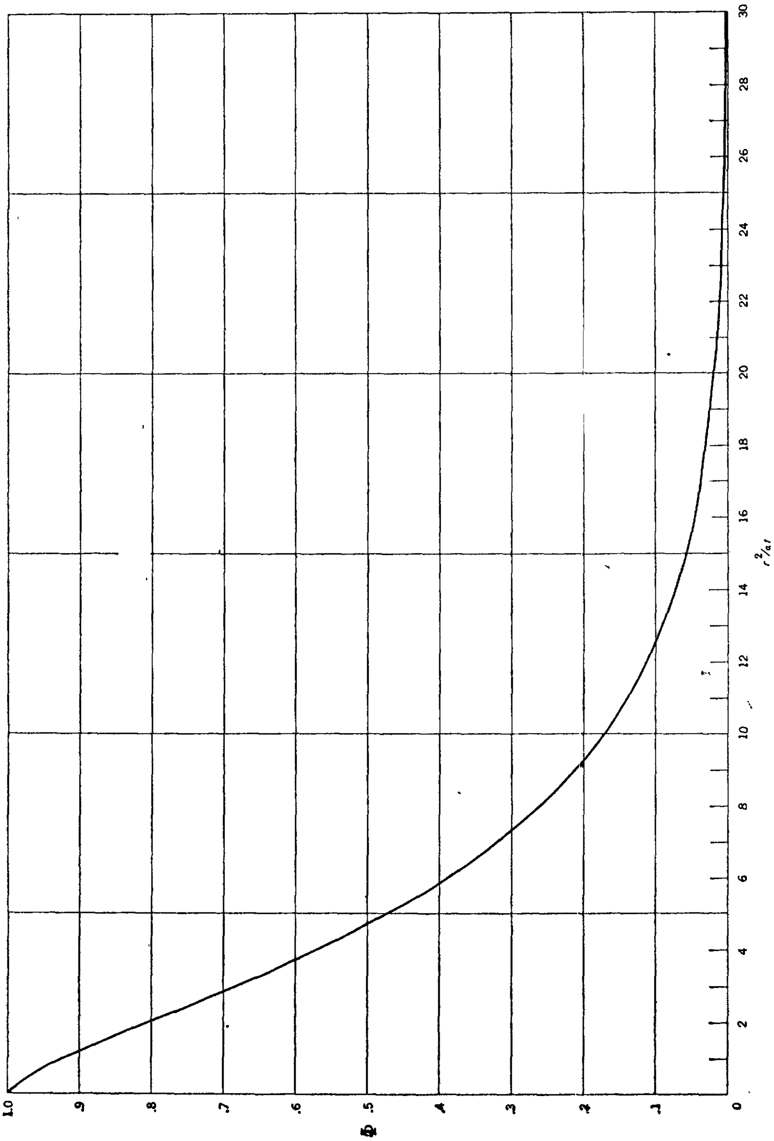

FIGURE 22.—Weighting factor $\left(\Phi=\frac{2}{\sqrt{\pi}}\frac{r}{2\sqrt{\alpha t}}e^{\frac{-r^2}{4\alpha t}}+\operatorname{erfc}\frac{r}{2\sqrt{\alpha t}}\right)$ as a function of $r^2/\alpha t$ for calculation of principal disturbance.

is positioned on the grid in such a way that the horizontal coordinates of the field point coincide with the center of the grid. If we want to calculate the principal disturbance under a corner of this building, the drawing is placed with this corner over the center of the grid. Next a value of αt is selected that depends upon the time for which temperature information is desired and the assumed diffusivity of the medium. The number of solid-angle elements subtended by the building in zone A is then counted. This number multiplied by 0.001 gives the contribution that this zone will make after equilibrium is established. Next multiply $z^2/\alpha t$ by the value of $1+(\bar{\rho}/z)^2$ for zone A given in column 4 of the table in plate 1. This product gives $\bar{r}^2/\alpha t$ for zone A. The corresponding value of the ordinate in figure 22 gives the amount by which this zone contribution must be weighted. This weighting factor multiplied by the steady-state contribution of the zone gives the contribution of zone A to the principal disturbance at (x, y, z) for the αt assumed. The process is repeated for all zones intersected by the building's outline. The contributions of all zones are then added to get the principal disturbance.

The accuracy of the graphical integration with respect to Ω can be checked quickly by comparison of the effect under the center of a circular region calculated by equation 30 with that obtained by the use of the grid. This was done for several cases and for a variety of assumed values for R, z, and αt. The effects calculated by the two methods agreed within 0.005 B. The error introduced by counting solid-angle increments on the grid can be checked by comparing the total with the computed solid angle subtended by the region at the field point if the geometry of the region permits (for example, for a rectangular region equation 32 may be used).

As an example of the application of the graphical method, we shall calculate the principal disturbance at 2 points 20 feet beneath a surface assumed to be disturbed over a rectangular region 40 feet by 100 feet. To fix ideas we may think of the effect of a building 40 feet by 100 feet in which the mean annual floor temperature is constant and exceeds the mean annual temperature of the ground surface outside the building by B°C. Let position 1 correspond to a point 20 feet beneath the center of such a building, and position 2 to a point 20 feet beneath a point on the transverse center line 10 feet from the building. The appropriate scale and positioning of the region's outline for these two calculations are shown in plate 2.

Table 1 illustrates the steps in the calculation of the principal disturbance for an assumed value $\alpha t=5\times10^5$ cm^2. Table 2 gives a summary of the results of table 1 and a comparison of the equilibrium principal disturbance as computed with the grid and with the solid-angle formula (eq 32). The discrepancy corresponds to a net error in counting of about one solid-angle unit on the grid (0.002π steradians).

The third column in table 1 shows the degree of advance toward equilibrium for each zone for the αt assumed. The portion of the disturbed region in zone J contributes on the order of only 10 percent of its full equilibrium effect while that in zones A through D contributes more than 90 percent. Because in position 2 the region subtends more of the outer zones, it is approximately 15 percent farther from equilibrium for the assumed αt (table 2). To contour the temperature at the 20-foot depth for a given αt, simply shift the tracing over the grid, changing only the values corresponding to columns 4 and 5 in table 1. Although the first calculation might take about 20 minutes, subsequent ones may each be made in about 5 minutes. To study the temperatures at the 10-foot depth, the scale of the region's outline is doubled and the process repeated. To study the change of temperature with time at a given point, values corresponding to those in columns 2 and 3 are changed but those corresponding to column 4 are not, and the calculation is again accomplished speedily. The change in principal disturbance with time, as calculated with the grid method, is shown graphically for positions 1 and 2 in figure 23.

TABLE 1.—*Calculation of principal disturbance*

[See pl. 2; fig. 22]

Zone	$\alpha t=5\times10^5$ cm^2		Position 1 $z=20$ ft, $x=0$, $y=0$		Position 2 $z=20$ ft, $x=30$ ft, $y=0$	
	$\frac{\bar{r}^2}{\alpha t}=\frac{z^2}{\alpha t}\left[1+\left(\frac{\rho}{z}\right)^2\right]$	Weighting factor, Φ	Number of Ω units subtended	Zone contribution (units of $10^{-3}B$)	Number of Ω units subtended	Zone contribution (units of $10^{-3}B$)
A	0.752	0.945	20	18.9	0	0
B	.804	.939	40	37.6	0	0
C	.896	.930	60	55.8	1¾	1.6
D	1.053	.914	80	73.1	14½	13.3
E	1.326	.880	99	87.1	31	27.3
F	1.773	.830	67	55.6	36	29.9
G	2.491	.740	46	34.0	39½	29.2
H	3.752	.600	34	20.4	42	25.2
I	6.309	.370	9	3.3	27	10.0
J	12.852	.093	0	.0	2	.2
Principal disturbance, units $10^{-3}B$			455 (equilibrium)	385.8 (transient)	193¾ (equilibrium)	136.7 (transient)

TABLE 2.—*Summary of principal-disturbance values*

Position	$\alpha t=\infty$		$\alpha t=5\times10^5$ cm^2	Percent of equilibrium at $\alpha t=5\times10^5$ cm^2
	Table 1	Equation 32		
1	0.455B	0.4559B	0.386B	84.8
2	.194B	.1945B	.137B	70.6

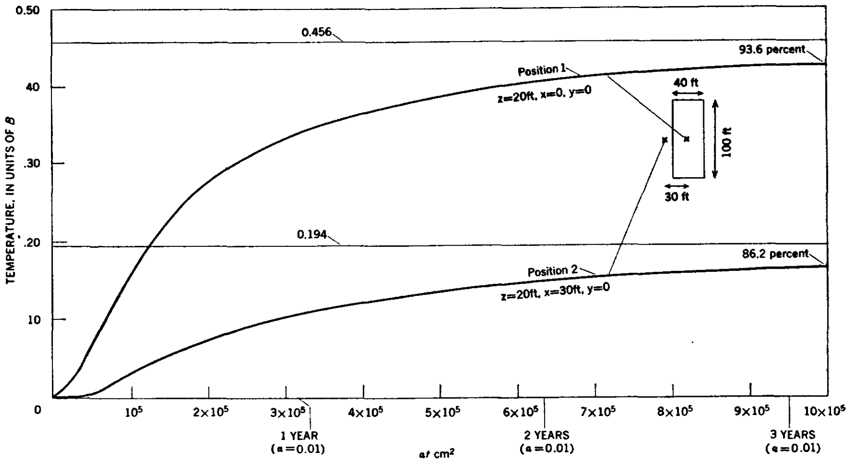

FIGURE 23.—Principal disturbance 20 feet beneath a 40- by 100-foot region with temperature $B°$ C above surrounding surface. Abscissa is $\alpha t\text{cm}^2$.

In the example given, the mean annual temperature disturbance, B, was assumed to be independent of position throughout the disturbed surface region. If it were different over some subregion (for example, in the vicinity of a furnace or an unheated wing), the number of solid angle units subtended by such a subregion would be weighted proportionally and the calculation carried out as before.

THE SEASONAL DISTURBANCE, Θ_2

The seasonal disturbance, Θ_2, as previously defined, is the temperature resulting in the ground ($z>0$) from a temperature in S of $D \sin \omega t$ while the temperature of the surface outside S is maintained at zero.

Equation 27 can be rewritten

$$\Theta_2(x, y, z, t)=\frac{2D}{\pi^{3/2}}\iint\left\{\int_0^\infty \int \sin \omega\left(t-\frac{r^2}{4\alpha\beta^2}\right)e^{-\beta^2}\beta^2 d\beta\right\}d\Omega-$$

$$\frac{2D}{\pi^{3/2}}\iint\left\{\int_0^{\frac{r}{2\sqrt{\alpha t}}} \sin \omega\left(t-\frac{r^2}{4\alpha\beta^2}\right)e^{-\beta^2}\beta^2 d\beta\right\}d\Omega. \quad (34)$$

The first integral in equation 34 represents the steady seasonal disturbance that will obtain after the surface disturbance in S has persisted for some time ($t \gg r^2/\alpha$). The second integral describes the manner in which the seasonal disturbance builds up for small values of time. The expression in braces in this integral will have to be evaluated numerically before the grid of the previous section can be used to perform the integration with respect to Ω.

For most problems the first integral in equation 34 is the more important quantity, as the second may eventually be neglected. The expression in braces in the first integral can be evaluated as follows:

$$\int_0^\infty \sin\omega\left(t-\frac{r^2}{4\alpha\beta^2}\right)e^{-\beta^2}\beta^2 d\beta$$
$$=\sin\omega t\int_0^\infty \cos\frac{\omega r^2}{4\alpha\beta^2}\, e^{-\beta^2}\beta^2 d\beta-\cos\omega t\int_0^\infty \sin\frac{\omega r^2}{4\alpha\beta^2}\, e^{-\beta^2}\beta^2 d\beta. \quad (35)$$

By integrating each of the expressions on the right side of equation 35 by parts and using the relations

$$\int_0^\infty e^{-x^2}\sin\frac{2q^2}{x^2}\frac{dx}{x^2}=\frac{\sqrt{\pi}}{4q}e^{-2q}[\cos 2q+\sin 2q]$$

(Bierens de Haan, 1939, table 369, formula 1)

$$\int_0^\infty e^{-x^2}\cos\frac{2q^2}{x^2}\frac{dx}{x^2}=\frac{\sqrt{\pi}}{4q}e^{-2q}[\cos 2q-\sin 2q]$$

(idem, table 369, formula 2)

$$\int_0^\infty e^{-x^2}\sin\frac{2q^2}{x^2}dx=\frac{\sqrt{\pi}}{2}e^{-2q}\sin 2q$$

(idem, table 263, formula 12)

$$\int_0^\infty e^{-x^2}\cos\frac{2q^2}{x^2}dx=\frac{\sqrt{\pi}}{2}e^{-2q}\cos 2q$$

(idem, table 263, formula 13)

we obtain

$$\int_0^\infty \sin\omega\left(t-\frac{r^2}{4\alpha\beta^2}\right)e^{-\beta^2}\beta^2 d\beta$$
$$=\frac{\sqrt{\pi}}{4}e^{-r\sqrt{\frac{\omega}{2\alpha}}}\left\{\sin\left(\omega t-r\sqrt{\frac{\omega}{2\alpha}}\right)+r\sqrt{\frac{\omega}{2\alpha}}\left[\cos\left(\omega t-r\sqrt{\frac{\omega}{2\alpha}}\right)+\sin\left(\omega t-r\sqrt{\frac{\omega}{2\alpha}}\right)\right]\right\}. \quad (36)$$

By substituting this result in equation 34,

$$\Theta_2(x,y,z,t)=\frac{D}{2\pi}\iint e^{-r\sqrt{\frac{\omega}{2\alpha}}}\left\{\sin\left(\omega t-r\sqrt{\frac{\omega}{2\alpha}}\right)+r\sqrt{\frac{\omega}{2\alpha}}\left[\cos\left(\omega t-r\sqrt{\frac{\omega}{2\alpha}}\right)+\sin\left(\omega t-r\sqrt{\frac{\omega}{2\alpha}}\right)\right]\right\}d\Omega-\text{transient term.} \quad (37)$$

As was true for the integral in equation 26, the integration in equation 37 is easily performed for a point (x,y,z) under the vertex of a circular sector, S, of radius R and central angle λ.

$$\Theta_2(0,0,z,t)=\frac{D\lambda}{2\pi}\left\{e^{-z\sqrt{\frac{\omega}{2\alpha}}}\sin\left(\omega t-z\sqrt{\frac{\omega}{2\alpha}}\right)-\frac{z}{\sqrt{z^2+R^2}}e^{-\sqrt{R^2+z^2}\sqrt{\frac{\omega}{2\alpha}}}\sin\left(\omega t-\sqrt{z^2+R^2}\sqrt{\frac{\omega}{2\alpha}}\right)\right\}-\text{transient term.} \quad (38)$$

For a circular region this reduces to

$$\Theta_2(0,0,z,t)=De^{-z\sqrt{\frac{\omega}{2\alpha}}}\sin\left(\omega t-z\sqrt{\frac{\omega}{2\alpha}}\right)-\frac{z}{\sqrt{z^2+R^2}}De^{-\sqrt{z^2+R^2}\sqrt{\frac{\omega}{2\alpha}}}\sin\left(\omega t-\sqrt{z^2+R^2}\sqrt{\frac{\omega}{2\alpha}}\right)-\text{transient term.} \quad (39)$$

In direct analogy to equation 30 the first term in equation 39 corresponds to the effect that would result if S covered the entire plane $z=0$, and the second term represents edge effects arising as a result of the finite radius, R, of S.

As explained above, the seasonal effect is obtained by adding the seasonal disturbance, Θ_2, and the normal undisturbed seasonal variation, Θ_0 (eq 22). Thus, after the transient effects have disappeared, the seasonal effect under the vertex of a circular sector of central angle λ and radius R is

$$\Theta_0+\Theta_2=Ae^{-z\sqrt{\frac{\omega}{2\alpha}}}\sin\left(\omega t-z\sqrt{\frac{\omega}{2\alpha}}\right)+\frac{D\lambda}{2\pi}\left[e^{-z\sqrt{\frac{\omega}{2\alpha}}}\sin\left(\omega t-z\sqrt{\frac{\omega}{2\alpha}}\right)-\frac{z^2}{\sqrt{z^2+R^2}}e^{-\sqrt{z^2+R^2}\sqrt{\frac{\omega}{2\alpha}}}\sin\left(\omega t-\sqrt{z^2+R^2}\sqrt{\frac{\omega}{2\alpha}}\right)\right] \quad (40)$$

or, using equation 21,

$$\Theta_0+\Theta_2=\left[A-\frac{\lambda}{2\pi}(A-C)\right]e^{-z\sqrt{\frac{\omega}{2\alpha}}}\sin\left(\omega t-z\sqrt{\frac{\omega}{2\alpha}}\right)+\frac{\lambda}{2\pi}(A-C)\frac{z}{\sqrt{z^2+R^2}}e^{-\sqrt{z^2+R^2}\sqrt{\frac{\omega}{2\alpha}}}\sin\left(\omega t-\sqrt{z^2+R^2}\sqrt{\frac{\omega}{2\alpha}}\right). \quad (41)$$

As in equation 30, equation 41 can be used to estimate effects under straight edges and corners of buildings for depths that are small relative to dimensions of the building. At such depths under the edge of a building, the seasonal temperature variation would be approximated by that produced by linear heat flow under an undisturbed surface whose seasonal temperature variation had an amplitude of ½ $(A+C)$. The surface amplitude in this case is the average of that inside the building and that outside. Under a square corner the effect would be approximated by that produced in linear heat flow by a surface temperature of amplitude ¼ $(3A+C)$.

The important special case of no seasonal temperature variation in S is obtained by setting $C=0$. Thus the seasonal temperature effect under the center of a circular region of constant temperature is seen from equation 41 to be

$$A\frac{z}{\sqrt{z^2+R^2}}\,e^{-\sqrt{z^2+R^2}\sqrt{\frac{\omega}{2\alpha}}}\sin\left(\omega t-\sqrt{z^2+R^2}\sqrt{\frac{\omega}{2\alpha}}\right). \tag{42}$$

Corresponding to equation 33, the seasonal effect under the vertex of the sector of an annulus with central angle λ, inner radius R_1, and outer radius R_2, is seen from equation 38 to be

$$\Delta\Theta_2=\frac{D\lambda}{2\pi}\left\{\frac{z}{\sqrt{z^2+R_1^2}}\,e^{-\sqrt{z^2+R_1^2}\sqrt{\frac{\omega}{2\alpha}}}\sin\left(\omega t-\sqrt{z^2+R_1^2}\sqrt{\frac{\omega}{2\alpha}}\right)-\right.$$
$$\left.\frac{z}{\sqrt{z^2+R_2^2}}\,e^{-\sqrt{z^2+R_2^2}\sqrt{\frac{\omega}{2\alpha}}}\sin\left(\omega t-\sqrt{z^2+R_2^2}\sqrt{\frac{\omega}{2\alpha}}\right)\right\}. \tag{43}$$

Equation 43 can be used to evaluate graphically the equilibrium seasonal disturbance at any point, resulting from a temperature $D\sin\omega t$ in an arbitrarily shaped region of the surface $z=0$.

As in the case discussed in the previous section, the calculation is somewhat simpler, though less precise, if the integration with respect to Ω is performed graphically with the aid of the grid of plate 1. The procedure is the same as that used to compute the principal disturbance, except that in place of figure 22, we now use plate 3, which gives the value of the integrand, Ψ, in equation 37 as a function of r^2/α for four values of ωt. Inasmuch as the seasonal disturbance is sinusoidal, with a period of 1 year at any point, its evaluation for only 2 values of ωt would be enough to determine its behavior throughout the annual cycle. The four curves are given in plate 3, so that additional points on the seasonal-effect-versus-time curve can be evaluated as a check.

As an extension of the example in the previous section, the equilibrium seasonal effect has been computed for positions 1 and 2, on the assumption that the temperature in the disturbed region is independent of time and that the temperature of the surface outside the region varies as $A\sin\omega t$. This seasonal effect is obtained by adding to Θ_0

(eq 22) the seasonal disturbance resulting from a temperature of $-A \sin \omega t$ in the disturbed region and zero outside (eq 37 with $C=0$). This seasonal effect superposed on the equilibrium principal disturbance is shown graphically in figure 24. These results may be pictured as an approximation to the temperature beneath a surface supporting a heated building whose floor temperature does not vary seasonally and is everywhere $B°$ C above the mean annual temperature of the surface outside. The numerical results in figure 24 are based upon an assumed diffusivity of 0.01 cm^2 per sec. The bottom curve in this illustration represents the temperature to be expected 20 feet beneath an undisturbed surface. A comparison of the curves representing the undisturbed case and position 2 in figure 24 reveals that the "building" has the effect of raising the mean annual temperature about $0.2B$, diminishing the amplitude of seasonal variation about 20 percent, and causing the seasonal temperature wave to lead the undisturbed wave by a phase angle corresponding to about 1 week. At position 1 the building has the effect of raising the mean annual temperature about $0.45B$, diminishing the amplitude by more than 85 percent, and causing the seasonal wave to lag the undisturbed wave by almost 2 months. As an example of the orders of magnitude involved, both A and B are close to 15° C for most heated buildings in the vicinity of Barrow, Alaska.

Cases in which the amplitude of seasonal variation of temperature in the disturbed region varies with position may be treated by the method discussed for variable B (p. 62).

Cases in which the phase of sinusoidal variation of surface temperature is different inside and outside the disturbed region may be treated by superposing the seasonal disturbance caused by a temperature $D' \sin \omega t$ in S on that of a temperature $D'' \cos \omega t$, where D' and D'' are adjusted to give the required phase difference and amplitude.

EFFECT OF IDEALIZING THE PROBLEM

In order to obtain the above results it was necessary to assume that the medium was homogeneous and to neglect the effects of latent heat absorbed (or liberated) by the change of state of ice (or water).

In many engineering problems connected with permafrost, a thermal disturbance of the surface results in a descent of the freezing isotherm and the consequent thawing of frozen ground. Because the heat required to melt the ice causes no change in temperature, the establishment of equilibrium conditions is retarded, and because the rate of descent of the freezing isotherm is, in general, a function of position, the amount by which the buildup of the temperature disturbance is retarded is also a function of position. When equilibrium is finally reached, however, the freezing isotherm is stationary and no latent

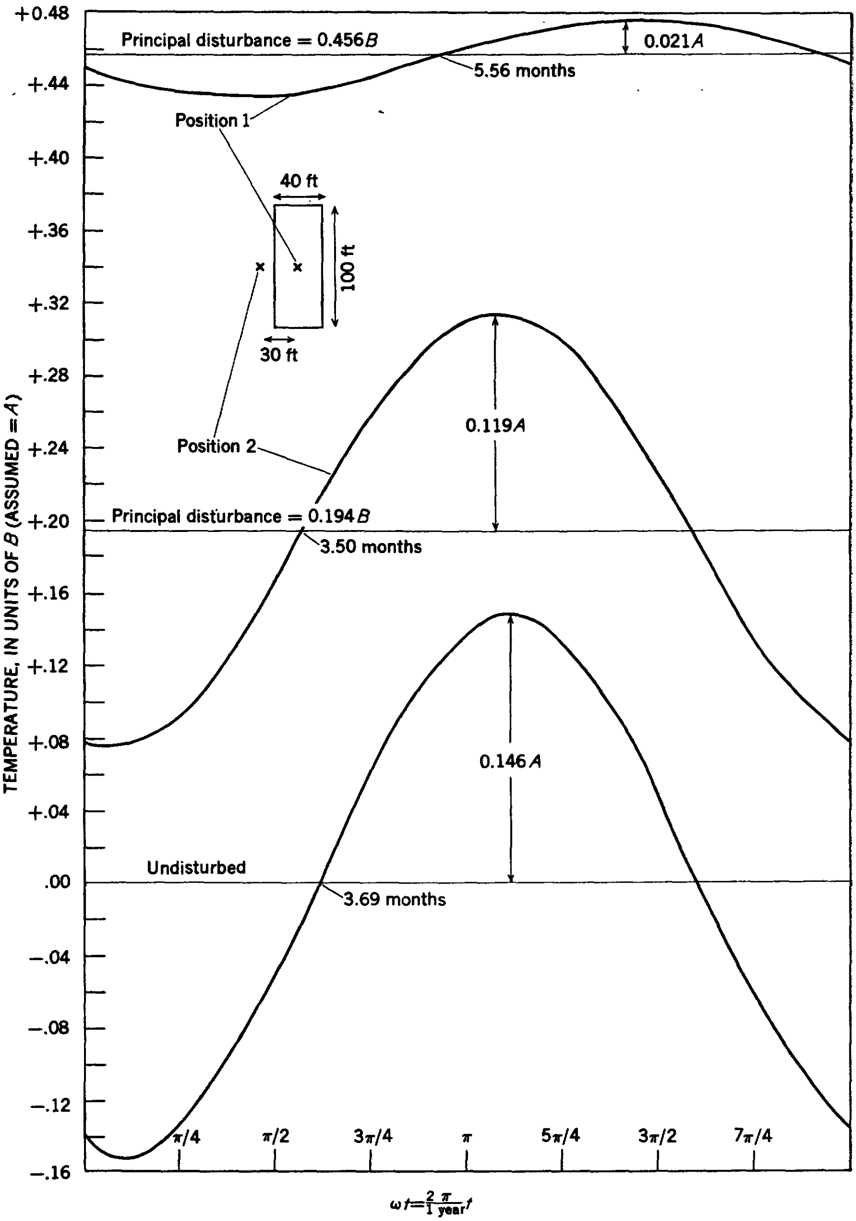

Figure 24.—Calculated temperature deviation from the undisturbed mean for large values of time at depth of 20 feet for $C=0$, $\alpha=0.01$ cm^2 per sec. B assumed equal to A to simplify presentation.

heat is absorbed or liberated. Thus, if the effect of latent heat alone is considered, the equilibrium principal disturbance would be unaltered but the establishment of equilibrium would be somewhat slower than that described above (for example, in figure 23). The shape of the isotherms in the transient stages would also be altered. Latent heat will alter the seasonal effects by diminishing the amplitude and introducing some asymmetry for points near the freezing isotherm. It will be observed that for most problems of foundation engineering, neglecting the effects of latent heat provides a margin of safety.

Effective homogeneity is rarely found in nature, and in general the ground is stratified with respect to thermal properties. For purposes of calculating the buildup of the principal disturbance in time or the equilibrium seasonal effects, an engineering approximation can probably be obtained in most cases by assuming an average value for the diffusivity and by using the results determined for the homogeneous case. Bracketing situations can commonly be obtained by assuming extreme values of diffusivity. It will be observed from equation 31 that the steady-state principal disturbance is independent of thermal properties, provided the material is homogeneous. In a stratified medium this is not so, although the homogeneous case will still probably give results satisfactory for engineering purposes in most cases.

SUMMARY

With the above results it is possible to evaluate the transient principal temperature disturbance and the equilibrium seasonal effects in a semi-infinite homogeneous medium in which surface temperature varies sinusoidally, but with different mean temperatures and amplitudes inside and outside a surface region of arbitrary shape. Two graphical methods are available. The first involves summing integrated contributions to the temperature; the second, a graphical integration with respect to solid angle. The first method is more precise but takes a little more time to apply. Since these methods would probably be primarily of interest to engineers who would use them to obtain approximate results, the second was discussed in greater detail. In the application of these methods to problems of foundation engineering in permafrost areas, some uncertainty is introduced by neglecting the effects of stratification and latent heat. The latter, however, provides a margin of safety in cases of thawing permafrost.

Although the discussion in this paper is devoted primarily to the thermal disturbance caused by a heated building, the methods described apply directly to several other important thermal problems in permafrost. For example, the geothermal disturbance caused by bodies of water in high latitudes is similar to that caused by a heated

building in that the mean annual temperature beneath the water is anomalously high and the amplitude of seasonal temperature variation is low. The thermal disturbance caused by the presence of a roadway usually results in a slight shift in the mean annual temperature but a large increase in the amplitude of seasonal variation. The increased amplitude causes a thickening of the active layer beneath the roadway, which frequently results in destruction of the road. Another useful application of these methods is to predict the extent to which ground will freeze beneath cold-storage vaults in temperate regions.

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