Equation of Continuity in Geology with Applications to the Transport of Radioactive Gas

GEOLOGICAL SURVEY BULLETIN 1052-I

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Equation of Continuity in Geology with Applications to the Transport of Radioactive Gas

By A. Y. SAKAKURA, CAROLYN LINDBERG, *and* HENRY FAUL

EXPERIMENTAL AND THEORETICAL GEOPHYSICS

GEOLOGICAL SURVEY BULLETIN 1052-I

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SYMBOLS

A	Substance carried
a	Number of moles of A participating in a chemical reaction
D	Pay-zone thickness
f	Porosity of medium
γ	Density of carrier
$\vec{v}$	Effective flow velocity of carrier
$\vec{F}$	Mass flux $=\gamma\vec{v}$
k_1	Reaction constant of a chemical reaction
k_2	Reaction constant of reverse reaction
k	Permeability
L_A	Rate of production of substance A per unit volume of medium, through chemical interaction or radioactive decay
M_0'	Radon density ($\mu\mu$c/unit volume) at STP from cylindrical shell source
p	pressure $=\sqrt{\Delta p^2 \dfrac{\log \frac{r}{r_w}}{\log \frac{r_e}{r_w}}+p_w^2}$
p_e	Reservoir pressure
p_w	Well head pressure
Δp^2	Squared pressure difference $=p_e^2-p_w^2$
p_0	Standard pressure
Q	Volume production at well-head conditions
Q_0	Volume production at STP
Q^*	Cumulative production at well-head conditions when radon first appears
Q_0^*	Cumulative production at STP when radon first appears
r_e	Reservoir radius
r_w	Well radius
r_0	Radius of cylindrical shell source
R	Inner radius of extended source
S_A	Rate of production of A by the medium per unit volume of the medium
T_0	Standard absolute temperature

EQUATION OF CONTINUITY IN GEOLOGY WITH APPLICATIONS TO THE TRANSPORT OF RADIOACTIVE GAS

By A. Y. Sakakura, Carolyn Lindberg, and Henry Faul

ABSTRACT

The transport of matter by fluids percolating through a porous medium is described by setting up a mass-conservation equation, analogous to that of hydrodynamics, for each of the substances in question. The sources and sinks of these equations serve to take into account the contribution from the medium and the interactions with other transported substances. The transport of radon by natural gases is treated in detail for the case of two important sources: the extended source and the cylindrical-shell source. From the solutions of the steady-state cases, the two source types can be distinguished, provided source density is uniform. From the solution of the transient case, the boundary of the source nearest to the gas well can be established. The results are applied to some selected data from the Texas Panhandle gas field.

INTRODUCTION

By the term "transport phenomena" we mean those situations in which the materials of interest are present in such small quantities that their individual diffusive properties are overwhelmed by the motion of the carrier. Such situations are common in geology. There are many instances where fluids flow, picking up, carrying, and depositing substances. Waters may transport and deposit various minerals. Gases and liquids may be tagged artificially with stable or radioactive tracers so that their flow may be observed. Sometimes gas may pick up radon, the gaseous decay product of radium contained in the pore space, and carry it as a natural tracer.

Under these conditions we can solve for the behavior of the carrier separately, and then set up an equation (or equations) of continuity for the substance of interest. These problems are greatly simplified by the availability of a large number of solutions to the various flow problems (Muskat, 1946). In the first part of this paper the equation for a general situation will be derived; the remainder of the paper will be devoted to the specific problem of radon transport.

Let A=substance carried,

a=moles of A participating in a chemical reaction,

$\vec{F}$=mass flux of carrier (grams per unit area per second),

η_A=concentration of substance A per gram of carrier,

f=porosity of the medium,

k_1=chemical-reaction constant,
k_2=reverse-reaction constant,
γ=density of the carrier (grams per unit volume),
S_A=rate of production of substance A by the porous medium per unit volume of the medium,
L_A=rate of production (or loss) of substance A per unit volume of the porous medium through chemical interaction or radioactive decay,
λ_A=decay constant of A (per second);

then the equation of continuity (mass conservation) becomes

$$\nabla \cdot (\vec{F}\eta_A) + f\frac{\partial}{\partial t}(\gamma\eta_A) = S_A + L_A,$$

$$\nabla \cdot (\vec{F}\eta_B) + f\frac{\partial}{\partial t}(\gamma\eta_B) = S_B + L_B \qquad (1.1)$$

and so on for other substances.

If the rate determining chemical reaction is of the form

$$aA + bB + cC + \ldots\ldots \leftrightarrows dD + eE\ldots\ldots,$$

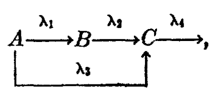
the L's can be shown easily to be

$$L_A = -f[k_1(\gamma\eta_A)^a(\gamma\eta_B)^b(\gamma\eta_C)^c\ldots - k_2(\gamma\eta_D)^d(\gamma\eta_E)^e\ldots]$$

$$L_B = \frac{b}{a}L_A,$$

$$L_D = -\frac{d}{a}L_A. \qquad (1.2)$$

In the radioactive chain



the L's become

$$L_A = -f\gamma(\lambda_1 + \lambda_3)\eta_A$$

$$L_B = f\gamma(\lambda_1\eta_A - \lambda_2\eta_B)$$

$$L_C = f\gamma(\lambda_3\eta_A + \lambda_2\eta_B - \lambda_4\eta_C), \text{ etc.} \qquad (1.3)$$

We may well have a combination of both processes. The flux F and the density γ of the carrier (if gases) are well known from solutions of Muskat's phenomenological equations of flow through porous media. The equations in general are nonlinear and will present great difficulty in solution except by approximate methods. Of course, if the importance of the problem warrants it, they may be solved by modern high-speed computers.

It should be noted that the reaction constants k_1 and k_2 in equation 1.2 are not necessarily the ones encountered in the laboratory, for the surface effects of the porous medium might be considerable.

ACKNOWLEDGMENTS

The writers wish to acknowledge the friendly cooperation of the personnel of various gas companies operating in the Panhandle area: Colorado Interstate Gas Co., Panhandle Eastern Pipeline Co., and Phillips Petroleum Co. This paper concerns work done by the U.S. Geological Survey on behalf of the Division of Research of the U.S. Atomic Energy Commission.

STEADY-STATE TRANSPORT OF RADON BY GASES

CYLINDRICAL SOURCE

We shall consider only sources possessing cylindrical symmetry about a gas well and extending throughout the vertical dimension of the gas pay zone. This problem is of paramount interest in a study of the radioactivity associated with some natural gases (Faul and others, 1954).

The gas sample tapped at the well will represent the source averaged over the surface of a cylinder centered about the well. The equation of continuity then becomes

$$\nabla\cdot\left(\vec{F}\eta\right)=-\lambda f\gamma\eta+\lambda\sigma\delta(r-r_0) \tag{2.1}$$

where (Muskat, 1946)

$$\vec{F}=\gamma\vec{v},$$

$$\gamma=\gamma' P,$$

$$\vec{v}=-\frac{k}{\mu}\nabla p=-\frac{Q}{2\pi D}\left(\frac{p_w}{p}\right)\frac{1}{r}\left(\frac{\vec{r}}{r}\right),\quad \left|\vec{v}\right|=v(r),$$

D=pay-zone thickness,

$$p=\sqrt{\frac{\Delta p^2}{\log\frac{r_e}{r_w}}\log\frac{r}{r_w}+p_w^2}\qquad\text{(pressure)},$$

$$\Delta p^2=p_e^2-{}_w^2,$$

p_e=reservoir pressure,

p_w=well-head pressure,

r_e=reservoir radius,

r_w=well radius,

Q=volume production at well-head conditions and where

σ=micromicrocuries of radon per unit area emitted by a cylindrical source of radius r_0, height D. The total source strength is $\lambda 2\pi r_0 D\sigma$,

η=radon concentration (micromicrocuries per gram of carrier),

The equation of continuity,

$$\frac{1}{r}\frac{d}{dr}\left[r\left(-\frac{Q}{2\pi D}\frac{p_w}{p}\frac{1}{r}\right)\gamma' p\eta\right]=-\lambda f\gamma' p\eta+\lambda\sigma\delta(r-r_0), \tag{2.2}$$

is integrated to

$$e^{-\lambda\Omega(r)}\eta(r)-\eta(r_w)=-\frac{2\pi D}{Q}\frac{\sigma\lambda}{\gamma' p_w}r_0 e^{-\lambda\Omega(r_0)}st(r-r_0), \tag{2.3}$$

where

$$\Omega(r)=f\frac{2\pi D}{Q}\frac{1}{p_w}\int_{r_w}^{r} rp(r)dr. \tag{2.4}$$

As no radon can flow backwards, $\eta=0$ when $r>r_0$. Thus, we find

$$\eta(r_w)=\frac{2\pi D r_0}{Q}\frac{\lambda\sigma}{\gamma' p_w}e^{-\lambda\Omega(r_0)} \tag{2.5}$$

The significance of the exponential term can be seen if we rewrite equation 2.4,

$$\Omega(r_0)=\int_{r_w}^{r_0}\frac{f}{|v(r)|}dr \tag{2.6}$$

which is merely the time elapsed while the radon travels from r_0 to r_w. Thus, the exponential factor is the decay factor as radon travels from r_0 to r_w. Ω can be calculated through integration by parts, and we find

$$\Omega(r_0)=\frac{f2\pi D}{Qp_w}\sum_{n=0}^{N}\frac{pr^2}{2}\left(-\frac{\Delta p^2}{2\log\frac{r_e}{r_w}}\right)^n\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-n\right)}\frac{1}{p^{2n}}\Bigg|_{r_w}^{r0}+R_N=\sum_{n=0}^{N}S_n+R_N \tag{2.7}$$

where the remainder, R_N, after N terms satisfies

$$\left|\frac{R_N}{S_N}\right|\leqq\left|\frac{\left[1-\left(\frac{p_w}{p}\right)^{2N-1}\right]}{\left[\left(\frac{p_w}{p}\right)^{2N-1}-\left(\frac{r_w}{r_0}\right)^2\right]}\right|$$

which is generally of the order of one. Thus, if

$$\frac{\Delta p^2}{2p_w^2\log\frac{r_e}{r_w}}<1,$$

the series is semiconvergent. We find then that the following expression is an adequate approximation.

$$\Omega(r_0) \approx \left\{ p(r_0) r_0^2 \left[1 - \frac{\Delta p^2}{4p^2(r_0) \log \frac{r_e}{r_w}} \right] - p_w r_w^2 \left[1 - \frac{\Delta p^2}{4p_w^2 \log \frac{r_e}{r_w}} \right] \right\} \frac{\pi D f}{Q p_w} \tag{2.8}$$

The expression is exact at $r = r_w$.

It is most convenient to express the measured radon content in terms of the gas sample tapped at the well head. Then the density (M_0), in micromicrocuries per unit volume at standard conditions (STP), of the radon due to a cylindrical shell source of radius r_0 and surface density σ is

$$M_0 = \gamma(T_0, p_0)\eta(r_w) = \frac{2\pi D r_0}{Q_0} \lambda \sigma e^{-\lambda \Omega(r_0)} \tag{2.9}$$

where

Q_0 = production at STP,

p_0 = standard pressure,

T_0 = standard temperature,

T_w = well-head temperature.

It should be noted that σ differs from the actual surface source density by a factor owing to the incomplete sweeping by the gas. The degree of sweeping is a function of the emanating power of the radium or uranium mineral present in the pore space. In the Panhandle gas field the radioactive parents are probably present in what is described as "asphaltic petroleum residues." The emanating power of this material is estimated at 10 percent (F. J. Davis, Oak Ridge National Laboratory, and J. N. Rosholt, Jr., U.S. Geological Survey, oral communications).

We shall now examine the behavior of solution 2.9. An examination of equation 2.8 reveals that for $r_0 \lesseqgtr 100$ feet the first term in square brackets can be neglected, although for all cases the second term is essentially zero. Thus, we rewrite equation 2.9 as

$$M_0 \approx \frac{2\pi D r_0}{Q_0} \lambda \sigma \exp \left[-\frac{\lambda f \pi D}{Q p_w} p(r_0) r_0^2 \left(1 - \frac{\Delta p^2}{4p^2(r_0) \log \frac{r_e}{r_w}} \right) \right]. \tag{2.10}$$

This can be rewritten using the relation

$$p^2 = p_e^2 - \left(\frac{\mu}{k} \right) \left(\frac{T_w}{T_0} \right) \frac{\log \frac{r_e}{r_0}}{\pi D} p_0 Q_0 \tag{2.11}$$

and upon taking the square root and retaining the first two terms

$$p \approx p_e\left(1-\frac{1}{2}\frac{\mu}{k}\frac{T_w}{T_0}\frac{p_0Q_0}{p_e^2}\frac{\log\frac{r_e}{r_0}}{\pi D}\right). \tag{2.12}$$

Equation 2.10 finally becomes

$$M_0 \approx \frac{2\pi D r_0}{Q_0}\lambda\sigma \exp\left[-\frac{\lambda f \pi D}{Q_0}\frac{T_0}{T_w}\frac{p_e}{p_0}r_0^2\right]\exp\left[\frac{\lambda f}{2}\frac{\mu}{k}\frac{r_0^2}{p_e}\left(\frac{1}{2}+\log\frac{r_e}{r_0}\right)\right]. \tag{2.13}$$

Several points are now immediately evident. If we multiply M_0 (measured at series of steady states) by corresponding values of Q_0 and plot against $1/Q_0$ on a semilogarithmic paper, the result will be a straight line. If the measured values yield a straight line when treated in the foregoing manner, the slope will yield the radius of the equivalent shell source. Moreover $\lambda\sigma 2\pi r_0 D$ is the total effective source strength which will be equal to the vertical intercept of the plot.

EXTENDED SOURCE

We are also interested in radon distribution originating from a source extending from R to the edge of the reservoir. We then replace σ by $\zeta\, dr_0$ (in micromicrocuries per unit volume) and integrate M_0 from R to r_e,

$$M_0' = \int_R^{r_e}\frac{\zeta}{\sigma}M_0 dr_0 = \frac{2\pi D}{Q_0}\lambda\zeta\int_R^{r_e} e^{-\lambda\Omega(r_0)}\, r_0 dr_0. \tag{2.14}$$

The integration can be readily carried out to first order in $\dfrac{\Delta p^2}{2p_w^2\log\frac{r_e}{r_w}}$. To this order, equation 2.8 gives sufficiently accurate value of $\Omega(r_0)$, with p_w^2 substituted for $p^2(r_0)$ in the denominator of the first square bracketed term. Upon expanding the ratio $\dfrac{p(r_0)}{p_w}$ to first order, we have

$$\lambda\Omega(r_0) \approx \frac{\pi D f\lambda}{Q}\left\{1+\omega\log\frac{r_0}{r_w}\right\}\left\{1-\frac{\omega}{2}\right\}r_0^2$$

$$\approx \frac{\pi D f\lambda}{Q}r_0^2\left\{1+\omega\left(\log\frac{r_0}{r_w}-\frac{1}{2}\right)\right\}$$

$$=\left(\frac{r_0}{R_0}\right)^2\left\{1+\omega\left(\log\frac{r_0}{r_w}-\frac{1}{2}\right)\right\} \tag{2.15}$$

where

$$\omega = \frac{\Delta p^2}{2p_w^2\log\frac{r_e}{r_w}}$$

with the additional restriction that

$$\frac{\Delta p^2}{p_w^2} < \frac{1}{2}.$$

The logarithm is expanded as

$$\log \frac{r_0}{r_w} \approx \log \frac{R_0}{r_w} - \frac{R_0 - r_0}{R_0} \tag{2.16}$$

Note that the expansion has very large error near r_w and r_e. However, the absolute magnitude of the exponent is very small at r_w so that the exponential is substantially one, whereas at r_e the exponent is so large that the exponential does not contribute to the integral. Moreover, the value of the logarithm is closely matched near R_0, the most significant portion of the integration interval. Thus

$$\begin{aligned}\lambda\Omega(r_0) &\approx \left(\frac{r_0}{R_0}\right)^2 \left[1 + \omega\left(\log \frac{R_0}{r_w} + \frac{1}{2}\right)\right] - \left(\frac{r_0}{R_0}\right)\omega \\ &\approx \alpha\left(\frac{r_0}{R_0}\right)^2 - \omega\left(\frac{r_0}{R_0}\right)\end{aligned} \tag{2.17}$$

$$\begin{aligned}M_0' &= \frac{2\pi D\lambda}{Q_0} R_0^2 \zeta \int_{\frac{R}{R_0}}^{\frac{r_e}{R_0}} e^{-\alpha y^2 + \omega y}\, y\, dy \\ &= \frac{Q\zeta}{Q_0\alpha f}\left[(e^{-\lambda\Omega(r_0)} - e^{-\lambda\Omega(r_e)}) + \frac{\omega e^{\frac{\omega^2}{4\alpha}}}{\alpha^{1/2}} \int_{\alpha^{1/2}\frac{R}{R_0} - \frac{\omega}{2\alpha^{1/2}}}^{\alpha^{1/2}\frac{r_e}{R_0} - \frac{\omega}{2\alpha^{1/2}}} e^{-y^2}\, dy\right]\end{aligned} \tag{2.18}$$

Although the integral can be expressed in terms of *erf* or *erfc* functions, it is much more useful to expand it. It is only necessary to use the first term of the expansion as the entire term is first order in ω. We distinguish several cases.

A. $r_0 \leqq R \leqq R_0 \qquad [\Omega(r_e) \to \infty,\ \Omega(R) \to 0]$

$$M_0' \approx \frac{Q\zeta}{Q_0\alpha f}\left(1 + \frac{\omega}{2}\sqrt{\frac{\pi}{\alpha}}\right) \tag{2.19}$$

B. $R_0 \approx R \ll r_e \qquad [\Omega(r_e) \to \infty]$

$$M_0' \approx \frac{Q\zeta}{Q_0\alpha f}\left(1 + \frac{1}{\frac{2\alpha R}{\omega R_0} - 1}\right) e^{-\lambda\Omega(R)} \tag{2.20}$$

C. $R_0 \ll R \lessapprox r_e$

$$M_0' \approx \frac{Q}{Q_0}\frac{\zeta}{\alpha f}\left[e^{-\lambda\Omega(R)}\left(1 + \frac{\omega R_0}{2\alpha R}\right) - e^{-\lambda\Omega(r_e)}\left(1 + \frac{\omega R_0}{2\alpha r_e}\right)\right] \tag{2.21}$$

From C one can infer for an extended source of inner radius R and outer radius R_1 such that $R_0 \ll R_1 < R \leq r_e$, that the equivalent shell source located at R has the surface density

$$\lambda\sigma \approx \frac{\lambda\zeta}{\alpha}\left(1-\frac{\omega}{2}\right)\frac{(R_1^2-R^2)}{2R}\left(1+\frac{\omega R_0}{2\alpha R}\right) \tag{2.22}$$

A semilogarithmic plot of measured values of M_0' times Q_0 versus $1/Q_0$ will not yield a straight line as with the finite source. Therefore, it may be possible to determine whether the radon can be attributed to a finite cylindrical or an extended source, assuming the source concentration remains constant for the latter source.

TRANSIENT-STATE TRANSPORT OF RADON BY GASES

Considerable geologic information can be obtained from artificially induced transient motion of a gas. For example, a well may be shut down for 2 or 3 weeks, or long enough to allow dynamic and radioactive equilibria to establish themselves. The well is then opened and produced at a known (and fairly constant) rate. Radon content is measured at suitable intervals for several days.

The time-dependent transport equation takes the following form:

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\left|\vec{v}\right|\gamma\eta\right)+f\frac{\partial}{\partial t}(\gamma\eta)=-f\gamma\eta+\gamma\sigma\delta(r-r_0) \tag{3.1}$$

where the quantities have the same meaning as before, except that the gas properties are now functions of time. We shall solve the problem formally to show the general procedure for solving equations of this type. Let

$$r\gamma\eta=\lambda\sigma r_0 st(r_0-r)st(t-t_0)e^{-\lambda(t-t')}\phi \tag{3.2}$$

where if

$$\frac{dr}{dt}=-\frac{1}{f}\left|\vec{v}\right|$$

with solution $G(r,t)$=constant, then t_0 and t' satisfy the following boundary conditions,

$$G(r,t_0)=G(r_0,0), \qquad G(r,t)=G(r_0,t') \tag{3.4}$$

t_0 is the time at which the radon, originating at r_0 at $t=0$, first appears at r; the step function states the physical condition that the radon concentration is zero up to that time. t' is the time at which radon must start from r_0 in order to arrive at r at time t. Then the exponential merely denotes the decay of radon en route. The step function in r_0 denotes the fact that radon cannot drift backwards. Inserting equation 3.2 now into equation 3.1 we find,

$$\delta(t-t_0)st(r_0-r)e^{-\lambda(t-t')}\phi\left[f+v\frac{dt_0}{dr}\right]+\lambda st(r_0-r)st(t-t_0)e^{-\lambda(t-t')}\phi\left[-v\frac{\partial t'}{\partial r}+f\frac{\partial t'}{\partial t}\right]$$
$$+\delta(r-r_0)[vst(t-t_0)e^{-\lambda(t-t)}\phi-1]+st(r_0-r)st(t-t_0)e^{-\lambda(t-t')}\left[-\frac{\partial}{\partial r}(v\phi)+f\frac{\partial\phi}{\partial t}\right]=0 \tag{3.5}$$

The first term is zero because

$$\frac{dt}{dr}=-\frac{f}{v(r,\,t)}$$

The second term also can be shown to be zero by the following argument,

$$G(r,t)=G(r_0,t')\text{ or }t'=H[r_0,G(r_0,t)]$$
$$-v\frac{\partial t'}{\partial r}+f\frac{\partial t'}{\partial t}=\frac{\partial H}{\partial G}\left[-v\frac{\partial G}{\partial r}+f\frac{\partial G}{\partial t}\right]=\frac{\partial H}{\partial G}f\frac{dG}{dt}\equiv 0$$

The remaining terms contain singularities of different orders, so they must vanish separately. The vanishing of the third term gives us the boundary condition on ϕ, as when $r=r_0$, $t_0=0$ and $t'=t$. Thus, when $r=r_0$,

$$\phi=\frac{1}{v(r_0,\,t)} \tag{3.6}$$

The remaining term finally becomes the differential equation for ϕ.

$$\frac{\partial}{\partial r}(v\phi)=f\frac{\partial\phi}{\partial t} \tag{3.7}$$

We solve equation 3.7 by successive substitution. On the right hand side of equation 3.7 we set ϕ equal to $\frac{1}{v(r,\,t)}$, which satisfies the boundary condition in equation 3.6. Then we find, upon solving equation 3.7,

$$\phi=\frac{1}{v(r,t)}\left[1-f\int_{r_0}^{r}\frac{1}{v^2(r,\,t)}\frac{\partial v}{\partial t}\,dr\right] \tag{3.8}$$

This process, of course, can be repeated indefinitely, that is, substitute equation 3.8 into the right-hand side of equation 3.7 and integrate. However, if the second term in equation 3.8 is small compared to the first, then we can conclude that equation 3.8 furnishes a sufficiently good solution. It may be pointed out that the first approximation yields an exact answer when t approaches infinity, as the steady state is approached wherein the velocity is no longer time dependent. We shall now show that the equation 3.8 is an adequate approximation. It has been shown previously (Ritchie and Sakakura, 1956) that

$$|v(r,t)|=\frac{\frac{1}{2}\frac{k}{\mu r_w}\left(-\frac{\partial\psi}{\partial\rho}\right)}{\sqrt{p_e^2-\psi}} \tag{3.9}$$

where

$$\psi=\frac{\Delta p^2}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{K_0(\sqrt{z}\rho)}{K_0(\sqrt{z})}\frac{e^{z\tau}}{z}\,dz$$

$$\tau=\frac{k}{\mu f}\frac{p_e}{r^2_w}t$$

$$\rho=\frac{r}{r_w}$$

and τ is large for almost all values of t. It further has been shown that the resulting expressions for the quantities in question are semiconvergent series in τ, and cannot be differentiated term by term. Thus, the contour integrals must be differentiated first, then integrated. If we retain only the dominant terms, we find the second term in the brackets of equation 3.8 to be $\frac{1}{2}\left(\frac{\rho^2-\rho_0^2}{\tau}\right)$ which is small, as the validity of the semiconvergent series requires. Thus, we can state that equation 3.8 yields an adequate solution. The radon density (at STP), from equations 3.2 and 3.8, becomes

$$M_0(r_w,t)=\gamma(r_0T_0)\eta(r_w,t)$$

$$=\lambda\sigma\frac{r_0}{r_w}\frac{T_w}{T_0}\frac{p_0}{p_w}st(r_0-r)st(t-t_0)\exp\left[-\lambda(t-t')\right]$$

$$\times\frac{1}{v(r_w,t)}\left[1+f\int_{r_w}^{r_0}\frac{1}{v^2(r,t)}\frac{\partial v}{\partial t}dr\right] \tag{3.10}$$

Transient gas-flow-phenomena are strongly dependent on the characteristics of the reservoir. These parameters are never sufficiently well known to warrant numerical evaluation of equation 3.10. The principal purpose of this derivation is to determine the nearest boundary of the radioactive source from which the radon may come.

NEAREST BOUNDARY OF SOURCE

We shall now discuss the most important aspect of the transient problem, common to the cylindrical source problem and to the extended source problem, which results from integrating the cylinder over r_0 from R to r_e. When radon first appears, it must necessarily come from the point nearest to the well. Thus, we are interested in the solution to the equation

$$\frac{dr}{dt}=-\frac{vr(,t)}{f} \tag{4.1}$$

with the boundary conditions,

$$t=0, \qquad r=r_0$$
$$t=t_0, \qquad r=r_w$$

that is, we are looking for t_0. Using the substitutions for τ and ρ (equation 3.9) and letting

$$\xi=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{K_0(\sqrt{z}\rho)}{K_0(\sqrt{z})}\frac{e^{z\tau}}{z}dz$$
$$\alpha=\frac{\Delta p^2}{p_e^2} \tag{4.2}$$

equation 4.1 can be rewritten into a dimensionless form

$$\frac{d\rho}{d\tau}=-\frac{\partial}{\partial\rho}\sqrt{1-\alpha\xi}=\frac{\alpha}{2}\frac{1}{\sqrt{1-\alpha\xi}}\frac{\partial\xi}{\partial\rho}=-\frac{\alpha}{2}\frac{1}{\sqrt{1-\alpha\xi}}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{K_1(\sqrt{z}\rho)}{K_0(\sqrt{z})}\frac{e^{z\tau}}{\sqrt{z}}dz \tag{4.3}$$

with $\rho=\rho_0$. when $\tau=\tau$ (eventually we shall set $\tau=0$) and $\rho=1$ when $\tau=\tau_0$. If we consider values of r_0 sufficiently small and (or) τ sufficiently large, the radical can be set equal to its steady state value $\frac{p}{p_e}$ and only the first term of the expansion of the Bessel functions need be considered in the Laplace inversion integral. It has been shown that

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{K_1(\sqrt{z}\rho)}{K_0(\sqrt{z})}\frac{e^{z\tau}}{\sqrt{z}}dz=\frac{1}{\rho}\Big[-2I_{1,-1}(\tau)+\ldots\Big] \tag{4.4}$$

where

$$I_{n,-\nu}=\frac{1}{2\pi i}\int_{\infty}^{0+}\frac{z^{\nu}e^{z\tau}}{\left(\log \beta^2\frac{z}{4}\right)^n}dz;$$

$$\beta=e^{\gamma},\ \gamma=\text{Euler's constant.}$$

Thus equation 4.3 becomes

$$\frac{d\rho}{d\tau}\approx\frac{\alpha}{\rho}\frac{p_e}{p}I_{1-1} \tag{4.5}$$

$$\int_1^{\rho_0}\frac{p_e}{p}\rho d\rho=\alpha\int_{\tau_0}^{\tau}I_{1,-1}d\tau=\alpha[I_{1,-2}(\tau)-I_{1,-2}(\tau_0)]. \tag{4.6}$$

We could obtain t from the above equation by fixing τ and solving for τ_0. We shall not do so for the same reason given for not evaluating equation 3.10. Returning now to finding the limiting case when $t=0$, τ in equation 4.3 represents some earlier value of τ, as the series diverge for $\tau=0$. However, it is known also that the cumulative production is

$$\int_1^t Qdt=Q^*=-2\pi D\frac{\Delta p^2}{p_e p_w}fr_w^2 I_{1,-2}(\tau) \tag{4.7}$$

so we can write equation 4.6 as

$$\int_1^{\rho_0} p\rho d\rho = \frac{p_w}{2\pi D f r_w^2} [Q^*(\tau_0) - Q^*(\tau)] \approx \frac{p_w}{2\pi D f r_w^2} Q^*(\tau_0), \qquad (4.8)$$

when we have dropped the cumulative production at τ as it is zero when t is equal to 0. Furthermore, we rewrite this in terms of the function in equation 2.4,

$$\Omega(r_0) = \frac{Q^*}{Q} \qquad (4.9)$$

which relates the function, Ω, to Q^*, the cumulative production at which radon first appears. This must be corrected for the volume of gas standing in the well before opening. If V were the volume of the well, we replace Q^* by $Q^* - V\frac{P_e}{P_w}$, and upon correcting to STP, we find

$$\Omega(r_0) = \frac{Q^*_0}{Q_0} - V\frac{p_e}{p_0}\frac{T_0}{T_w}\frac{1}{Q_0} \qquad (4.10)$$

as the expression which will yield the nearest boundary of the source.

APPLICATION TO THE PROBLEM OF RADON FLOW IN THE TEXAS PANHANDLE

A series of calculations were made to find the source density (equation 2.19) for five wells in the Texas Panhandle gas field. The exponential term in the equation is a function of the source radius R and of the pressure at R, and these quantities cannot be measured. If our assumption that R is small enough compared to the reservoir boundary radius r_e and that R can be set equal to the well radius r_w is valid, calculations of the source density can be made with data obtained on the surface. To test the validity of this assumption, we investigated the location of the nearest boundary of the radioactive source. Equation 4.10 yields the boundary in terms of the time $\Omega(r_0)$ it takes the radon to travel from the source to the well and can be calculated from well data. In order to determine the actual distance to the source, we made use of equation 2.8 which expresses Ω in terms of r_0. For each well various values are assigned to r_0, the corresponding Ω is calculated (table 1) and plotted against r_0 giving a curve of

TABLE 1.—*Data for calculated values of nearest boundary of radon source for five Colorado Interstate gas wells in the Texas Panhandle area*

Well	Q_0^* (10^6ft^3)	Q_0 (10^6ft^3 per day)	V (ft^3)	Reservoir pressure, p_e (psi)	$\Omega(r_0)$ (days)	r_0 (ft)
Thompson B-7	2.1	2.50	1300	302	−0.001	r_w
Thompson A-1	3.5	2.66	1210	252	.006	7
Masterson A-2	1.7	2.17	1120	273	−.001	r_w
Kilgore A-11	1.9	1.84	1440	317	−.005	r_w
Thompson C-1	2.5	2.61	1270	313	.0002	r_w

the type shown in figure 89. The r_0 corresponding to Ω, calculated from equation 4.10 is found from these curves. No ambiguity arises from using the equations to find r_0 to justify setting R equal to r_e; the location of a cylindrical source will serve equally well in locating the nearest boundary of an extended source.

The graphs of $\Omega(r_0)$ versus r_0 were calculated from equation 2.8. Here, $p(r_0)$ was found from the relation

$$p^2(r_0)=p_w^2+\Delta p^2\frac{\log\frac{r_0}{r_w}}{\log\frac{r_e}{r_w}} \tag{5.1}$$

where p_w is measured at the well, Δp^2 is determined from $\Delta p^2=p_e^2-p_w^2$ where p_e was the measured shutdown pressure in October 1951, r_e was taken equal to 500 feet, and r_w equal to ⅜ foot. The values assigned to r_0 were r_w, 1, 2, 5, 10, 25, and 50 feet. The pay-zone thickness D was taken equal to 50 feet, as no exact data were available on the actual thickness. Q_0 is the production measured at STP at time of sampling, and T_w was taken as 300° K. The porosity of the pay zone f is known to vary between 8 and 13 percent in the Panhandle area; 10 percent was used in the calculations. The various values assumed are reasonable, and the error involved is negligible in the overall results.

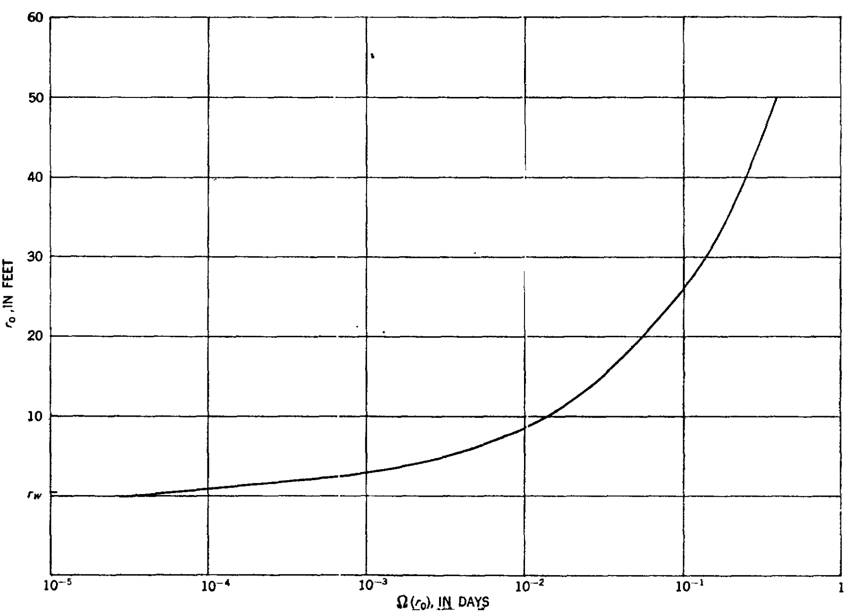


FIGURE 89.—Typical curve for computing distance of radon source from well.

The nearest boundary $\Omega(r_0)$ was computed from equation 4.10, where $V=r_w^2\pi h$ where h is the total depth of the drill hole and r_w, p_e, p_0, T_w, and T_0 are taken as before. Q_0^* was taken from the graphs (Q_0 versus M_0') of dynamic tests on the wells (figs. 90, 91, 92, 93, 94) at the point at which radon first appears; Q_0 was taken from the production curve at the same time as Q_0^* was chosen. The results are summarized in table 2. Graphs, similar to figure 89, for each well were used to find r_0.

TABLE 2.—*Total calculated uranium source density for wells in the Texas Panhandle gas field*

Company and well	Well-head pressure, p_w (psi)	M_0' ($\mu\mu c$ per l)	10^{-6} gU per g
Panhandle Eastern: Kilgore 1-16	232	522	9.08
Colorado Interstate:			
Thompson B-2 [1]	216	475	7.54
Thompson B-7	204	226	3.46
Thompson A-1	215	163	2.64
Masterson A-2	206	71	1.10
Kilgore A-11	240	45	.81
Thompson C-1	223	23	.38

[1] Average of 10 samples.

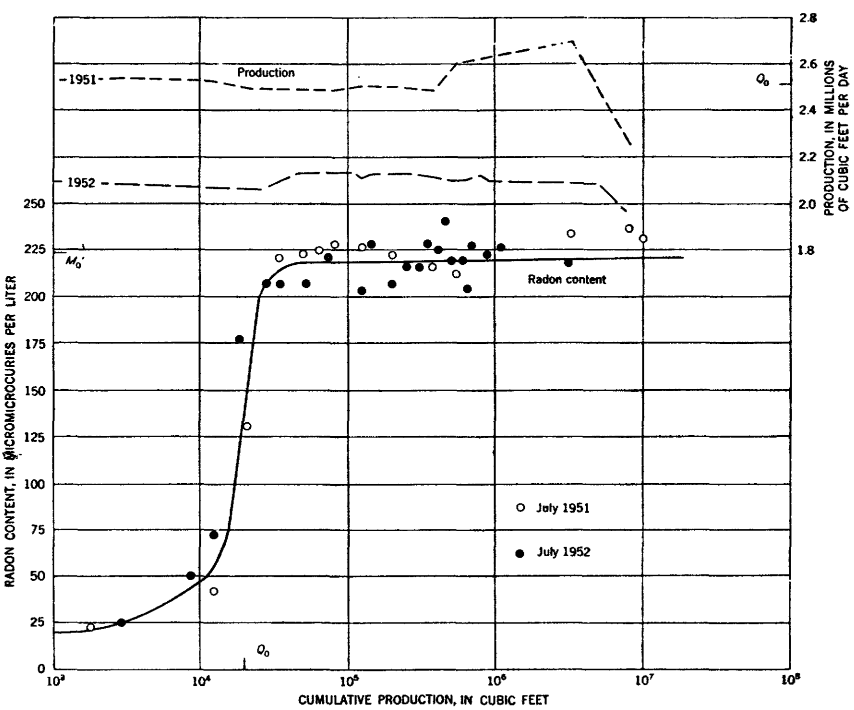


FIGURE 90.—Dynamic test data for Thompson B-7 well giving points at which data were taken for calculation.

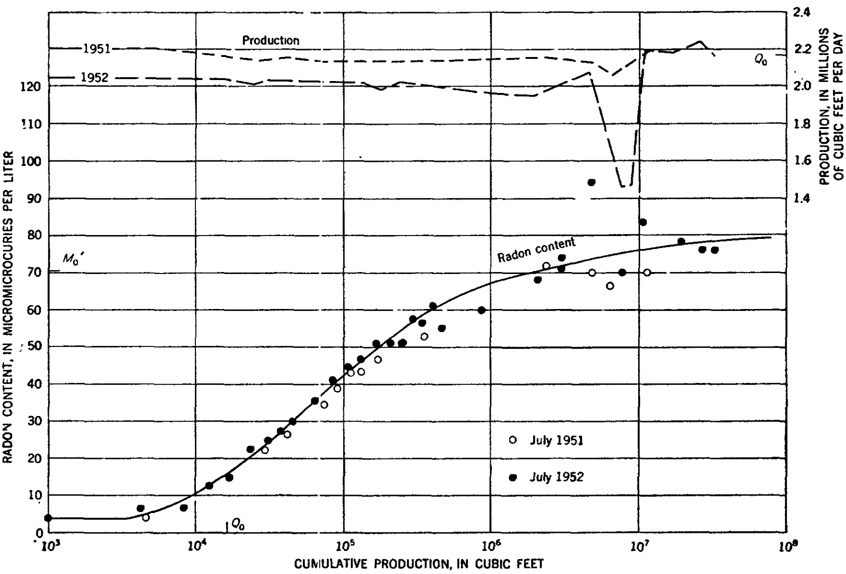


FIGURE 91.—Dynamic test data for Masterson A-2 well giving points at which data were taken for calculation.

The negative values of Ω can be attributed to various sources of error: (1) assumed values of the parameters; (2) neglect of higher order terms in the derivation of the formulas; (3) assumption that the gas in the well expands isothermally, which is not true in general; (4) the uncertain effect of acidizing on the well radius and volume—caliper logs in the Panhandle area show that the well diameter increases about an inch after acidizing; it seems logical that acidizing could produce near-cavernous regions at the well bottom—and (5) experimental error involved in measuring p_w, p_e, Q_0, M_0' and h. We have taken these negative values to mean zero time, giving $r_0=r_w$. These negative values also give us a rough measure of the error involved—that is, ± 0.005. Thus, even though the Ω (r_0) versus r_0 plot for Thompson A–1 gives $r_0=7$ feet, it would be more appropriate to assume $r_0=r_w$ here.

The effective source density τ, computed from

$$\xi=\frac{p_w}{p_0}\frac{T_0}{T_w}fM_0'\left[1+\frac{\Delta p^2}{2p_w^2\log\frac{r_e}{r_w}}\right] \tag{5.2}$$

as the other factor in equation 2.19, is always very nearly equal to one for the wells in question. M_0', the measured radon content at the well, was chosen after the radon content reaches equilibrium (figs. 90, 91, 92, 93, 94) and p_w is the pressure at the time of sampling.

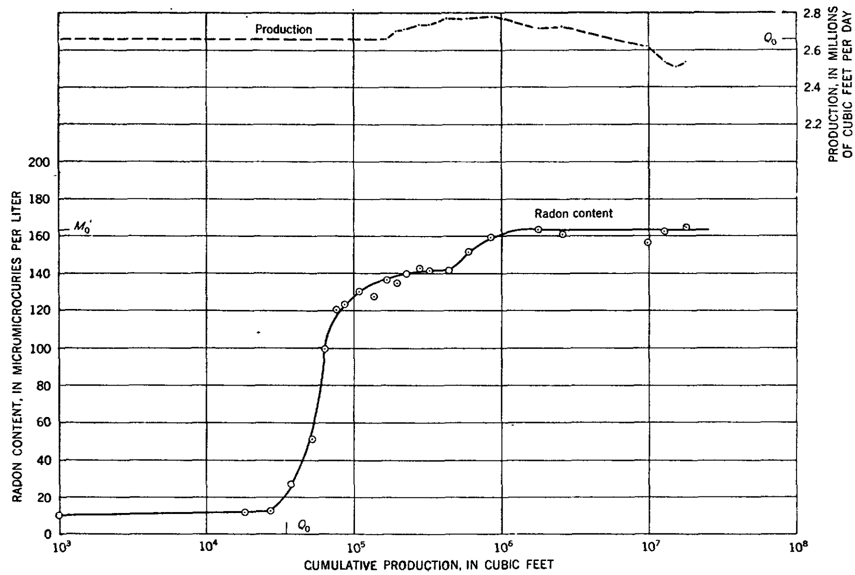


FIGURE 92.—Dynamic test data for Thompson A-1 well giving points at which data were taken for calculation.

The remaining parameters were chosen as before. The results were converted from micromicrocuries per liter to grams of uranium per gram of the medium. As the effective source is assumed to have an emanating power of 10 percent, ξ was corrected to give total source density ξ'. These source densities are shown in table 2.

The character of the graph of radon content versus cumulative production for the Masterson A-2 well is different from the other four graphs. Although the radon first appears relatively shortly after opening, thus giving $r_0=r_w$, the intensity continues to rise slowly and the equilibrium value is not reached until about 2 million cubic feet have been produced. This behavior was thought anomalous, and the tests were repeated a year later with the same result. The slow rise could be explained by assuming that the producing zone has several layers of varying permeability and that the less permeable rock contains more radioactive parent elements. An alternative, and possibly better, explanation would be a lack of symmetry in the radon source about the well. It is plausible that the radium concentration could increase gradually in any direction away from the hole. Similarly, the double knee in the graph for Thompson A-1 could be interpreted as evidence of a localized, sharply bounded concentration of radium. The value for ξ' shown in table 2 would then be a weighted average of the two sources, the farther (and stronger) source being less effective. It must be emphasized that all the source strengths in table 2 are minimum values. It is known from geological studies (J. W. Mytton,

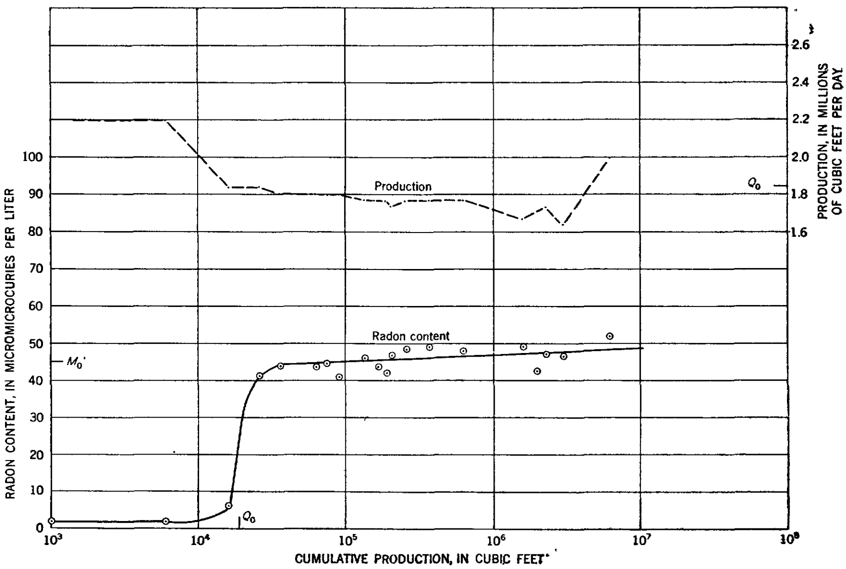


FIGURE 93.—Dynamic test data for Kilgore A-11 well giving points at which data were taken for calculation.

oral communication) in this area that the radioactive asphaltic residues generally tend to obstruct the permeability. Consequently, the more concentrated sources are likely to be swept less effectively by the gas.

DISCUSSION AND CONCLUSIONS

To recapitulate, we list here the various formulas derived in this paper under the restriction that $\omega < \dfrac{1}{2\log\dfrac{r_e}{r_w}}$

I. Steady state, cylindrical source at r_0:

$$M_0(r_w) = \frac{2\pi r_0 D}{Q_0}\lambda\sigma \exp\left[-\lambda\Omega(r_0)\right] \tag{2.9}$$

$$\approx \frac{2\pi r_0 D}{Q_0}\lambda\sigma \exp\left[\frac{-\lambda\pi f D}{Q p_w}\, p(r_0) r_0^2 \left(1 - \frac{\Delta p^2}{4p^2(r_0)\log\frac{r_e}{r_w}}\right)\right] \tag{2.10}$$

II. Steady state, extended source from R to r_e

$$A.\quad r_0 \leqq R \leqq R_0,\ M_0' \approx \frac{T_w}{T_0}\frac{p_0}{p_w}\frac{\zeta}{\alpha f}\left[1 + \frac{\omega}{2}\sqrt{\frac{\pi}{\alpha}}\right] \tag{2.19}$$

$$B.\quad R_0 \approx R << r_e,\ M_0' \approx \frac{T_w}{T_0}\frac{p_0}{p_w}\frac{\zeta}{\alpha f}\left[1 + \frac{1}{\frac{2\alpha R}{\omega R_0} - 1}\right] e^{-\lambda\Omega(R)} \tag{2.20}$$

$$C.\quad R_0 << R \lessapprox r_e,\ M_0' \approx \frac{T_w}{T_0}\frac{p_0}{p_w}\frac{\zeta}{\alpha f}\left[e^{-\lambda\Omega(R)}\left(1 + \frac{\omega R_0}{2\alpha R}\right) - e^{-\lambda\Omega(r_e)}\left(1 + \frac{\omega R_0}{2\alpha r_0}\right)\right. \tag{2.21}$$

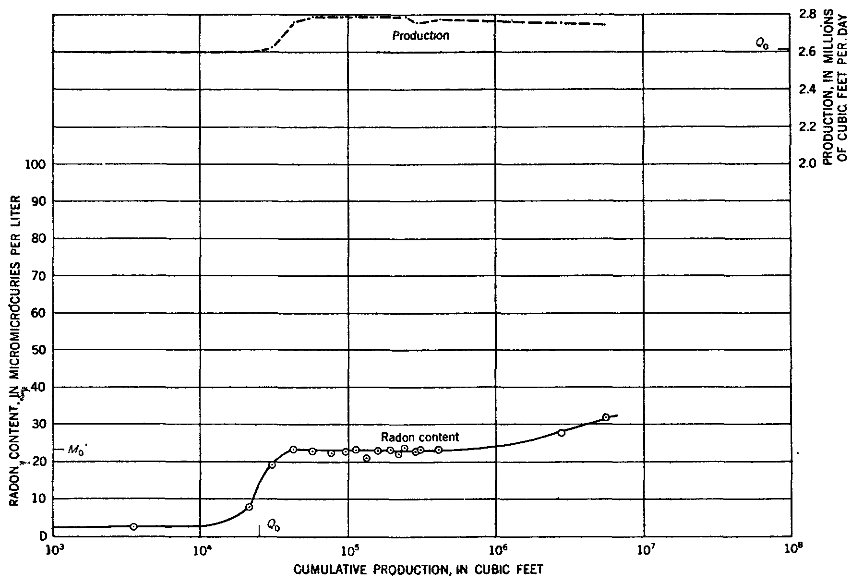


FIGURE 94.—Dynamic test data for Thompson C-1 well giving points at which data were taken for calculation.

III. Steady state, effective surface density of an extended source from R to R_1 where $R_0 << R < R_1 \lesssim R_0$

$$R_0 << R < R_1 \lesssim r_e, \quad \lambda\sigma \approx \frac{\lambda\zeta\left(1-\frac{\omega}{2}\right)(R_1^2-R^2)}{\alpha 2R}\left(1+\frac{\omega R_0}{2\alpha R}\right) \quad (2.22)$$

IV. Transient state, cylindrical source:

$$M_0(r_w,t) = \frac{1}{v(r_w,t)}\left[1+f\int_{r_w}^{r_0}\frac{1}{v^2(r,t)}\frac{\partial v}{\partial t}\,dr\right]\exp\left[-\lambda(t-t')\right]$$

$$\times\frac{\lambda\sigma r_0}{r_w}\frac{T_w}{T_0}\frac{p_0}{p_w}\,st(r_0-r)st(t-t_0) \quad (3.10)$$

V. Nearest boundary of radon source:

$$\Omega(r_0) = \frac{Q_0^*}{Q_0} - V\frac{p_e}{p_0}\frac{T_0}{T_w}\frac{1}{Q_0}$$

Geologic conditions are never so simple and clear cut as indicated here. Geologic structures certainly are not homogeneous or isotropic, nor are they in a convenient geometric configuration. The greatest source of error in the evaluation of source densities is the value of the emanating power. Moreover, it should be noted that the source distribution obtained will be an average, and localized high concentrations are likely to go unnoticed. The calculation of source densities is made on the basis of extended source distribution, assuming that a uniform distribution would be the most probable one. In the

Texas Panhandle gas field, from which examples were selected the edge of the source can be considered to be at the well according to the calculations on dynamic test data, although there is some ambiguity in determining the production at which radon first appears.

It is interesting to note that three of the wells (Masterson A–2, Kilgore A–11, and Thompson C–1) exhibit normal uranium density, whereas the others show abnormally high uranium content.

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