# Periodic Heat Flow in a Stratified Medium With Application to Permafrost Problems

GEOLOGICAL SURVEY BULLETIN 1083-A

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*By* ARTHUR H. LACHENBRUCH

EXPERIMENTAL AND THEORETICAL GEOPHYSICS

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*A discussion of temperature fluctuations in a stratified medium caused by a periodically varying surface temperature with special reference to the effects of seasonal temperature variation on gravel fills, the natural active layer and permafrost*

UNITED STATES GOVERNMENT PRINTING OFFICE, WASHINGTON : 1959

UNITED STATES DEPARTMENT OF THE INTERIOR

ROGERS C. B. MORTON, *Secretary*

GEOLOGICAL SURVEY

V. E. McKelvey, *Director*

First printing 1959

Second printing 1975

For sale by the U.S. Geological Survey, National Center, Reston, Virginia 22092

# CONTENTS



# ILLUSTRATIONS

[All plates in pocket]

## TABLES

## SYMBOLS

The following is a partial list of mathematical symbols for convenient reference. Others are explained in the text.

- $\theta$ Temperature (°C)
- $F$ Negative of the mean annual ground surface temperature in °C
- $\rho$ Density (gm/cm$^{-3}$)
- $c$ Specific heat (cal gm$^{-1}$ °C$^{-1}$)
- $K$ Thermal conductivity (cal cm$^{-1}$ sec$^{-1}$ °C$^{-1}$)
- $\alpha$ $=K/\rho c$, thermal diffusivity (cm$^2$ sec$^{-1}$)
- $\beta$ $=\sqrt{K\rho c}$, thermal contact coefficient (cal cm$^{-2}$ °C$^{-1}$ sec$^{-1/2}$)
- $\omega$ Angular frequency ($\omega=1.99\times10^{-7}$ radians per second for a period equal to 1 year)
- $\gamma$ $(2\alpha/\omega)^{1/2}$ for a period equal to 1 year
- $M$ $=\dfrac{\beta_1-\beta_2}{\beta_1+\beta_2}$
- $S$ $=1+2Me^{-2X\sqrt{\frac{\omega}{2\alpha_1}}}\cos 2X\sqrt{\frac{\omega}{2\alpha_1}}+M^2e^{-4X\sqrt{\frac{\omega}{2\alpha_1}}}$

EXPERIMENTAL AND THEORETICAL GEOPHYSICS

# PERIODIC HEAT FLOW IN A STRATIFIED MEDIUM WITH APPLICATION TO PERMAFROST PROBLEMS

By Arthur H. Lachenbruch


## ABSTRACT

Solutions to the Fourier heat equation for quasi-steady periodic flow in a stratified semi-infinite medium can be obtained readily by standard methods. The results have wide application to studies of earth-temperature variations induced by diurnal, annual, and other periodic variations in ground-surface temperature. Much of the previous work on this subject has been interpreted with reference to the solution for the homogeneous case; and this can be seriously in error when applied to stratified earth materials.

One application of the theory is to the important problem of determining the minimum thickness of gravel fill required to maintain the material on which it rests (the subgrade) in a perennially frozen state in permafrost areas. The results indicate that the required fill thickness is very sensitive to the thermal properties of the subgrade. If a thin layer of material with a low thermal contact coefficient, such as spruce logs, is placed between the fill and subgrade, the thickness of fill required to maintain undisturbed permafrost can be greatly reduced.

The thermal properties of the soil beneath the layer supporting plant growth can have an important influence on the temperatures in that layer. This effect, which cannot be explained by studies of the ground surface and the surficial layer, is likely to have important application to plant ecology in the arctic.

The theory yields an approximate method of estimating the effect of winter snow cover on the mean annual temperature of the ground surface.


## INTRODUCTION

When a simple harmonic temperature variation is impressed on the ground surface, the resulting thermal wave passes downward into the earth. If the periodic variation persists over a long period of time, initial transient effects die out, and the temperature at each depth oscillates with the same frequency as the surface temperature but with an amplitude and phase that diminish with increasing depth. This condition, which is sometimes referred to as a quasi-steady state, provides a very useful model for study of the effects of annual, diurnal, and other roughly harmonic variations of the ground-surface temperature. It has been customary to interpret these phenomena on the basis of the well-known solution to the Fourier heat equation for a homogeneous semi-infinite solid whose surface temperature varies sinusoidally with time. In most cases, however, it is more realistic to consider the ground as stratified, and important aspects of many problems are not treated adequately by the homogeneous case.

Solutions for periodic heat flow in a stratified semi-infinite medium are developed below together with a few examples of their application to ground temperature studies. Throughout this paper it is assumed that the dominant mode of heat transfer, beneath the ground surface, is conduction. Where appreciable amounts of heat are transferred by other means, such as the movement of water, these effects must be considered separately. A second assumption is that the water in the soil has a well-defined freezing point, which is taken here to be 0° C. This assumption should not result in any serious difficulty insofar as the major applications are concerned.

Lettau (1954) has presented an ingenious solution to the more general but inverse problem of finding the thermal parameters as a function of depth from a knowledge of the temperature as a function of depth, periodic in time. His results are useful in analyzing observational temperature data. In the present approach, the temperature as a function of depth and time is predicted from a knowledge of the surface temperature and thermal parameters as a function of depth. These results are useful in predicting the temperature in stratified media.

## ACKNOWLEDGMENTS

This work was carried out by the U.S. Geological Survey and was supported in part by the Office of Naval Research and the Arctic Research Laboratory; the Bureau of Yards and Docks, U.S. Navy; and the Snow, Ice, and Permafrost Research Establishment, U.S. Army.

I am grateful to Arthur Y. Sakakura for providing a careful check of the mathematical material in this chapter and to Professor Francis Birch for a critical reading of the manuscript. Max C. Brewer is responsible for the temperature data presented. Gordon W. Greene and B. Vaughn Marshall assisted ably with numerical computations and the preparation of illustrations.

## THE TWO-LAYER PROBLEM

### THEORY

In this section the ground is represented as the semi-infinite medium, $x>0$, composed of materials with one set of thermal properties (denoted by the subscript 1) in the stratum $0<x<X$ and a second set of properties (denoted by the subscript 2) in the region $x>X$, shown in figure 1. It will be assumed that the temperature of the surface

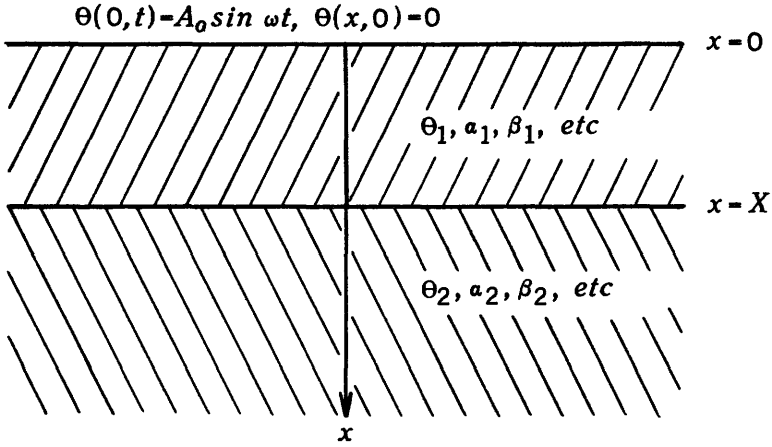


FIGURE 1.—Conditions of the two-layer problem.

$x=0$ varies as $A_0 \sin \omega t$, and that the temperature is initially zero for all $x$. (Any other initial condition would serve equally well in determining solutions valid for large values of time.) Stated formally, the problem is to find the temperature, $\Theta(x,t)$, which satisfies the following equations:

$$\Theta(x,t)=\Theta_1, 0<x<X$$

$$=\Theta_2, X<x<\infty$$

$$\frac{\partial^2\Theta_1}{\partial x^2}=\frac{1}{\alpha_1}\frac{\partial\Theta_1}{\partial t}, 0<x<X \tag{1}$$

$$\Theta_1(0,t)=A_0 \sin \omega t, x=0, t>0 \tag{2}$$

$$\frac{\partial^2\Theta_2}{\partial x^2}=\frac{1}{\alpha_2}\frac{\partial\Theta_2}{\partial t}, X<x<\infty \tag{3}$$

$$\Theta_1(X,t)=\Theta_2(X,t), x=X \tag{4}$$

$$K_1\frac{\partial\Theta_1}{\partial x}=K_2\frac{\partial\Theta_2}{\partial x}, x=X \tag{5}$$

$$\Theta_1=\Theta_2=0, t=0 \tag{6}$$

$$\Theta_2\rightarrow 0, x\rightarrow\infty. \tag{7}$$

The Laplace transform will be denoted by a bar, thus

$$\bar{f}(x,p)=\int_0^\infty e^{-pt}f(x,t)dt.$$

With this notation, the transformation of equations 1 through 7 leads to

$$\frac{d^2\bar{\Theta}_1}{dx}=q_1^2\bar{\Theta}_1, q_1^2=\frac{p}{\alpha_1}, \; 0<x<X \tag{8}$$

$$\bar{\Theta}_1(0)=A_0\frac{\omega}{p^2+\omega^2}, \; x=0 \tag{9}$$

$$\frac{d^2\bar{\Theta}_2}{dx^2}=q_2^2\bar{\Theta}_2, q_2^2=\frac{p}{\alpha_2}, \; X<x \tag{10}$$

$$\bar{\Theta}_1(X)=\bar{\Theta}_2(X), \; x=X \tag{11}$$

$$K_1\frac{d\bar{\Theta}_1}{dx}=K_2\frac{d\bar{\Theta}_2}{dx}, \; x=X \tag{12}$$

$$\bar{\Theta}_2(x)\rightarrow 0, \; x\rightarrow\infty. \tag{13}$$

Following a standard procedure, we assume solutions of the form

$$\bar{\Theta}_1(x,p)=C\,e^{q_1x}+D\,e^{-q_1x}, \; 0<x<X \tag{14a}$$

$$\bar{\Theta}_2(x,p)=E\,e^{-q_2x}, \; X<x \tag{14b}$$

where $C$, $D$, and $E$ are functions which are independent of $x$ and are to be determined from equations 8 through 13. Upon evaluating $C$, $D$, and $E$, and substituting in equation 14 we obtain:

$$\bar{\Theta}_1(x,p)=A_0\,\frac{\omega}{p^2+\omega^2}\,e^{-\sqrt{p}\,\frac{x}{\sqrt{\alpha_1}}}\left\{\frac{1+Me^{-2\sqrt{p}\,\frac{X-x}{\sqrt{\alpha_1}}}}{1+Me^{-2\sqrt{p}\,\frac{X}{\sqrt{\alpha_1}}}}\right\}, \; 0<x<X \tag{15a}$$

$$\bar{\Theta}_2(x,p)=A_0\,\frac{\omega}{p^2+\omega^2}\,e^{-\sqrt{p}\left[\frac{X}{\sqrt{\alpha_1}}+\frac{x-X}{\sqrt{\alpha_2}}\right]}\left\{\frac{1+M}{1+Me^{-2\sqrt{p}\,\frac{X}{\sqrt{\alpha_1}}}}\right\}, \; X<x \tag{15b}$$

where

$$M=\frac{\beta_1-\beta_2}{\beta_1+\beta_2}$$

$$\beta=\sqrt{K\rho C}.$$

The thermal parameter $\beta$ will be referred to as the "contact coefficient," it is also known as "thermal inertia" and "conductive capacity." It plays a very important role in the theory of transient heat flow through stratified media.

There are two convenient ways to manage the inversion of equations 15a and 15b. The first is to expand the denominator in a Taylor series. The terms of the resulting series are each of the same form as the transform of the solution for the homogeneous case, and hence the term by term inversion can be written down directly. Thus for the steady oscillating part of the solution this method leads to

$$\Theta_1(x,t)=A_0 e^{-x\sqrt{\frac{\omega}{2\alpha_1}}}\sin\left[\omega t-x\sqrt{\frac{\omega}{2\alpha_1}}\right]$$

$$+A_0\sum_{n=1}^{\infty}(-M)^n\left\{e^{-(2nX+x)\sqrt{\frac{\omega}{2\alpha_1}}}\sin\left[\omega t-(2nX+x)\sqrt{\frac{\omega}{2\alpha_1}}\right]\right.$$

$$\left.-e^{-(2nX-x)\sqrt{\frac{\omega}{2\alpha_1}}}\sin\left[\omega t-(2nX-x)\sqrt{\frac{\omega}{2\alpha_1}}\right]\right\},\ 0<x<X \qquad (16a)$$

$$\Theta_2(x,t)=A_0(1+M)\sum_{n=0}^{\infty}(-M)^n e^{-\sqrt{\frac{\omega}{2}}\left[\frac{x-X}{\sqrt{\alpha_2}}+\frac{(2n+1)X}{\sqrt{\alpha_1}}\right]}$$

$$\times\sin\left[\omega t-\sqrt{\frac{\omega}{2}}\left(\frac{x-X}{\sqrt{\alpha_2}}+\frac{(2n+1)X}{\sqrt{\alpha_1}}\right)\right],x>X. \qquad (16b)$$

The first term in equation 16a represents the solution for the homogeneous case, while the series represents the correction due to the contrast in contact coefficients at the interface, $x=X$. Equations 16a and 16b are useful in analyzing problems in which the stratification produces only small departures from the homogeneous case.

When the effects of stratification are great, it is usually more convenient to use expressions for $\Theta_1$ and $\Theta_2$ in closed form. These can be obtained by direct application of the Laplace transform inversion theorem to equation 15.

Inasmuch as $|M|<1$, $\overline{\Theta}_1$ and $\overline{\Theta}_2$ have poles only at $p=\pm i\omega$ on the principal branch and branch points at the origin. They are therefore single valued within and on the contour shown in figure 2. Hence

$$\Theta(x,t)=\frac{1}{2\pi i}\int_{u_0-i\infty}^{u_0+i\infty}\overline{\Theta}(x,p)e^{pt}dp$$

$$=\text{Res}\,(\overline{\Theta}_{p=+i\omega})+\text{Res}\,(\overline{\Theta}_{p=-i\omega})-\frac{1}{2\pi i}\left\{\lim_{R\to\infty}\left[\int_{I}+\int_{V}\overline{\Theta}e^{pt}dp\right]\right.$$

$$\left.+\lim_{\delta\to 0}\left[\int_{II}+\int_{IV}\overline{\Theta}e^{pt}dp\right]+\lim_{\epsilon\to 0}\int_{III}\overline{\Theta}e^{pt}dp\right\} \qquad (17)$$

where

$$u_0=Re(p_0)>0.$$

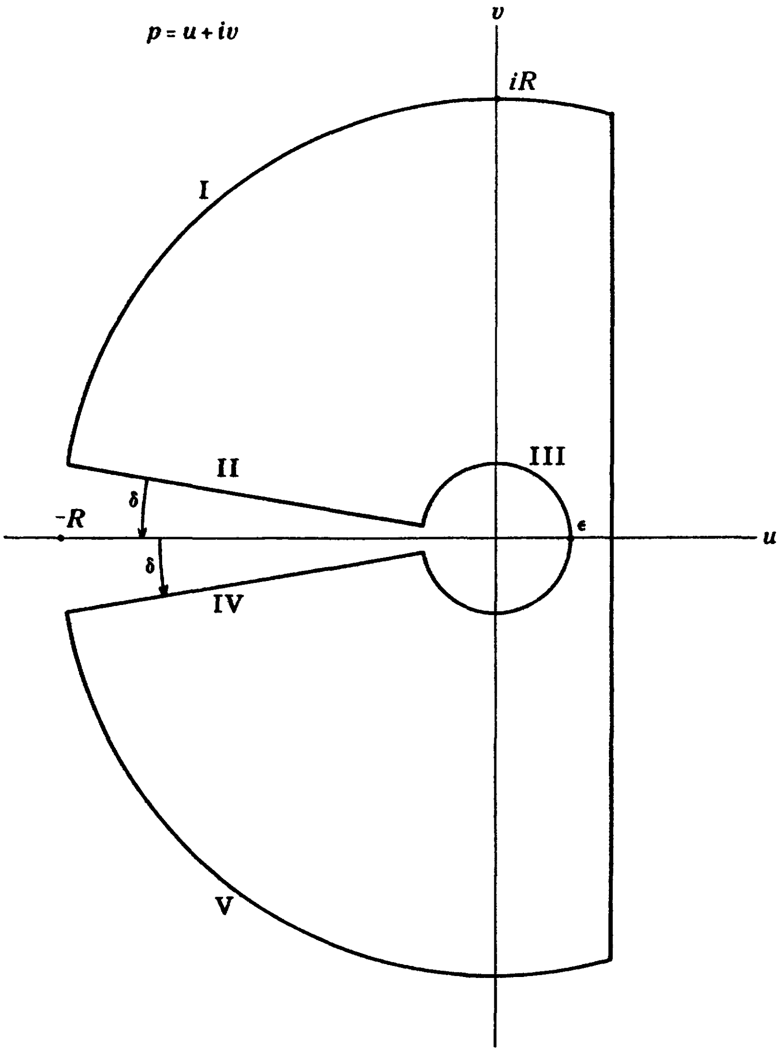


FIGURE 2.—Contour for inversion of $\bar{\Theta}(x, p)$.

In the limit as $R \rightarrow \infty$ and $\epsilon \rightarrow 0$ the integral over the contours I, V, and III vanish. As $\delta \rightarrow 0$ the integrand in the integral over contours II and IV each contain as a factor a descending exponential in $t$, while the remaining factor is bounded. Hence, the contribution of these integrals to $\Theta$ $(x,t)$ vanishes as $t$ becomes infinite. They, therefore, represent transient effects associated with the initial conditions assumed. For reasons that will be discussed below, we are interested only in the steady oscillating part of the solution which must be given simply by the sum of the residues at the two poles $p = \pm i\omega$.

On evaluating the residues and neglecting the initial transient effects, we obtain

$$\Theta_1(x,t)=A_0e^{-x\sqrt{\frac{\omega}{2\alpha_1}}}\frac{1}{S}\Bigg\{\sin\left[\omega t-x\sqrt{\frac{\omega}{2\alpha_1}}\right]$$

$$+Me^{-2(X-x)\sqrt{\frac{\omega}{2\alpha_1}}}\sin\left[\omega t-(2X-x)\sqrt{\frac{\omega}{2\alpha_1}}\right]$$

$$+Me^{-2X\sqrt{\frac{\omega}{2\alpha_1}}}\sin\left[\omega t+(2X-x)\sqrt{\frac{\omega}{2\alpha_1}}\right]$$

$$+M^2e^{-2(2X-x)\sqrt{\frac{\omega}{2\alpha_1}}}\sin\left[\omega t+x\sqrt{\frac{\omega}{2\alpha_1}}\right]\Bigg\} \qquad (18a)$$

$$\Theta_2(x,t)=A_0(1+M)e^{-X\sqrt{\frac{\omega}{2\alpha_1}}-(x-X)\sqrt{\frac{\omega}{2\alpha_2}}}\times\frac{1}{S}\Bigg\{\sin\left[\omega t-\left(X\sqrt{\frac{\omega}{2\alpha_1}}\right.\right.$$

$$\left.\left.+(x-X)\sqrt{\frac{\omega}{2\alpha_2}}\right)\right]+Me^{-2X\sqrt{\frac{\omega}{2\alpha_1}}}\sin\left[\omega t+\left(X\sqrt{\frac{\omega}{2\alpha_1}}-(x-X)\sqrt{\frac{\omega}{2\alpha_2}}\right)\right]\Bigg\} \qquad (18b)$$

where

$$S=1+2Me^{-2X\sqrt{\frac{\omega}{2\alpha_1}}}\cos 2X\sqrt{\frac{\omega}{2\alpha_1}}+M^2e^{-4X\sqrt{\frac{\omega}{2\alpha_1}}}.$$

It is seen that these results reduce to the corresponding solution for the homogeneous case (Carslaw and Jaeger, 1947)

$$\Theta(x,t)=A_0e^{-x\sqrt{\frac{\omega}{2\alpha}}}\sin\left(\omega t-x\sqrt{\frac{\omega}{2\alpha}}\right) \qquad (19)$$

by setting

$$\beta_1=\beta_2 \text{ and } \alpha_1=\alpha_2.$$

## APPLICATION TO THE GRAVEL-FILL PROBLEM

The problem that will now be considered is that of finding the minimum thickness of gravel fill necessary to keep the material on which it rests (the subgrade) perennially frozen in permafrost areas. This problem is of great practical importance inasmuch as the thawing of the subgrade produces mechanical effects which can result in costly damage to a roadway or airfield. In approaching this problem analytically, it is customary to treat the ground as homogeneous with thermal properties selected to approximate those of the fill material.

In what follows, we shall consider the dependence of the minimum fill thickness on the thermal properties of the subgrade as well as the properties of the fill itself.

The problem is to find the minimum thickness, $X$, of gravel fill with properties $K_1$, $\rho_1$, $c_1$, $\alpha_1$, required to maintain the material on which it rests (with properties $K_2$, $\rho_2$, $c_2$, $\alpha_2$) in a perennially frozen state when the surface temperature of the gravel varies annually as $A_0 \sin \omega t$. In other words, we seek the thickness of fill that will reduce the amplitude of annual temperature variation at its base ($x=X$) to $F$; where $F$ is the freezing temperature of water referred to the mean annual surface temperature as zero. A fortunate feature of the problem formulated in this way is that the complicating effects of latent heat can often be neglected in the first analysis without serious error. This is because superfreezing temperatures occur only in the gravel, which is generally highly permeable and well drained and hence has a very low moisture content. If the fill material is emplaced in the winter or early in spring, initial transient effects can be neglected, and only the periodic part of the solution need be considered.

At the interface $x=X$, equations 18a and 18b give

$$\Theta(X,t)=A_0(1+M)e^{-X\sqrt{\frac{\omega}{2\alpha_1}}}\times$$

$$\frac{1}{S}\left\{\sin\left(\omega t-X\sqrt{\frac{\omega}{2\alpha_1}}\right)+Me^{-2X\sqrt{\frac{\omega}{2\alpha_1}}}\sin\left(\omega t+X\sqrt{\frac{\omega}{2\alpha_1}}\right)\right\}. \quad (20)$$

The amplitude of the expression in braces in equation 20 is equal to $\sqrt{S}$. Therefore equation 20 can be written

$$\Theta(X,t)=A(X)\sin(\omega t-\phi(X))$$

where

$$A(X)=A_0e^{-X\sqrt{\frac{\omega}{2\alpha_1}}}\left\{\frac{1+M}{\sqrt{S}}\right\} \quad (21)$$

$$\tan[\phi(X)]=\left\{\frac{1-Me^{-2X\sqrt{\frac{\omega}{2\alpha_1}}}}{1+Me^{-2X\sqrt{\frac{\omega}{2\alpha_1}}}}\right\}\tan\left[X\sqrt{\frac{\omega}{2\alpha_1}}\right]. \quad (22)$$

Numerical results for equation 21 are presented in terms of the three dimensionless parameters, $X\sqrt{\frac{\omega}{2\alpha_1}}$, $\beta_1/\beta_2$, and $A(X)/A_0$, in plate 1. The range of values selected for $\beta_1/\beta_2$ includes most cases likely to be encountered in ground temperature studies. The phase relationships given by equation 22 are shown in figure 3. From plate 1 it is seen

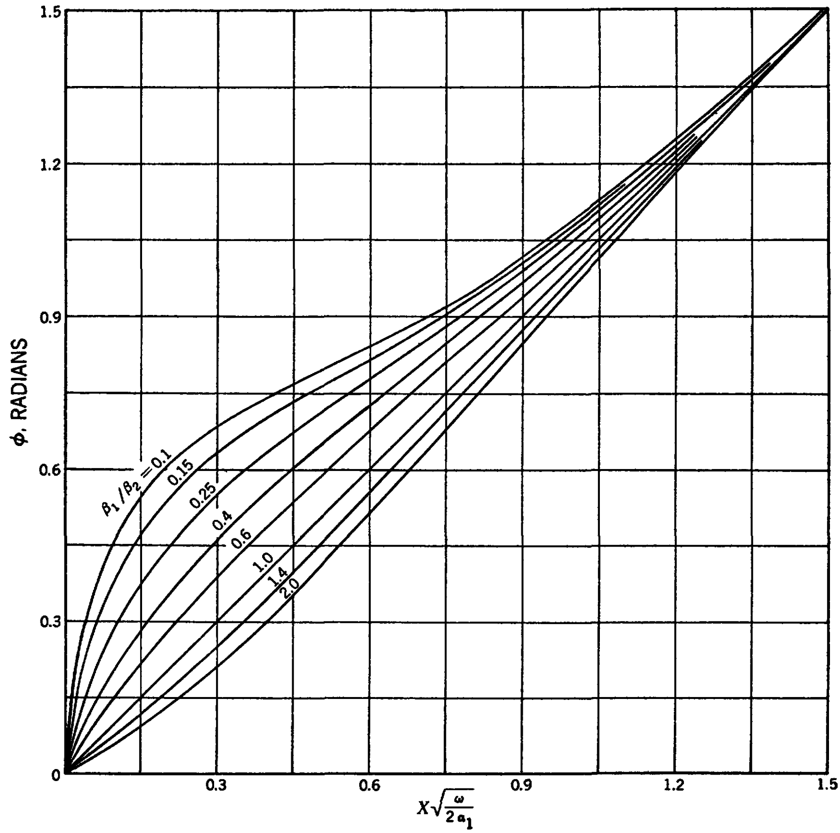


FIGURE 3.—Two-layer case. Effect of thermal properties and thickness of surficial layer on phase at the interface ($x=X$).

that, at the interface, the amplitude can be strongly affected by the deeper layer, and that the only parameter of the deeper layer involved is its contact coefficient, $\beta_2$. The damping is greater or less than in the homogeneous case according to whether $\beta_2$ is greater or less than $\beta_1$. At large depths, $S\rightarrow 1$, and the amplitude differs from that given by the homogeneous case by a factor of

$$1+M=\frac{2\beta_1}{\beta_1+\beta_2}.$$

The phase lag $\phi$, however, approaches that for the homogeneous case for large $X$ and is identically equal to it whenever

$$X\sqrt{\frac{\omega}{2\alpha_1}}=(2n-1)\frac{\pi}{2},\ n=1,2,3\ \ldots$$

In most applications the values of $\phi$ for $X\sqrt{\frac{\omega}{2\alpha_1}}<\frac{\pi}{2}$ are of principal importance. In this quadrant, $\phi$ is greater or less than $X\sqrt{\frac{\omega}{2\alpha_1}}$, according to whether $\beta_2$ is greater or less than $\beta_1$.

In most of the applications discussed below, we are concerned only with rough approximations in which only the first harmonic of the surface temperature and the associated amplitude are considered. The above information on phase is included here so that these results can be applied directly to more refined approximations taking account of higher order harmonics.

Table 1 gives typical values for the thermal parameters of various materials. They are taken from measurements made in place by the writer and W. J. Maher in arctic Alaska (Lachenbruch, 1957a).

It should be emphasized that this table is presented only for illustrative purposes, and that different materials with the same geologic or engineering designation, such as gravel, can vary widely with respect to these parameters.

The gravel-fill problem can be solved with the aid of plate 1 by setting $A(X)=F$. For example, suppose we wanted to know how much gravel (medium (1), table 1) would be required to keep a subgrade of icy silt (medium 3, table 1) from thawing by conduction effects in the summer. For this problem the curve $\beta_1/\beta_2=0.6$, shown in plate 1, is used. In a region where $F/A_0$ for the surface of the gravel is about 0.5, it is found from plate 1 that $X/\gamma=0.45$ or $X=(0.45)(290)=130$ cm or 4¼ feet. Comparable conditions might be found along the Arctic coast of Alaska where $F$ might be close to 9 centigrade degrees and $A_0$ about 18 centigrade degrees. The exact values of $A_0$ and $F$, of course, depend upon the nature of the surface, and they are subject to considerable variation locally for this reason. They are also subject to variation from year to year at a given place. Making

TABLE 1.—*Typical thermal parameters for selected materials*

[Given in cgs units. See list of symbols on page IV]

| Medium | $K$ | $\rho$ | $c$ | $\beta$ | $\alpha$ | $\gamma$ |
|---|---|---|---|---|---|---|
| 1. Gravel | 0.003 | 2.0 | 0.18 | 0.033 | 0.0083 | 290 |
| 2. Sandy gravel | .006 | 2.1 | .20 | .050 | .014 | 375 |
| 3. Icy silt | .006 | 1.6 | .31 | .055 | .012 | 345 |
| 4. Frozen organic silty clay | .003 | 1.35 | .32 | .036 | .007 | 265 |
| 5. Spruce logs | .0004 | .5 | .4 | .009 | .002 | 140 |
| 6. Dry peat | .0004 | .4 | .5 | .009 | .002 | 140 |
| 7. Icy peat | .0045 | .9 | .4 | .040 | .0125 | 355 |
| 8. Wet peat (thawed) | .0013 | 1.0 | 1.0 | .036 | .0013 | 115 |
| 9. Snow (fresh drift) | .0002 | .2 | .45 | .0042 | .0022 | 150 |
| 10. Snow (packed drift) | .0006 | .35 | .45 | .0097 | .0038 | 195 |

a similar calculation based on the homogeneous case, $\beta_1/\beta_2=1$, shown in figure 3, we find $X/\gamma=0.69$, which gives $X=(0.69)\ (290)=200$ cm or 6½ feet. Thus in this problem, neglecting the thermal properties of the subgrade would lead to an overestimation of the required fill-thickness by 50 percent. If $F/A_0$ were 0.3, a condition that could be realized in northern interior Alaska, the 0.6 curve gives $X=261$ cm or about 8½ feet. The homogeneous case would indicate a value for $X$ of 378 cm or about 11½ feet.

Although gravel fill usually has a lower contact coefficient ($\beta$) than frost-susceptible subgrade materials, this is not always so. For example, suppose the fill material was a sandy gravel (medium 2, table 1), and the subgrade was frozen organic silty clay (medium 4, table 1). For this case we use the curve labeled $\beta_1/\beta_2=1.4$, shown in plate 1. Now when $F/A_0=0.5$, then $X=(375)\ (0.84)=315$ cm (or 10.3 feet), and the preservation of permafrost by the artificial emplacement of such fill is not practicable even in an extremely cold climate. The homogeneous case applied to this problem would have given $X=(0.69)\ (375)=259$ cm or 8.5 feet.

These numerical examples indicate that even with ideal fill materials, a perennially frozen subgrade can be maintained by gravel fill only under extremely cold climatic regimes. Examination of equation 21 shows that the damping effect of the layer of thickness $X$ is increased by a low diffusivity (as in the homogeneous case) through the exponential factor and by a low contact coefficient through the factor $(1+M)=2\beta_1/(\beta_1+\beta_2)$. This suggests that more effective damping might be accomplished by inserting a thin stratum with a very low contact coefficient between the fill layer and the subgrade. To investigate the possibilities of such an approach, the corresponding problem of periodic heat flow through a three-layered semi-infinite medium will now be considered.

## THE THREE-LAYER PROBLEM

### THEORY

In this section the ground will be represented as the semi-infinite medium, $x>0$, composed of materials with thermal properties

$$K_1, \alpha_1, \beta_1 \text{ in } 0<x<X$$

$$K_\Delta, \alpha_\Delta, \beta_\Delta \text{ in } X<x<X+\Delta$$

$$K_2, \alpha_2, \beta_2 \text{ in } X+\Delta<x<\infty .$$

As in the previous problem, it will be assumed that the temperature at the surface $x=0$ varies as $A_0 \sin \omega t$, and that the temperature is

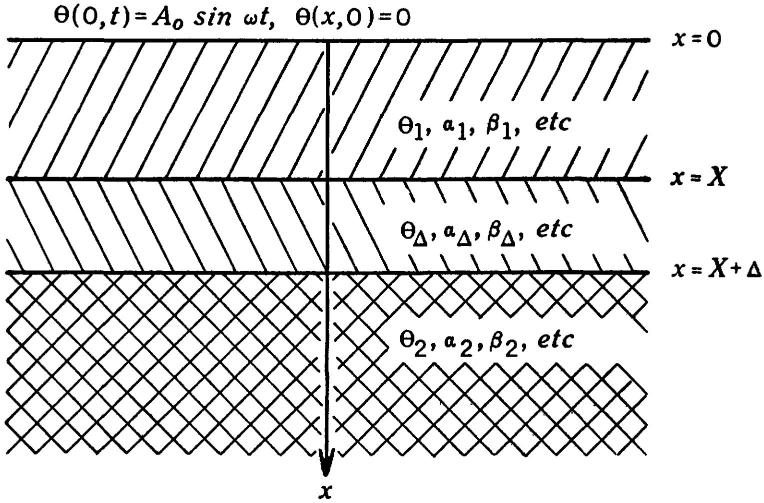


FIGURE 4.—Conditions of the three-layer problem.

initially zero for all $x$ (fig. 4). Stated formally, we are to find the temperature, $\Theta(x,t)$, which satisfies the following equations:

$$\begin{aligned}\Theta(x,t)&=\Theta_1, 0<x<X\\&=\Theta_\Delta, X<x<X+\Delta\\&=\Theta_2, X+\Delta<x<\infty\end{aligned}$$

$$\frac{\partial^2\Theta_1}{\partial x^2}=\frac{1}{\alpha_1}\frac{\partial\Theta_1}{\partial t},\ 0<x<X \tag{23}$$

$$\Theta_1(0,t)=A_0 \sin \omega t, x=0, t>0 \tag{24}$$

$$\frac{\partial^2\Theta_\Delta}{\partial x^2}=\frac{1}{\alpha_\Delta}\frac{\partial\Theta_\Delta}{\partial t},\ X<x<X+\Delta \tag{25}$$

$$\frac{\partial^2\Theta_2}{\partial x^2}=\frac{1}{\alpha_2}\frac{\partial\Theta_2}{\partial t},\ X+\Delta<x<\infty \tag{26}$$

$$\Theta_1=\Theta_\Delta=\Theta_2=0,\ t=0,\ x>0 \tag{27}$$

$$\Theta_1(X,t)=\Theta_\Delta(X,t),\ x=X \tag{28}$$

$$\Theta_\Delta(X+\Delta,t)=\Theta_2(X+\Delta,t),\ x=X+\Delta \tag{29}$$

$$K_1\frac{\partial\Theta_1}{\partial x}=K_\Delta\frac{\partial\Theta_\Delta}{\partial x},\ x=X \tag{30}$$

$$K_\Delta\frac{\partial\Theta_\Delta}{\partial x}=K_2\frac{\partial\Theta_2}{\partial x},\ x=X=\Delta \tag{31}$$

$$\Theta_2(x,t)\rightarrow 0,\ x\rightarrow\infty. \tag{32}$$

Applying the Laplace transformation to equations 23 through 32 we obtain

$$\frac{d^2\overline{\Theta}_1}{dx^2}=q_1^2\overline{\Theta}_1,\ q_1^2=\frac{p}{\alpha_1},\ 0<x<X \tag{33}$$

$$\frac{d^2\overline{\Theta}_\Delta}{dx^2}=q_\Delta^2\overline{\Theta}_\Delta,\ q_\Delta^2=\frac{p}{\alpha_\Delta},\ X<x<X+\Delta \tag{34}$$

$$\frac{d^2\overline{\Theta}_2}{dx^2}=q_2^2\overline{\Theta}_2,\ q_2^2=\frac{p}{\alpha_2},\ X+\Delta<x \tag{35}$$

$$\overline{\Theta}_1(0,p)=A_0\frac{\omega}{p^2+\omega^2},\ x=0 \tag{36}$$

$$\overline{\Theta}_1(X,p)=\overline{\Theta}(X,p),\ x=X \tag{37}$$

$$\overline{\Theta}_\Delta(X+\Delta,p)=\overline{\Theta}_2(X+\Delta,p),\ x=X+\Delta \tag{38}$$

$$K_1\frac{d\overline{\Theta}_1}{dx}=K_\Delta\frac{d\overline{\Theta}_\Delta}{dx},\ x=X \tag{39}$$

$$K_\Delta\frac{d\overline{\Theta}_\Delta}{dx}=K_2\frac{d\overline{\Theta}_2}{dx},\ x=X+\Delta \tag{40}$$

$$\lim\overline{\Theta}_2=0,\ x\rightarrow\infty. \tag{41}$$

We seek a solution to equations 33 through 41 of the following form:

$$\overline{\Theta}(x,p)=\overline{\Theta}_1(x,p)=Fe^{q_1x}+Ge^{-q_1x},\ 0<x<X$$

$$=\overline{\Theta}_\Delta(x,p)=He^{q_\Delta x}+Je^{-q_\Delta x},\ X<x<X+\Delta$$

$$=\overline{\Theta}_2(x,p)=Re^{-q_2x},X+\Delta<x$$

where $F$, $G$, $H$, $J$, and $R$ are functions independent of $x$ to be determined. Inasmuch as the expressions for these functions are cumbersome, only the result for $\overline{\Theta}_2$ will be considered as this is sufficient for the present application. The result is

$$\overline{\Theta}_2=A_0\frac{\omega}{p^2+\omega^2}P_1e^{-\sqrt{p}\left[\frac{X}{\sqrt{\alpha_1}}+\frac{\Delta}{\sqrt{\alpha_\Delta}}+\frac{(x-X-\Delta)}{\sqrt{\alpha_2}}\right]}$$

$$\times\left\{1+P_2e^{-2\sqrt{p}\left[\frac{X}{\sqrt{\alpha_1}}+\frac{\Delta}{\sqrt{\alpha_\Delta}}\right]}+P_3e^{-2\sqrt{p}\frac{X}{\sqrt{\alpha_1}}}+P_4e^{-2\sqrt{p}\frac{\Delta}{\sqrt{\alpha_\Delta}}}\right\}^{-1},x<X+\Delta \tag{42}$$

where

$$P_1=\frac{4\beta_1\beta_\Delta}{\beta_2\beta_\Delta+\beta_\Delta^2+\beta_2\beta_1+\beta_\Delta\beta_1}$$

$$P_2=\frac{-\beta_2\beta_\Delta+\beta_\Delta^2-\beta_2\beta_1+\beta_\Delta\beta_1}{\beta_2\beta_\Delta+\beta_\Delta^2+\beta_2\beta_1+\beta_\Delta\beta_1}$$

$$P_3=\frac{-\beta_2\beta_\Delta-\beta_\Delta^2+\beta_2\beta_1+\beta_\Delta\beta_1}{\beta_2\beta_\Delta+\beta_\Delta^2+\beta_2\beta_1+\beta_\Delta\beta_1}$$

$$P_4=\frac{\beta_2\beta_\Delta-\beta_\Delta^2-\beta_2\beta_1+\beta_\Delta\beta_1}{\beta_2\beta_\Delta+\beta_\Delta^2+\beta_2\beta_1+\beta_\Delta\beta_1}. \tag{43}$$

The inversion may again be performed by integration along the contour of figure 2. As in the two-layer case the integrals along contours I, III, and V vanish in the limit. In the Appendix it is shown that the integrand has no singularities along contours II and IV and hence the contributions from II and IV represent initial transient effects. Thus the steady oscillating part must again be given by the residues at the poles of the integrand.

It is necessary to show that the expression in braces in equation 42 has no zeros within the contour shown in figure 2. If any roots exist, they would be most likely to occur on the positive real axis, where each of the exponential factors must be less than unity.

Denoting

$$e^{-2\sqrt{u}\frac{X}{\sqrt{\alpha_1}}} \text{ by } A^{-1},\ e^{-2\sqrt{u}\frac{\Delta}{\sqrt{\alpha_\Delta}}} \text{ by } B^{-1},$$

we must prove

$$\frac{P_2}{AB}+\frac{P_3}{A}+\frac{P_4}{B}>-1, A, B>1. \tag{44}$$

The expression for $P_2$, $P_3$, and $P_4$ in equations 43 leads to

$$AB[\beta_2\beta_\Delta+\beta_\Delta^2+\beta_2\beta_1+\beta_\Delta\beta_1]>(1+B-A)\beta_2\beta_\Delta$$
$$+(-1+B+A)\beta_\Delta^2+(1-B+A)\beta_2\beta_1+(-1-B-A)\beta_1\beta_\Delta \tag{45}$$

and equation 45 is true by the inequality $AB=A+B-1+(A-1)(B-1)>A+B-1$. It is difficult to give a formal proof that roots do not occur for $p$ not on the real axis, but it can be seen that if one did, an infinitesimal change in thermal parameters would remove it. Such a situation is incompatible with the physics of heat flow. Thus, it will be assumed that, as in the two-layer case, poles occur only at $p=\pm i\omega$. The inverted result for large $t$ is at $x=X$,

$$\Theta(X+\Delta,t)=A_0P_1e^{-\sqrt{\frac{\omega}{2}}\left[\frac{X}{\sqrt{\alpha_1}}+\frac{\Delta}{\sqrt{\alpha_\Delta}}\right]}\frac{1}{\Sigma}\left\{\sin\left[\omega t-\sqrt{\frac{\omega}{2}}\left(\frac{X}{\sqrt{\alpha_1}}+\frac{\Delta}{\sqrt{\alpha_\Delta}}\right)\right]\right.$$

$$+P_2e^{-2\sqrt{\frac{\omega}{2}}\left[\frac{X}{\sqrt{\alpha_1}}+\frac{\Delta}{\sqrt{\alpha_\Delta}}\right]}\sin\left[\omega t+\sqrt{\frac{\omega}{2}}\left(\frac{X}{\sqrt{\alpha_1}}+\frac{\Delta}{\sqrt{\alpha_\Delta}}\right)\right]$$

$$+P_3e^{-2X\sqrt{\frac{\omega}{2\alpha_1}}}\sin\left[\omega t+\sqrt{\frac{\omega}{2}}\left(\frac{X}{\sqrt{\alpha_1}}-\frac{\Delta}{\sqrt{\alpha_\Delta}}\right)\right]$$

$$\left.+P_4e^{-2\Delta\sqrt{\frac{\omega}{2\alpha_\Delta}}}\sin\left[\omega t-\sqrt{\frac{\omega}{2}}\left(\frac{X}{\sqrt{\alpha_1}}-\frac{\Delta}{\sqrt{\alpha_\Delta}}\right)\right]\right\} \qquad (46a)$$

where

$$\Sigma=1+2P_2e^{-2\sqrt{\frac{\omega}{2}}\left(\frac{X}{\sqrt{\alpha_1}}+\frac{\Delta}{\sqrt{\alpha_\Delta}}\right)}\cos\left[2\sqrt{\frac{\omega}{2}}\left(\frac{X}{\sqrt{\alpha_1}}+\frac{\Delta}{\sqrt{\alpha_\Delta}}\right)\right]$$

$$+2P_3e^{-2X\sqrt{\frac{\omega}{2\alpha_1}}}\cos 2X\sqrt{\frac{\omega}{2\alpha_1}}+2P_4e^{-2\Delta\sqrt{\frac{\omega}{2\alpha_\Delta}}}\cos 2\Delta\sqrt{\frac{\omega}{2\alpha_\Delta}}$$

$$+2P_2P_3e^{-2\sqrt{\frac{\omega}{2}}\left(\frac{2X}{\sqrt{\alpha_1}}+\frac{\Delta}{\sqrt{\alpha_\Delta}}\right)}\cos 2\Delta\sqrt{\frac{\omega}{2\alpha_\Delta}}$$

$$+2P_2P_4e^{-2\sqrt{\frac{\omega}{2}}\left(\frac{X}{\sqrt{\alpha_1}}+\frac{2\Delta}{\sqrt{\alpha_\Delta}}\right)}\cos\left[2X\sqrt{\frac{\omega}{2\alpha_1}}\right]$$

$$+2P_3P_4e^{-2\sqrt{\frac{\omega}{2}}\left(\frac{X}{\sqrt{\alpha_1}}+\frac{\Delta}{\sqrt{\alpha_\Delta}}\right)}\cos\left[2\sqrt{\frac{\omega}{2}}\left(\frac{X}{\sqrt{\alpha_1}}-\frac{\Delta}{\sqrt{\alpha_\Delta}}\right)\right]$$

$$+P_2^2e^{-4\sqrt{\frac{\omega}{2}}\left(\frac{X}{\sqrt{\alpha_1}}+\frac{\Delta}{\sqrt{\alpha_\Delta}}\right)}+P_3^2e^{-4X\sqrt{\frac{\omega}{2\alpha_1}}}+P_4^2e^{-4\Delta\sqrt{\frac{\omega}{2\alpha_\Delta}}}. \qquad (46b)$$

## APPLICATION TO THE GRAVEL-FILL PROBLEM

In the application to the fill problem, the amplitude of $\Theta$ at the surface of the subgrade, $x=X+\Delta$, is of primary interest. It can be shown that the amplitude of the expression in braces in equation 46a is equal to $\Sigma^{1/2}$. Thus

$$A(X+\Delta)=A_0e^{-\sqrt{\frac{\omega}{2}}\left[\frac{X}{\sqrt{\alpha_1}}+\frac{\Delta}{\sqrt{\alpha_\Delta}}\right]}\left\{\frac{P_1}{\sqrt{\Sigma}}\right\} \qquad (47)$$

where

$A(X+\Delta)$ = steady amplitude of $\Theta(X+\Delta,t)$

The exponential part of equation 47 represents the damping to be anticipated from the theory of the homogeneous case, while the expression in braces represents the interaction due to the dissimilarity of contact coefficients.

It was pointed out earlier that a thin layer of material with a low contact coefficient placed between the gravel layer and the subgrade might greatly increase the damping effect, making it possible to preserve the subgrade in a frozen state with less fill material. For example, suppose a 1-foot layer ($\Delta$) of spruce logs[1] (medium 5, table 1) is placed on top of a subgrade of icy silt (medium 3, table 1). The problem is to find how much gravel, $X$, must be placed on top of the logs to keep the original surface, $x=X+\Delta$, frozen—that is, by using $\Delta=$ 1 foot and the appropriate values for $\alpha_1, \alpha_\Delta, \alpha_2$; $\beta_1, \beta_\Delta, \beta_2$ from table 1, we must find $X$ as a function of $F/A_0$.

This is given by $A(X+\Delta)=F$, or

$$e^{-\sqrt{\frac{\omega}{2}}\left[\frac{X}{\sqrt{\alpha_1}}+\frac{\Delta}{\sqrt{\alpha_\Delta}}\right]}\left\{\frac{P_1}{\sqrt{\Sigma}}\right\}=\frac{F}{A_0}. \tag{48}$$

The results for this case are presented as curve $C$, shown in figure 5. The results for the corresponding case without the intermediate layer are presented for comparison as curve $B$, shown in figure 5. Curve $A$, shown in figure 5, represents the result that would be obtained if the calculation had been made on the basis of the theory of the homogeneous case. It is seen from curve $C$ that about 3½ feet of gravel on top of 1 foot of corduroy would maintain a frozen subgrade for $F/A_0=$ 0.3. Without the corduroy (curve $B$), 8½ feet of fill would be required. From the homogeneous case we would have estimated that more than 11 feet would be required. These data indicate that with the proper selection of materials the so-called passive method of road and airfield construction might be used for climatic conditions under which it is presently considered not feasible.

The other two-layer case discussed in the previous section is represented by curve $B$, shown in figure 6. The corresponding three-layer case—that is, 1 foot of spruce logs between $X$ feet of sandy gravel (medium 2, table 1) and a subgrade of frozen organic silty clay (medium 4, table 1)—is represented by curve $C$, shown in figure 6. The homogeneous case is shown as curve $A$ in figure 6. In this case when $F/A_0=0.3$, about 11 feet of fill is required over 1 foot of corduroy, and the passive method is not practical. However at $F/A_0=0.4$, less than

[1] Logs are mentioned here for the purpose of giving a concrete example. The selection of the appropriate medium will, of course, be dictated by various engineering considerations, not the least of which is local availability. Pumice, which is available locally in some parts of the subarctic, and antarctic would probably serve equally well. Under conditions of poor drainage these media will be much less effective owing to the effects of water-saturation on the thermal properties.

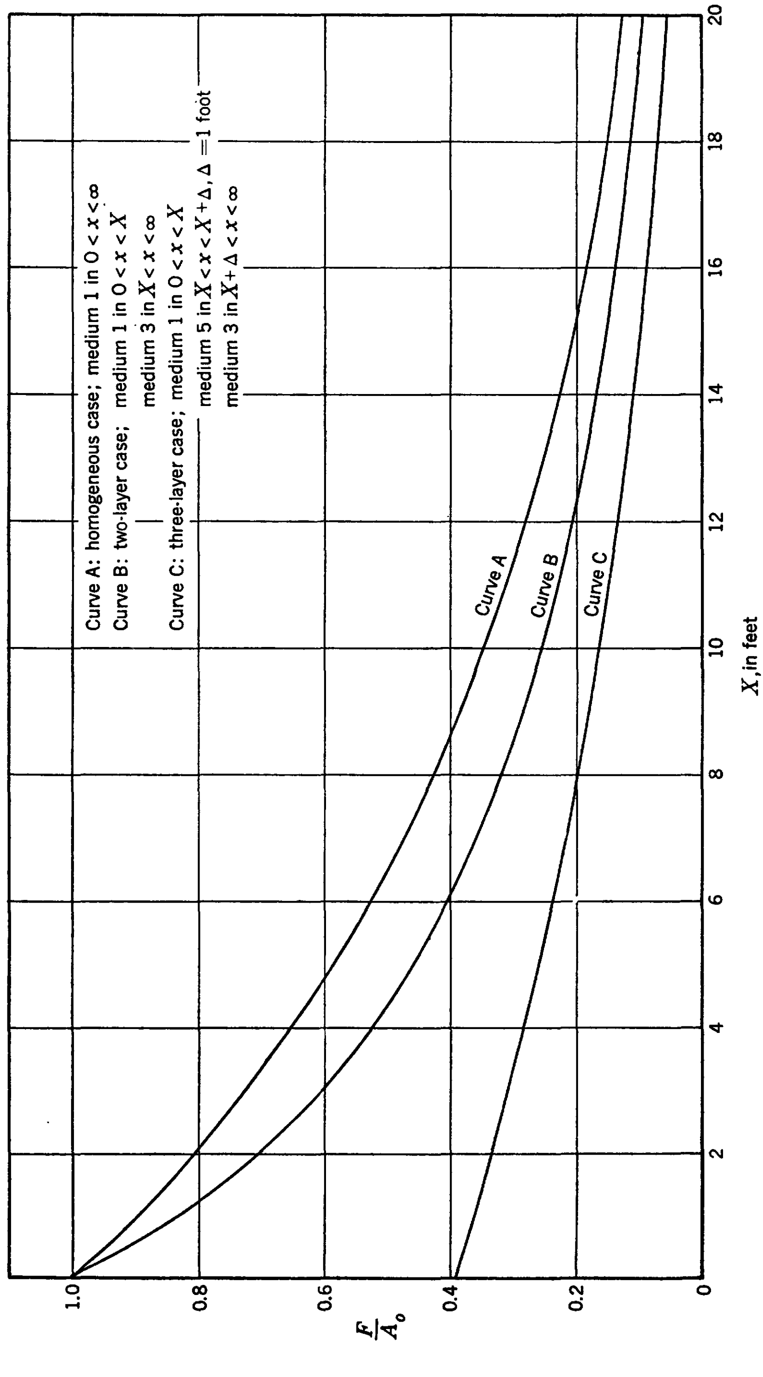


FIGURE 5.—Amplitude at the subgrade surface as a function of depth of fill ($X$), media 1, 3 and 5.

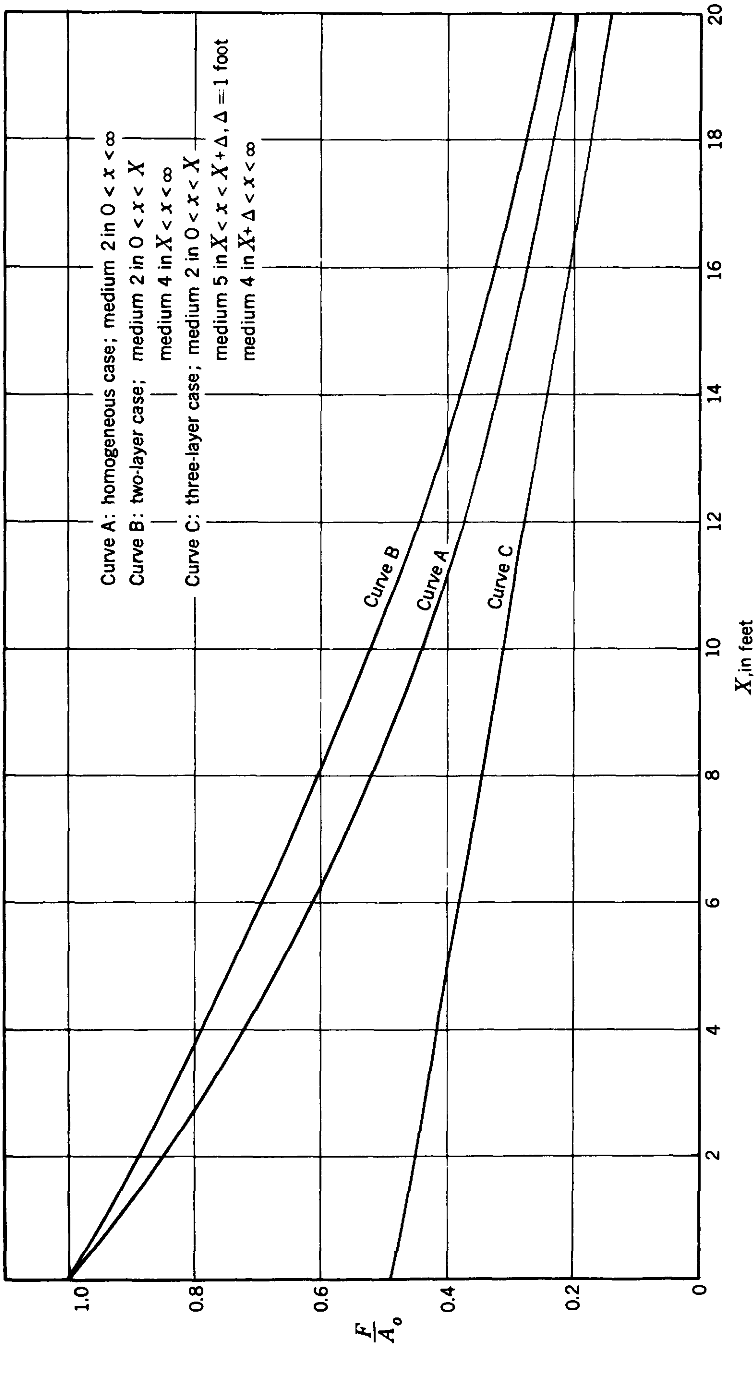


FIGURE 6.—Amplitude at the subgrade surface as a function of depth of fill ($X$), media 2, 4, and 5. (An error in the position of curve C, pointed out by D. J. Anderson, has been corrected in this second printing.)

5 feet of fill plus 1 foot of logs (curve *C*, fig. 6) produces the effect which would have required 13 feet of fill alone (curve *B*, fig. 6), a saving of 8 feet of fill.

Curve *C*, shown in figure 5, intercepts the axis of $F/A_0$ at about 0.4, indicating that 1 foot of logs with no fill is adequate to keep a subgrade of icy silt frozen when $F/A_0=0.4$. Similarly, we find from Curve *C*, shown in figure 6, that the same 1 foot of logs would not preserve a frozen subgrade of organic silty clay unless $F/A_0$ were about 0.5. From the homogeneous case, shown in plate 1, we see that a great thickness of logs would thaw to greater than 1 foot unless $F/A_0$ were greater than 0.8. This provides a good example of the effect of the contrast in contact coefficients on amplitude.

## EFFECT OF LATENT HEAT

In the foregoing application it was stated that the moisture content of the surficial layer is small, and the thermal effects of latent heat associated with periodic freezing and thawing of this moisture were neglected. Although neglecting latent heat provides a margin of safety, inasmuch as it leads to an overestimation of the minimum fill thickness, it is desirable to have a method of estimating the effect.

Differentiating equation 16a with respect to $x$, setting $x=0$, and integrating with respect to time throughout the thawing period, we find that the heat, $Q$, entering the ground surface at temperatures above freezing is

$$Q=\sqrt{A_0^2-F^2}\beta_1\sqrt{\frac{2}{\omega}}\left\{1+2\sum_{n=1}^{\infty}(-M)^n e^{-2n\frac{X}{\gamma}}\left[\cos 2n\frac{X}{\gamma}+\sin 2n\frac{X}{\gamma}\right]\right\}. \quad (49)$$

The series in this equation represents the amount by which the flux of summer heat at the surface is modified by the presence of a second medium in $x>X$. The total amount of latent heat absorbed during the thawing of the surficial layer is

$$L\rho_1 X$$

where $L$ is the latent heat per unit mass of the surficial layer. If we simply decrease $Q$ by $L\rho_1X$, the effect of latent heat would be overestimated, however, because the latent heat retards the penetration of the freezing isotherm and hence increases the gradient and heat flux at the surface. The latent heat behaves as a continuous plane heat sink at the position of the freezing isotherm which descends from $x=0$ to $x=X$ in time $\tau$ with a mean strength of

$$L\rho_1X/\tau.$$

We therefore approximate this gradient effect by that of a stationary continuous plane sink at $x=(1/2)X$ with a strength of $L\rho_1X/\tau$. The sink is supposed to persist throughout the time interval that the surface temperature remains above freezing. Field observations show that this interval is approximately equal to $\tau$. For this approximate correction stratification is neglected. Integrating the effect of this sink on the gradient at $x=0$, we find that the extra heat drawn into the ground by virtue of the presence of thawing moisture is, on the basis of these assumptions

$$2L\rho_1 X i^2 erfc \frac{X}{4\sqrt{\alpha_1 \tau}}$$

where

$$i^2 erfc\, y = \int_y^\infty d\beta \int_\beta^\infty erfc\, \gamma\, d\gamma.$$

Therefore to get the approximate effect of latent heat on $X$, we can reduce $Q$ in equation 49 by

$$\Delta Q = L\rho_1 X \left[1 - 2i^2 erfc \frac{X}{4\sqrt{\alpha_1 \tau}}\right]. \tag{50}$$

This may be done by reducing the amplitude by $\Delta A_0$ according to the following relation

$$Q - \Delta Q = \beta_1 \sqrt{\frac{2}{\omega}} \sqrt{(A_0 - \Delta A_0)^2 - F^2} \left\{1 + \sum_{n=1}^{\infty} (-M)^n e^{-2n\frac{X}{\gamma}} \left[\cos 2n\frac{X}{\gamma} + \sin 2n\frac{X}{\gamma}\right]\right\}. \tag{51}$$

The amplitude reduction, $\Delta A_0$, may be computed by combining equations 49, 50, and 51. Thus,

$$A_0 - \Delta A_0 = \left\{F^2 + \left[\sqrt{A_0^2 - F^2} - \frac{L\rho_1 X}{\lambda(X)}\left(1 - 2i^2 erfc \frac{X}{4\sqrt{\alpha_1 \tau}}\right)\right]^2\right\}^{\frac{1}{2}} \tag{52}$$

where

$$\lambda(X) = \beta_1 \sqrt{\frac{2}{\omega}} \left\{1 + \sum_{n=1}^{\infty} (-M)^n e^{-2n\frac{X}{\gamma}} \left[\cos 2n\frac{X}{\gamma} + \sin 2n\frac{x}{\gamma}\right]\right\} \tag{53}$$

$$\tau = \frac{2}{\omega}\left[\frac{\pi}{2} - \sin^{-1}\frac{F}{A_0}\right]. \tag{54}$$

It is seen from these results that when latent heat is involved, the minimum thickness of fill, $X$, depends upon the amplitude, $A_0$, and mean surface temperature, $-F$, individually and not simply on their ratio.

To obtain the correct value of $\Delta A_0$, trial values of $X$ are introduced in equation 52. The corresponding trial value of $\Delta A_0$ is then used to form the ratio $F/(A - \Delta A_0)$; $X$ is then read from plate 1, using $F/(A - \Delta A_0)$ as ordinate. If this value of $X$ agrees with the trial value, the problem is solved. Knowing the value of $X$ from the case neglecting latent heat, the new value of $X$ can generally be obtained with 1 or 2 trials in equation 52.

As an example of the calculation of the effects of small amounts of moisture, we shall consider the two-layer case discussed in previous sections in which $X$ feet of medium 1 overlies medium 3. Gravel with the properties of medium 1 would normally have a moisture content of about 2 percent of the wet weight. Using this value and taking $A_0$ as 18°C, from equation 52 it is found that the estimate of $X$ is reduced from 4¼ feet to 3¾ feet by the correction for latent heat, when $F/A_0=0.5$. (Neglecting stratification gave a value of 6½ feet.) When $F/A_0=0.3$, the correction for latent heat reduces $X$ from 8½ feet to about 7 feet. The homogeneous case lead to 11½ feet in this instance. Thus, small amounts of moisture can affect an appreciable reduction in the minimum thickness of fill required to maintain a frozen subgrade. However, in selecting fill materials it is important to bear in mind that as the moisture-retaining capacity of the fill is increased, its density, $\rho_1$, conductivity, $K_1$, and specific heat, $C_1$, are generally increased; hence the contact coefficient, $\beta_1=\sqrt{K_1\rho_1C_1}$, increases rapidly. This in turn can lead to a large increase in the required thickness of fill by increasing the ratio $\beta_1/\beta_2$.

## OTHER APPLICATIONS

### TEMPERATURES IN A WELL-DRAINED ACTIVE LAYER

In permafrost areas sedimentary materials that were laid down under conditions of saturation are commonly drained locally by natural, and sometimes artificial, processes. If the material is permeable, excess water drains from the seasonally thawed stratum, while the original high ice content persists in the underlying permafrost. The sharp discontinuity in water (ice) content at the permafrost table leads to a corresponding change in thermal properties there. Because the active layer materials are relatively dry, the effect of seasonal change of state can often be neglected and the two-layer problem can be applied to anticipate the thickness of the active layer in terms of thermal properties and climatic factors.

As an example, we shall consider in some detail a locality near Barrow, Alaska, at which a great accumulation of peat was deposited under bog conditions. Subsequent recession of a nearby river terrace lead to very effective local drainage and disappearance of the bog. Today there occurs here a surficial layer about 25 cm thick of loose dry peat in which virtually all the water is contained within the vegetal fibers. At depths greater than 25 cm, the material is a perennially frozen peat in which virtually all the space between the vegetal fragments is filled with ice.

The thermal parameters of the surficial peat and the permafrost peat are represented, respectively, by those given for media 6 and 7 in table 1. If the ground-surface temperature did, in reality, vary as $A_0 \sin \omega t$ with a period of 1 year, the interface would be expected to lie ultimately at the position given by the curve $\beta_1/\beta_2=0.25$, shown in plate 1, for the appropriate value of $F/A_0$. This is the maximum depth to which the thaw could ultimately penetrate under these conditions and hence the maximum depth to which the surface layer could be drained of its interstitial moisture. From table 1 it is seen that $X=25$ cm corresponds to an abscissa of $X/140=0.18$ in plate 1. Thus the value of $F/A_0$ characteristic of the surface temperature under these conditions, as given by plate 1, is 0.54. Actually, of course, the surface temperature is not strictly sinusoidal, and the ratio of annual mean to amplitude of the first harmonic ($F/A_0$) is subject to variation from year to year. From 3 years of continuous temperature recordings just beneath the surface at this site, it was found that for 1 year the ratio of the mean to amplitude of the first harmonic ($F/A_0$) was close to 0.54 while for the other 2 years it was greater than 0.70. From a comparison of these data with air temperature measurements made by the weather bureau during the past 25 years, it seems reasonable to conclude that the ratio $F/A_0$ at this site is rarely appreciably less than 0.54 although it probably approaches this value fairly frequently, or about every 3 or 4 years. During 4 of the 6 years from the spring of 1949 to the spring of 1955, $F/A_0$ was probably close to 0.54. Thus $F/A_0=0.54$ might be said to represent a frequently occurring warm year, and it is not unreasonable that the position of the interface should be identified with such a year. An isolated anomalously warm year would result in only a small increment of seasonally thawed permafrost due to the retarding effects of latent heat, and the amount of moisture drained from this increment during these isolated years would probably be restored by the freezing of summer rainwater during the more normal years that occur between the anomalous years.

If the active layer thickness of 25 cm was to be accounted for by the homogeneous case, it is seen from plate 1 that a value of $F/A_0 \approx 0.83$ would be required. Such large values probably occur only a few times in a century.

An example of actual surface-temperature data at the site under discussion is given in plate 2. The simple harmonic approximation (curve $A$) is the one with the same maximum number of cumulative degree-days above and below 0°C. (same freezing and thawing index) and the same mean as the observed values. For problems involving a change of properties at the top of permafrost, this approximation has certain advantages over using the first term in a trigonometric series. The freezing-index method is much easier to apply, and it takes better account of the critical role played by the freezing point. Amplitudes obtained by the two methods generally agree to within a few percent however, and the chief advantage of the first method is its convenience. A harmonic analysis taking account of many terms is of course preferable to any simple harmonic approximation, but approximate results can usually be obtained quickly by the simple method, with a minimum of data.

A comparison of the simple harmonic approximation made by the freezing index method (curve $A$), with the results of a six-term harmonic analysis (curve $B$), is given in plate 2. The simple method leads to $A_0=17.5$, $F=9.45$ in centigrade degrees, and $F/A_0=0.54$. The parameters in the trigonometric-series approximation are given in the second and third columns of table 2. Each of these approximations to the surface temperature has been projected to the interface $x=X=25$ cm with the aid of plate 1 and figure 3. The results are represented by curves $A'$ and $B'$ in plate 2. It will be noted that both curves are tangent to the 0° C. line, indicating that both the simple and refined approximation lead to the same value of $X$ in this case.

TABLE 2.—*Parameters in the harmonic analysis of ground temperatures at special hole 35, Barrow, Alaska, May 11, 1954 to May 11, 1955*

[Terms of form $A(X) \sin [\omega t-\phi(x)]$]

| Term in trigonometric series | Surface $x=0$ | | Two-layer case, $x=X=25$ cm | | Homogeneous case, $x=25$ cm, $X=\infty$ | |
|---|---|---|---|---|---|---|
| | $A$ (0) | $\phi$ (0) | $A$ ($X$) | $\phi$ ($X$) | $A$ ($x$) | $\phi$ ($x$) |
| 1st | 16.90 | 0.05 | 9.13 | 0.47 | 14.11 | 0.23 |
| 2d | 2.37 | .65 | 1.05 | 1.15 | 1.84 | .90 |
| 3d | 1.41 | .16 | .55 | .72 | 1.03 | .47 |
| 4th | 1.20 | 3.45 | .42 | 4.05 | .84 | 3.81 |
| 5th | 1.07 | .88 | .35 | 1.52 | .72 | 1.28 |
| 6th | 1.43 | 3.80 | .43 | 4.47 | .92 | 4.24 |

It is interesting to compare these results for the two-layer case with those that would have been predicted on the assumption that the surficial dry layer extended to great depth—that is, the homogeneous case. This is done in plate 3 where curve $B$ and curve $B'$ are reproduced from plate 2. Curve $B''$ represents the temperature that would result at $x=25$ cm from a surface temperature represented by curve $B$, if the change in properties at the top of permafrost is neglected. In each case the thermal wave passes through 25 cm of dry peat, but in the homogeneous case the maximum temperature at 25 cm. is greater than 5° C, whereas in the stratified case it is 0° C. In fact, from plate 1 it is seen that in the homogeneous case the thaw would penetrate to about 85 cm. This illustration points out again the incompleteness of statements that a given amount of a specified surficial material will preserve permafrost in a given climate. The insulating quality of the surficial layer is actually quite sensitive to the thermal properties of the material it insulates.

For the two-layer case under discussion the expression in braces in equation 49 has the value of 3.0. Hence the paradox that three times as much summer heat is drawn through the ground surface, x=0, in the two-layer case as in the corresponding homogeneous case, although the depth of thaw is less than one-third as great.

To get some idea of the magnitude of the neglected effects of higher order harmonics, we note from plate 3 that for purposes of estimating the position of the top of permafrost, the most significant of these can be represented roughly by a sine wave with a period of about 10 days and an amplitude, $A'_0$, of about 2½° C. From plate 1 it is seen that at $X=25$ cm the amplitude would have diminished to about $0.12A'_0 \approx 0.02A_0$, and the maximum effect of this higher frequency term would be to increase the estimate of $X$ by only 1 or 2 cm.

The effect of latent heat on $X$ can be estimated from equation 52. The available moisture in the surficial dry peat represents about 50 percent of the wet-sample weight. Using this value and $A_0=17.5°$ C, $F/A_0=0.54$, equation 52 yields $X=24$ cm. Thus the effect of latent heat would be to decrease the estimate of depth of thaw by only 1 cm in the present case.

Inasmuch as virtually all the living organisms at this locality must reside within this 25-cm seasonally-thawing stratum, the temperatures within it are important to ecological and physiological studies. Insofar as these temperatures are controlled by conducted heat, they can be investigated with the aid of the theory developed above.

If the temperature in the active layer is represented by

$$\Theta_1(x)=A(x)\sin[\omega t-\varphi(x)],\quad 0\leq x\leq X$$

it is found from equation 18a that

$$A(x)=A_0 e^{-x\sqrt{\frac{\omega}{2\alpha_1}}}\left\{\frac{\Psi}{S}\right\},\quad 0\leq x\leq X \tag{55}$$

where

$$\begin{aligned}\Psi^2=1+M^2e^{-4(X-x)\sqrt{\frac{\omega}{2\alpha_1}}}+M^2e^{-4X\sqrt{\frac{\omega}{2\alpha_1}}}+M^4e^{-4(2X-x)\sqrt{\frac{\omega}{2\alpha_1}}}\\ +2M^2e^{-2(2X-x)\sqrt{\frac{\omega}{2\alpha_1}}}\cos 2x\sqrt{\frac{\omega}{2\alpha_1}}+2\left[M^3e^{-2(3X-2x)\sqrt{\frac{\omega}{2\alpha_1}}}\right.\\ \left.+Me^{-2X\sqrt{\frac{\omega}{2\alpha_1}}}\right]\cos 2X\sqrt{\frac{\omega}{2\alpha_1}}+2\left[M^3e^{-2(3X-x)\sqrt{\frac{\omega}{2\alpha_1}}}+Me^{-2(X-x)\sqrt{\frac{\omega}{2\alpha_1}}}\right]\\ \times\cos 2(X-x)\sqrt{\frac{\omega}{2\alpha_1}}+2M^2e^{-2(2X-x)\sqrt{\frac{\omega}{2\alpha_1}}}\cos 2(2X-x)\sqrt{\frac{\omega}{2\alpha_1}}\end{aligned} \tag{56}$$

and $\phi(x)$ is given by

$$\tan\phi = \frac{\left[1-M^2e^{-2(2X-x)\sqrt{\frac{\omega}{2\alpha_1}}}\right]\sin x\sqrt{\frac{\omega}{2\alpha_1}}+\left[e^{+2x\sqrt{\frac{\omega}{2\alpha_1}}}-1\right]Me^{-2X\sqrt{\frac{\omega}{2\alpha_1}}}\sin(2X-x)\sqrt{\frac{\omega}{2\alpha_1}}}{\left[1+M^2e^{-2(2X-x)\sqrt{\frac{\omega}{2\alpha_1}}}\right]\cos x\sqrt{\frac{\omega}{2\alpha_1}}+\left[e^{+2x\sqrt{\frac{\omega}{2\alpha_1}}}+1\right]Me^{-2X\sqrt{\frac{\omega}{2\alpha_1}}}\cos(2X-x)\sqrt{\frac{\omega}{2\alpha_1}}}. \tag{57}$$

Based on equation 55, curve *B* shown in figure 7, shows how the amplitude of a wave with a period of 1 year diminishes through the 25 cm layer. Curve *A*, the homogeneous case, and curve *C*, the result obtained from plate 1, are also shown for comparison. It is seen from curve *B* that for $F/A_0>0.54$, $X=25$ cm, superfreezing temperatures penetrate somewhat deeper than would be predicted on the

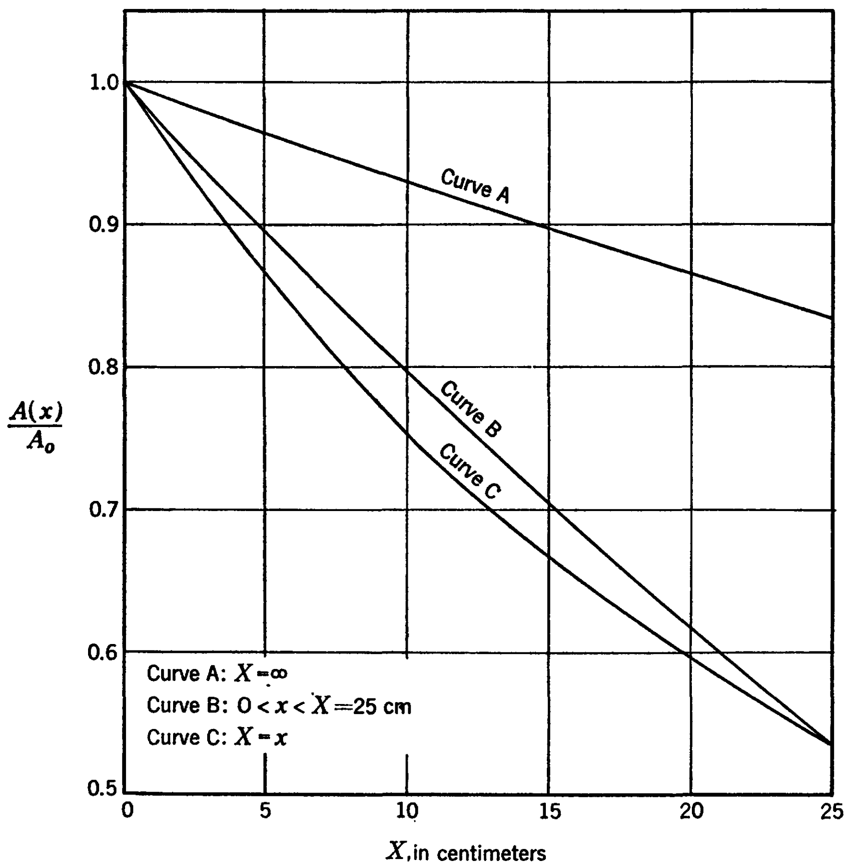


FIGURE 7.—Amplitude of annual wave in surficial layer for medium 6 in $0<x<X$; medium 7 in $x>X$.

assumption that the interface occurred at the position of the maximum depth of the freezing isotherm (curve *C*, fig. 7).

For the data plotted in plate 2, the simple harmonic approximation to the surficial temperature has an amplitude, $A_0$, of 17.5° C and a mean temperature, $-F$, of $-9.45°$ C. This simple harmonic approximation will be used to show the effect of stratification on annual temperature variation within the active layer. Curve *B* in figure 7 shows that the amplitude of the sinusoid is reduced to about $0.74\ A_0 \approx 13.0°$ C, at a depth of 13 cm. Thus the annual maximum temperature at the 13-cm depth is $13.0-9.45=3.6°$ C. In general, if the material below 25 cm had a lower contact coefficient than it does in this case, the maximum temperature at the 13-cm depth would be greater. In particular, if the deeper material had the same contact coefficient as the surficial 25-cm layer, the amplitude of the annual wave, as given by curve *A* of figure 7, would be about $0.91\ A_0 \approx 15.9°$ C, and the annual maximum temperature would be about 6.5° C at 13 cm. Knowing the amplitude and mean in the two cases, we can easily compute other quantities of probable importance in ecological studies. The results are summarized in table 3. The example shows that for identical thermal conditions at the surface the thermal properties of the material underlying the layer supporting plant growth can have a profound effect on the temperatures of the plants at root level. The particular media used in this example were selected as a matter of convenience, and they are probably of minor importance insofar as the roots of vascular plants are concerned. Nevertheless the principal is quite general, and it is likely that these effects have an appreciable influence on the local distribution and characteristics of plant types in the arctic.

TABLE 3.—*Effect of deep layer* $(x>X)$ *on temperatures in surficial layer* $(0<x<X=25$ *cm) at a depth of 13 cm in medium 6 (of table 1).*

| | $\beta_1/\beta_2=0.25$ | $\beta_1/\beta_2=1.00$ |
|---|---|---|
| Maximum summer temperature °C | +3.6 | +6.5 |
| Period with temperature >0° C days | 90 | 110 |
| Maximum cumulative degree-days above 0° C | 200 | 450 |

A wave with a period of 10 days would diminish in amplitude as indicated by curve *B*, shown in figure 8. This provides an interesting example of how an increasing depth to the interface increases the effect of higher order harmonics in the surficial layer. Thus, if $F/A_0$ were consistently 0.7, the surficial dry layer would be about 13 cm thick according to curve *C*, shown in figure 7. As a result, the 10-day wave would have an amplitude of about 25 percent of its surface value at 13 cm (curve *C*, fig. 8). For $F/A_0=0.54$ however, the

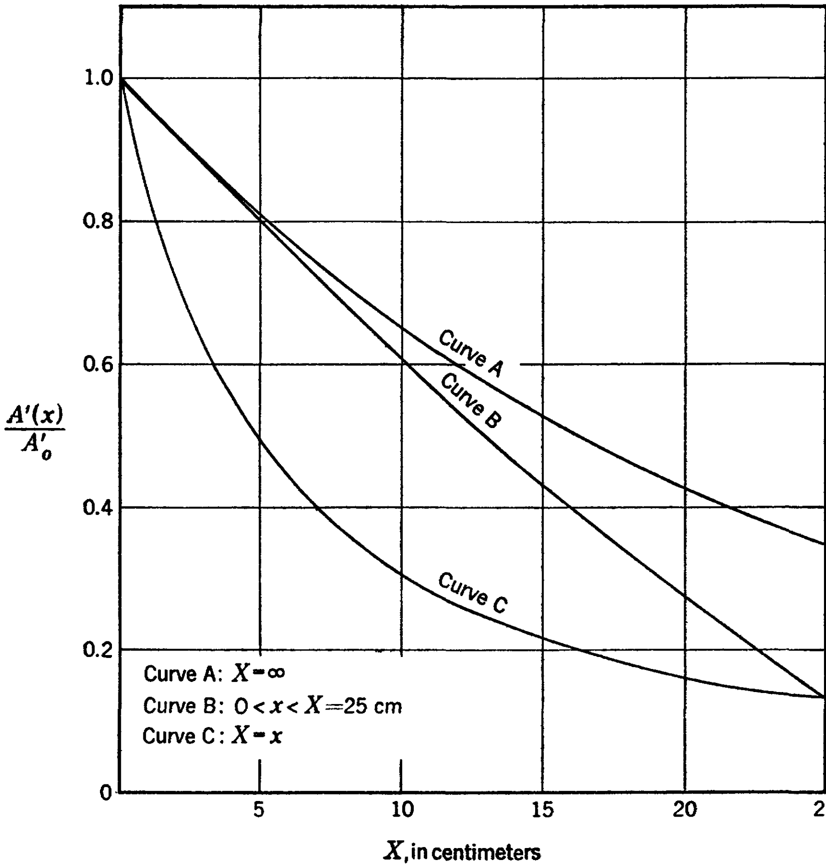


FIGURE 8.—Amplitude of 10-day wave in surficial layer for medium 6 in $0<x<X$, medium 7 in $x>X$.

interface lies close to 25 cm, and the 10-day wave has an amplitude of about 50 percent of its surface value at 13 cm (curve *B*, fig. 8). An additional point of interest in figure 8 is that the amplitude computed from equation 55, curve *B*, differs by less than 1 percent from that computed by the result for the homogeneous case, curve *A*, for depths less than 3 or 4 cm. That is, when $\beta_1/\beta_2=0.25$, the effect of the contrast in contact coefficients at 25 cm is not felt at depths less than 3 or 4 cm for a wave with a period of 10 days. By multiplying the abscissal scale by $\sqrt{\frac{365}{10}}\approx 6$, it is seen that the damping of an annual wave could be treated by the homogeneous case in the upper 20 to 25 cm if the interface occurred at 150 cm. These and similar results can be predicted quickly by evaluating the term corresponding to $n=1$ in equation 16a.

## EFFECT OF SEASONAL SNOW COVER ON MEAN ANNUAL GROUND TEMPERATURES

Although the mean annual temperature of the ground surface is one of the most important parameters in permafrost problems, it is one for which very little information is available. A great deal of attention has been devoted to the problem of estimating this quantity from air-temperature measurements and other data commonly recorded in routine meteorological observations. The relationship between air temperature and ground-surface temperature is extremely complex, and several modes of energy transfer are involved. However, in high latitudes one of the most important factors influencing the difference between the mean annual temperature of the air and ground is the insulating effect of the continuous winter snow cover. We shall attempt to apply the results from the two-layer case to obtain an order of magnitude for this effect.

It will be assumed that the temperature of the bare-ground surface in the summer and the upper surface of the snow in the winter can be represented by a sinusoid with amplitude $A^*$ and a period of 1 year. We shall assume that the ground is covered by a blanket of snow of thickness $X$, for a winter period, assumed for convenience to be of 6 months duration. (This provides a reasonable representation of conditions at many sites in Arctic Alaska.) The radiative transfer in the snow of short wave incoming radiation will be neglected. This is probably justified during most of the Arctic winter.

The mean summer temperature at the bare ground surface is then given by

$$\tilde{\theta}\text{ summer}=A^*\frac{1}{\pi}\int_0^\pi \sin\zeta\, d\zeta$$
$$=\frac{2}{\pi}A^*.$$

If the transient effects due to the movement of the boundary surface at $\omega t=0$ and $\pi$ are neglected, the mean winter temperature at the ground surface is

$$\tilde{\theta}\text{ winter}=-A(X)\frac{2}{\pi}$$

where $A(X)$ is the steady amplitude that would result from a wave with amplitude $A^*$ passing through thickness $X$ of snow. Neglecting the transients in this way is justified if $X^2/4\alpha_1<<\pi/\omega$, where $\alpha_1$=diffusivity of the snow. With these simplifications the shift in mean annual temperature ($\Delta\tilde{\theta}$) due to the snow is given by

$$\Delta\tilde{\theta}=\frac{1}{\pi}A^*[1-A(X)/A^*]. \quad (58)$$

It is important to bear in mind that this expression represents only the amount by which the mean-annual temperature of the ground surface will exceed that of the air-solid interface as a result of conduction effects in the snow. In precise work the latter cannot be identified directly with mean values computed from routinely measured air temperatures. In the Arctic the mean winter temperature of the snow surface is generally probably somewhat lower than the corresponding air temperature because of the high emissivity of snow for long wave lengths. In the summer the mean temperature of the bare ground surface can be higher or lower than that of the air, according to local radiation and moisture relationships.

When a third layer is not involved, the ratio $A(X)/A^*$ in equation 58 can be computed for particular cases from plate 1. It is seen that the insulating effect of the snow is more pronounced when the surficial ground layers have a high contact coefficient ($\beta_2$). In addition, the effect increases not only with increasing snow depth, $X$, and decreasing snow diffusivity, $\alpha_1$, but also with a decreasing snow-contact coefficient, $\beta_1$, and increasing amplitude, $A^*$.

When 1 foot of packed, drifted snow (medium 10, table 1) overlies a considerable thickness of sandy gravel (medium 2, table 1), $\beta_1/\beta_2 \approx 0.2$, $X/\gamma \approx 0.15$; and from plate 1, $A(X)/A^* \approx 0.52$. Then, according to equation 58, the shift in mean temperature from the air-solid interface to the ground surface would be expected to be about $0.15A^*$. Under these conditions at Barrow, Alaska, a reasonable value for $A^*$ is 20° C, which yields a mean shift of about 3° C. Under the same situation at Umiat, about 80 miles inland from the Arctic Coast, $A^*$ is probably about 50 percent greater and hence so is $\Delta\tilde{\Theta}$. The mean air temperatures and snow conditions are very nearly the same at Umiat and Barrow, and the greater permafrost temperatures which are recorded at Umiat are probably due largely to this amplitude effect. If the same snow conditions occurred over several feet of dry peat (medium 6, table 1), $\Delta\tilde{\Theta}$ would be expected to be about $0.04A^*$, and the insulating effect would be reduced to about one-fourth of that in the previous case. In areas where wind drifting is less intense, the snow can have a lower $\beta$, and the mean shift is accentuated. Conditions that could easily be realized in the Alaskan Arctic might be represented by 1 foot of fresh snow drift (medium 9, table 1) overlying icy peat (medium 7, table 1). In this case $\beta_1/\beta_2 \approx 0.1$, which leads to $\Delta\tilde{\Theta} \approx 0.23A^*$. At Barrow this might result in elevating mean ground temperatures by 4 or 5° C and at Umiat by as much as 7° C. Without more precise observational information these values cannot be verified directly. However, the magnitude of the mean shifts predicted by this method is in general agreement with the observed differences in the mean annual temperatures of the air and ground surface.

Approximate values for $A^*$ have been estimated from known values of the amplitude of air-temperature variation and from ground-surface temperature observations at a limited number of sites. The assumption of constant snow depth is thought to be reasonable for this degree of approximation, as the snow in this area commonly builds up rapidly early in winter, and during that time it has not packed from drifting. Thus, although $X$ is smaller, $\alpha_1$ and $\beta_1$ are also smaller, and a more refined approximation based upon monthly steps would lead to approximately the same results.

The examples serve only to illustrate that this crude approximation for the effect of snow cover on mean ground temperatures seems to yield results that are reasonable. The extent to which the approximation is valid can be determined only after a much more detailed study.

An interesting correlary to this discussion is that if, over a period of time, the mean annual temperature and the time distribution of snow fall did not change systematically, but the amplitude of seasonal variation increased, geothermal evidence would indicate a secular increase in surface temperatures. In fact, this effect might be expected in the case of a regressing shoreline in the arctic as the climate changed from oceanic to continental. In the example discussed above, Umiat, which is about 80 miles inland, has an amplitude of seasonal variation about 50 percent greater than that at Barrow on the Arctic coast.

## MEASUREMENT OF THE THERMAL PROPERTIES OF SOILS AND ROCKS

Higashi (1953) described an apparatus that generated a sinusoidal temperature on the surface of a soil sample, and he used it successfully to measure the thermal diffusivity of soils in the laboratory. The temperature was measured as a function of time at two points at different distances from the controlled surface, and the diffusivity was computed by reference to the solution for the homogeneous semi-infinite medium—equation 19.

The theoretical results discussed in the preceding sections suggest a modification of this procedure which may prove useful. A sinusoidal temperature is impressed on the upper surface of a glass or plastic reference disc of known thermal properties, and thermally sensitive elements are placed on the upper and lower surfaces of this disc. Then when the lower surface of the reference disc is in thermal contact with the medium to be tested, measurements of the amplitude on the two surfaces of the disc will give the contact coefficient, $\beta$, of the test medium. The result is obtained from equation 21. Such a device could be made portable and used for measurements in place.

If the medium is consolidated, determination of the volume specific heat can usually be made in the laboratory without difficulty. If the medium is unconsolidated, however, the determination of the bulk density is sometimes very difficult. In this case the following modification of the device might be used to measure the volume specific heat in place. A small thermally sensitive element placed on the tip of a thin poorly conducting plastic spike protruding from the axis of the reference disc would be embedded in the test medium as the instrument is positioned on its surface. A third measurement of the amplitude by this element would give the thermal diffusivity of the test medium from equation 19. Thus the two independent thermal parameters would be determined from a single field test.

The use of a controlled sinusoidal temperature for field and laboratory measurements has the advantage of transient methods in that it permits a determination of both the conductivity and the volume specific heat. Most transient methods, however, have the disadvantage that the theoretical models used for interpretation are based on idealizations neglecting certain finite dimensions of the instrument; and this introduces a time-dependent error. Thus, considerable uncertainty is often involved in fixing the optimum duration of an experiment and the absolute accuracy of the measurements. Although the sinusoidal method does not result in a steady state in the true sense, the wave form approaches an equilibrium configuration. Thus the amplitude and phase approach steady values at each point. This gives the method the advantage of steady-state techniques. The oscillating part of the edge effects $\Theta_e$, at the element imbedded in the test medium also approaches this quasi-steady state. They may be computed from Lachenbruch (1957b)

$$\Theta_e = -A \frac{x}{\sqrt{x^2+R^2}} e^{-\sqrt{x^2+R^2}\sqrt{\frac{\omega}{2\alpha}}} \sin\left(\omega t - \sqrt{x^2+R^2}\sqrt{\frac{\omega}{2\alpha}}\right)$$

where $R$ is the radius of the disc.

A further advantage of this method for field use is that the results are relatively insensitive to temperature drift in the test medium because they are obtained from the oscillating part of the temperature variation. This drift effect is often difficult to control in field measurements in near-surface media inasmuch as the test disturbs the natural temperatures and obviates the possibility of a drift measurement at the completion of the test.

There are three obvious difficulties in the use of this method. The first is that good thermal contact between the reference disc and the test medium is necessary. However, the thermal effects of a semifluid coating introduced to reduce contact resistance can be estimated from the two- or three-layer cases discussed above.

The second difficulty is the restriction on the grain size of the test medium, which is imposed by the maximum allowable horizontal gradients in the neighborhood of the thermal element at the base of the reference disc.

The third difficulty involves the disturbance of the temperature field by the spike supporting the thermal element imbedded in the test medium. The error due to conduction along the leads, however, can be estimated by the method described by Jeager (1955). One method of greatly reducing that error is to make the heater and reference disc in the form of an annulus with the spike mounted on a hub at the axis. In this method the measurements from the element on the spike would be interpreted by reference to (Lachenbruch, 1957b)

$$\Theta(x,t)=A\left\{\frac{x}{\sqrt{x^2+R_1^2}}e^{-\sqrt{x^2+R_1^2}\sqrt{\frac{\omega}{2\alpha}}}\sin\left[\omega t-\sqrt{x^2+R_1^2}\sqrt{\frac{\omega}{2\alpha}}\right]\right.$$
$$\left.-\frac{x}{\sqrt{x^2+R_2^2}}e^{-\sqrt{x^2+R_2^2}\sqrt{\frac{\omega}{2\alpha}}}\sin\left[\omega t-\sqrt{x^2+R_2^2}\sqrt{\frac{\omega}{2\alpha}}\right]\right\} \quad (59)$$

where $R_1$ and $R_2$ are the inner and outer radii, respectively, of the annulus.

The minimum thickness of a stratum to which this method can be applied may be estimated from equation 16a.

## CONCLUSION

Many interesting and important phenomena associated with periodic oscillations of ground temperature are related to the fact that the ground is stratified with respect to thermal properties. These effects, of course, cannot be studied by reference to the classical solution of the heat equation for the homogeneous case. However, solution for a stratified semiinfinite medium with sinusoidal surface temperature can be obtained directly by the Laplace transform method. The result for two layers is relatively simple. For a greater number of layers the solutions are more cumbersome, but they present no great analytical difficulty.

The insulating effect of a surficial stratum, or its ability to protect the underlying material from extremes of temperature, does not depend on its conductivity alone as in the case of steady flow, or upon its diffusivity alone as in the case of periodic flow in a homogeneous medium, and serious errors can result from neglecting this fact. It depends upon both the diffusivity and contact coefficient of the surficial stratum and the contact coefficients of all underlying strata sufficiently close to produce appreciable effects. As in the homogeneous case, an exponential amplitude decrease with depth is produced by the diffusivity, $\alpha = \frac{K}{\rho c}$.

In the two-layer case the amplitude in the surficial stratum decreases more rapidly with depth than in the homogeneous case if $\beta_1/\beta_2 < 1$, where $\beta_1$ and $\beta_2$ are the contact coefficients in the top and bottom strata, respectively. If $\beta_1/\beta_2 > 1$ the insulating effect of the surficial stratum is less than in the homogeneous case. Where $\beta_1/\beta_2 = 1$, temperatures in the surficial stratum may be computed from the homogeneous case without error. The diffusivity and contact coefficient can vary quite independently of one another. For example, a water-saturated peat (medium 8, table 1) experiences very little change in its contact coefficient as a result of freezing (medium 7, table 1), although freezing increases its diffusivity by a factor of 10.

These factors are probably important in determining the distribution of vegetation in the arctic. Summer temperatures in the neighborhood of plant roots in the surficial layer supporting vegetal growth are sensitive to the thermal properties of the subjacent layer. These root temperatures would be higher or lower according to whether the contact coefficient of the deeper layer were relatively low or high. The large magnitude of these effects is indicated by the graphs presented above. Thus in two areas where all surface conditions were equal, identical surface layers might support plant assemblages of quite different characteristics if the contact coefficients of the subjacent strata differed widely between the two areas. The explanation for this would never be found from surface observation.

From the two-layer case it is found that the minimum thickness of gravel fill necessary to maintain a perennially frozen subgrade is strongly influenced by the thermal properties of the subgrade. Except under favorable conditions the amount of material required for permafrost preservation by a single layer of fill is too great to be practicable. From the three-layer case it is found that a thin layer of material of relatively low contact coefficient, such as logs or pumice, placed between the fill and the subgrade can greatly reduce the amount of fill required. Thus, the subgrade can probably be maintained in a perennially frozen state under a variety of conditions by practicable methods when appropriate materials are available locally.

The foregoing results can also be applied to obtain an estimate of the effect of winter snow cover on the mean annual temperature of surficial permafrost layers. This effect may prove to be important in geothermal studies of climatic history.

The results from the two-layer case indicate a method for simultaneous measurement of the two independent thermal parameters, diffusivity and contact coefficient, by use of an artificial sinusoidal temperature wave.

The discussion in this paper is confined primarily to problems in which the neglected effects of latent heat are relatively unimportant, although a means of correcting for these effects is presented. It is important to bear in mind that under some conditions the movement of water results in the nonconductive transfer of appreciable amounts of heat and a change in thermal parameters with time. These effects are not considered here. As in any mathematical study of natural phenomena the models discussed above represent idealizations, but they give a useful insight into the natural processes associated with conductive heat transfer in stratified ground.

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## APPENDIX—PROOF THAT $\bar{\theta}_2$ IS BOUNDED ON THE NEGATIVE REAL AXIS, THREE-LAYER CASE

It must be proved that

$$1+P_2e^{-2\sqrt{p}\left[\frac{X}{\sqrt{\alpha_1}}+\frac{\Delta}{\sqrt{\alpha_\Delta}}\right]}+P_3e^{-2\sqrt{p}\frac{X}{\sqrt{\alpha_1}}}+P_4e^{-2\sqrt{p}\frac{\Delta}{\sqrt{\alpha_\Delta}}}\neq 0,\ p=ue^{\pm\pi i} \quad (60)$$

on contours II and IV. On contour II $\sqrt{p}=i\sqrt{|u|}$, and on IV $\sqrt{p}=-i\sqrt{|u|}$. The 2 resulting expressions will be conjugates, hence only 1 case need be considered. Suppose that equation 60 is false. Then, setting

$$A=2\sqrt{|u|}\frac{X}{\sqrt{\alpha_1}}$$

$$B=2\sqrt{|u|}\frac{\Delta}{\sqrt{\alpha_\Delta}}$$

and using equation 43, there exist some values of $A$ and $B$ such that

$$\begin{aligned}\beta_2\beta_\Delta\,[e^{i(A+B)}-1+e^{iA}-e^{iB}]\\ +\beta_2^2\,[-e^{i(A+B)}-1+e^{iA}+e^{iB}]\\ +\beta_2\beta_1\,[e^{i(A+B)}-1-e^{iA}+e^{iB}]\\ +\beta_\Delta\beta_1\,[-e^{i(A+B)}-1-e^{iA}-e^{iB}]=0.\end{aligned} \quad (61)$$

Denoting the expressions in brackets by the radius vectors $\hat{Z}_1$, $\hat{Z}_2$, $\hat{Z}_3$, $\hat{Z}_4$ in the complex plane we have

$$\beta_2\beta_\Delta\hat{Z}_1+\beta_\Delta^2\hat{Z}_2+\beta_2\beta_1\hat{Z}_3+\beta_\Delta\beta_1\hat{Z}_4=0.$$

It is seen that

$$\hat{Z}_1\times\hat{Z}_3=\hat{Z}_2\times\hat{Z}_4=0$$

$$\hat{Z}_1\cdot\hat{Z}_2=\hat{Z}_3\cdot\hat{Z}_4=0.$$

Hence

$$\hat{Z}_1//\hat{Z}_3\perp\hat{Z}_2//\hat{Z}_4.$$

And since the sum of two orthogonal vectors vanishes only if each vector vanishes

$$\beta_2\beta_\Delta\hat{Z}_1+\beta_2\beta_1\hat{Z}_3=0$$

$$\beta_\Delta^2\hat{Z}_2+\beta_\Delta\beta_1\hat{Z}_4=0.$$

Because the "$\beta$'s" are positive,

$\hat{Z}_1$ and $\hat{Z}_3$ have opposite sense (62)

$\hat{Z}_2$ and $\hat{Z}_4$ have opposite sense. (63)

But

$$e^{i(A+B)}-1=2\sin\tfrac{1}{2}(A+B)[-\sin\tfrac{1}{2}(A+B)+i\cos\tfrac{1}{2}(A+B)]$$

$$e^{iA}-e^{iB}=2\sin\tfrac{1}{2}(A-B)[-\sin\tfrac{1}{2}(A+B)+i\cos\tfrac{1}{2}(A+B)].$$

Hence, $e^{i(A+B)}-1$ and $e^{iA}-e^{iB}$ are colinear and by 62 the latter must dominate. This leads to

$$|\sin\tfrac{1}{2}(A-B)|>|\sin\tfrac{1}{2}(A+B)|. \quad (64)$$

Similarly,

$$e^{i(A+B)}+1=2\cos\tfrac{1}{2}(A+B)[\cos\tfrac{1}{2}(A+B)+i\sin\tfrac{1}{2}(A+B)]$$

$$e^{iA}+e^{iB}=2\cos\tfrac{1}{2}(A-B)[\cos\tfrac{1}{2}(A+B)+i\sin\tfrac{1}{2}(A+B)]$$

and 63 requires

$$|\cos\tfrac{1}{2}(A-B)|>|\cos\tfrac{1}{2}(A+B)|. \quad (65)$$

Squaring equations 64 and 65 and adding leads to the contradiction

$$1>1.$$

Therefore equation 60 is true.

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