Computer-Assisted Map Projection Research

By J. P. Snyder

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PREFACE

The expanding ability of computers to solve problems for which the solutions have been impractical in the past has been applied to several areas of map-projection research by the U.S. Geological Survey. Since the principle of least squares applies to several of the topics researched, some of the earlier usage of least squares in the development of map projections, with or without computers, is also reviewed.

Two general areas of development are described. The first area, treating efficient data transfer between maps, is subdivided into two principal investigations. In the first, the principle of least squares is applied to the development of polynomials to be used in place of analytical equations in transferring large quantities of data from one map to another map or transferring data to or from a data base in geodetic or rectangular coordinates. This use of least squares was described in large part in a paper by Wu and Yang (1981). The derivations contained herein are more extensive, but were inspired by that paper.

In the second investigation, the computer is used to determine which, if any, of several common map projections fit a map for which the projection or set of parameters is not known. The user must carefully measure rectangular coordinates for a certain matrix of nine points of known latitude and longitude. The computer program tests these for successive types of projections. The answer is not exact, limited by paper expansion, accuracy of cartography, accuracy of measurement, and the fact that, on some large-scale maps, projection difference is almost undetectible; for the latter, a second-order polynomial fits satisfactorily. The result normally is sufficiently accurate to permit transfer of other data from the map.

The second general area of study discussed in this paper concerns minimum-error projections. Here the computer is used to provide a least-squares fit of a conformal or equal-area map projection to a large number of weighted points representing the region to be mapped. Thus the central meridian and central scale factor of the Transverse Mercator or the various parameters of oblique aspects of other projections may be chosen analytically, rather than by subjective judgment, to reduce overall scale error on the map. With the careful use of complex transformations, still less error can usually be achieved in a given region, for example, to prepare a 50-State map of the United States.

I appreciate the helpful counsel of several individuals in and out of the Survey in reviewing the manuscript or inspiring some of the research. Especially do I thank Atef A. Elassal and Allen J. Pope of the National Ocean Service, Paul R. Wolf of the University of Wisconsin/ Madison, and Lee U. Bender, Joel L. Morrison, and John F. Waananen of the U.S. Geological Survey. In addition, I appreciate the devotion of K. Susan Bruckschen and Cynthia L. Cunningham in formatting, typing, and repeatedly changing the equation-laden manuscript. The remaining errors and limitations are entirely my responsibility.

It is hoped that these applications of well-known principles will be found useful to prepare maps with still less distortion than those designed with earlier conventions.

John P. Snyder

Reston, Va.
1984

CONTENTS

Page

ILLUSTRATIONS

Page

Page

TABLES

Page

SYMBOLS

If a symbol is not listed here, it is used only briefly and identified near the formulas in which it is given.

a = semimajor axis of Earth at map scale

$\arctan_2$ = arc tangent with quadrant adjustment like that of Fortran ATAN2 function. Otherwise, arctan corresponds to ATAN.

e = eccentricity of reference ellipsoid (generally).
= base of natural logarithms (2.718 ...)(if specially indicated)

f = function of

i = square root of -1

ln = natural logarithm based on e=2.718...

n = cone constant in conic projections; otherwise integer serving as upper limit.

R = radius of Earth as sphere at map scale

RMSE = root-mean-square error

s = distance

x = rectangular coordinate: distance to the right of the vertical line (Y axis) passing through the origin or center of a projection (if negative, it is distance to the left).

y = rectangular coordinate: distance above the horizontal line (X axis) passing through the origin or center of a projection (if negative, it is distance below).

(x', y') = rectangular coordinates in a different reference frame, such as on another map.

z = angular distance from the pole of the projection.

Δ = finite change in

∂ = partial derivative of

θ = angle, usually between meridians

λ = longitude east of Greenwich (for longitude west of Greenwich, a minus sign is used).

λ_p = longitude east of Greenwich of the pole of a transformed projection.

λ_o = longitude east of Greenwich of the central meridian of the map, or of the origin of the rectangular coordinates.

SYMBOLS--continued

ϕ = north geodetic or geographic latitude (if latitude is south, a minus sign is applied).

ϕ_p = north latitude of pole of transformed projection.

ϕ_o = latitude of central point on miscellaneous map projections

ϕ_1 = single standard parallel of latitude on cylindrical or conic projections; latitude of central point on azimuthal projections.

ψ = isometric latitude

Σ = sum of

COMPUTER—ASSISTED MAP PROJECTION RESEARCH

By John P. Snyder

ABSTRACT

Computers have opened up areas of map projection research which were previously too complicated to utilize, for example, using a least-squares fit to a very large number of points. One application has been in the efficient transfer of data between maps on different projections. While the transfer of moderate amounts of data is satisfactorily accomplished using the analytical map projection formulas, polynomials are more efficient for massive transfers. Suitable coefficients for the polynomials may be determined more easily for general cases using least squares instead of Taylor series.

A second area of research is in the determination of a map projection fitting an unlabeled map, so that accurate data transfer can take place. The computer can test one projection after another, and include iteration where required.

A third area is in the use of least squares to fit a map projection with optimum parameters to the region being mapped, so that distortion is minimized. This can be accomplished for standard conformal, equal-area, or other types of projections. Even less distortion can result if complex transformations of conformal projections are utilized.

This bulletin describes several recent applications of these principles, as well as historical usage and background.

INTRODUCTION

The use of the computer for map plotting has, in less than two decades, become standard for many governmental agencies as well as an increasing number of private firms. Many of the map projections used,

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however, remain those of long standing, although formulas have not been available for some until recently because they had been plotted geometrically. Until computers were available, the formulas for some projections were too complicated to be practical for computation of coordinates.

Computers and even pocket calculators have opened up large areas of research on new projections which were previously too complicated for practical computation of coordinates for plotting. One area has been in the numerical solutions of problems, such as those involving hundreds of points in least-squares fitting, or iteration involving several simultaneous equations. Previously these problems could defeat even the most persistent mathematician because of the sheer volume of work and the ease with which errors could be made.

A leader in this field from its earlier days (soon after 1960) has been Waldo R. Tobler. During his tenure on the faculty of the University of Michigan, where he remained until 1977, he was a major influence in moving the field from its infancy to accepted practice. Among Tobler's numerous special studies were applications of spatial effects to mapping so that the areas could be automatically altered in accordance with, for example, population or retail sales (Tobler, 1963a, 1963b). In other studies he used a computer to determine possible projections for old maps and to determine by least squares an "optimum" map projection of the United States (Tobler, 1966, 1977).

Another recent product of map-projection research with computers was the mathematical development of the Space Oblique Mercator projection. This was pursued by the writer using only programmable pocket calculators for the numerous calculations and checking of formulas (Snyder, 1978, 1981a). On a more conventional basis, the writer used the same calculators to check numerous existing map-projection formulas as well as newly derived ones before including them in other published papers, or in a computerized USGS map-projection package called the General Cartographic Transformation Package (U.S. Geological Survey, 1981).

EFFICIENT DATA TRANSFER BETWEEN MAPS OF DIFFERENT PROJECTIONS: OVERVIEW

Since data are being digitized at an increasing rate from existing maps into a data base, and are being increasingly plotted from one map onto another, it is essential that the mathematical relationship between the projections of the maps involved be properly understood. Several common situations occur that affect the nature of these relationships. The map projection of the existing map may be unidentified, identified only by name, or completely identified with parameters. The projection of the map to which data are to be transferred will more likely be known in detail, but this may not be the case. The transfer of data may involve only a few points or a great many points. Several options are available to perform the transfer of data based on the situations that exist.

Judging from the notations on many maps, there is a widespread belief that the name of the projection is sufficient identification. Generally this is not true. For example, unless the standard parallels for a conic projection are given, data cannot be properly transferred from this map to another. The Lambert Conformal Conic label tells the user that there is conformality and therefore correct local shape, and that the meridians are straight, but only if the standard parallels are known can the scale be used to determine accurately the geographic positions of points on the map. Similarly, the central meridian of a Transverse Mercator projection must be known together with any central scale reduction.

Measurements may be made to determine standard parallels or central meridians, as discussed in a later section, but this procedure can be technically complicated, and it is limited by errors of map drafting, measurement, and paper expansion. On the other hand, the choice by the mapmaker for the origin of rectangular coordinates on any map, or for the central meridian of a conic projection, although used for the original plotting, is not needed for the transfer of data, since another choice merely rotates or translates the map and does not affect relative positions.

Because of the increased emphasis on data transfer, the USGS has recently undertaken research to develop additional capabilities in transferring data automatically from one map to another. The standard approach in data transfer, namely analytical transfer using exact formulas for the projections involved, is briefly mentioned below. The mention is brief only because it is standard. It remains the approach to consider first. This is followed by a discussion of the use of polynomial approximations for the same purpose. The third phase under the same heading "Efficient Data Transfer ..." involves computer techniques for identification of the map projection for an incompletely labeled map, and therefore one from which data cannot be readily transferred. This identification is developed with sufficient parameters to permit data transfer from the map to a data base or to another map.

ANALYTICAL TRANSFER OF DATA

Assuming that the projection of the existing map is adequately identified, the transfer of a moderate number of points from one map to another is most satisfactorily accomplished by the use of the exact trigonometric formulas, usually called "inverse" formulas, for conversion of rectangular to geographic coordinates for the first map, followed by conversion of geographic to rectangular coordinates with exact "forward" formulas for the second map. These formulas apply to maps extending over small or large portions of the Earth. While a few projections are computed using approximate series, such as the Space Oblique Mercator and the ellipsoidal form of the Transverse Mercator, the series are sufficiently accurate for the entire normal ranges of the projections.

There are several computer packages containing the forward formulas for one or more projections, such as Philip Voxland's WORLD package at the University of Minnesota (Voxland, 1981). It contains over a hundred projections. The inverse formulas are much less common in computer programs. The USGS General Cartographic Transformation Package (GCTP) has forward and inverse formulas for 17 major projections, including the ellipsoidal or spherical versions for 8 of them (U.S. Geological Survey, 1981). This package is also incorporated into GS-CAM, a plotting package modified from CAM (the Cartographic Automatic Mapping program of the Central Intelligence Agency) by USGS (U.S. Geological Survey, 1982). A recent USGS bulletin lists the forward and inverse formulas for all projections used by the Survey in its past and present published maps (Snyder, 1982).

The exactness of the analytic formulas is offset by the fact that most involve several trigonometric calculations and similar time-consuming operations which may unnecessarily increase the computer time and therefore the cost of data transfer, depending on the quantity and type of transformation required.

POLYNOMIAL APPROXIMATIONS FOR DATA TRANSFER

If thousands of points are to be transferred between two maps of known projection parameters, the use of a polynomial approximation should be considered to reduce computation time. However, this is not a panacea. If the region of the map is continent-sized, a polynomial of sufficient accuracy will usually contain an excessive number of terms and will not save computer time. If the area is smaller, however, a low-order polynomial may be more efficient. Since the calculation of the polynomial coefficients may require moderate computer storage, and the coefficients apply only to a limited area, the number of points to be transferred must normally be very large for this technique to be effective.

The actual computer time saved varies with the projection, the accuracy required, and the type of transformation. For forward or inverse conversion of Lambert Conformal Conic coordinates, even a nested third-order polynomial can take longer to compute than the analytical equations, and is more limiting. Polynomials are normally faster than the ellipsoidal Transverse Mercator series. To convert from rectangular coordinates of the Lambert Conformal Conic to those of the Transverse Mercator, or to transfer data between two Transverse Mercator projections, it is usually faster to use polynomials, if enough transformations are required to justify computation of coefficients suitable for the region of interest.

In the analyses given later, polynomial equations for converting one type of coordinates to another are described for two categories of map projections, namely general and conformal. The general equations also apply to conformal map projections, but for conversion from one conformal projection to another, the polynomial may be expressed in complex algebra, using fewer coefficients for a given accuracy.

The computation of polynomial coefficients can take one of at least two forms. The conventional pattern is to develop coefficients from a Taylor series. This becomes increasingly unwieldly and error prone as the size of the geographic region increases, or when the conversion involves rectangular to rectangular coordinates with at least one nonconformal projection. A more generally useful technique for developing coefficients involves the use of least squares. The Taylor series approach is briefly described, but the application of the least-squares method is discussed more completely.

IDENTIFYING AN UNMARKED MAP PROJECTION

The foregoing approaches are based upon adequately knowing the map projection together with its parameters for the source map. If they are not known, data cannot be properly transferred from this map to another base. For these transfers to take place, it may not be necessary to determine the original projection parameters precisely, but a close approximation is needed. An experienced observer can frequently determine a possible projection and its parameters by beginning with elementary checks and ending with careful measurements. Initially the questions may include these: Are meridians straight or curved? Are parallels straight or curved? If parallels are curved, are they concentric circular arcs? How are parallels spaced along meridians? Without measurements, however, an unlabeled map of the United States according to the Lambert Conformal Conic cannot be distinguished from one according to the Albers Equal-Area Conic, and the standard parallels are even more difficult to determine. For large-scale maps such as topographic quadrangles covering small areas, ascertaining the projection is still more difficult, and a low-order polynomial or more than one projection may fit the points as accurately as normal measurement permits. In any case, the identity of the projection for purposes of data transfer is incomplete without parameters such as scale and standard parallels or central meridian.

While the computer lacks an ability to give the map an overview, it can be programmed to make some of the normal human tests, and also to make checks which are too subtle even for experienced observers, much less those less familiar with map projection design. Even the computer is limited by the problem of unequal expansion and contraction of paper maps, the usual form in which maps with unknown parameters are supplied.

A computer, however, can make a large number of computations for an undefined map to determine which projection fits reference points measured. In view of this, a program has been developed and tested to determine the suitability of a second-order polynomial or of some 18 projections. Of these projections eight are also tested for the Earth taken as an ellipsoid. The program then permits calculation of geographic coordinates of other points on the map based on the parameters determined. These projections are as follows:

Regular Cylindrical:
- Mercator (spherical or ellipsoidal)
- Miller Cylindrical (spherical)
- Equirectangular (spherical)
- Gall's (spherical)

Regular Conic (spherical or ellipsoidal):
- Lambert Conformal Conic
- Albers Equal-Area Conic
- Equidistant Conic

Azimuthal (spherical for any aspect, except ellipsoidal form also for polar aspects asterisked (*)):
- Stereographic*
- Lambert Azimuthal Equal-Area*
- Azimuthal Equidistant*
- Orthographic
- Gnomonic
- General Vertical Perspective

Pseudocylindrical (spherical only):
- Sinusoidal
- Mollweide

Others (spherical or ellipsoidal):
- Polyconic
- Transverse Mercator
- Tilted Perspective
- Second-order polynomial

Additional projections may be added with varying difficulty if the projections have curved meridians and curved parallels, but minimal difficulty for others (regular cylindricals, conics, and pseudocylindricals).

LIMITATIONS

If coordinates are artificially calculated rather than measured, the program determines the projection parameters almost exactly. For 25 actual maps, using a coordinate-measuring machine and a skilled technician, the final projection determined by the program generally fits within 0.5 mm. Determining map parameters in this manner is not ideal. It is often not possible to determine the original parameters of an existing map correctly. It is only possible to determine parameters of sufficient accuracy to permit transfer of data.

GENERAL FORMAT OF PROGRAM

The program as developed is designed to handle data supplied in the form of a matrix of nine known points along three meridians and three parallels; the second meridian and second parallel are not necessarily midway between the first and third. Knowing the latitude, longitude, and rectangular coordinates of these nine points relative to an arbitrary set of X and Y axes, the program first checks for the straightness of meridians and parallels.

If meridians and parallels are all straight, and one set perpendicular to the other, the spherical Mercator projection is tested for fitting the nine points. If it fails to fit, the ellipsoidal Mercator is tested, and then the Miller, Equirectangular, and Gall's, in order. If none fits, the program reports that the projection is cylindrical, but does not fit projections currently programmed.

If fit is achieved for one of these cylindrical projections, it is so reported, giving the scale of the map and, for the Equirectangular, the standard parallels. The program then applies the scale and unreported parameters, including rotation and translation of the axes used for measurement, to other pairs of rectangular coordinates, computing and reporting latitude and longitude.

If meridians and parallels are not all straight, the suitability of a second-order bivariate polynomial (the highest order which may be generally determined from nine points) is checked. It would exactly fit the nine-point matrix for a cylindrical projection without fitting other points well, so it is not tested if all meridians and parallels are straight. If the polynomial does not fit well, projection tests continue.

If meridians are straight, but parallels are not, meridians are checked for parallelism. If parallel, the Transverse Mercator, Polyconic and equatorial Gnomonic projection are tested for fit; if not parallel, meridians are tested for convergence at a common point. (Although meridians of the Transverse Mercator and Polyconic are curved, they are nearly straight on many large-scale maps of small areas.) If there is no convergence, the program cannot find the projection. If there is convergence, and if parallels are not concentric about this point, the Transverse Mercator, Polyconic, and oblique Gnomonic are tested.

If parallels are concentric, successive conics are tested, spherical and ellipsoidal, Lambert Conformal, Albers Equal-Area, and Equidistant. If the cone constant (the ratio of meridian spacing on the map to true spacing) is ±1, the conic is reported in its polar form (Stereographic, Lambert Azimuthal Equal-Area, and Azimuthal Equidistant, respectively) if fit occurs. If these projections do not fit, and the cone constant is not ±1, the program reports that the projection is an unprogrammed conic; if the cone constant is ±1, the Orthographic, Gnomonic, and Vertical Perspective are successively tested before the program reports an unprogrammed polar azimuthal.

If parallels are straight and meridians are not, the equatorial Orthographic and pseudocylindrical projections are tested. If neither set of lines is straight, the program tries successively the Transverse Mercator and Polyconic in both spherical and ellipsoidal form, and then the various oblique or equatorial azimuthals, including the Vertical Perspective, and finally the Tilted Perspective. The mathematics for the all-curved group is more difficult, and programming involved several false starts because of iteration which seemed feasible in principle, but which failed to converge or which converged to the wrong answers due to the difficulty of choosing the first approximations.

In any of these cases, errors in measurements and errors due to dimensional instability of the paper map must be considered. Therefore a tolerance is required in comparison checks such that a reasonable projection solution is not rejected. The tolerances are discussed later. The general flowsheet is shown in figures 1 and 2. Once the projection is determined, the computation of latitude and longitude from rectangular coordinates of additional points involves previously published inverse projection formulas combined with translation and rotation of coordinates based on parameters of the nine points already used.

These testing approaches are converted to mathematical expressions with the derivations following those for polynomials. They are shown approximately in the order the tests are used. Additional formulas are included in the Appendix (Section 6).

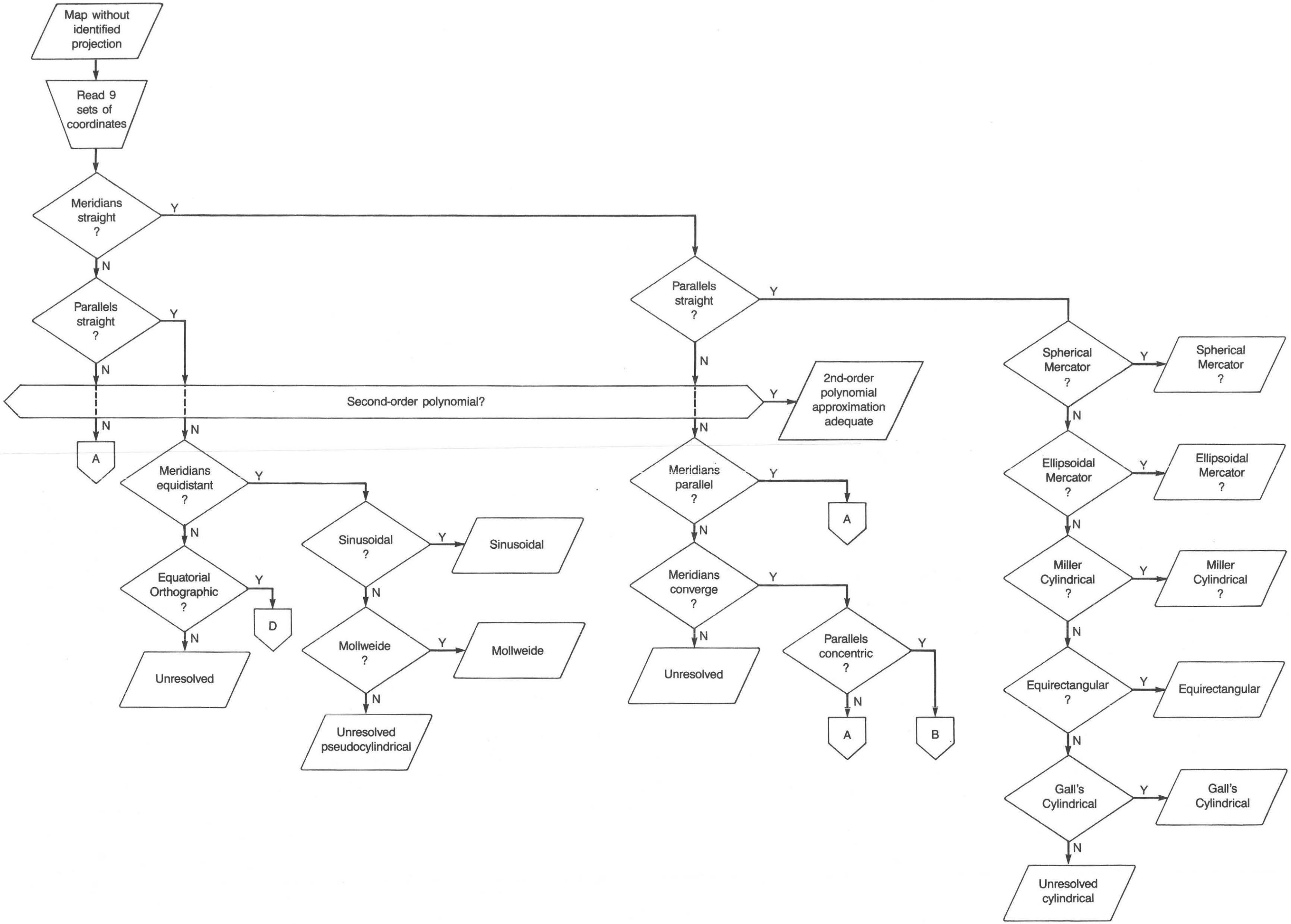

Figure 1.--General flowsheet of program to identify map projection (part 1).

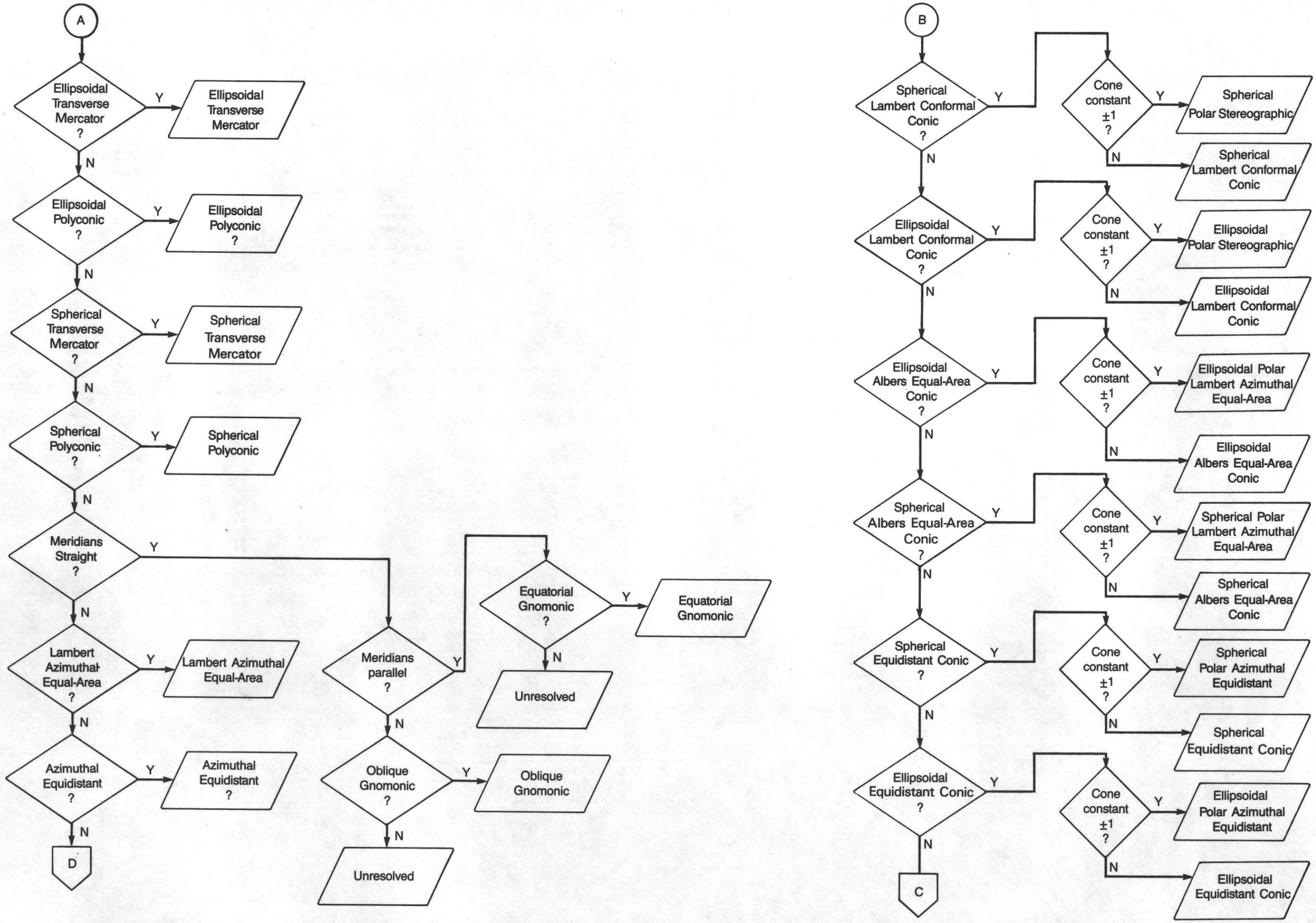

Figure 2a.--General flowsheet (part a & b).

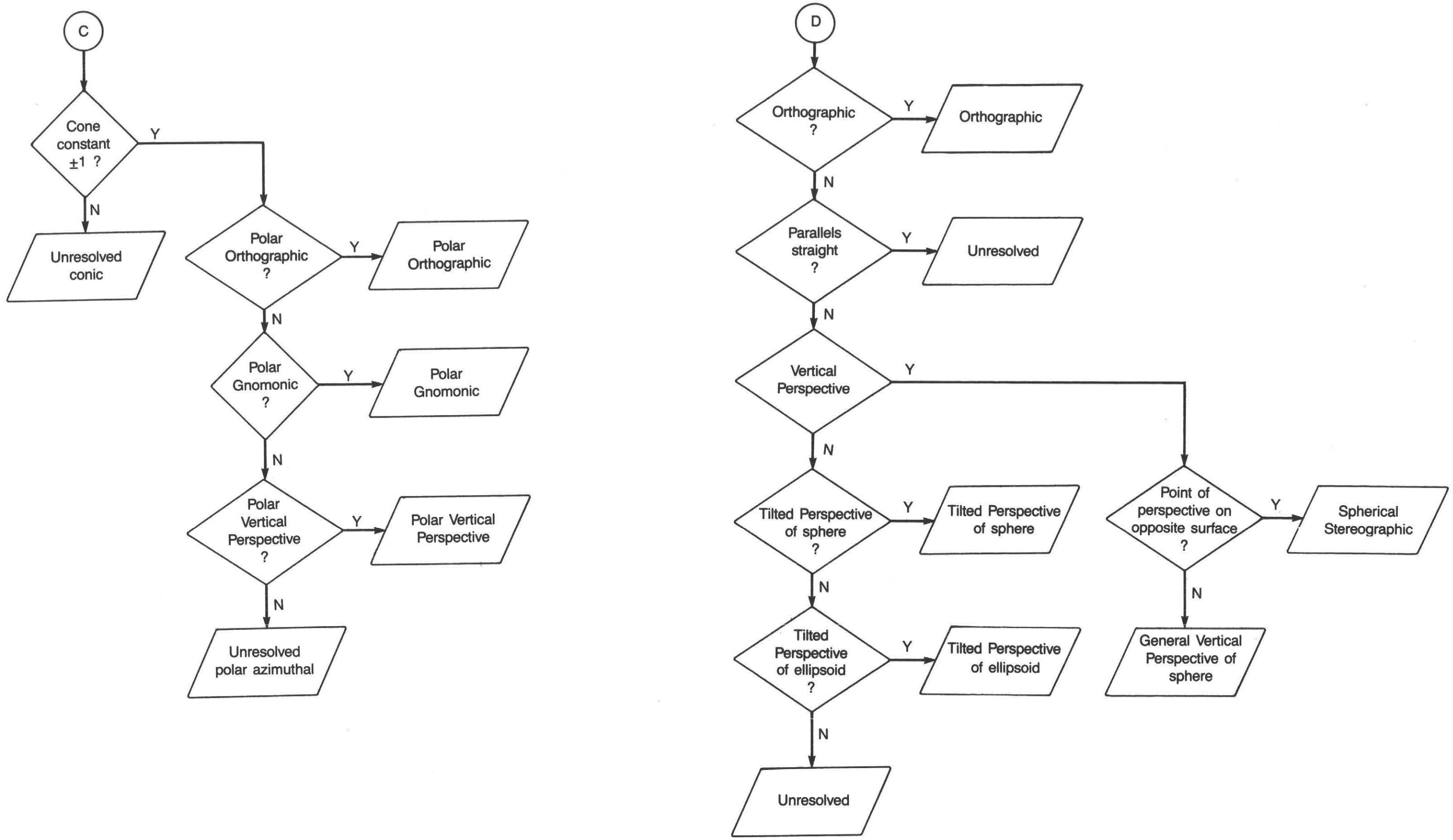

Figure 2b.--General flowsheet (part c & d).

EFFICIENT DATA TRANSFER: ANALYSIS AND DERIVATIONS

1. ANALYTICAL TRANSFER OF DATA

The use of exact trigonometric formulas (or approximating series if required) is important enough to be given a separate section number. For the actual formulas, the reader is referred to Snyder (1982), and for the derivations to some of the references listed therein. Except for the use of some of these equations in connection with the derivations presented in this bulletin, they are not repeated here.

2. POLYNOMIAL APPROXIMATIONS FOR DATA TRANSFER

Before reading through the following derivations, the paragraphs under this same heading on p. 5 should be read first.

a. BASIC EQUATIONS

The general bivariate transformation polynomial for converting longitude λ and latitude ϕ to rectangular coordinates x and y is as follows:

$$x = C_1 + C_2\lambda + C_3\phi + C_4\lambda^2 + C_5\lambda\phi + C_6\phi^2 + C_7\lambda^3 + C_8\lambda^2\phi + C_9\lambda\phi^2 + C_{10}\phi^3 + \ldots \quad (2\text{-}1)$$

$$y = \text{the same equation but with } C' \text{ in place of } C \quad (2\text{-}2)$$

where C_j and C_j' are constant coefficients.

To transform coordinates from (x,y) to (λ,ϕ), equations (2-1) and (2-2) are rewritten, interchanging x with λ and y with ϕ and using new coefficients.

To transform from rectangular coordinates (x',y') of one projection to (x,y) of another projection, equations (2-1) and (2-2) are written with x' and y' in the place of λ and ϕ respectively, using new coefficients.

Although equations (2-1) and (2-2) are left in the above form for further analysis below, they should be nested for more efficient actual computation, once the coefficients are determined, to avoid repeated exponentiation. For example, equation (2-1) can become

$$x = C_1 + \phi(C_3 + \phi(C_6 + C_{10}\phi)) + \lambda(C_2 + \phi(C_5 + C_9\phi) + \lambda(C_4 + C_8\phi + C_7\lambda)) + \ldots \quad (2\text{-}3)$$

This can save 20 to 30 percent in computation time for a fifth-order polynomial.

If the projection is symmetrical about the Y axis, which is normally made to lie along the central meridian, some of the above coefficients are zero: In equation (2-1) the coefficients of even powers of λ (including the zero power) are zero; in equation (2-2) the coefficients of odd powers of λ are zero. If the projection is symmetrical about the X axis within the mapped region, the coefficients of odd powers of ϕ in equation (2-1) and even powers of ϕ in equation (2-2) are zero. For symmetry about both axes, both consequences apply.

b. CONFORMAL TRANSFORMATIONS

If the projections involved are conformal, complex algebra may be used instead of the real equations above. Wherever ϕ is involved, however, the isometric latitude, often given the symbol ψ, must be used in the polynomial (Snyder, 1982, p. 18-19). Equations (2-1) and (2-2) may be replaced as follows, where $i^2 = -1$:

$$x + iy = (K_0 + iK_0') + (K_1 + iK_1')(\lambda + i\psi) + (K_2 + iK_2')(\lambda + i\psi)^2 + (K_3 + iK_3')(\lambda + i\psi)^3 + \ldots \quad (2\text{-}4)$$

or

$$x + iy = \sum_{j=0}^{n} (K_j + iK_j')(\lambda + i\psi)^j \quad (2\text{-}5)$$

For the inverse, x may be interchanged with λ, and y with ψ, using new coefficients. For transformation from rectangular coordinates (x',y') of one projection to (x,y) of another projection, λ and ψ in (2-4) or (2-5) are replaced with x' and y', respectively.

If there is symmetry about the Y axis in equations (2-4) or (2-5), K_j is zero if j is even, and K_j' is zero if j is odd. With symmetry about the X axis, K_j' is zero for all values of j. With symmetry about both axes, then, K_j is zero if j is even, and K_j' is zero for all values of j. By interchanging x with y and λ with ψ in equation (2-4), for the fairly common case of symmetry about the Y axis, it is found that all coefficients are real, and equation (2-5) may be written for this case only:

$$y + ix = \sum_{j=o}^{n} K_j(\psi + i\lambda)^j \qquad (2\text{-}6)$$

The new K_o, K_1, K_2, and K_3 are equal to the old K_o', K_1, $-K_2'$, and $-K_3$, respectively.

Expanding (2-4) or (2-5), and separating the real and imaginary portions,

$$x = K_o + K_1\lambda - K_1'\psi + K_2\lambda^2 - K_2\psi^2 - 2K_2'\lambda\psi + K_3\lambda^3$$
$$- 3K_3\lambda\psi^2 - 3K_3'\lambda^2\psi + K_3'\psi^3 + \ldots \qquad (2\text{-}7)$$

$$y = K_o' + K_1\psi + K_1'\lambda + 2K_2\lambda\psi + K_2'\lambda^2 - K_2'\psi^2 + 3K_3\lambda^2\psi$$
$$- K_3\psi^3 + K_3'\lambda^3 - 3K_3'\lambda\psi^2 + \ldots \qquad (2\text{-}8)$$

Comparison with equations (2-1) and (2-2) shows that, if ψ were used instead of φ in the latter two equations,

$C_1 = K_o$	$C_1' = K_o'$
$C_2 = K_1$	$C_2' = K_1'$
$C_3 = -K_1'$	$C_3' = K_1$
$C_4 = K_2$	$C_4' = K_2'$
$C_5 = -2K_2'$	$C_5' = 2K_2$
$C_6 = -K_2$	$C_6' = -K_2'$
$C_7 = K_3$	$C_7' = K_3'$
$C_8 = -3K_3'$	$C_8' = 3K_3$
$C_9 = -3K_3$	$C_9' = -3K_3'$
$C_{10} = K_3'$	$C_{10}' = -K_3$

For transformation from rectangular coordinates of one conformal projection to those of another, only (x,y) and (x',y') need be compared, so the complex coefficients may be directly compared to the real coefficients without involving ψ.

To convert geodetic latitude φ to isometric latitude ψ,

$$\psi = \ln \{\tan (\pi/4 + \phi/2)[(1 - e \sin \phi)/(1 + e \sin \phi)]^{e/2}\} \quad (2\text{-}9)$$

where e is the eccentricity of the ellipsoid.

Nesting of equation (2-5) for improved final calculating efficiency leads to the following:

$$x + iy = [[(K_3 + iK_3')(\lambda + i\psi) + K_2 + iK_2'](\lambda + i\psi)$$
$$+ K_1 + iK_1'](\lambda + i\psi) + K_o + iK_o', \text{ etc.} \quad (2\text{-}10)$$

It is still more efficient to use Knuth's algorithm for evaluation of equation (2-5) (Knuth, 1969):

$$\text{Let } r = 2\lambda;\ s' = \lambda^2 + \psi^2;\ g_o = 0;\ g_f = K_f + iK'_f;\ a_1 = g_n;$$
$$b_1 = g_{n-1};\ a_j = b_{j-1} + ra_{j-1};\ b_j = g_{n-j} - s'a_{j-1}.$$

After j is given the value of successive integers from 2 to n,

$$x + iy = (\lambda + i\psi)a_n + b_n \quad (2\text{-}11)$$

Equation (2-5) and the Knuth algorithm are used again in this paper, but to solve a different problem with equation (4-65) and following equation (5-28).

c. COMPUTATION OF COEFFICIENTS

(1) Conventional

A standard method of determining the coefficients in equations (2-1) and (2-2) is by the development of a Taylor series. If a central point has coordinates ϕ_o and λ_o, and x_o and y_o, and if $x = f_1(\lambda,\phi)$, $y = f_2(\lambda,\phi)$, $\Delta x = x - x_o$, $\Delta y = y - y_o$, $\Delta\lambda = \lambda - \lambda_o$, and $\Delta\phi = \phi - \phi_o$, then equation (2-1) may be rewritten using a bivariate Taylor series,

$$\Delta x = (\partial f_1/\partial\lambda)(\Delta\lambda) + (\partial f_1/\partial\phi)(\Delta\phi) + (1/2!)[(\partial^2 f_1/\partial\lambda^2)(\Delta\lambda)^2$$
$$+ 2(\partial^2 f_1/\partial\lambda\partial\phi)(\Delta\lambda)(\Delta\phi) + (\partial^2 f_1/\partial\phi^2)(\Delta\phi)^2] + (1/3!)$$
$$[(\partial^3 f_1/\partial\lambda^3)(\Delta\lambda)^3 + 3(\partial^3 f_1/\partial\lambda^2\partial\phi)(\Delta\lambda)^2(\Delta\phi) + 3(\partial^3 f_1/$$
$$\partial\lambda\partial\phi^2)(\Delta\lambda)(\Delta\phi)^2 + (\partial^3 f_1/\partial\phi^3)(\Delta\phi)^3] + \ldots \quad (2\text{-}12)$$

and (2-2) may be written identically but with Δy in place of Δx and f_2 in place of f_1. Without a difficult bivariate inversion of this series, the coefficients for the inverse (x,y to ϕ,λ) must be determined by separate differentiation of the more complicated inverse formulas for the projection. The differentiation, forward or inverse, is normally very tedious, and successive derivatives are increasingly subject to errors in derivation for each projection involved. To determine coefficients to transform rectangular coordinates from one projection to another is still more complicated, unless one pair of series based on the inverse formulas of one projection is followed by a pair based on the forward formulas of the second projection. Use of numerical bivariate differentiation also requires inverse formulas, or bivariate series inversion.

If only conformal projections are involved, as is often the case, the handling of Taylor series is simplified. Equation (2-5) becomes

$$\Delta x + i\Delta y = \sum_{j=o}^{n} [f_j(\lambda_o,\psi_o)/j!](\Delta\lambda + i\Delta\psi)^j \qquad (2\text{-}13)$$

This is described by Jordan-Eggert (1962, p. 160-163, 225-228), Lee (1974a), and others. To change from (x',y') of one projection to (x,y) of another, where both projections are conformal, the constant coefficients may be evaluated using only the forward formulas, as discussed for two Transverse Mercator projections by Jordan-Eggert (1962, p. 199-217), and for the Transverse Mercator and Lambert Conformal Conic projections by Shmutter (1981) and Doytsher and Shmutter (1981). Numerical monovariate or bivariate differentiation using a computer subroutine is probably a safer means of accomplishing the above objectives on a universal basis, but this was not attempted by the writer, in favor of a least-squares approach.

(2) Least Squares

For general purposes, the least-squares approach provides a system relatively free of many separate complicated derivations and differentiations. To permit transformation involving additional projections in a computer program based on this approach, only the forward formulas need to be added, even though the program computes forward, inverse or rectangular-to-rectangular transformations.

The techniques described here are used to determine a least-squares fit of a given matrix of known points to a given-size polynomial, described in large part by Wu and Yang (1981).

To determine the coefficients in the general polynomial equations (2-1) and (2-2) by least squares, the exact forward map projection equations are used to calculate rectangular coordinates for a matrix of m points distributed over the region for which points are to be transformed.

There must be at least as many distinct points as the number of coefficients desired. For example, if there is symmetry about the Y axis, symmetrical pairs of points should count as only one per pair, in determining matrix size. The m x n matrix [A] is then developed, where n is the number of coefficients to be computed, using the various values of ϕ and λ, one pair to a row:

$$[A]=\begin{bmatrix} 1 & \lambda_1 & \phi_1 & \lambda_1^2 & \lambda_1\phi_1 & \phi_1^2 & \lambda_1^3 & \lambda_1^2\phi_1 & \lambda_1\phi_1^2 & \phi_1^3\ldots \\ 1 & \lambda_2 & \phi_2 & \lambda_2^2 & \lambda_2\phi_2 & \phi_2^2 & \lambda_2^3 & \lambda_2^2\phi_2 & \lambda_2\phi_2^2 & \phi_2^3\ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \lambda_m & \phi_m & \lambda_m^2 & \lambda_m\phi_m & \phi_m^2 & \lambda_m^3 & \lambda_m^2\phi_m & \lambda_m\phi_m^2 & \phi_m^3\ldots \end{bmatrix} \tag{2-14}$$

Using a standard means of solving for the coefficients, an n x m matrix [D] is developed (see Appendix, Section 7, for derivation):

$$[D] = [A^T\ A]^{-1} A^T \tag{2-15}$$

from which

$$\begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix} = [D] \bullet \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \tag{2-16}$$

and

$$\begin{bmatrix} C_1' \\ C_2' \\ \vdots \\ C_n' \end{bmatrix} = [D] \bullet \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \tag{2-17}$$

where (x_m, y_m) are the rectangular coordinates calculated for the respective values of (λ_m, ϕ_m) from the map-projection formulas.

For complex variables, equation (2-14) takes the form

$$[A_c] = \begin{bmatrix} 1 & \zeta_1 & \zeta_1^2 & \zeta_1^3 & \cdots \\ 1 & \zeta_2 & \zeta_2^2 & \zeta_2^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \\ 1 & \zeta_m & \zeta_m^2 & \zeta_m^3 & \cdots \end{bmatrix} \qquad (2\text{-}18)$$

where $\zeta_m = \lambda_m + i\psi_m$ (compare equation (2-4)). Equation (2-15) is rewritten

$$[D_c] = [A_c^T A_c]^{-1} A_c^T \qquad (2\text{-}19)$$

Equations (2-16) and (2-17) may be combined:

$$\begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} = [D_c] \bullet \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{bmatrix} \qquad (2\text{-}20)$$

where $k_n = K_n + iK_n'$ and $z_m = x_m + iy_m$. Wu and Yang did not describe the complex alternative, although they discussed conversion between two conformal projections, the Mercator and the Lambert Conformal Conic. Yang (1982) addressed the complex approach in a later paper, however.

To decrease rounding errors, it is important to subtract the average of the various values of λ and ϕ from the individual values before calculating the coefficients, and therefore before using the coefficients to calculate coordinates. Coefficients for the inverse or for transformation from one projection to another are determined from equations (2-14) through (2-20), using the proper substitutions for ϕ_m, λ_m, x_m, and y_m as described above. The forward formulas may be used in each case, even though λ and ϕ or ψ will not appear in equations (2-14) through (2-20) for rectangular to rectangular conversion.

The accuracy of the coefficients with various levels of polynomials (1st, 2nd, 3rd order, etc.) may be checked by using the coefficients to recalculate (x,y), comparing these values with the correct values obtained from analytical equations. The root-mean-square error (RMSE), r, is found as follows:

$$r = \left\{ \sum_{j=1}^{m} [(x_c - x_t)^2 + (y_c - y_t)^2]/m \right\}^{1/2} \qquad (2\text{-}21)$$

where (x_c, y_c) are computed for each of the m points from equations (2-1), (2-2), or (2-5), using the calculated coefficients, (x_t, y_t) are found from the true formulas, and Σ indicates the sum of the squares of each residual. If the residual is less than the desired accuracy limit, the coefficients are accepted. It was found for examples tested that the use of complex algebra for conformal projections resulted in residuals about twice those found using real coefficients derived from equations (2-14) through (2-17) for a given order polynomial. This is largely due to the reduced number of complex coefficients at a given level (not over 2 for each order if complex, but up to (n+1) for the nth-order terms if all are real). It is also necessary to calculate coefficients for a given order polynomial by separate computation, rather than to obtain them by truncating a higher order series, because of errors introduced by the latter approach.

It is also desirable to weight the various points used to calculate polynomial coefficients in proportion to the expected use in data transfer of the region surrounding the point. The most elementary weighting is by area. On maps of small regions, a graticule of uniform spacing in degrees is almost uniform in spacing by area. In a larger region, weighting in proportion to the cosine of the latitude on such a graticule would compensate exactly for the sphere, satisfactorily for the ellipsoid. This may be done by multiplying every term in the first row of the matrix of equation (2-14) by $\cos^{1/2}\phi_1$, in the second row by $\cos^{1/2}\phi_2$, and in the ith row by $\cos^{1/2}\phi_i$, etc., and each term in the (m x 1) matrices at the right of equations (2-16) and (2-17) by $\cos^{1/2}\phi_1, \cos^{1/2}\phi_2, \ldots \cos^{1/2}\phi_m$, respectively. Equation (2-21) becomes

$$r = \left\{ \sum_{j=1}^{m} \cos\phi_j[(x_c - x_t)^2 + (y_c - y_t)^2]/\sum_{j=1}^{m} \cos\phi_j \right\}^{1/2} \qquad (2\text{-}22)$$

Table 1 shows examples of real and complex coefficients determined for a 6° x 6° range, transforming geodetic to rectangular coordinates for the ellipsoidal Lambert Conformal Conic projection.

Table 1.--Examples of polynomial coefficients for map projection transformations

Example 1:

Conversion: Geodetic to rectangular coordinates

Projection: Ellipsoidal Lambert Conformal Conic
Standard parallels: Latitudes 33° and 45°N.
Central Meridian: Longitude 95°W.
Origin of rectangular coordinates: Latitude 23°N.; Longitude 95°W.
Ellipsoid: Clarke 1866

Scale: Full, meters

Range: Longitudes 98° to 92°W., Latitudes 40° to 46°N.

Matrix Intervals: 1° Longitude x 1° Latitude (number of points in matrix: m = 49)

Sample Coordinates:

	Input		Output	
Point	λ (radians)	ψ	x	y
(1,1)	-1.71042266	0.75855478	-254775.581	1901261.028
(1,2)	-1.69296937	0.75855478	-169867.529	1898924.567
(1,3)	-1.67551608	0.75855478	-84938.907	1897522.578
(1,4)	-1.65806279	0.75855478	0.000	1897055.229
(1,5)	-1.64060949	0.75855478	84938.907	1897522.578
(1,6)	-1.62315620	0.75855478	169867.529	1898924.567
(1,7)	-1.60570291	0.75855478	254775.581	1901261.028
(2,1)	-1.71042266	0.78141800	-251129.288	2011672.055
⋮	⋮	⋮	⋮	⋮
(7,7)	-1.60570291	0.90140082	232832.464	2565706.325

Average input coordinates: $\lambda_o = -1.65806279$, $\psi_o = 0.82900061$

Note: λ = longitude (west is -)
ψ = isometric latitude (see equation (2-9))
(x,y) = rectangular coordinate, Y axis increasing northerly along central meridian.

Complex polynomial coefficients, for input λ in radian,

$$x + iy = \sum_{j=o}^{n} (K_j + iK_j')[(\lambda-\lambda_o) + i(\psi - \psi_o)]^j$$

If polynomial is 2nd order,
$K_o' = 2232395.619$
$K_1 = 4653848.161$
$K_2' = 1469623.762$
rms error = 150.124 m

If polynomial is 3rd order,
$K_o' = 2232394.967$
$K_1 = 4655308.609$
$K_2' = 1467865.207$
$K_3 = -308173.969$
rms error = 1.210 m

If polynomial is 4th order,
$K_o' = 2232394.424$
$K_1 = 4655308.791$
$K_2' = 1467577.828$
$K_3 = -308446.444$
$K_4' = -48609.0346$
rms error = 0.011 m

Table 1.--Examples of polynomial coefficients for map projections transformations (cont'd.)

Example 2:

Same as Example 1, but real instead of complex. Instead of ψ, ϕ is used as an input coordinate (if ψ = 0.75855478, ϕ = 0.69813170 radians). Average input ϕ_o = 0.75049158.

$$x = C_1 + C_2(\lambda - \lambda_o) + C_3(\phi - \phi_o) + C_4(\lambda - \lambda_o)^2 + C_5(\lambda - \lambda_o)(\phi - \phi_o) + C_6(\phi - \phi_o)^2 + C_7(\lambda - \lambda_o)^3 + C_8(\lambda - \lambda_o)^2(\phi - \phi_o) + \ldots$$

$$y = C_1' + C_2'(\lambda - \lambda_o) + C_3'(\phi - \phi_o) + C_4'(\lambda - \lambda_o)^2 + C_5'(\lambda - \lambda_o)(\phi - \phi_o) + C_6'(\phi - \phi_o)^2 + C_7'(\lambda - \lambda_o)^3 + C_8'(\lambda - \lambda_o)^2(\phi - \phi_o) + \ldots$$

If polynomial is 2nd order,	If polynomial is 3rd order,	If the polynomial is 4th order,
C_2 = 4656747.857	C_2 = 4657602.103	C_2 = 4657602.097
C_5 =-4001532.769	C_5 =-4001670.100	C_5 =-4000712.736
C_1' = 2228756.675	C_7 = -308852.687	C_7 = -308551.350
C_3' = 6346215.322	C_9 = -160790.692	C_9 = -161313.447
C_4' = 1469530.890	C_1' = 2228757.414	C_{12} = 265137.469
C_6' = 254996.621	C_3' = 6345337.588	C_{14} = -714444.096
rms error = 58.256 m	C_4' = 1468097.120	C_1' = 2228757.634
	C_6' = 255825.654	C_3' = 6345337.185
	C_8' =-1261532.493	C_4' = 1468300.751
	C_{10}' = 1133030.948	C_6' = 254842.304
	rms error = 0.832 m	C_8' =-1261575.789
		C_{10}' = 1133244.655
		C_{11}' = -48683.4726
		C_{13}' = -50691.2433
		C_{15}' = 358537.886
		rms error = 0.012 m

In some cases, the weighting can be adjusted to include only points on the map which are within the desired country or on land rather than water, etc. Points near a boundary or a shoreline can be weighted in proportion to the area of interest. In each case $\cos^{1/2}\phi_j$ in the above weighting is replaced by the square root of the area, and $\cos\phi_j$ by the area.

A common basis for weighting in the use of least squares is to account for variation in accuracy of measurement of points using given equipment or operators. Since the above polynomials are computed from rigorously determined values, this criterion is not involved here.

3. IDENTIFYING AN UNMARKED MAP PROJECTION

a. GENERAL MATHEMATICAL FORMAT

Before reading through the following derivations, the paragraphs under this heading on p. 6-13 should be read first.

Before showing the mathematical analysis for specific types of projections, the more general tests are derived below. These consist of tests for the straightness of meridians and parallels, for the fit of a second-order polynomial, and for convergence of meridians and concentricity of parallels.

The matrix of nine points for which coordinates are to be measured is numbered as shown in figure 3. Longitude may be stated relative to Greenwich (or any other prime meridian), and (x,y) may be measured relative to any perpendicular coordinate axes. To determine whether the meridians are straight, first the angle σ_{1-2} of slope of a straight line

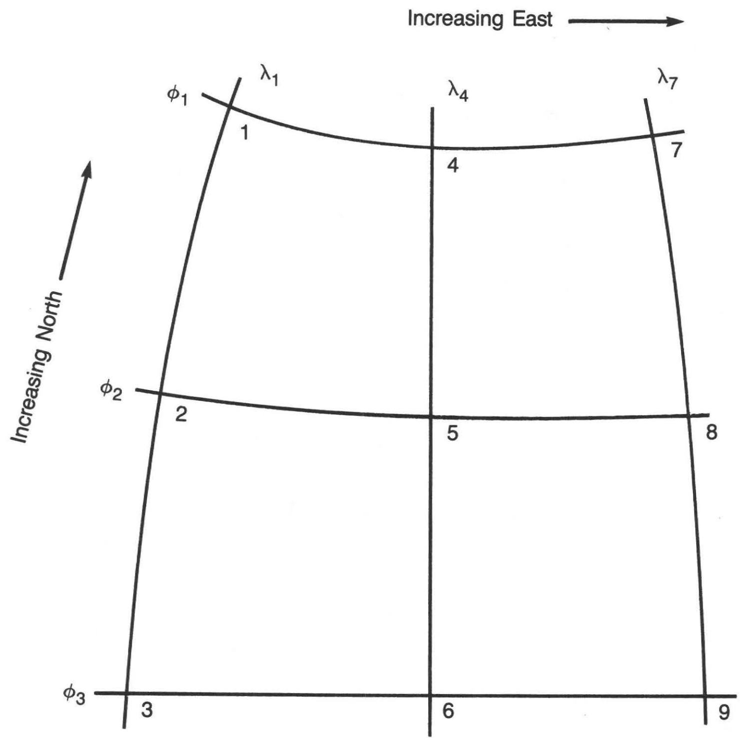

Figure 3.--Matrix of points for which coordinates are measured for map projection identification. Example: Point 5 has geodetic latitude ϕ_5, longitude λ_5, and rectangular coordinates (x_5,y_5).

between points 1 and 2 is compared with the angle σ_{2-3} of slope of the line from 2 to 3. If x and y are rectangular coordinates of these points relative to any fixed axes,

$$\sigma_{1-2} = \arctan_2[(y_1 - y_2)/(x_1 - x_2)] \quad (3\text{-}1)$$

$$\sigma_{2-3} = \arctan_2[(y_2 - y_3)/(x_2 - x_3)] \quad (3\text{-}2)$$

If the absolute value of $(\sigma_{1-2} - \sigma_{2-3})$ is greater than a preset convergence factor, the meridians are considered curved. If it is less, the bending of line 4-5-6 is similarly tested, and then line 7-8-9. All three must be straight for the meridians to be considered straight, since one meridian may be a straight central meridian. Similar calculations are made for the parallels. Obviously, with this arrangement, the computer can be fooled with doubly curved lines which fall on the same nine points as straight meridians, but this would be highly coincidental and could be resolved by measuring nine other points.

If not all meridians and parallels are straight, a second-order bivariate polynomial is determined for the nine points. Such a polynomial requires six coefficients for each variable. Since a third-order polynomial requires ten coefficients, and there are only nine points given, this determination is limited to second order. While two or three of the third-order terms could be used, the significance of the least-squares residuals would be reduced in determining the general accuracy of the polynomial fit. The second-order equations for transforming rectangular to geodetic coordinates for nine points are as follows, adapting general polynomial equations (2-1) and (2-2) and subtracting the average coordinates of the given matrix:

$$\lambda_j = C_1 + C_2\Delta x_j + C_3\Delta y_j + C_4(\Delta x_j)^2 + C_5(\Delta x_j)(\Delta y_j) + C_6(\Delta y_j)^2 \quad (3\text{-}3)$$

$$\phi_j = C_1' + C_2'\Delta x_j + C_3'\Delta y_j + C_4'(\Delta x_j)^2 + C_5'(\Delta x_j)(\Delta y_j) + C_6'(\Delta y_j)^2 \quad (3\text{-}4)$$

where

$$\Delta x = x - x_o \quad (3\text{-}5)$$

$$\Delta y = y - y_o \quad (3\text{-}6)$$

$$x_o = (\sum_{j=1}^{9} x_j)/9 \tag{3-7}$$

$$y_o = (\sum_{j=1}^{9} y_j)/9 \tag{3-8}$$

To determine the coefficients, standard least-squares formulas are used, adapting equations (2-14) through (2-17), which were used to compute coefficients for polynomials of any order:

$$\begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_6 \end{bmatrix} = [D] \bullet \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_9 \end{bmatrix} \tag{3-9}$$

$$\begin{bmatrix} C_1' \\ C_2' \\ \vdots \\ C_6' \end{bmatrix} = [D] \bullet \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_9 \end{bmatrix} \tag{3-10}$$

where

$$[D]=[A^T A]^{-1} A^T \tag{3-11}$$

$$[A]= \begin{bmatrix} 1 & \Delta x_1 & \Delta y_1 & (\Delta x_1)^2 & (\Delta x_1)(\Delta y_1) & (\Delta y_1)^2 \\ 1 & \Delta x_2 & \Delta y_2 & (\Delta x_2)^2 & (\Delta x_2)(\Delta y_2) & (\Delta y_2)^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \quad \vdots & \vdots \\ 1 & \Delta x_9 & \Delta y_9 & (\Delta x_9)^2 & (\Delta x_9)(\Delta y_9) & (\Delta y_9)^2 \end{bmatrix} \tag{3-12}$$

Once the coefficients are determined, longitudes and latitudes calculated using the coefficients with the given nine pairs of (x_j, y_j) coordinates are compared with the given (λ_j, ϕ_j), and the RMSE r is calculated as an angle converted to distance at map scale, using $(\lambda_{cj}, \phi_{cj})$ for the coefficient-calculated geographic coordinates:

$$r = \left\{ \sum_{j=1}^{9} [(\lambda_{cj} - \lambda_j)^2 \cos^2\phi_j + (\phi_{cj} - \phi_j)^2]/9 \right\}^{1/2}$$

$$[(x_4 - x_5)^2 + (y_4 - y_5)^2]^{1/2}/(\phi_4 - \phi_5) \qquad (3\text{-}13)$$

If r exceeds 1.5 mm (an arbitrary limit based on testing experience) for actual maps used in program tests, the computer begins to test projections which are not regular cylindricals. If less, the polynomial is used to calculate (λ,ϕ) for other points of known (x,y).

If all meridians and parallels are straight, the polynomial test is omitted, and it is then determined whether meridians are parallel to each other by comparing slope angles σ_{1-2} and σ_{4-5} of meridians λ_1 and λ_4, respectively (see equations (3-1) and (3-2)). The equidistance of straight meridians along straight parallels, even if not measured at equal intervals on the map, can be determined by dividing the distance between meridians along a given parallel by the difference in longitude. If spacing between points 1 and 4 is s_{1-4} and between points 4 and 7 is s_{4-7},

$$s_{1-4} = [(x_1 - x_4)^2 + (y_1 - y_4)^2]^{1/2}/(\lambda_4 - \lambda_1) \qquad (3\text{-}14)$$

$$s_{4-7} = [(x_4 - x_7)^2 + (y_4 - y_7)^2]^{1/2}/(\lambda_7 - \lambda_4) \qquad (3\text{-}15)$$

To compare s_{1-4} and s_{4-7} on a unit basis, unaffected by map scale, the absolute value of

$$\Delta = (1 - s_{1-4}/s_{4-7}) \qquad (3\text{-}16)$$

is compared to the convergence tolerance.

Perpendicularity of meridians to parallels, if straight and parallel to each other, is determined by comparing the angle between lines 1-2 and 1-4 to 90°. If all these conditions fall within tolerances, the projection is a regular cylindrical (i.e., not oblique or transverse), and specific cylindricals are tried one by one. The procedure for this is described later.

If parallels are curved but meridians are straight, the meridians are first tested for being parallel in the above manner, and then tested further for the equatorial Gnomonic if they are parallel. If they are not parallel, the point of convergence of meridians λ_1 and λ_4 is determined, calling the rectangular coordinates of this point (x_o,y_o):

$$x_o = \frac{(x_4 - x_6)(y_1x_3 - x_1y_3) - (x_1 - x_3)(y_4x_6 - x_4y_6)}{(x_4 - x_6)(y_1 - y_3) - (x_1 - x_3)(y_4 - y_6)} \quad (3\text{-}17)$$

$$y_o = \frac{(y_4 - y_6)(y_1x_3 - x_1y_3) - (y_1 - y_3)(y_4x_6 - x_4y_6)}{(x_4 - x_6)(y_1 - y_3) - (x_1 - x_3)(y_4 - y_6)} \quad (3\text{-}18)$$

Using this x_o and the coordinates of points 7 and 9, the y of meridian λ_7 at x_o may be compared with y_o:

$$y = [x_o(y_7 - y_9) - y_7x_9 + x_7y_9]/(x_7 - x_9) \quad (3\text{-}19)$$

If y and y_o are the same (within tolerance), it is concluded that the meridians converge to a common point, and the parallels may then be tested for concentricity. The radius ρ of a circular arc for ϕ_1 must be constant as measured between (x_o,y_o) and each of the points 1, 4, and 7, or

$$\rho = [(x_o - x_j)^2 + (y_o - y_j)^2]^{1/2} \quad (3\text{-}20)$$

for j=1, 4, and 7. Similarly, a different constant ρ should be found for j=2, 5, and 8, and another for j=3, 6, and 9. If this test fails, the oblique Gnomonic is still a possibility. If parallels are concentric, the equidistance of meridians is checked by determining the cone constant n based on two pairs of meridians. For meridians λ_1 and λ_7, the angle θ_{1-7} between them, using the principle of equations (3-1) and (3-2), is

$$\theta_{1-7} = \sigma_{1-3} - \sigma_{7-9} \quad (3\text{-}21)$$

and the cone constant n is

$$n = \theta_{1-7}/(\lambda_7 - \lambda_1) \quad (3\text{-}22)$$

The angle and cone constant are similarly calculated between meridians λ_1 and λ_4. If the two calculated cone constants are nearly enough equal, they are averaged, the projection is assumed to be a regular conic (or polar azimuthal, if n=±1), and specific conics are tested one by one. This procedure is described later.

If meridians are curved but parallels are straight, the parallels are checked for parallelism, and the meridians are checked for equidistant spacing along a given parallel. If they are not equidistant, the equatorial Orthographic is tested; if equidistant, specific pseudocylindrical projections are tested one by one.

Last but certainly not least is the condition in which both meridians and parallels are curved. This includes many of the most widely used projections such as the Transverse Mercator, the Polyconic and the azimuthals (other than the Gnomonic and the polar aspects). In theory, it would appear that these projections can be tested as groups without having to test every projection in each group in sequence until a satisfactory one is found. In practice, this approach involves too many uncertainties in selecting initial estimates for iteration to produce the correct answers. Therefore, the testing of projections in this category takes place one by one. If these projections fail to fit the map, the Tilted Perspective is tested last.

b. TESTING CYLINDRICAL PROJECTIONS

Once the category of cylindrical, conic, pseudocylindrical, or "other" is established as described above, the individual projection tests follow patterns similar to the examples given below. The determination of the projection could have been based upon either forward or inverse formulas, since all coordinates are supplied, but the forward formulas are used, although rearranged, because they are normally simpler than the inverse.

In any regular cylindrical projection,

$$x' = ak_o(\lambda - \lambda_o) \tag{3-23}$$

$$y' = af(\phi) \tag{3-24}$$

where k_o is a scale factor equal to or less than 1, λ_o is the central meridian, $f(\phi)$ is some function of ϕ, a is the equatorial radius of the Earth at map scale, and (x',y') are rectangular coordinates relative to standard axes, the X' axis lying along the Equator and the Y' axis along meridian λ_o. These coordinates (x',y') are related to coordinates (x,y) measured on the given map as follows:

$$x = x' \cos \Theta - y' \sin \Theta + x_o \tag{3-25}$$

$$y = y' \cos \Theta + x' \sin \Theta + y_o \tag{3-26}$$

where Θ is the counterclockwise inclination of the X' axis to the X axis, and (x_o, y_o) are the coordinates of the origin of the (X',Y') axes in the (x,y) coordinate frame. The constants Θ, x_o, and y_o are unknown, and are related to the choice of λ_o, which is indeterminate for a given cylindrical map. Furthermore, k_o is indeterminate for a given conformal map, unless the scale at a given point or along a given line is known or assumed. Since the program described is based upon complete ignorance of the projection and parameters, including scale, λ_o for regular cylindrical or conic projections is assumed to be zero, and the scale factor of conformal projections is assumed to be 1.0 at the central line or point - the equator of the regular Mercator, the central meridian of the Transverse Mercator, a single standard parallel of the Lambert Conformal Conic, or the center of projection of the Stereographic.

In order to find the values of the unknown parameters, equations (3-23) and (3-24) are substituted into (3-25) and (3-26):

$$x = ak_o(\lambda - \lambda_o) \cos \Theta - af(\phi) \sin \Theta + x_o \tag{3-27}$$

$$y = af(\phi) \cos \Theta + ak_o(\lambda - \lambda_o) \sin \Theta + y_o \tag{3-28}$$

Measurements x_1, y_1, x_4, y_4, λ_1, and λ_4 (see figure 3) along parallel ϕ_1 are substituted in (3-27) and (3-28) for x, y, and λ to obtain four equations. Then x_o and y_o as well as $f(\phi_4)$ and $f(\phi_1)$, since $\phi_1 = \phi_4$, cancel when subtracting each new pair of equations, and ak_o and Θ may be found by squaring the two remaining equations and adding to obtain

$$ak_o = [(y_4 - y_1)^2 + (x_4 - x_1)^2]^{1/2}/(\lambda_4 - \lambda_1) \tag{3-29}$$

and by dividing the two equations to obtain

$$\tan \Theta = (y_4 - y_1)/(x_4 - x_1) \tag{3-30}$$

These parameters are averaged by recalculating for other pairs of given coordinates.

While these values of ak_o and Θ are used for all regular cylindrical projections tested, x_o and y_o are computed for the specific projection. This is done by transposing equations (3-27) and (3-28), solving for x_o and y_o using one of the measured points, such as $(x_1, y_1, \phi_1, \lambda_1)$, and letting λ_o equal zero as stated above:

$$x_o = x_1 - ak_o\lambda_1 \cos\Theta + af(\phi_1)\sin\Theta \tag{3-31}$$

$$y_o = y_1 - af(\phi_1)\cos\Theta - ak_o\lambda_1 \sin\Theta \tag{3-32}$$

More rigorously, x_o could be found as the constant $(x_o - ak_o\lambda_o \cos\Theta)$ and y_o as $(y_o - ak_o\lambda_o \sin\Theta)$.

Specifically, for the spherical Mercator projection,

$$f(\phi) = k_o \ln\tan(\pi/4 + \phi/2) \tag{3-33}$$

For the ellipsoidal Mercator,

$$f(\phi) = k_o \ln[\tan(\pi/4 + \phi/2)((1 - e\sin\phi)/(1 + e\sin\phi))^{e/2}] \tag{3-34}$$

where e is the eccentricity of the Earth ellipsoid. In equations (3-33) and (3-34), k_o is taken as 1.0. For the Miller Cylindrical, $k_o=1$, and

$$f(\phi) = 1.25 \ln\tan(\pi/4 + 2\phi/5) \tag{3-35}$$

For the Equirectangular, $k_o = \cos\phi_o$, where ϕ_o is the unknown standard parallel (N. and S.) and

$$f(\phi) = \phi \tag{3-36}$$

To determine k_o for the Equirectangular,

$$k_o = s_{1-4}(\phi_1 - \phi_2)/[(x_1 - x_2)^2 + (y_1 - y_2)^2]^{1/2} \tag{3-37}$$

where s_{1-4} is found from equation (3-14).

For Gall's, $k_o = 2^{1/2}/2$, and

$$f(\phi) = (1 + 1/2^{1/2})\tan(\phi/2) \tag{3-38}$$

With all the parameters now calculated for a given projection, the forward formulas (3-27) and (3-28) are used to determine the fit of the projection to the nine given points (see section 3i. Tolerances, on p. 56). If the residuals are unsatisfactory, the next projection is tested. If satisfactory, (x',y') may be found for other points (x,y) by inverting equations (3-25) and (3-26):

$$x' = (x - x_o) \cos \Theta + (y - y_o) \sin \Theta \tag{3-39}$$

$$y' = (y - y_o) \cos \Theta - (x - x_o) \sin \Theta \tag{3-40}$$

From x' and y', ϕ and λ may be determined from inverses of equations (3-23), (3-24), and (3-33) through (3-38) for the particular projection. For example, for the spherical Mercator,

$$\phi = \pi/2 - 2 \arctan (e^{-y'/a})$$

$$\lambda = x'/a$$

Since k_o is taken as 1.0 and λ_o is taken as zero; note that e is 2.718..., the base of natural logarithms. The other formulas are not given here, but they may be rather readily derived from equations (3-34) through (3-37).

c. TESTING CONIC PROJECTIONS

If the unidentified map does not pass the general test for a regular cylindrical projection and does not fit a second-order polynomial, but does appear to conform to regular conic requirements, tests are made to determine the specific conic projection. For any regular conic projection,

$$x' = \rho \sin \Theta' \tag{3-41}$$

$$y' = \rho_o - \rho \cos \Theta' \tag{3-42}$$

where

$$\rho = a\ f(\phi) \tag{3-43}$$

$$\Theta' = n(\lambda - \lambda_o) \tag{3-44}$$

ρ and Θ' are polar coordinates, ρ_o is the radius of the parallel of the origin, a is again the equatorial radius of the Earth at map scale, and n is the cone constant. As with the cylindricals, λ_o is indeterminate on a given map. This may arbitrarily be made zero.

The coordinates of the center of the parallels have already been determined as (x_o, y_o) from equations (3-17) and (3-18); thus ρ_o is zero, and equation (3-43) may be used with the given measurements to determine the suitability of a specific conic projection. The radius is calculated as follows:

$$\rho_j = \pm[(x_o - x_j)^2 + (y_o - y_j)^2]^{1/2} \tag{3-45}$$

where j=1, 2, and 3, successively, and the ± takes the sign of n, positive for a map centered in the Northern Hemisphere and negative if Southern, as calculated from equation (3-22). To find the map scale, from (3-43),

$$a = \rho_j / f(\phi_j) \tag{3-46}$$

For the spherical Lambert Conformal Conic, using a single standard parallel ϕ_s = arcsin n, because the identity of two standard parallels is indeterminate,

$$f(\phi) = k' \tan^n(\pi/4 - \phi/2) \tag{3-47}$$

where the constant

$$k' = 1/\tan \phi_s \tan^n(\pi/4 - \phi_s/2) \tag{3-48}$$

does not need to be calculated until ak' for each parallel along a given meridian λ_1 has been determined from $\rho_j/[f(\phi_j)/k']$ (see equation (3-46)).

For the ellipsoidal Lambert Conformal Conic, ϕ_s is also taken as arcsin n, and

$$f(\phi) = k'[\tan(\pi/4 - \phi/2)/((1 - e \sin \phi)/(1 + e \sin \phi))^{e/2}]^n \tag{3-49}$$

where

$$k' = 1/\{\tan \phi_s (1 - e^2 \sin^2 \phi_s)^{1/2} [\tan(\pi/4 - \phi_s/2)/ ((1 - e \sin \phi_s)/(1 + e \sin \phi_s))^{e/2}]^n \tag{3-50}$$

If the three values of ak' are nearly enough equal, they are averaged, and k' and a are separated by using equations (3-48) or (3-50). The forward formulas (3-41), (3-42), and related equations are used with least squares (see sections 3i and 3j) to determine how well the projection fits the nine given points. Since, for the conic projections, the critical tests have already been made, this fit should be satisfactory. If not, the next projection is tested, If so, inverse computations may take place, but only after determining Θ, the remaining required parameter.

In order to compute Θ, equations (3-25), (3-26), (3-41), (3-42), and (3-44) may be combined as follows, with λ_o and ρ_o both taken as zero, as stated above:

$$x = \rho \sin (n\lambda + \Theta) + x_o \quad (3\text{-}51)$$

$$y = -\rho \cos (n\lambda + \Theta) + y_o \quad (3\text{-}52)$$

The value of Θ may be determined from point (x_1, y_1), combining equations (3-51) and (3-52),

$$\Theta = \arctan_2 [\pm(x_1 - x_o)/\pm(y_o - y_1)] - n\lambda_1 \quad (3\text{-}53)$$

in which each ± sign takes the sign of n. The value of a determined from (3-46) provides the equatorial radius of the Earth model to supply true scale along ϕ_s. With a, k', Θ, x_o from (3-17), y_o from (3-18), and n from (3-22), ϕ and λ may be determined for any other given (x,y) as follows from the following transformations of (3-43), (3-51), (3-52), and, for the ellipsoidal Lambert Conformal Conic, (3-49):

$$\lambda = \{\arctan_2 [\pm(x - x_o)/\pm(y_o - y)] - \Theta\}/n \quad (3\text{-}54)$$

$$\phi = \pi/2 - 2 \arctan \{t[(1 - e \sin \phi)/(1 + e \sin \phi)]^{e/2} \quad (3\text{-}55)$$

where $t = (\rho/ak')^{1/n}$ (3-56)

ρ is found from equation (3-45), deleting subscripts j, (3-55) is solved by iteration using successive substitution, and the ± signs in equations (3-45) and (3-54) take the sign of n.

For testing the spherical case of the Albers Equal-Area Conic, if ϕ_s is one of the (unknown) standard parallels, the function for equation (3-43) is as follows:

$$f(\phi) = (C - nq)^{1/2}/n \quad (3\text{-}57)$$

where C is a constant, and

$$q = 2 \sin \phi \quad (3\text{-}58)$$

Since a in equation (3-43) is unknown, equation (3-57) is substituted into (3-46), for j = 1 and 2:

$$a = \rho_1 n/(C - nq_1)^{1/2} = \rho_2 n/(C - nq_2)^{1/2} \quad (3\text{-}59)$$

Eliminating a and solving for C,

$$C = n(q_2\rho_1^2 - q_1\rho_2^2)/(\rho_1^2 - \rho_2^2) \tag{3-60}$$

If the value of C calculated from points 1 and 2 is close enough to C as calculated from points 2 and 3 (replacing subscripts accordingly), the projection is considered to be a spherical Albers, subject to the least-squares-residual check mentioned for the Lambert, and a may then be found from (3-59), using one of the points.

Although the values of standard parallels are not needed in order to find (ϕ,λ) for other values of (x,y) on the given map, they are of interest in identifying the parameters of the projection, and may be found for the Albers. The formula for scale factor k along a parallel of any conic projection is as follows:

$$k = \rho n/am \tag{3-61}$$

where $$m = \cos\phi/(1 - e^2\sin^2\phi)^{1/2} \tag{3-62}$$

for the ellipsoid, or

$$m = \cos\phi \tag{3-63}$$

for the sphere. At the standard parallels, k = 1, or

$$1 = \rho_s n/am_s \tag{3-64}$$

where s refers to either standard parallel.

For the spherical Albers, substituting from equations (3-43), (3-57), (3-58), and (3-63) into (3-64) and solving for ϕ_s,

$$\sin\phi_s = n \pm (n^2 - C + 1)^{1/2} \tag{3-65}$$

in which the "+" provides one standard parallel and the "-" the other.

For inverse computations, equation (3-53) applies in calculating the remaining parameter Θ, and (3-54) may be used to find λ, but (3-55) is replaced with the following:

$$\phi = \arcsin[(C - \rho^2n^2/a^2)/2n] \tag{3-66}$$

with ρ found from equation (3-45) without subscripts j.

Testing for the ellipsoidal Albers and for the ellipsoidal or spherical Equidistant Conic projections follows patterns similar to those given above. The standard parallels for all these projections may be determined, unlike the Lambert. Some iteration is involved. The polar azimuthals are handled in the same manner. The polar Stereographic, Lambert Azimuthal Equal-Area, and Azimuthal Equidistant aspects are limiting forms of these conic projections. The polar Orthographic, Gnomonic, and Vertical Perspective are not thus related to useful conic projections, but they are analogous. Formulas for all these projections are given in the Appendix.

d. TESTING PSEUDOCYLINDRICAL PROJECTIONS

The other remaining important category of projections in which either the meridians or the parallels remain straight is the pseudocylindrical, such as the Sinusoidal or Mollweide.

For regular pseudocylindrical projections, the fundamental equations analogous to (3-23) and (3-24) (for the cylindricals) are as follows:

$$x' = a(\lambda - \lambda_o)f_2(\phi) \tag{3-67}$$

$$y' = af_1(\phi) \tag{3-68}$$

where f_1 and f_2 are different functions of ϕ.

As with the general cylindrical projection equations (3-23) and (3-24), incorporation of rotation-translation equations (3-25) and (3-26) leads to equations like (3-27) through (3-30), except that k is replaced with $f_2(\phi)$, and $f(\phi)$ becomes $f_1(\phi)$. In order to solve for a, formulas from analytic geometry are adapted to provide the distance between parallels ϕ_1 and ϕ_2. The distance on the measured map is

$$d_{1-2} = \frac{(x_1 - x_2)(y_2 - y_5) - (y_1 - y_2)(x_2 - x_5)}{[(x_2 - x_5)^2 + (y_2 - y_5)^2]^{1/2}} \tag{3-69}$$

In equation (3-68), this distance is the difference between y' for ϕ_1 and for ϕ_2. Equating and solving for a,

$$a = d_{1-2}/[f_1(\phi_1) - f_1(\phi_2)] \tag{3-70}$$

By obtaining a based on the distance between parallels ϕ_2 and ϕ_3 in like manner (adding 1 to each subscript in (3-69)), an average value of a may be used for further computation.

For the Sinusoidal projection,

$$f_1(\phi) = \phi \tag{3-71}$$

$$f_2(\phi) = \cos \phi \tag{3-72}$$

For the Mollweide projection, as an example involving iteration,

$$f_1(\phi) = 2^{1/2} \sin \omega \tag{3-73}$$

$$f_2(\phi) = (8^{1/2} \cos \omega)/\pi \tag{3-74}$$

where $$2\omega + \sin 2\omega = \pi \sin \phi \tag{3-75}$$

Equation (3-75) may be solved for ω using a Newton-Raphson iteration.

It is then necessary to determine λ_o, x_o, and y_o. For a pseudo-cylindrical projection, λ_o is not indeterminate, as it was for cylindrical projections. Solving equation (3-25) for x_o, and substituting from (3-67) and (3-68),

$$x_o = x - af_2(\phi)(\lambda - \lambda_o) \cos \Theta + af_1(\phi) \sin \Theta \tag{3-76}$$

Using x_1, ϕ_1, λ_1, and Θ and a as calculated from (3-30) and (3-70), respectively, for one equation, and x_2, ϕ_2, and λ_1 for a second equation, x_o may be eliminated by subtraction and the difference solved for λ_o:

$$\lambda_o = \frac{\sin \Theta \, [f_1(\phi_1)] + (x_2 - x_1)/a}{\cos \Theta \, [f_2(\phi_1) - f_2(\phi_2)]} + \lambda_1 \tag{3-77}$$

Then x_o is found from (3-76) and y_o from a similar transposition of equation (3-26). By calculating the values of λ_o using ϕ_2 and ϕ_3 at λ_1, and repeating the calculations for the three parallels using λ_4 and λ_7, all with the proper subscripts and the corresponding x, the six values of λ_o may be averaged. These parameters may then be applied to computations of (x,y) for (ϕ,λ) of the nine given points. These coordinates are compared with the given (x,y) by a least-squares fit as described later, to determine whether the projection applies. If satisfactory, the same parameters are applied to inverse forms of equations (3-67), (3-68), (3-71), and (3-72), together with (3-39) and (3-40), to find values of (ϕ,λ) for other (x,y) values.

e. TESTING FOR THE TRANSVERSE MERCATOR AND POLYCONIC PROJECTIONS

Projections in which meridians and parallels are generally curved, such as the Transverse Mercator, the Polyconic, and azimuthal projections, were incorporated into this testing package only after several false starts. Most disappointing was being unable to test all transverse cylindrical projections with one package, all azimuthal projections with another package, etc. Ostensibly, this may be done by eliminating the functions which vary from one projection to another within each such package. For example, all spherical azimuthal projections fit the formulas,

$$x' = ak'\cos\phi \sin(\lambda - \lambda_o) \qquad (3\text{-}78)$$

$$y' = ak'[\cos\phi_o \sin\phi - \sin\phi_o \cos\phi \cos(\lambda - \lambda_o)] \qquad (3\text{-}79)$$

where k' is a function of ϕ, λ, ϕ_o, and λ_o, different for each azimuthal. By eliminating ak' and incorporating equations (3-25) and (3-26), it would appear that the five parameters ϕ_o, λ_o, x_o, y_o, and θ may be found from five simultaneous equations using five measured points and Newton-Raphson iterations. In testing this approach, it was found that the proper initial values of these parameters are critical, and certain projections in the group had to be omitted due to lack of convergence regardless of initial estimates. The initial values chosen often led to parameters which were incorrect (when checked using the other four points, or based upon known examples), and attempts to overcome this problem were unsuccessful until projections were treated individually and parameters x_o, y_o, and θ were eliminated from the simultaneous iteration.

The approach finally used relies on the fact that the difference in slope between two lines joining measured points is independent of a, x_o, y_o, and θ. This was found successful along meridians, but generally not along parallels, and not for the Gnomonic with its straight meridians. In the latter case, an approach found satisfactory uses the principle that the ratio of lengths of the meridian line segments is also independent of a, x_o, y_o, and Θ, but this is not sufficiently sensitive to use generally for other azimuthal projections.

The Transverse Mercator and Polyconic projections (spherical or ellipsoidal) are somewhat simpler to resolve than the azimuthal, after eliminating a, x_o, y_o, and Θ, because only λ_o must be found by iteration for the former two projections, but both ϕ_o and λ_o must be found for the latter. For the general perspective azimuthal, P, the location of the point of perspective in radii from the Earth's center, should be found as well, but a different approach was found necessary. For the Transverse Mercator and Polyconic, where g and h are specific functions given later,

$$x' = ah(\phi,\lambda,\lambda_o) \tag{3-80}$$

$$y' = ag(\phi,\lambda,\lambda_o) \tag{3-81}$$

The angle F between straight lines joining points 1 and 2 and points 2 and 3 (figure 3) may be determined from the data given and is

$$F = \arctan_2[(y_1 - y_2)/(x_1 - x_2)] - \arctan_2[(y_2 - y_3)/(x_2 - x_3)] \tag{3-82}$$

As indicated above, this angle is also

$$F = \arctan_2[(y_1' - y_2')/(x_1' - x_2')] - \arctan_2[(y_2' - y_3')/(x_2' - x_3')] \tag{3-83}$$

Adapting equation (3-83) to prepare for a Newton-Raphson iteration, and dividing through by a,

$$f(\lambda_o) = F - \arctan_2(G_{1-2}/H_{1-2}) + \arctan_2(G_{2-3}/H_{2-3}) \tag{3-84}$$

where $H_{1-2} = h_1 - h_2$, $G_{1-2} = g_1 - g_2$, $H_{2-3} = h_2 - h_3$, and $G_{2-3} = g_2 - g_3$, with h_j and g_j the respective functions of ϕ, λ, and λ_o in equations (3-80) and (3-81) at point j. Function $f(\lambda_o)$ should converge to near zero with iteration.

Differentiating with respect to λ_o, and letting primes on g and h denote differentials,

$$f'(\lambda_o) = -[H_{1-2}(g_1' - g_2') - G_{1-2}(h_1' - h_2')]/(G_{1-2}^2 + H_{1-2}^2) + [H_{2-3}(g_2' - g_3') - G_{2-3}(h_2' - h_3')]/(G_{2-3}^2 + H_{2-3}^2) \tag{3-85}$$

The differentials are given in detail in the Appendix (section 6) by equations (6-29) through (6-39), (6-45) through (6-53), and (6-58) through (6-61).

Applying the Newton-Raphson iteration formula,

$$\Delta\lambda_o = -f(\lambda_o)/f'(\lambda_o) \tag{3-86}$$

with an initial estimate of λ_o along or near the middle meridian of the measurements, iteration is carried out toward a desired convergence, changing the previous λ_o by $\Delta\lambda_o$ at the end of each iteration. For the Transverse Mercator for the sphere, for example,

$$h(\phi,\lambda,\lambda_o) = (\tfrac{1}{2})k \ln \{[1 + \cos\phi \sin(\lambda - \lambda_o)]/[1 - \cos\phi \sin(\lambda - \lambda_o)]\} \tag{3-87}$$

$$g(\phi,\lambda,\lambda_o) = k \arctan_2[\tan\phi/\cos(\lambda - \lambda_o)] \tag{3-88}$$

(The indeterminate k for a conformal projection is again assumed to be 1.0.) When the iteration is completed, a similar iteration is performed using λ_4 instead of λ_1, and then λ_7. The three values of λ_o are averaged.

The fact that the lengths of segments between any two points on the map are proportional to the scale but independent of Θ, x_o, and y_o may now be used to calculate scale. Since

$$[(x_1 - x_2)^2 + (y_1 - y_2)^2]^{1/2} = [(x_1' - x_2')^2 + (y_1' - y_2')^2]^{1/2} \tag{3-89}$$

substituting from (3-80) and (3-81) and solving for a,

$$a = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{1/2}/[(h_1 - h_2)^2 + (g_1 - g_2)^2]^{1/2} \tag{3-90}$$

using measured values of x and y in the numerator and values of h and g calculated from equations (3-87) and (3-88) in the denominator, or the equivalent for other projections, based upon the given values of ϕ and λ.

For the remaining parameters, rotation-translation equations (3-25) and (3-26) may be converted to the following by substitution of measurements and functions for points 1 and 2 and subtracting pairs of equations:

$$x_1 - x_2 = (x_1' - x_2') \cos \Theta - (y_1' - y_2') \sin \Theta \tag{3-91}$$

$$y_1 - y_2 = (y_1' - y_2') \cos \Theta + (x_1' - x_2') \sin \Theta \tag{3-92}$$

Solving for cos Θ and then sin Θ by determinants or the equivalent, and then dividing the resulting equations, it will be found that a can be factored out, x' replaced with h, and y' with g:

$$\tan \Theta = \frac{-(x_1 - x_2)(g_1 - g_2) + (y_1 - y_2)(h_1 - h_2)}{(x_1 - x_2)(h_1 - h_2) + (y_1 - y_2)(g_1 - g_2)} \tag{3-93}$$

By transposing and substituting in (3-25) and (3-26),

$$x_o = x_1 - a(h_1 \cos \Theta - g_1 \sin \Theta) \tag{3-94}$$

$$y_o = y_1 - a(g_1 \cos \Theta + h_1 \sin \Theta) \tag{3-95}$$

Since the parameters λ_o, a, Θ, x_o, and y_o are now known, the forward formulas (3-25), (3-26), (3-80), (3-81) and the appropriate functions such as (3-87) and (3-88) may be used to calculate (x,y) for the nine given pairs of (φ,λ). These coordinates are compared with the given values of (x,y) by least squares (see section 3i, p. 52). If satisfactory, (φ,λ) for other points (x,y) may be determined from equations (3-39) and (3-40) (the inverses of (3-25) and (3-26)) and the inverses of equations (3-80) and (3-81) for the given projection (see equations (6-40) through (6-44), (6-54) through (6-57), and (6-62) through (6-67) in the Appendix, Section 6).

f. TESTING FOR AZIMUTHAL PROJECTIONS

Although azimuthal projections generally have curved meridians and parallels as do the Transverse Mercator and Polyconic, an additional parameter must be known, namely the latitude of the center of projection. While the principles used in the foregoing derivations may be used, here they lead to iteration for both latitude and longitude instead of for longitude only.

Equations (3-80) and (3-81) may be revised as follows for azimuthal projections:

$$x' = ah(\phi,\lambda,\phi_o,\lambda_o) \quad (3\text{-}96)$$

$$y' = ag(\phi,\lambda,\phi_o,\lambda_o) \quad (3\text{-}97)$$

After comparing with equations (3-78) and (3-79), for the spherical form,

$$h = k' \cos\phi \sin(\lambda - \lambda_o) \quad (3\text{-}98)$$

$$g = k' [\cos\phi_o \sin\phi - \sin\phi_o \cos\phi \cos(\lambda - \lambda_o)] \quad (3\text{-}99)$$

From various standard references,

$k' = 2/(1 + \cos z)$ for the Stereographic projection, (3-100)

$k' = 1$ for the Orthographic, (3-101)

$k' = z/\sin z$ for the Azimuthal Equidistant, (3-102)

and

$k' = [2/(1 + \cos z)]^{1/2}$ for the Lambert Azimuthal Equal Area, (3-103)

where z, the great circle distance, is found from the formula

$$\cos z = \sin\phi_o \sin\phi + \cos\phi_o \cos\phi \cos(\lambda - \lambda_o) \quad (3\text{-}104)$$

While the Stereographic may be tested in the following manner, it is tested in the program as part of the perspective package to eliminate iteration. To find ϕ_o and λ_o, equations (3-82) through (3-85) apply as shown, except that $f(\lambda_o)$ in (3-84) is written $f(\phi_o,\lambda_o)$, and (3-84) must also be differentiated with respect to ϕ_o to provide a new equation (3-85a) identical with (3-85) except that the left-hand term is $f'(\phi_o)$ and the right-hand prime terms refer to differentiation with respect to ϕ_o. These differentials are detailed in equations (6-68) through (6-81).

Now the Newton-Raphson iteration shown in (3-86) is modified to require two simultaneous equations of the form

$$\Delta\lambda_o f'(\lambda_o) + \Delta\phi_o f'(\phi_o) = -f(\phi_o,\lambda_o) \quad (3\text{-}105)$$

With assumed initial values of ϕ_o and λ_o near the center of the map, the unknowns $\Delta\phi_o$ and $\Delta\lambda_o$ may be found for the first iteration by evaluating f and f' from equations (3-84), (3-85), and (3-85a) based on points 1, 2, and 3 of Figure 3 for one equation (3-105), and based on points 4, 5, and 6 for a second equation (3-105), the two equations being linear in the unknowns and solvable by determinants. The $\Delta\phi_o$ and $\Delta\lambda_o$ obtained are added to the initial values, and the operations in this paragraph are repeated until the changes in both ϕ_o and λ_o are minimal. These values are averaged with those obtained by iteration from points 1, 2, 3, 7, 8, and 9 instead of 1 through 6.

To determine a, Θ, x_o, and y_o for the azimuthals, equations (3-90), (3-93), (3-94), and (3-95) apply without change. Forward formulas are then used to check the validity of the projection with least squares, as in the case of the Transverse Mercator and Polyconic. If the projection applies, values of (ϕ,λ) for other points (x,y) may then be found from equations (3-39), (3-40) and the inverses of (3-96) through (3-104) (see equations (6-82) through (6-92)).

To test the equatorial Gnomonic projection, with its parallel meridians, the spacing of the meridians on the map is compared to the correct spacing. Basic formulas for the equatorial Gnomonic are as follows:

$$x' = a \tan (\lambda - \lambda_o) \qquad (3\text{-}106)$$

$$y' = a \tan \phi / \cos (\lambda - \lambda_o) \qquad (3\text{-}107)$$

Adapting equation (3-69), the distance between straight meridians λ_1 and λ_4 is

$$d_{1-4} = \frac{(x_1 - x_4)(y_4 - y_5) - (y_1 - y_4)(x_4 - x_5)}{[(x_4 - x_5)^2 + (y_4 - y_5)^2]^{1/2}} \qquad (3\text{-}108)$$

From equation (3-106), this distance is the difference between x' for λ_1, and λ_4. An equation similar to (3-108), called (3-108a), may be prepared by replacing subscripts 1, 4, and 5 with 4, 7, and 8, respectively. Although λ and a are unknown, a can be eliminated by dividing (3-108) and its equivalent in (3-106) by (3-108a) and its equivalent in (3-106). The resulting ratio, which will be called F_1, is

$$F_1 = \frac{d_{1-4}}{d_{4-7}} = \frac{\tan(\lambda_1 - \lambda_o) - \tan(\lambda_4 - \lambda_o)}{\tan(\lambda_4 - \lambda_o) - \tan(\lambda_7 - \lambda_o)} \quad (3\text{-}109)$$

After expanding the terms with the identity for the tangent of the difference between two angles, equation (3-109) can be solved for $\tan \lambda_o$:

$$\tan \lambda_o = - \frac{\tan \lambda_1 - \tan \lambda_4 + F_1(\tan \lambda_7 - \tan \lambda_4)}{\tan \lambda_7(\tan \lambda_1 - \tan \lambda_4) + F_1 \tan \lambda_1(\tan \lambda_7 - \tan \lambda_4)} \quad (3\text{-}110)$$

The $\arctan_2$ function does not seem to resolve the quadrant selection for λ_o; the latter must be within 90° of say λ_4 by adding or subtracting 180° if necessary.

For a, transposing the relationship between (3-106) and (3-108),

$$a = d_{1-4}/[\tan(\lambda_1 - \lambda_o) - \tan(\lambda_4 - \lambda_o)] \quad (3\text{-}111)$$

For Θ, taking the slope of one of the meridians,

$$\tan \Theta = -(x_4 - x_5)/(y_4 - y_5) \quad (3\text{-}112)$$

while x_o and y_o may be determined from equations (3-94) and (3-95), substituting for h and g by relating equations (3-80), (3-81), (3-106), and (3-107). For the oblique Gnomonic projection, as stated before, the ratio of distances along meridians is used instead of curvature, since meridians are straight. Otherwise the principle used for other azimuthal projections above is followed. In equations (3-98) and (3-99),

$$k' = 1/\cos z \quad (3\text{-}113)$$

where z is found from (3-104). The square of the ratio of distances, which will be called F_2, is

$$F_2 = \frac{(x_1 - x_2)^2 + (y_1 - y_2)^2}{(x_2 - x_3)^2 + (y_2 - y_3)^2} \quad (3\text{-}114)$$

Also, in the (x',y') coordinate system,

$$F_2 = (G_{1-2}^2 + H_{1-2}^2)/(G_{2-3}^2 + H_{2-3}^2) \tag{3-115}$$

By analogy with equation (3-84),

$$f(\phi_o,\lambda_o) = F_2 - (G_{1-2}^2 + H_{1-2}^2)/(G_{2-3}^2 + H_{2-3}^2) \tag{3-116}$$

and its differential with respect to λ_o is, using primes as in (3-85),

$$f'(\lambda_o) = -2\{(G_{2-3}^2 + H_{2-3}^2)[G_{1-2}(g_1' - g_2') + H_{1-2}(h_1' - h_2')] - (G_{1-2}^2 + H_{1-2}^2)[G_{2-3}(g_2' - g_3') + H_{2-3}(h_2' - h_3')]\}/ (G_{2-3}^2 + H_{2-3}^2)^2 \tag{3-117}$$

The differential with respect to ϕ_o is written identically, but with ϕ_o instead of λ_o, and g' and h' related to ϕ_o. Equations (3-105), (3-90), (3-93), (3-94), and (3-95) are used with these revised functions to establish the values of the six parameters.

g. TESTING FOR THE GENERAL PERSPECTIVE PROJECTION

After experiencing failure when applying the above principles to iteration for the general Vertical Perspective projection with its additional unknown parameter denoting the location of the point of perspective, the non-iterative projective formulas for the general tilted perspective projection of the ellipsoid were applied successfully. These equations are as follows (Snyder, 1981b):

$$x = (XK_1 + YK_2 + ZK_3 + K_4)/(XK_5 + YK_6 + ZK_7 + 1) \tag{3-118}$$

$$y = (XK_8 + YK_9 + ZK_{10} + K_{11})/(XK_5 + YK_6 + ZK_7 + 1) \tag{3-119}$$

where K_1 through K_{11} are constants and X, Y, and Z are the rectangular coordinates of the point on the Earth's surface in the Earth reference system. For the ellipsoid (omitting height above the ellipsoid in map projection considerations),

$$X = N \cos \phi \cos \lambda \tag{3-120}$$

$$Y = N \cos \phi \sin \lambda \tag{3-121}$$

$$Z = N(1 - e^2)\sin \phi \tag{3-122}$$

$$N = a/(1 - e^2\sin^2\phi)^{1/2} \tag{3-123}$$

For the sphere, N = a and Z = a sin ϕ.

Given the measured (x,y) for 5 ½ of the nine points (x for six points, y for five), the values of K_1 through K_{11} may be found from the above equations by solving eleven simultaneous linear equations with standard algorithms. For the six x equations, transposing equation (3-118),

$$X_jK_1 + Y_jK_2 + Z_jK_3 + K_4 - (x_jX_j)K_5 - (x_jY_j)K_6 - (x_jZ_j)K_7 + 0K_8 + 0K_9 + 0K_{10} + 0K_{11} = x_j \quad (3\text{-}124)$$

For the five y equations, transposing equation (3-119),

$$0K_1 + 0K_2 + 0K_3 + 0K_4 - (y_jX_j)K_5 - (y_jY_j)K_6 - (y_jZ_j)K_7 + X_jK_8 + Y_jK_9 + Z_jK_{10} + K_{11} = y_j \quad (3\text{-}125)$$

In the program, first the sphere is assumed for the calculation of K_1 through K_{11}. After these constants are obtained, a test is made to determine whether the perspective projection is vertical, by calculating seven parameters as follows based on constants K_3 through K_7, K_{10}, and K_{11}. The relationship of K_n to the parameters may be determined by combining equations (3-25), (3-26), (3-96) through (3-99), and (3-126), expanding and combining in somewhat the manner described in Snyder (1981b), but retaining the derivation of the constants. The function k' for the Vertical Perspective is as follows:

$$k' = (P - 1)/(P - \cos z) \quad (3\text{-}126)$$

where z is found from equation (3-104), and P is described just before equation (3-80). After several steps of algebra, it is found that, for the Vertical Perspective,

$$\lambda_o = \arctan_2[(-K_6)/(-K_5)] \quad (3\text{-}127)$$

$$\phi_o = \pm\arctan [-K_7/(K_5^2 + K_6^2)^{1/2}] \quad (3\text{-}128)$$

$$P = \pm 1/(K_5^2 + K_6^2 + K_7^2)^{1/2} \quad (3\text{-}129)$$

$$\theta = -\arctan_2[(K_3 - K_4K_7)/(K_{10} - K_{11}K_7)] \quad (3\text{-}130)$$

$$x_o = K_4 \quad (3\text{-}131)$$

$$y_o = K_{11} \quad (3\text{-}132)$$

$$a = [(K_3 - K_4K_7)^2 + (K_{10} - K_{11}K_7)^2]^{1/2}/(K_5^2 + K_6^2)^{1/2}(|P - 1|) \quad (3\text{-}133)$$

If the λ_o determined above, and ϕ_o using the "+" sign, are more than 90° from (ϕ_2,λ_4) using equation (3-104), 180° must be added to (or subtracted from) λ_o, while ϕ_o and P must be given a "-" sign instead of "+". This indicates that the point of perspective is opposite the center of the Earth with respect to the center of the map projection, as in the case of the Stereographic (P = -1). Gnomonic and Orthographic projections, although they are both Vertical Perspectives, cannot be tested with this approach without modifications to the formulas.

Using the seven parameters found above in the forward formulas for the Vertical Perspective, (3-126), (3-104), (3-96) through (3-99), (3-25), and (3-26), (x,y) for the given nine points are calculated from (ϕ,λ). If these agree with the given (x,y) for the points, the projection is reported as Vertical Perspective of the sphere. If they do not agree, the constants K_1 through K_{11} already calculated are used in equations (3-118) through (3-123) (for the sphere) to attempt to duplicate the coordinates given. If there is agreement, the projection is reported as a Tilted Perspective of the sphere. If there is no agreement, K_1 through K_{11} are calculated for the ellipsoid, using equations (3-120) through (3-125), and then used to calculate coordinates for all nine given points. If accurate, the ellipsoidal Tilted Perspective is reported as the projection; if not, the program reports that no solution is programmed. The program does not include the Vertical Perspective of the ellipsoid as such, but the Tilted Perspective does include this, although the parameters (center, scale, etc.) are not computed.

h. TOLERANCES IN MEASUREMENTS ON THE MAP SUPPLIED

It proved to be difficult in many cases to set meaningful tolerances within which the program can determine the correct (or a reasonable) map projection for an actual map. It was necessary to permit the cartography to be slightly but not unreasonably inaccurate, to permit the paper base to expand and contract, and to permit the matrix of points to be measured with normal care, but not perfectly.

After a number of tests, it was found that for a given error in placement and measurement of points on the graticule, differentiation could only be satisfactorily used to establish tolerances in matching of values in the case of cylindrical and conic projections. For a projection in which both meridians and parallels are curved, this approach could not be satisfactorily applied. In these situations, the final solution, even in several cases where differentiation was satisfactory, was to calculate the rectangular coordinates for the nine given points according to the projection and parameters being considered, and then to determine whether a least-squares fit of the given rectangular coordinates into the calculated coordinates gives a small enough RMSE. If so, the constants determined for the least-squares fit are used for other points being transformed. If not, the next projection is tested. Differentiation of equations to establish tolerances is used only to determine whether the projection is cylindrical, conic, or in another category, and then to distinguish between the conics.

In practice, for determining the projection category, a measurement accuracy of 0.01 inch is used for actual maps and 10^{-4} inch is used for hypothetical maps for which coordinates are mathematically calculated for entry into the computer program to 10^{-6} inch. With this general tolerance, which is called dm here, individual tolerances are determined by elementary differentiation of functions used in analysis of the maps for projections. Specifically, the first check made is that for straight meridians (see equations (3-1) and (3-2)). Instead of differentiating them, one may simplify the mathematics if one intuitively thinks of the maximum error in angle σ_{1-2} as the result of misplacing each end of the chord from point 1 to point 2 by 0.01" (dm) in a direction perpendicular to the chord. Then, the possible error in σ_{1-2} is twice dm divided by the length of the chord, or

$$d\sigma_{1-2} = 2dm/[(x_1 - x_2)^2 + (y_1 - y_2)^2]^{1/2} \qquad (3\text{-}134)$$

Similarly

$$d\sigma_{2-3} = 2dm/[(x_2 - x_3)^2 + (y_2 - y_3)^2]^{1/2} \qquad (3\text{-}135)$$

The maximum error in the angle between the two chords, or the maximum allowable deviation of the two chords from a straight line, is the sum of the absolute values of the two. If this allowance is greater than the calculated value of $(\sigma_{1-2} - \sigma_{2-3})$, using equations (3-1) and (3-2), the meridian may be considered straight. If not it is assumed curved.

The same determination is made for each of the other meridians. All three must pass the test if meridians are to be considered straight. An analogous test is made for each parallel of latitude.

To test equidistance of meridians along a given straight parallel, a similar intuitive relationship is used: The greatest error in measuring the length of a segment of a parallel between two meridians is a lengthwise error of dm at each end, or 2dm total. In equations (3-14) through (3-16), if s_{1-4} and s_{4-7} are both subject to this error, the tolerance for Δ, taking absolute values, is as follows:

$$ds_{1-4} = 2dm/(\lambda_4 - \lambda_1) \tag{3-136}$$

$$ds_{4-7} = 2dm/(\lambda_7 - \lambda_4) \tag{3-137}$$

$$d\Delta = [|s_{4-7} ds_{1-4}| + |s_{1-4} ds_{4-7}|]/s^2_{4-7} \tag{3-138}$$

The tolerances for perpendicularity of meridians to parallels and for skewness of meridians are calculated using the same concept as equations (3-137) and (3-138). The tolerance for convergence of the meridians at x_o and y_o was determined by direct differentiation of equations (3-17) and (3-18), with lengthy results. Extracting dx_n and dy_n from all expressions before changing each of them to dm, to minimize the accumulation of absolute errors, and calling the identical denominators of equations (3-17) and (3-18) D, and the numerators N and N', respectively,

$$\begin{aligned} dx_o = dm[&|D(y_1x_3 - x_1y_3 + x_1y_6 - x_3y_6) - N(y_1 - y_3)| \\ &+ |D(y_1x_3 - x_1y_3 + x_1y_4 - x_3y_4) - N(y_1 - y_3)| \\ &+ |(Dx_3 - N)(x_4 - x_6)| + |D(x_4y_3 - x_6y_3 + y_4x_6 - x_4y_6) \\ &- N(y_4 - y_6)| + |D(x_4y_1 - x_6y_1 + y_4x_6 - x_4y_6) \\ &- N(y_4 - y_6)| + |(Dx_1 - N)(x_4 - x_6)| + |(Dx_6 - N)(x_1 - x_3)| \\ &+ |(Dx_4 - N)(x_1 - x_3)|]/D^2 \end{aligned} \tag{3-139}$$

$$\begin{aligned} dy_o = dm[&|D(y_1x_3 - x_1y_3 - y_1x_6 + y_3x_6) + N'(y_1 - y_3)| \\ &+ |D(y_1x_3 - x_1y_3 - x_4y_1 + x_4y_3) + N'(x_1 - x_3)| \\ &+ |D(-x_3y_4 + x_3y_6 + y_4x_6 - x_4y_6) + N'(x_4 - x_6)| \\ &- |(Dy_3 - N')(y_4 - y_6)| + |(Dy_1 - N')(y_4 - y_6)| \\ &+ |D(-x_1y_4 + x_1y_6 + y_4x_6 - x_4y_6) + N'(x_4 - x_6)| \\ &+ |(Dy_6 - N')(y_1 - y_3)| + |(Dy_4 - N')(y_1 - y_3)|]/D^2 \end{aligned} \tag{3-140}$$

Differentiating equation (3-19) likewise,

$$dy = \{dx_o|(x_7 - x_9)(y_7 - y_9)| + dm[|(x_7 - x_9)(x_o - x_9)| + |(x_7 - x_9)(x_o - x_7)| + |(y_7 - y_9)(x_o - x_9)| + |(y_7 - y_9)(x_o - x_7)|]\}/(x_7 - x_9)^2 \qquad (3\text{-}141)$$

Comparing y_o with y,

$$\Delta = y_o - y \qquad (3\text{-}142)$$

and the tolerance is

$$d\Delta = |dy_o| + |dy| \qquad (3\text{-}143)$$

To check concentricity, strict differentiation of equation (3-20) would result in excessive tolerance in radius due to the relatively large value of $d\Delta$ from equation (3-143):

$$d\rho = -[(dx_o - dx_j)(x_o - x_j) + (dy_o - dy_j)(y_o - y_j)]/\rho$$

By taking absolute values and letting dx_o, dy_o, dx_j, and dy_j all equal dm,

$$d\rho = 2dm(|x_o - x_j| + |y_o - y_j|)/\rho \qquad (3\text{-}144)$$

Although this is not rigorous, it has been found satisfactory. This value of $d\rho$ is used as the allowable variation in ρ calculated for j = 1, 4, and 7 from equation (3-20).

The tolerance was also not practical when derived rigorously for spacing of meridians on the conics. Instead, the error in the angle of slope of each meridian is found by applying equation (3-134) to point 3 and the point of convergence 0. From equations (3-21) and (3-22), since the distance between points 0 and 3 equals the distance between points 0 and 9,

$$dn = (d\sigma_{1-3} - d\sigma_{7-9})/(\lambda_7 - \lambda_1)$$
$$= 4dm/\{[(x_o - x_3)^2 + (y_o - y_3)^2]^{1/2}(\lambda_7 - \lambda_1)\} \qquad (3\text{-}145)$$

The calculation for dn based on λ_1 and λ_4 is the same, but using λ_4 in place of λ_7, assuming an equal radius vector. The tolerance for the difference between the two calculated n's is the sum of the absolute values, or

$$dn = \{4dm/[(x_o - x_3)^2 + (y_o - y_3)^2]^{1/2}\}[1/|\lambda_1 - \lambda_4| + 1/|\lambda_1 - \lambda_7|] \qquad (3\text{-}146)$$

i. TOLERANCES FOR INDIVIDUAL PROJECTIONS

For the tested projections with curved meridians and curved parallels, differentiation of equation (3-82) to calculate tolerance as a function of F is relatively straightforward. It appeared that the variability of λ_o and ϕ_o could then be determined from this. The results were unsatisfactory. After several tests and other approaches, it was decided not to derive a tolerance for these projections (as well as for many others) but instead to calculate the rectangular coordinates of each of the nine points of the given matrix, using the given latitude and longitude and the parameters determined for the projection under consideration, with the forward formulas for the projection rather than the inverse. The measured coordinates given for these same nine points are translated, rotated, and changed in scale affinely to find the closest least-squares fit to the rectangular coordinates just calculated. The RMSE or r of distances at map scale between each of the nine adjusted given points (x_{aj}, y_{aj}) and the corresponding calculated points (X_j,Y_j) is determined:

$$r = \left\{\sum_{j=1}^{9} [X_j - x_{aj})^2 + (Y_j - y_{aj})^2]/9\right\}^{1/2} \qquad (3\text{-}147)$$

If the residual is less than the experimentally established 1.5 mm for a real map (0.01 mm on a mathematically calculated set of given coordinates), the projection is accepted, and the constants for the final fit are used to modify any other given rectangular coordinates before calculating latitude and longitude from projection parameters. If not, the program similarly tests the next projection in order.

The formulas used for this operation are adapted from traditional ones (see Appendix, section 8 for derivation): If the given coordinates are (x_j,y_j) and the calculated coordinates are (X_j,Y_j), where j = 1 to 9, the constants a_1 to a_6 for the affine transformation (equations (3-164) and (3-165)) are calculated as follows (each summation Σ is taken for $j = 1$ to 9):

$$a_1 = (AE - BF)/D \quad (3\text{-}148)$$

$$a_2 = (CF - BE)/D \quad (3\text{-}149)$$

$$a_3 = (\Sigma X_j/9) - a_1\bar{x} - a_2\bar{y} \quad (3\text{-}150)$$

$$a_4 = (AG - BH)/D \quad (3\text{-}151)$$

$$a_5 = (CH - BG)/D \quad (3\text{-}152)$$

$$a_6 = (\Sigma Y_j/9) - a_4\bar{x} - a_5\bar{y} \quad (3\text{-}153)$$

where

$$A = \Sigma(y_j - \bar{y})^2 \quad (3\text{-}154)$$

$$B = \Sigma(x_j - \bar{x})(y_j - \bar{y}) \quad (3\text{-}155)$$

$$C = \Sigma(x_j - \bar{x})^2 \quad (3\text{-}156)$$

$$D = AC - B^2 \quad (3\text{-}157)$$

$$E = \Sigma X_j(x_j - \bar{x}) \quad (3\text{-}158)$$

$$F = \Sigma X_j(y_j - \bar{y}) \quad (3\text{-}159)$$

$$G = \Sigma Y_j(x_j - \bar{x}) \quad (3\text{-}160)$$

$$H = \Sigma Y_j(y_j - \bar{y}) \quad (3\text{-}161)$$

$$\bar{x} = \Sigma x_j/9 \quad (3\text{-}162)$$

$$\bar{y} = \Sigma y_j/9 \quad (3\text{-}163)$$

Then, for the adjusted values (x_{aj}, y_{aj}) of (x_j, y_j),

$$x_{aj} = a_1x_j + a_2y_j + a_3 \quad (3\text{-}164)$$

$$y_{aj} = a_4x_j + a_5y_j + a_6 \quad (3\text{-}165)$$

Because this sort of test proved generally applicable for any projection, it was also finally used for cylindrical projections and certain polar azimuthals, giving results equal to or better than individual differentiation applied to the specific projection.

j. TOLERANCES FOR CONIC PROJECTIONS

Using the least-squares approach on the conic projections was not sufficiently sensitive to choose correctly between conformal, equal-area and equidistant conic projections when real maps were involved. More satisfactory answers were obtained using tolerances obtained by differentiation.

For the Lambert Conformal Conic, the tolerance for ak', as calculated from equations (3-46) and (3-47), or (3-46) and (3-49), is

$$d(ak_j') = d\rho_j/[f(\phi_j)/k_j']$$

$$= dm/[f(\phi_j)/k_j'] \qquad (3\text{-}166)$$

since (x_o,y_o) is considered fixed so that the radius ρ_j varies only by dm. Then if

$$\Delta = 1 - ak_j'/ak_1' \qquad (3\text{-}167)$$

the tolerance of the fit,

$$d\Delta = [|ak_1'd(ak_j')| + |ak_j'd(ak_1')|]/(ak_1')^2 \qquad (3\text{-}168)$$

where j = 2,3 as defined for equation (3-45).

For the Albers Equal-Area Conic, by differentiation of equation (3-60),

$$dC_{1-2} = 2n(dm)\rho_1\rho_2(q_2 - q_1)(|\rho_1| + |\rho_2|)/(\rho_1^2 - \rho_2^2)^2 \qquad (3\text{-}169)$$

assuming $d\rho$=dm and adding the absolute values since $d\rho$ may be + or –. As in (3-167) and (3-168),

$$\Delta = 1 - C_{2-3}/C_{1-2} \qquad (3\text{-}170a)$$

$$d\Delta = [|C_{1-2}dC_{2-3}| + |C_{2-3}dC_{1-2}|]/C_{1-2}^2 \qquad (3\text{-}171)$$

where subscripts refer to the points involved along a given meridian.

For the Equidistant Conic, by differentiation of equation (6-13) (see Appendix),

$$dG_{1-2} = dm(M_2 - M_1)(|\rho_1| + |\rho_2|)/(\rho_2 - \rho_1)^2 \qquad (3\text{-}172)$$

$$\Delta = 1 - G_{2-3}/G_{1-2} \qquad (3\text{-}173)$$

$$d\Delta = [|G_{1-2}dG_{2-3}| + |G_{2-3}dG_{1-2}|]/G_{1-2}^{\,2} \qquad (3\text{-}174)$$

MINIMUM—ERROR MAP PROJECTIONS

In principle, the selection of a map projection for any given application usually is based on an attempt to minimize distortion, but the term "minimum-error" is normally applied to a projection which has been derived by applying the principle of least squares to a given set of parameters. Commonly, the type of projection will be established, but certain constants will be allowed to vary until the distortion, as determined by an often arbitrary standard, becomes a minimum. Thus, a minimum-error azimuthal projection has been developed, as well as a minimum-error perspective azimuthal, a minimum-error equidistant conic, etc. It is possible to derive, for example, different minimum-error equidistant conics by using different criteria for calculating the amount of distortion at any given point on the map.

Most published minimum-error projections were devised between 1850 and 1950, after the development of the least-squares principle in the early 1800's, but before the availability of high-capacity computers. Therefore, they are limited to projections such as regular conics in which the parameters can be more easily altered. Since then, investigators such as Tobler, Reilly, and this writer have developed projections which would have been nearly impossible without modern computers.

Before discussing recent developments by the U.S. Geological Survey, it is appropriate to review several earlier minimum-error projections. These help to set the stage for later studies, inspired by the earlier concepts. They are varied both in approaches and in nationality, with British, German, Russian, New Zealander, and American contributions detailed, including formulas.

The various historical minimum-error approaches in map projections are outlined just below and mathematically described in the section following. Then the least-squares approach is used to develop a low-error map projection for the 50 States, substantially reducing the range of scale as contrasted with standard projections available for the purpose. Following a discussion of the 50-State projection, a least-squares fit is used to find parameters giving the minimum-error Oblique Conformal Conic, Oblique Mercator, or other conformal projections for North America, Alaska, and South America.

Figure 4.--Airy minimum-error azimuthal projection - oblique aspect of one hemisphere, centered at Washington, D.C. (latitude 39° N., longitude 77° W.)

Finally, this least-squares principle is applied to a very different type of projection, a pseudocylindrical equal-area projection of the Earth, primarily to show the versatility of the approach even for the single subject of map projections.

4. HISTORY OF MINIMUM-ERROR PROJECTIONS

a. OVERVIEW

(1) Azimuthal Projections

In 1861, George Biddell Airy (1801-1892), a British geodesist and astronomer, presented an azimuthal projection with a minimum "total misrepresentation" determined "by balance of errors" (Airy, 1861). He

Figure 5.--Azimuthal Equidistant projection - oblique aspect of one hemisphere, centered at Washington, D.C. (latitude 39° N., longitude 77° W.) Very similar to Airy projection if the bounding circle is given the same spherical radius.

achieved a sort of mean between the Lambert Azimuthal Equal-Area and the Stereographic conformal projections, based on least squares. Applied to a hemisphere (or less), the projection (figure 4) resembles an Azimuthal Equidistant projection of the same area (figure 5). Fellow British geodesists A.R. Clarke, whose name is best remembered in the United States because of the Clarke 1866 and 1880 ellipsoids, and Henry James corrected an error in Airy's constraints the next year (figure 4 is based on the correction), and at the same time also applied Airy's approach with the additional restraint of producing a perspective projection onto a secant plane. Overall scale errors are reduced using a secant rather than tangent plane for either projection.

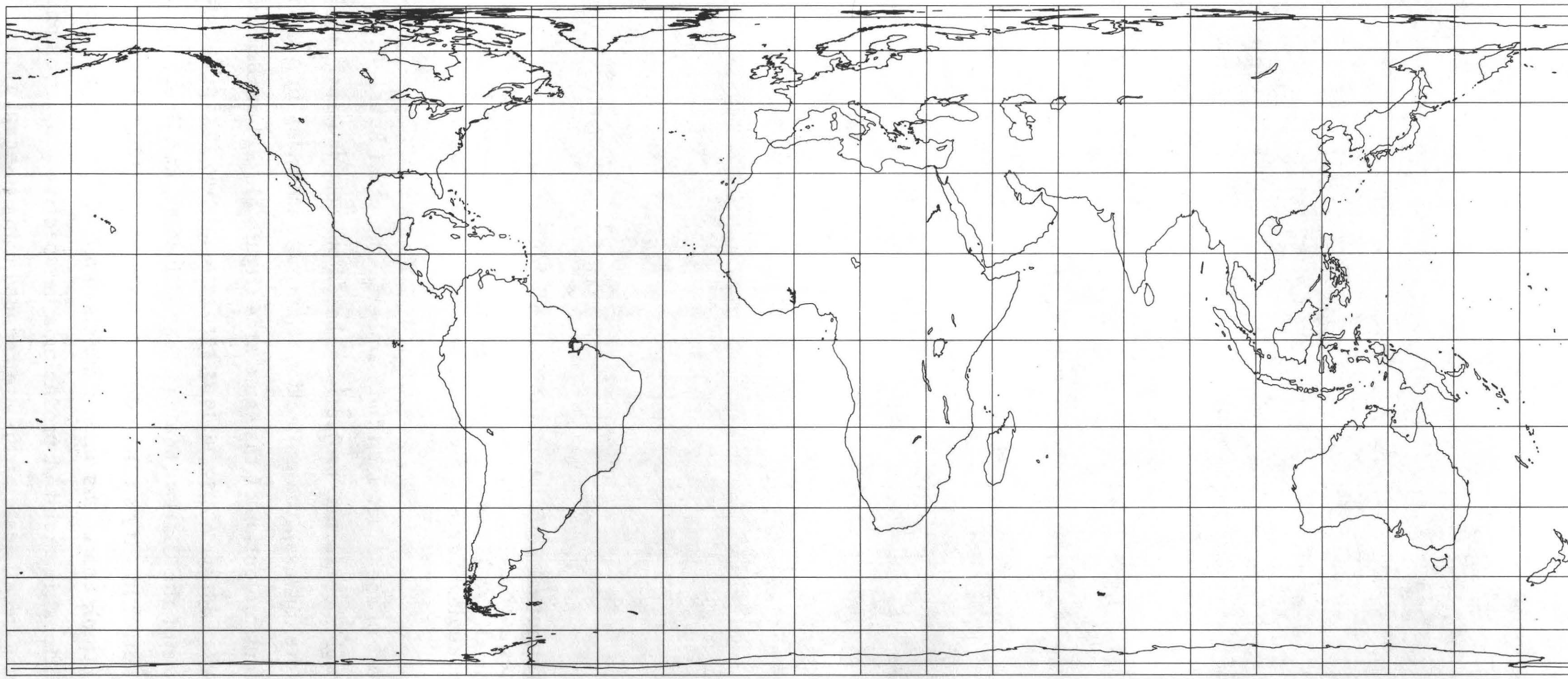

Figure 6.--Behrmann's cylindrical equal-area projection, with standard parallels at latitudes 30° N. and S.

(2) A Cylindrical Projection

Walther Behrmann (1910) of Germany sought a minimum-error equal-area world map, but did not use the least-squares approach as described by Airy. He apparently found the arithmetic average of the maximum angular deformation as determined for uniformly spaced intersections of latitude and longitude, weighting these angles in proportion to the cosine of the latitude. By comparing this average for various projections used as world maps, he found that the Cylindrical Equal-Area projection gave the least average value, provided that latitudes 30° N. and S. were made standard parallels rather than the Equator (figure 6).

(3) Conic Projections

Alfred Ernest Young (1920) of the Royal Geographical Society, London, made a careful study of several low-error map projections, especially azimuthal and conic. He applied the least-squares principle with such restraints on azimuthal projections as the "minimum-error" conformal, equal-area, equidistant, or tangent perspective projection.

In some cases this resulted in a change in the nominal map scale to balance errors, without any other change in the map. By applying the principle involved in Airy's projection to a minimum-error conic projection, Young devised a projection which is almost an Equidistant Conic, but which has very complicated formulas.

The computations are less complicated if the minimum-error analysis is confined to determining the standard parallels for a conformal, equal-area, or equidistant conic projection:

The standard parallels define the shape of a conformal conic projection of the sphere or given ellipsoid. The scale or region shown may be varied, but the distortion pattern of a conformal conic is solely a function of the standard parallels. The same is true of equal-area or equidistant conics, although some other conics, such as Young's minimum-error conic, are neither conformal, equal-area, nor equidistant. Several different-named conformal conic projections vary only in the manner in which standard parallels are chosen. One may make the maximum scale factor the reciprocal of the minimum scale factor, or the greatest scale error (scale factor minus one) between the standard parallels may be made equal and opposite in sign to the greatest scale error beyond the standard parallels, and so on. These approaches reduce the range of the scale factors, but do not apply the principle of least squares to minimizing the overall scale error throughout the map, and therefore do not have minimum error.

A conformal conic projection for which standard parallels are chosen by a least-squares analysis was derived in 1916 by N.J. Tsinger (1842-1918) of Russia (Tsinger, 1916; Graur, 1956, p. 157-170; Maling, 1960, p. 264)[1]. Since scale is a function only of latitude on a regular conic projection, the projection constants are chosen to make the overall root-mean-square scale error a minimum based on the linear scale error at each latitude. The scale errors are weighted, however, in proportion to the area being shown along the particular parallel. This area may be based on the total width of the map, or on the area of the country or region of interest which lies within say a 1°-wide band centered on the parallel. Tsinger (1916) also applied the same principles to the equal-area conic, choosing the standard parallels to suit best the area being mapped.

While Tsinger applied the least-squares principle on conic projections only to the conformal and equal-area forms, V.V. Kavrayskiy (1884-1954) (or Kavraisky), also of Russia, extended it to the Equidistant Conic projection in 1934 (Kavrayskiy, 1934; Graur, 1956, p. 159-160; Maling, 1960, p. 263-265). This projection was then used for maps of Russia published by the Soviet Union, and became confused in some literature in the United States with the general Equidistant Conic with two standard parallels, devised two centuries earlier. As in the Tsinger variations, the Kavrayskiy projection (actually his fourth projection) involves only a least-squares technique for determining the standard parallels (or other constants). Once this is done, the standard projection just mentioned is used.

(4) Projections Using Complex-Algebra Transformations

One of the intriguing mathematical relationships in mapping is found in the Cauchy-Riemann equations, which state, as applied to map projections, that any map projection which is conformal may be reshapen or transformed to another conformal map projection if certain simple relationships occur between the respective coordinates. This occurs with the use of a certain general polynomial series involving real and imaginary coefficients, although there are practical limitations on the use of the series. Miller (1953, 1955), Reilly (1973), and Stirling (1974) have used

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[1] Tsinger is sometimes spelled Zinger in a different transliteration.

this concept to construct, in Miller's case, Prolated and Oblated Stereographic projections for low-error maps of Africa, Europe, Asia, and Australasia, and, in Reilly's and Stirling's case, a "minimum-error" projection for New Zealand.

Miller used the conformal relationship in a relatively simple form to transform the oblique Stereographic projection, with small spherical circles of constant scale, to a new projection with ovals of constant scale. The New Zealand projection began with the regular Mercator, transforming it to produce lines of constant scale roughly following the irregular outlines of the two main islands of the nation.

The Cauchy-Riemann equations have also been used within the USGS to develop a new low-error map projection for the 50 States. This will be discussed at length later in this paper (p. 79-92).

(5) Other Existing Minimum-Error Projections

Waldo R. Tobler (1977), then of the University of Michigan, devised an empirical minimum-error projection based on minimum overall error for all great-circle distances between selected points covering the region under consideration. For his map of the United States, he chose as points the 65 graticule intersections at every 5° of longitude between longitudes 125° and 65° W. and every 7.5° of latitude between latitudes 22.5° and 52.5°N. The number of distances to be minimized in error was then (65 x 64/2) or 2080, using the sphere for "true" distances, although the same principle can be applied to the ellipsoid.

b. A MINIMUM-ERROR AZIMUTHAL PROJECTION

Airy (1861) applied least squares to the development of his minimum-error azimuthal projection by minimizing the sum of the squares of the errors in scale both along and perpendicular to the radii from the center, that is,

$$\text{error} = \int_0^{\beta}[(h' - 1)^2 + (k' - 1)^2]\sin z \, dz$$
$$= \text{minimum} \qquad (4\text{-}1)$$

where, for an azimuthal projection of the sphere,

β = the angular distance z from the center of projection to the rim of the circular map region to which minimum error is being applied.

h' = the scale factor at a given point along the radius from the center

$$= d\rho/R\,dz \tag{4-2}$$

k' = the scale factor perpendicular to the radius

$$= \rho/(R \sin z) \tag{4-3}$$

z = the angular distance of the given point from the projection center, as measured from the center of the Earth.

ρ = the radius from the projection center to the given point, as measured on the map.

R = the radius of the Earth at map scale.

and scale factor is the ratio of the scale on the map at a given point to the nominal scale of the map.

Airy had made an error in his constraints, but this was corrected by James and Clarke (1862), resulting in the following formulas in polar coordinates:

$$\rho = 2R\,[\cot(1/2)z \ln \sec(1/2)z + \tan(1/2)z \cot^2(1/2)\,\beta \ln \sec(1/2)\,\beta] \tag{4-4}$$

$$\Theta = \arctan_2\{\cos\phi \sin(\lambda - \lambda_0)/[\cos\phi_1 \sin\phi - \sin\phi_1 \cos\phi \cos(\lambda - \lambda_0)]\} \tag{4-5}$$

where

$$\cos z = \sin\phi \sin\phi_1 + \cos\phi \cos\phi_1 \cos(\lambda - \lambda_0) \tag{4-6}$$

and (ϕ,λ) = the latitude and longitude, respectively, of the given point

(ϕ_1,λ_0) = the latitude and longitude, respectively, of the center of the projection

(ρ,Θ) = polar coordinates: radius and azimuth east of north, respectively.

When converted to rectangular coordinates,

$$x = Rk' \cos\phi \sin(\lambda - \lambda_0) \tag{4-7}$$

$$y = Rk' [\cos \phi_1 \sin \phi - \sin \phi_1 \cos \phi \cos (\lambda - \lambda_o)] \tag{4-8}$$

where

$$k' = - \{[\ln [(1 + \cos z)/2]]/(1 - \cos z) + 2[\ln \cos (\beta/2)]/[\tan^2(\beta/2) (1 + \cos z)]\} \tag{4-9}$$

If z = 0, equation (4-9) is indeterminate, but

$$k' = R[\tfrac{1}{2} - (\ln \cos (\beta/2))/\tan^2(\beta/2)] \tag{4-10}$$

This projection is not perspective. It is also projected onto a secant plane rather than a tangent plane. If $\beta = 90°$, thus applying the minimum-error constraint to one hemisphere, the projection resembles an Azimuthal Equidistant projection (figures 4 and 5) as stated previously, but h' (h for the polar aspect described) is about 18 percent greater at the limit of the hemisphere than at the center on the Azimuthal Equidistant (see Table 2). It is calculated as follows:

$$h' = 1 + [\ln [(1 + \cos z)/2]]/(1 - \cos z) - 2[\ln \cos (\beta/2)]/ [\tan^2(\beta/2) (1 + \cos z)] \tag{4-11}$$

while k' is found from equations (4-9) or (4-10). It may be noted that many terms in (4-9) and (4-11) are identical.

The projection was used for an Ordnance Survey map of the United Kingdom at ten miles per inch, but has rarely been used otherwise (Airy, 1861; Hinks, 1912, p. 36-37; Close and Clarke, 1911, p. 660; Young, 1920, p. 2-7; Andrews, 1938).

c. A MINIMUM-ERROR PERSPECTIVE AZIMUTHAL PROJECTION

In analyzing and correcting Airy's formulas, James and Clarke (1862) explored minimum-error perspective projections. The general formulas for polar coordinates of any secant vertical perspective projection of the sphere are as follows:

$$\rho = RP' \sin z/(P + \cos z) \tag{4-12}$$

with Θ found from equation (4-5). Symbols ρ, R, and z are defined above, while P is the distance of the point of projection from the center of the sphere in terms of R, and P' is the distance of the plane of projection from the point of projection in terms of R (see figure 7).

Table 2.--Comparison of Airy's minimum-error azimuthal projection with Azimuthal Equidistant projection

[Based on $\beta = 90°$, or minimum error within one hemisphere; polar aspect; Earth taken as sphere, with radius = 1.0.; h = scale factor along meridian of longitude; k = scale factor along parallel of latitude.]

Lat.	Airy's Minimum-Error			Azimuthal Equidistant		
	Radius	h	k	Radius	h	k
90°	0.00000	0.84657	0.84657	0.00000	1.0	1.00000
80	.14780	.84732	.85114	.17453	1.0	1.00510
70	.29586	.84966	.86504	.34907	1.0	1.02060
60	.44450	.85392	.88899	.52360	1.0	1.04720
50	.59408	.86074	.92423	.69813	1.0	1.08610
40	.74516	.87113	.97273	.87266	1.0	1.13918
30	.89847	.88673	1.03746	1.04720	1.0	1.20920
20	1.05514	.91014	1.12285	1.22173	1.0	1.30014
10	1.21686	.94555	1.23563	1.39626	1.0	1.41780
0	1.38629	1.00000	1.38629	1.57080	1.0	1.57080

Applying equations (4-1) through (4-3) to (4-12), Clarke found that H_2^2/H_1 must be a maximum,

where $$H_2 = B - (P + 1) \ln (A + 1) \qquad (4\text{-}13)$$

$$H_1 = A(2 - B + B^2/3)/(P + 1) \qquad (4\text{-}14)$$

$$A = (1 - \cos \beta)/(P + \cos \beta) \qquad (4\text{-}15)$$

$$B = A(P - 1) \qquad (4\text{-}16)$$

and β is defined after equation (4-1). For each β desired, various values of P are tested to obtain a maximum H_2^2/H_1 (this can, of course, also be done by calculus). Then the corresponding P' is found thus:

$$P' = - H_2/H_1 \qquad (4\text{-}16a)$$

Clarke obtained various constants depending on the β chosen (Close and Clarke, 1911, p. 655-656). Minor differences are obtained with modern calculators or computers. For example, Clarke's constants (followed by recalculated constants in parentheses) for a map of Africa or South America, in which $\beta = 40°$ are as follows: P = 1.625 (1.626); P' = 2.543 (2.544). For Asia, using $\beta = 54°$, P = 1.61 (1.594), P' not given (2.443).

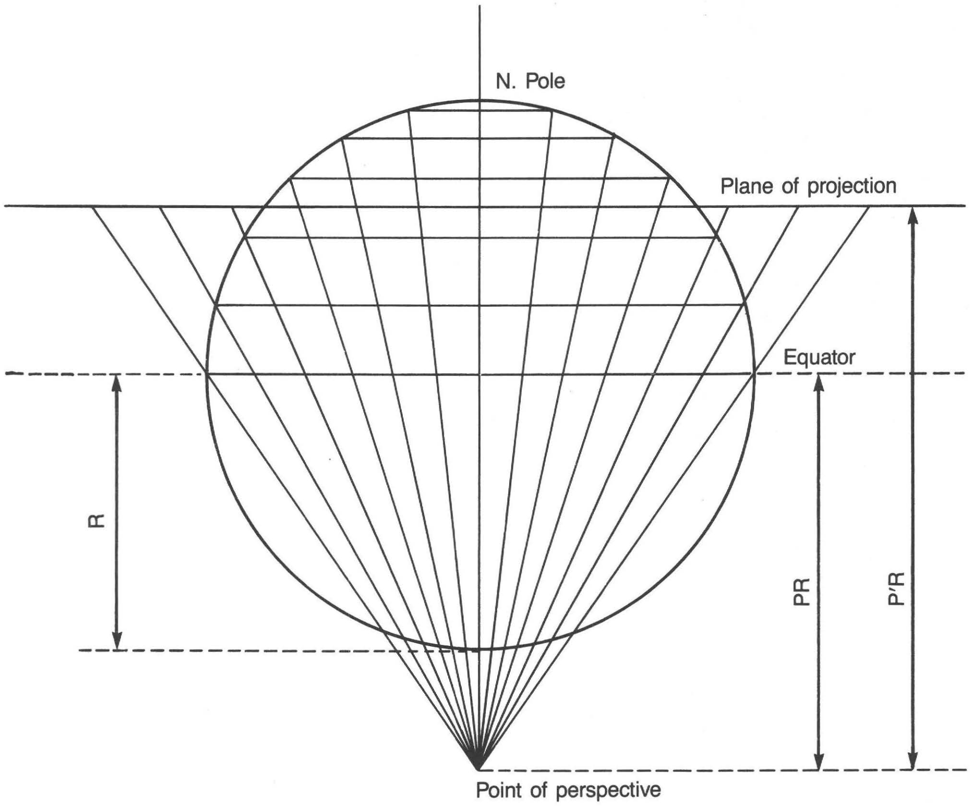

Figure 7.--Geometry of Clarke's minimum-error perspective projection, shown for a north polar aspect.

For a hemisphere, $\beta = 90°$, P = 1.47 (1.472), P' = 2.038 (2.040). James proposed $\beta = 113\ ½°$, so that P = 1.367 (1.367), P' = 1.663 (1.662). The latter was based on 90° plus the 23 ½° latitude of the tropic lines. With the Tropic of Cancer at longitude 15° E. as the center, all the larger continental masses are shown complete on the James projection, although Australia and Antarctica are missing (Craig, 1882, p. 95; Young, 1920, p. 13-16).

A few years later Clarke (1879) presented the "Twilight" projection (figure 8), with $\beta = 108°$, obtaining P = 1.4 (1.393), P' = 1.7572 (1.759). This was so named because astronomical twilight officially ends when the sun is 18° below the horizon (108° from the zenith).

d. A MINIMUM-ERROR EQUAL-AREA CYLINDRICAL PROJECTION

Behrmann's minimum-error equal-area projection, with standard parallels at latitudes 30° N. and S., may be constructed by compressing

Figure 8.--Clarke's "Twilight" perspective projection, centered at latitude 23½° N., longitude 0°, projected from a point 1.393 times the radius of the globe from the center of the globe, and extending 108° from the center of the map. The error is minimum for a perspective projection with this range.

the regular Cylindrical Equal-Area projection from east to west and expanding it from north to south in the same proportion, specifically,

$$x = R(\lambda - \lambda_o) \cos 30° \tag{4-17}$$

$$y = R \sin \phi / \cos 30° \tag{4-18}$$

where symbols are as defined with equations (4-1) through (4-8) for Airy's projection. A graticule of Behrmann's projection with Tissot indicatrices superimposed is shown in figure 9. At latitudes 30° N. and S. the indicatrices are circles, indicating that there is no local shape (or angular) distortion. The other indicatrices are ellipses with the same area as the circles, indicating distortion in shape but not area.

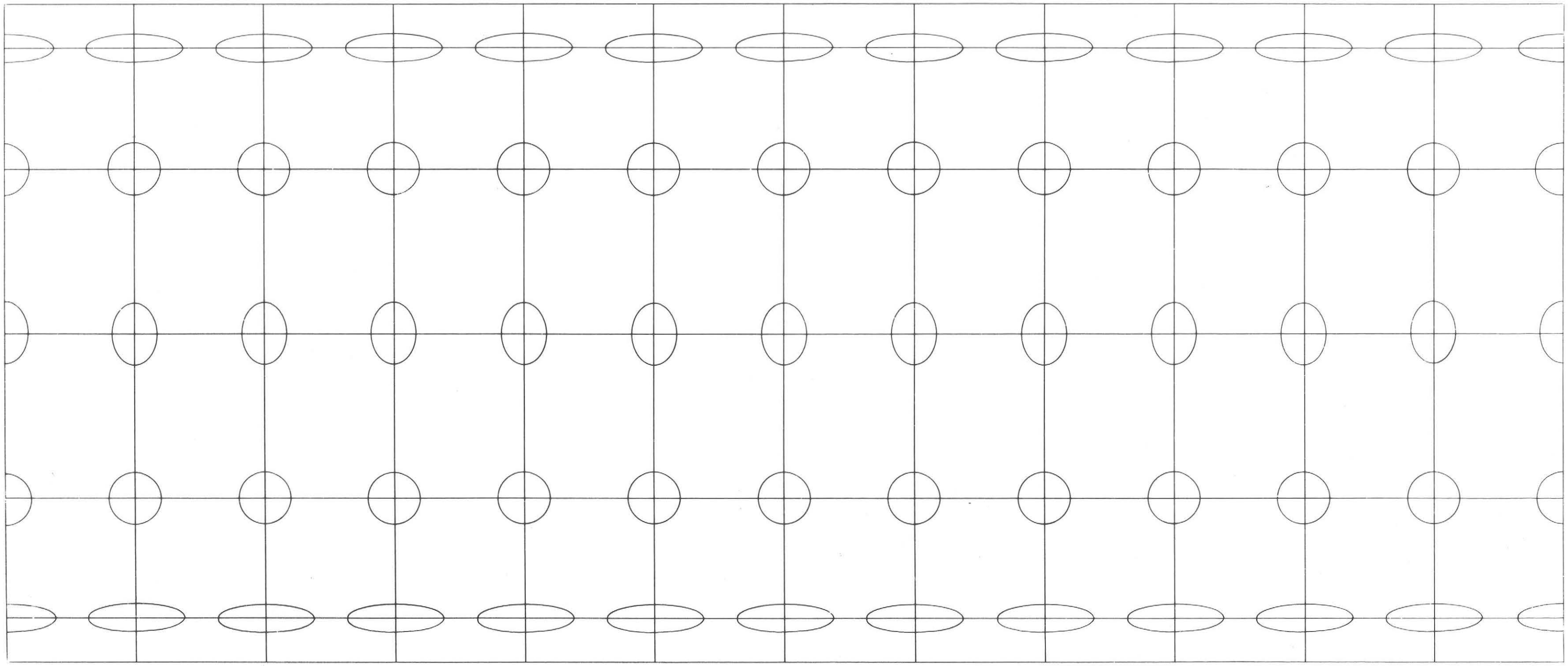

Figure 9.--Behrmann's cylindrical equal-area projection with Tissot indicatrices using a 30° graticule.

There are other ways to define minimum error for a projection of this type: For example, the minimum mean maximum angular deformation may be determined by least squares, or the principle of equation (4-1) can be applied in the following manner:

For the Cylindrical Equal-Area projection, as shown in equations (4-17) and (4-18),

$$h = K \cos \phi \tag{4-19}$$

$$k = 1/(K \cos \phi) \tag{4-20}$$

where $K = 1/\cos \phi_1$, ϕ_1 is the standard parallel, and h and k are scale factors along the meridian and parallel, respectively, at a given point. If equation (4-1) is applied to this projection,

$$E = \text{error} = \int_0^\beta [(h-1)^2 + (k-1)^2] \cos \phi \, d\phi \tag{4-21}$$

$$= (1/3)K^2 \cos^2\beta \sin \beta + (2/3)K^2 \sin \beta - K\beta - (1/2)K \sin 2\beta$$
$$+ 2 \sin \beta + (1/K^2) \ln (\sec \beta + \tan \beta) - 2\beta/K \tag{4-22}$$

where β is the latitude limit, north and south.

For a minimum E, equation (4-22) is differentiated with respect to K, and set equal to zero:

$$(2/3)K^4 \sin \beta (\cos^2\beta + 2) - K^3(\beta + (1/2) \sin 2\beta) + 2K\beta$$
$$- 2 \ln (\sec \beta + \tan \beta) = 0 \tag{4-23}$$

If $\beta = 90°$, $K = \infty$, and there is no meaningful solution. If $\beta = 80°$, $K = 1.417$, or $\phi_1 = 45.1°$. If $\beta = 40°$, $K = 1.083$, or $\phi_1 = 22.6°$.

Other alternatives to Behrmann's "best-known equal-area" world map (translating the title of his paper) are mentioned in Section 5c (pp. 120-131).

e. A MINIMUM-ERROR CONIC PROJECTION

Young's minimum-error conic projection has such lengthy formulas that there is almost no justification for its use over the very similar Equidistant Conic (Young, 1920, p. 22-23). The equations are presented here as an example of the results of a complicated earlier analysis. Now, of course, these equations can be programmed and routinely computed, if desired.

$$\rho/R = A \tan^n(z/2) + \cot^n(z/2)\int_0^z \tan^n(z/2)dz + B \cot^n(z/2) \quad (4\text{-}24)$$

$$A = [\int_\alpha^\beta \tan^n(z/2)dz]/[\tan^{2n}(\beta/2) - \tan^{2n}(\alpha/2)] \quad (4\text{-}25)$$

$$B = [\tan^{2n}(\alpha/2)\int_0^\beta \tan^n(z/2)dz - \tan^{2n}(\beta/2)\int_0^\alpha \tan^n(z/2)dz]/$$

$$[\tan^{2n}(\beta/2) - \tan^{2n}(\alpha/2)] \quad (4\text{-}26)$$

α and β are the limiting co-latitudes of the map (90° - latitude), and n is the cone constant, such that, if

$$f = n[\int_\alpha^\beta \tan^n(z/2)dz]^2/[\tan^{2n}(\beta/2) - \tan^{2n}(\alpha/2)] \quad (4\text{-}27)$$

then f is to be made a maximum. Young derived a series from (4-27) for n, but its lack of terms results in a ½-percent error in n in one example checked. However, by differentiating (4-27) with respect to n and setting the result equal to zero, a lengthy expression is obtained which may be iterated by false position to obtain a more accurate n:.

$$(E - F)\int_\alpha^\beta \tan^n(z/2)dz - 2n[[E \ln \tan(\beta/2) - F \ln \tan(\alpha/2)]$$

$$\int_\alpha^\beta \tan^n(z/2)dz - (E - F)\int_\alpha^\beta \tan^n(z/2) \ln \tan(z/2)dz] = 0 \quad (4\text{-}28)$$

where $E = \tan^{2n}(\beta/2)$ (4-29)

$$F = \tan^{2n}(\alpha/2) \quad (4\text{-}30)$$

Actually equations (4-24) through (4-26) provide the minimum-error conic for any given n, but (4-28) through (4-30) permit the calculation of the minimum-error conic of them all for a given α and β.

If n = 1 and $\alpha = 0$, the formulas reduce to the polar form of Airy's minimum-error azimuthal (see equations (4-4) through (4-6)). In table 3 an example of Young's conic, using the more accurate n, is compared with an Equidistant Conic for which the standard parallels are selected with Kavrayskiy's minimum-error technique.

f. A MINIMUM-ERROR CONFORMAL CONIC PROJECTION

Mathematically, Tsinger's minimum-error conformal conic discussed earlier is based upon the following least-squares relationship:

$$E = (\Sigma\epsilon^2 P/\Sigma P)^{1/2} = \text{minimum} \quad (4\text{-}31)$$

Table 3.--Comparison between Kavrayskiy's minimum-error Equidistant Conic and Young's minimum-error conic projections

[Map range: latitudes 25° to 49° N. Earth taken as sphere. Map width for Kavrayskiy computation taken as same longitude width throughout the map. n = cone constant. Standard parallels for Kavrayskiy: 30.220° and 44.125°.]

Lat.	Radius of Parallel			
	Young	difference	Kavrayskiy	difference
50°	1.0883967		1.0884016	
		0.0872830		0.0872665
45	1.1756797		1.1756681	
		.0873073		.0872664
40	1.2629870		1.2629345	
		.0872665		.0872665
35	1.3502535		1.3502010	
		.0872390		.0872665
30	1.4374925		1.4374675	
		.0872411		.0872665
25	1.5247336		1.5247340	
n	0.602724		0.602736	

where E is the overall root-mean-square scale error, ϵ is the linear scale error at each latitude, and P is the area of each element.

For conformal projections, the following version of ϵ was used:

$$\epsilon = \ln k \tag{4-32}$$

where k is the scale factor at a given point; thus ϵ is approximately equal to (k-1) when k is near 1, as it normally is. Since Tsinger's approach is merely a means of determining standard parallels for a conformal conic, k may be computed from the ellipsoidal Lambert Conformal Conic projection formulas (Snyder, 1982, p. 107),

$$k = m_1 t^n / m t_1^n \tag{4-33}$$

$$m = \cos\phi/(1 - e^2 \sin^2\phi)^{1/2} \tag{4-34}$$

$$t = \tan(\pi/4 - \phi/2)/[(1 - e\sin\phi)/(1 + e\sin\phi)]^{e/2} \tag{4-35}$$

where n is the cone constant to be determined, and m_1 and t_1 are found from equations (4-34) and (4-35), substituting one of the standard parallels in place of ϕ, although they are not yet determined. To determine the standard parallels, certain related constants are found first. Combining (4-32) and (4-33),

$$\epsilon = \ln (m_1 t^n / m t_1^n)$$

$$= \ln m_1 + n \ln t - \ln m - n \ln t_1 \tag{4-36}$$

Let $$v = \ln m_1 - n \ln t_1 \tag{4-37}$$

$$u = \ln t \tag{4-38}$$

$$s = \ln m \tag{4-39}$$

Then $$\epsilon = v + nu - s \tag{4-40}$$

in which v and n are projection constants, while u and s are functions of the latitude. For a minimum E in equation (4-31), since the denominator, or the total area ΣP, is a constant, the derivatives of the numerator with respect to v and n are set equal to zero:

$$\partial \Sigma \epsilon^2 P / \partial v = 0 \tag{4-41}$$

$$\partial \Sigma \epsilon^2 P / \partial n = 0 \tag{4-42}$$

Differentiating and cancelling out the common factor of 2,

$$\Sigma Pun + \Sigma Pv - \Sigma Ps = 0 \tag{4-43}$$

$$\Sigma Pu^2 n + \Sigma Puv - \Sigma Pus = 0 \tag{4-44}$$

Solving these linear simultaneous equations for cone constant n and combined constant v,

$$n = [(\Sigma Ps)(\Sigma Pu) - (\Sigma Pus)(\Sigma P)]/[(\Sigma Pu)^2 - (\Sigma Pu^2)/(\Sigma P)] \tag{4-45}$$

$$v = [(\Sigma Pus)(\Sigma Pu) - (\Sigma Ps)(\Sigma Pu^2)]/[(\Sigma Pu)^2 - (\Sigma Pu^2)(\Sigma P)] \tag{4-46}$$

with each Σ equivalent to $\sum_{j=1}^{w}$ and each P, u, and s having subscript j. The map is divided into w zones 1° of latitude wide (or as desired), and the central latitude of each zone is used for the successive values of ϕ_j.

The standard parallels ϕ_1 and ϕ_2 may then be found by trial and error as the two values of ϕ_1 which fit (4-37), using (4-34) and (4-35). The above derivation and the one following are essentially those given in Graur (1956, p. 170, 178).

g. A MINIMUM-ERROR EQUAL-AREA CONIC PROJECTION

The standard parallels for Tsinger's minimum-error equal-area conic are determined using an analysis similar to that for his conformal conic, and the standard Albers formulas (Snyder, 1982, p. 97) may be used to compute coordinates. To determine the parallels and equivalent constants, instead of equation (4-32) Tsinger used the following approximate relationship for scale error about a single point on an equal-area conic projection (Graur, 1956, p. 177):

$$\epsilon = (1/2)(k^2 - 1) \qquad (4\text{-}47)$$

The equations corresponding to (4-33) and (4-35) for the ellipsoidal Albers Equal-Area Conic are as follows:

$$k = (C - nq)^{1/2}/m \qquad (4\text{-}48)$$

$$C = m_1^2 + nq_1 \qquad (4\text{-}49)$$

$$q = (1 - e^2)\{\sin\phi/(1 - e^2 \sin^2\phi) - [1/(2e)] \ln [(1 - e \sin\phi)/(1 + e \sin\phi)]\} \qquad (4\text{-}50)$$

in which m is found from (4-34), and m_1 and q_1 are found from (4-34) and (4-50) substituting one of the standard parallels (not yet determined) in place of ϕ. If $e = 0$, (4-50) is indeterminate, but $q = 2 \sin\phi$.

Substituting from (4-48) and (4-49) into (4-47),

$$\epsilon = C/2m^2 - nq/2m^2 - 1/2 \qquad (4\text{-}51)$$

Let

$$u = - q/m^2 \qquad (4\text{-}52)$$

$$s = 1/m^2 \qquad (4\text{-}53)$$

C and n are projection constants to be determined, and u and s vary with latitude.

Performing steps analogous to those in equations (4-40) through (4-46), using equations (4-41) and (4-42) with C in place of v,

$$n = [(\Sigma Ps^2)(\Sigma Pu) - (\Sigma Pus)(\Sigma Ps)]/[(\Sigma Ps^2)(\Sigma Pu^2) - (\Sigma Pus)^2] \quad (4\text{-}54)$$

$$C = [(\Sigma Ps)(\Sigma Pu^2) - (\Sigma Pu)(\Sigma Pus)]/[(\Sigma Ps^2)(\Sigma Pu^2) - (\Sigma Pus)^2] \quad (4\text{-}55)$$

The comments following (4-46) also apply here.

The cone constant is n, and the standard parallels may be found by trial and error as the two values of ϕ_1 fitting (4-49).

h. A MINIMUM-ERROR EQUIDISTANT CONIC PROJECTION

For the derivation of Kavrayskiy's minimum-error Equidistant Conic projection, similar in principle to Tsinger's work, the equivalent of equations (4-32) and (4-47) for the Equidistant Conic projection is

$$\epsilon = (k - 1)/2^{1/2} \quad (4\text{-}56)$$

The equations corresponding to (4-33) and (4-35) are as follows (Snyder, 1979, p. 71):

$$k = (G - nM)/m \quad (4\text{-}57)$$

$$G = m_1 + nM_1 \quad (4\text{-}58)$$

$$M = a[(1 - e^2/4 - 3e^4/64 - 5e^6/256 - \ldots)\phi - (3e^2/8 + 3e^4/32 + 45e^6/1024 + \ldots) \sin 2\phi + (15e^4/256 + 45e^6/1024 + \ldots) \sin 4\phi - (35e^6/3072 + \ldots) \sin 6\phi + \ldots] \quad (4\text{-}59)$$

in which m is found from (4-34), and m_1 and M_1 are found from (4-34) and (4-59) using one of the standard parallels (to be determined) in place of ϕ.

Substituting from (4-57) into (4-56),

$$\epsilon = (G/m - nM/m - 1)/2^{1/2} \quad (4\text{-}60)$$

Letting

$$u = - M/m \quad (4\text{-}61)$$

$$s = 1/m \quad (4\text{-}62)$$

and performing steps analogous to those in equations (4-40) through (4-46), with equations (4-41) and (4-42) intact but using G in place of v, the final equations are found to be identical with (4-54) and (4-55), except that G replaces C, and u and s represent the above functions. This time the standard parallels are found from equation (4-58), by trial and error. Kavrayskiy's projection is compared with Young's minimum-error conic in table 3.

i. A MINIMUM-ERROR CONFORMAL PROJECTION FOR NEW ZEALAND

One mathematical expression of the Cauchy-Riemann equations referred to in the introduction to this portion of this paper states that any map which is conformal and represented by a set of rectangular coordinates (x',y') is also conformal when transformed to another set of rectangular coordinates (x,y), provided that

$$\partial x/\partial y' = -\partial y/\partial x' \qquad (4\text{-}63)$$

and

$$\partial x/\partial x' = \partial y/\partial y' \qquad (4\text{-}64)$$

A general equation which fits these conditions is the long-established formula (used in analogous form in equation (2-5) when discussing polynomials for conformal projections),

$$x + iy = \sum_{j=1}^{n} (A_j + iB_j)(x' + iy')^j \qquad (4\text{-}65)$$

where $i^2 = -1$, n is any positive integer, and A_j and B_j are any real constants. It can be fairly readily differentiated to prove that it agrees with (4-63) and (4-64). In theory, it might appear that A_j, B_j, and n can be determined to make the scale factors on the new projection follow almost any prescribed pattern to minimize distortion in certain regions. In practice, this can be very difficult or impossible in many cases. The use of least squares allows some freedom in determining coefficients A_j and B_j which may not be possible if the exact locations of points with given scale factors are specified. There have been few earlier applications of the least-squares principle to equation (4-65) for developing new map projections.

Reilly (1973) and Stirling (1974) used these equations to develop a new conformal projection for the topographic mapping of the irregularly shaped islands of New Zealand. Using the Mercator projection of the

International ellipsoid as the basis for the (x',y') coordinates of equation (4-65), Reilly made a least-squares fit with a sixth order (n = 6) complex polynomial to 228 points at half-degree intervals of latitude and longitude spread over New Zealand.

Since the derivation is quite lengthy and is published in English, its inclusion here will not be attempted, but the final formulas for calculating coordinates are given below. For the Mercator projection, relative to the origin and an ellipsoid of unit radius, and interchanging (not rotating) axes for consistency with Reilly,

$$x' = \psi - \psi_o \tag{4-66}$$

$$y' = \lambda - \lambda_o \tag{4-67}$$

where ψ is isometric latitude which may be calculated as

$$\psi = -\ln t \tag{4-68}$$

t is found from equation (4-35), and ψ_o is ψ calculated for $\phi = 41°$ S. latitude, $\lambda_o = 173°$ E. longitude. Reilly and Stirling provide a 10th-order polynomial instead of (4-35) and (4-68) for calculating ψ from ϕ, and a 9th-order polynomial for the inverse, giving 10-place accuracy between 34° and 48° S. latitude. To obtain only positive coordinates, using N and E instead of x and y, respectively, equation (4-65) is in effect rewritten

$$N + iE = a \sum_{j=1}^{6} B_j(x' + iy')^j + N_o + iE_o \tag{4-69}$$

where B_j is a set of complex constants, and a is 6,378,388 m, the semimajor axis of the International ellipsoid. Note again that the x' and y' axes are interchanged from the orientation elsewhere in this paper. For the constants,

	Real		Imaginary
$N_o + iE_o$ =	6023150	+	2510000
B_1 =	0.7557853228	+	0
B_2 =	0.249204646	+	0.003371507
B_3 =	-0.001541739	+	0.041058560
B_4 =	-0.10162907	+	0.01727609
B_5 =	-0.26623489	-	0.36249218
B_6 =	-0.6870983	-	1.1651967

The finite series (4-69) is inverted for inverse computations to provide a theoretically infinite complex series of which the first six terms are used for a first approximation to (x',y') for a given (N,E). A rational-function equation using the first approximation and the above coefficients B_1 through B_6 provides a second, then a third approximation which "gives sufficient accuracy at any point within the land area of New Zealand." From x' is obtained ψ and then ϕ, using the inverse series mentioned above, and λ is obtained directly from y'. Other formulas give the scale factor and convergence (Stirling, 1974). The error in scale using this projection is less than ±0.02 percent for the land area of New Zealand.

j. A MINIMUM-ERROR PROJECTION BASED ON FINITE DISTANCES

Tobler's (1977) approach to an empirical minimum-error projection minimized overall error for all great circle distances between a matrix of points, such as intersections of meridians and parallels distributed over the region being mapped. If D_{ij} represents the various true distances on the Earth, and d_{ij} represents the corresponding distances as measured on the map, the development of the map projection requires that, for overall error E,

$$E = \Sigma(D_{ij} - d_{ij})^2 = \text{minimum} \qquad (4\text{-}70)$$

where

$$d_{ij} = [(x_i - x_j)^2 + (y_i - y_j)^2]^{1/2} \qquad (4\text{-}71)$$

$$D_{ij} = R \arccos [\sin \phi_i \sin \phi_j + \cos \phi_i \cos \phi_j \cos (\lambda_i - \lambda_j)] \qquad (4\text{-}72)$$

and R is the radius of the sphere representing the Earth.

Differentiation of E in equation (4-70) with respect to map coordinates x_i, y_i, x_j, and y_j for the 65 points leads to simultaneous equations which may be solved for changes in the rectangular coordinates, starting with the initial guesses. Successive iteration leads to convergence in coordinate adjustments. As Tobler states, points can also be weighted in (4-70) in accordance with importance or area of the region surrounding each point. The resulting map projection is defined not by formulas but by rectangular coordinates, and does not fall into any of the standard classifications such as a conic.

Instead of minimizing distances, Tobler showed that a different constraint may be applied to obtain a projection which is "nearly conformal in the large," minimizing the error in azimuths between the finite points.

Tobler also computed a map projection for the Mediterranean Sea, minimizing distances by loxodrome (line of constant compass bearing or rhumb line) rather than great circle, as a modern simulation of a possible principle of construction of 13th-century portolan charts. Portolan charts were prepared for navigators and showed seacoasts, ports, and numerous diagonal straight lines radiating from compass roses dispersed over the map.

5. CURRENT STUDIES OF MINIUMUM-ERROR PROJECTIONS

a. A LOW-ERROR CONFORMAL PROJECTION FOR A 50-STATE MAP OF THE UNITED STATES

As stated previously (equation (4-65)), a finite complex series of varied order and coefficients can be used to create an infinite set of precisely conformal map projections. The Prolated (or Oblated) Stereographic projection by O.M. Miller (1953) of the American Geographical Society, and developed for a reduced-distortion conformal map of Europe and Africa, was mentioned previously as an early example. The (x',y') coordinates of equation (4-65) are based on the oblique Stereographic projection, n is made 3 and all but two coefficients are made zero in the Oblated Stereographic. The remaining coefficients A_1 and A_3 are chosen to provide a line of constant scale which is an oval instead of the circle of the Stereographic. In essence, Miller chose $A_1 = 0.9245$, $A_3 = 0.01943$, and centered the Stereographic at latitude 18° N. and longitude 20° E. Equation (4-65) simplifies to

$$x = 0.9245x'\{1 - (0.2522/12)[3(y')^2 - (x')^2]\} \tag{5-1}$$

$$y = 0.9245y'\{1 + (0.2522/12)[3(x')^2 - (y')^2]\} \tag{5-2}$$

and x' and y' may be found from equations (5-3) through (5-8), (5-26), and (5-27), simplified for the sphere (e = 0). L.P. Lee (1974b) adapted an oblique aspect of this projection, also third order, but with different constants, to a map of the Pacific Ocean.

In an unpublished manuscript report, Miller (1955) extended the principle to other continents of the Eastern Hemisphere, with two additional Oblated Stereographic projections linked by non-conformal "fill-in" projections. His 1953 Oblated Stereographic projection was used without change, except that the central meridian was moved from longitude 20° E. to 18° E. The American Geographical Society prepared continuous maps of all of Africa, Europe, Asia, and Australasia, using this "Miller Oblated Stereographic" Projection at a scale of 1:5,000,000.

The Oblated Stereographic projections are not based upon the least-squares principle but were derived by selecting suitable values for the scale factor at the center and various limits of the map. The scale factors are symmetrical about two perpendicular axes, and the constants were determined without iteration.

It seemed appropriate to apply equation (4-65) to the development of a projection which would show all 50 States of the United States with as little scale variation as possible. Prior to the achievement of statehood by Alaska and Hawaii in 1959, these two territories were commonly shown as insets on maps of the 48 conterminous States. Alaska was normally reduced considerably in scale for the inset. After 1959, there was concern that the largest State was shown smaller than several other States and to the south rather than the north. Rand McNally promptly published a map for wall and atlas use showing the 50 States in their relative positions and sizes. The projection used was apparently the Lambert Azimuthal Equal-Area. In 1975, the U.S. Geological Survey issued a map with similar coverage (although omitting the smaller islands of Hawaii) at scales of 1:10,000,000 and 1:6,000,000, but cast on the Lambert Conformal Conic projection with standard parallels of latitudes 37° and 65° N. (see figure 10).

While both of these standard projections accomplished the basic purpose for visual purposes, the scale distortion or error (scale factor minus one) on the USGS maps, for example, varies from +12 percent on the island of Hawaii, +7 percent in southern Florida, and +4 percent in northern Alaska to -3 percent at the 49th parallel (see figure 11). Since the scale along any given parallel on the Lambert Conformal Conic projection is constant, the scale error is zero or minimal not only in central Alaska and the middle of the 48 conterminous States, but in regions of northern Canada and the northern Pacific Ocean, at the expense of other regions which are parts of the United States.

When equation (4-65) was applied to this problem without using least squares, but with the stipulation that the scale factor be 1.0 through several specific points, no satisfactory solution was found. While coefficients could be obtained (in quantity equal to the number of points), the scale at other points on the map fluctuated too widely, and it was too difficult to guess where to move the reference points. On the other hand, when the same equation was applied to a large number of points using least squares, and developing a moderate number of coefficients, far more satisfactory solutions were found, and simple adjustments of the location of points led to a map projection on which the scale error does not exceed ±2 percent for all land forms of the 50 States as well as adjacent waters and the connecting routes between Hawaii, Alaska, and the West Coast of the 48 conterminous States (see figures 12 and 13).

This range of scale error includes all the islands of Alaska and of Hawaii to its western limit at Kure I., as well as northern Mexico, adjacent Canada and Cuba. These regions, adjacent Siberia, the rest of Mexico, and the rest of mainland Canada (except for Labrador) appear normal to the eye, providing an esthetically acceptable map covering the general region. The northern Pacific Ocean varies from true scale by 3.2 percent or less. The projection has been given the name GS50.

These conditions were obtained by using the least-squares fit to obtain 20 coefficients ($n = 10$ but $B_1 = 0$) for equation (4-65), beginning with an oblique Stereographic projection, and fitting to 44 points. After the 44 points were adjusted to produce the final coefficients, their locations are as shown in table 4.

The 50-State region is not surrounded by the ideal line of constant scale to satisfy Chebyshev's principle for a minimum-error map projection; instead there are 18 prongs of true scale extending away from the heart of the map. This map is not claimed to be the theoretically best solution to the problem, so it is called low-error rather than minimum-error. It is believed, however, to be a good practical solution. It is minimum-error for the number of coefficients and locations of points, but these are somewhat arbitrary. More coefficients lead to too much variation in scale between points; fewer coefficients do not provide the desired accuracy range. Because of the low scale-error range in the relevant regions of the GS50, corrections for the ellipsoid (up to 0.6 percent in shape) are worth incorporating. While the projection is much more complicated to program and compute than standard projections, once programmed it may be plotted by the computer operator at any scale almost as if it were a projection like the Lambert. The latitude and longitude ranges must be limited, as discussed below.

The derivation of the formulas used to determine the coefficients of the GS50 or other similar projections is given in the appendix (section 9) and in Snyder (1984). Here the equations are given in the order needed for computing the coefficients. The equations also are based on the ellipsoid only, although the sphere is obtained by letting the eccentricity $e = 0$, and simplifying.

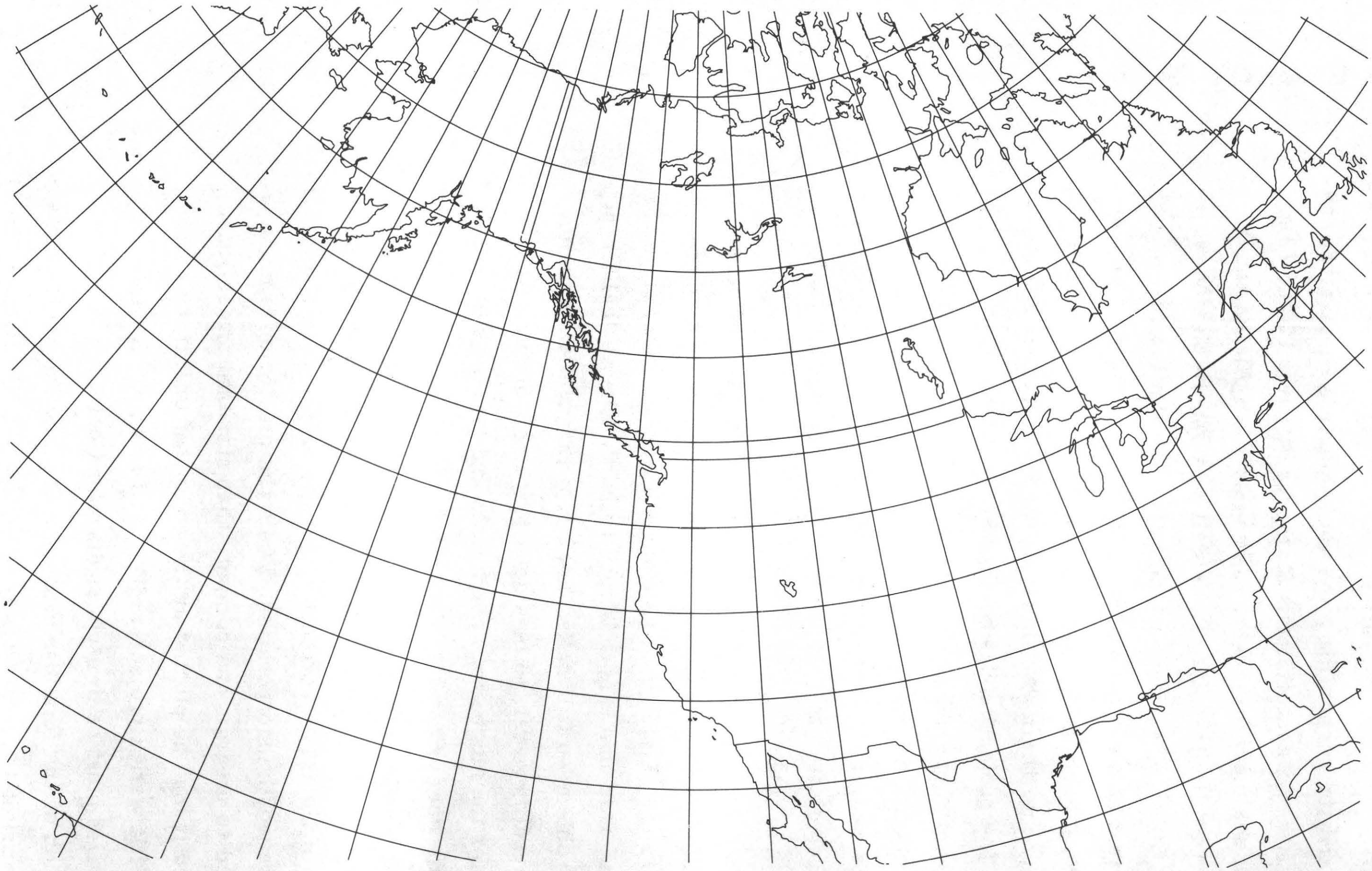

Figure 10.--The United States: A computer-generated map using the projection of a 50-State map published by the U.S. Geological Survey, 1975, at a much larger scale. The projection is Lambert Conformal Conic with standard parallels at latitudes 37° and 65° N.

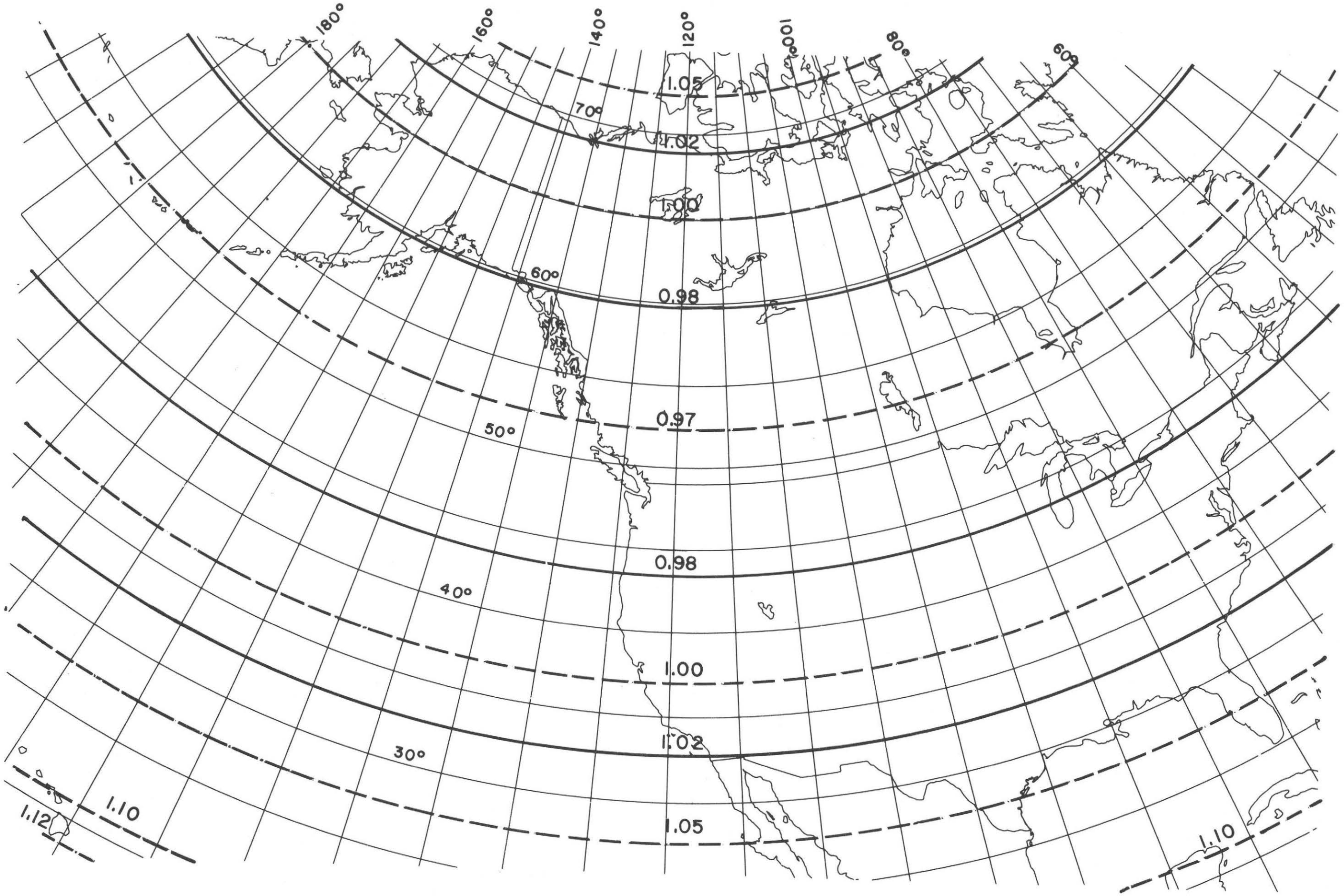

Figure 11.--The map of figure 10 with lines of constant scale factor superimposed. The scale factors for the 50 States vary from 1.12 at the south tip of the island of Hawaii to 0.97 at the 49th parallel.

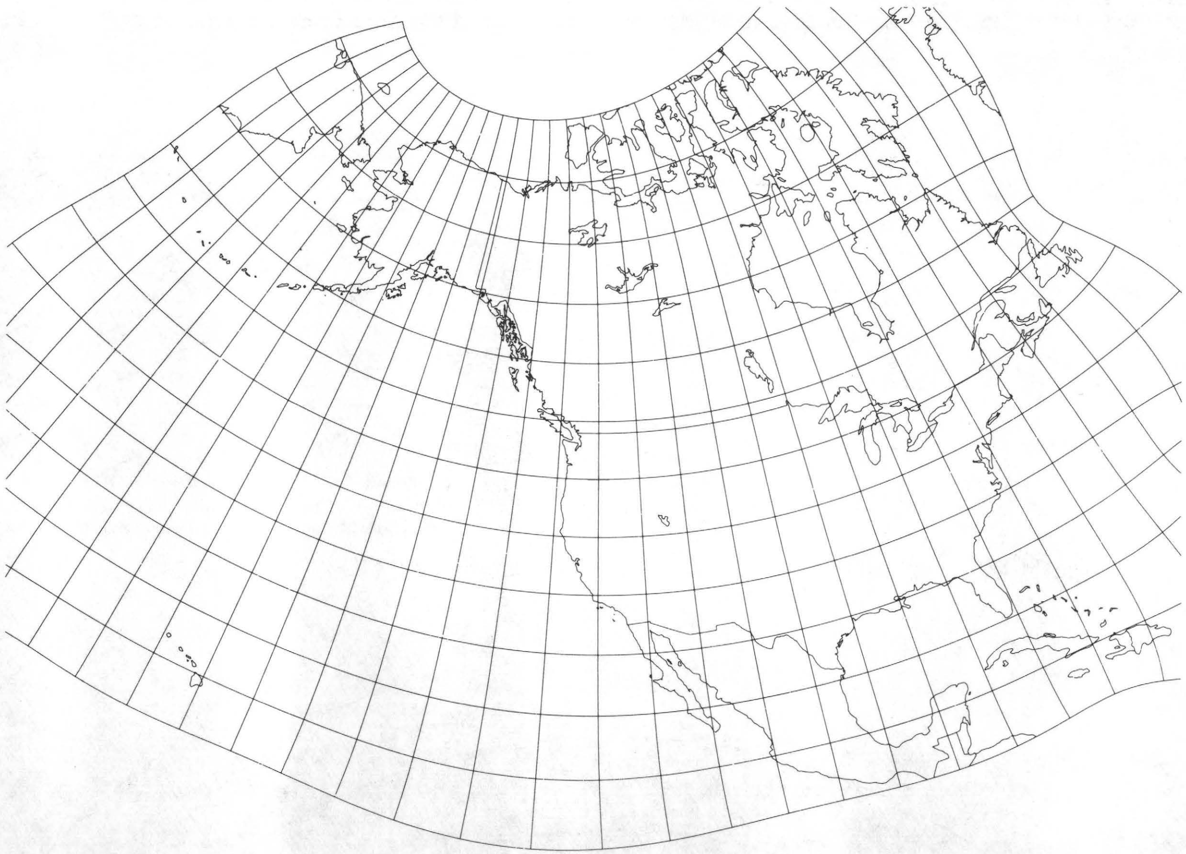

Figure 12.--The United States: A 50-State outline map generated by computer using a complex conformal transformation of the oblique Stereographic projection with 20 coefficients. While meridians and parallels near the edge are visually more distorted than those of figure 10, the region of the 50 States is much less distorted. The more distorted regions may be omitted on the final map.

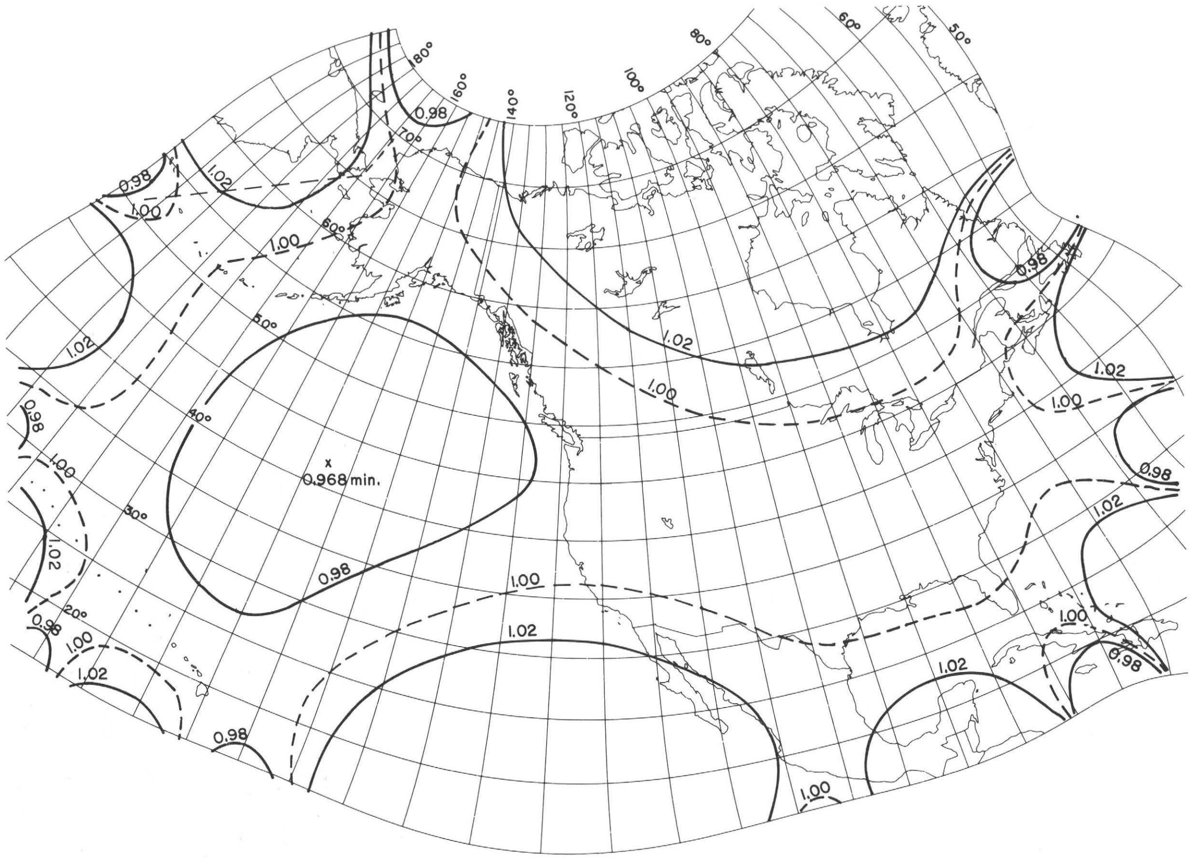

Figure 13.--The map of figure 12 with lines of constant scale factor superimposed. All 50 States, including islands and passages between Alaska, Hawaii, and the conterminous 48 States are shown with scale factors ranging only from 1.02 to 0.98.

Table 4.--The 44 points used for least-squares fit of the 50-State conformal map projection GS50

[Center of the base oblique Stereographic Projection: latitude 45°N., longitude 120°W. The order of the points does not matter. The points were equally weighted.]

Point No.	Lat.	Long.	Point No.	Lat.	Long.
1	70°N.	165°W.	23	48°N.	65°W.
2	70	150	24	45	65
3	70	138	25	40	180
4	65	170	26	40	125
5	65	150	27	40	110
6	65	140	28	40	100
7	60	170	29	40	90
8	58	150	30	40	70
9	58	140	31	34	120
10	55	170°E.	32	32	72
11	55	178°W.	33	30	180
12	55	165	34	30	135
13	55	130	35	30	110
14	50	170°E.	36	28	90
15	50	178°W.	37	27	175
16	50	165	38	25	170
17	50	110	39	25	98
18	50	100	40	25	80
19	50	90	41	22	145
20	50	80	42	21	165
21	50	70	43	18	160
22	49	127	44	17	155

It should be stressed that most transformations resulting from the use of equation (4-65) can only be used within a limited range, depending on the number and values of coefficients. As the distance from the projection center increases, meridians, parallels, and shorelines begin to exhibit loops, overlapping, and other undesirable curves. A world map using the GS50 projection is nearly illegible, with the meridians and parallels intertwined like wild vines (see also figure 19).

To determine coefficients:

(I) Given are the latitude ϕ_o and longitude λ_o of the projection center of the base oblique Stereographic projection, the number n of complex coefficients $(A_j + iB_j)$ to be determined, and the latitude ϕ and longitude λ of each of the m points to be fitted by least squares. For the GS50 projection, $n = 10$, $m = 44$, ϕ_o, λ_o, ϕ, and λ are found in table 4, and a (semimajor axis of ellipsoid) may be taken as 1 for computation of coefficients. For the first trial, A_1 is set equal to 1, and A_2 through A_n and B_1 through B_n are made zero.

For ϕ_o, the following is calculated:

$$\chi_o = 2\arctan\{\tan(\pi/4 + \phi_o/2)[(1 - e\sin\phi_o)/$$
$$(1 + e\sin\phi_o)]^{e/2}\} - \pi/2 \qquad (5\text{-}3)$$

where e is the eccentricity of the ellipsoid, and χ_o is the conformal latitude corresponding to ϕ_o.

(II) The following equations (5-4) through (5-15) are calculated for each of the m points,

$$\chi = \text{same as equation (5-3), but using } \phi \text{ in place of } \phi_o \qquad (5\text{-}4)$$

$$m = \cos\phi/(1 - e^2\sin^2\phi)^{1/2} \qquad (5\text{-}5)$$

$$g = \sin\chi_o \sin\chi + \cos\chi_o \cos\chi \cos(\lambda - \lambda_o) \qquad (5\text{-}6)$$

$$s = 2/(1 + g) \qquad (5\text{-}7)$$

$$k' = s\cos\chi/m \qquad (5\text{-}8)$$

where k' is the scale factor on the base Stereographic map, and χ is the conformal latitude for ϕ. A constant factor included in other references (Thomas, 1952, p. 139; Snyder, 1982, p. 162) to produce a central scale factor of 1 is omitted, since this is adjusted by the complex coefficients.

$$\rho = 2a(s - 1)^{1/2} \qquad (5\text{-}9)$$

$$\Theta = \arctan_2[(\cos\chi_o \sin\chi - \sin\chi_o \cos\chi \cos(\lambda - \lambda_o))/$$
$$(\cos\chi \sin(\lambda - \lambda_o))] \qquad (5\text{-}10)$$

$$F_1 = \sum_{j=1}^{n} j\rho^{j-1}[A_j \sin(j - 1)\Theta + B_j \cos(j - 1)\Theta] \qquad (5\text{-}11)$$

$$F_2 = \sum_{j=1}^{n} j\rho^{j-1}[A_j \cos(j - 1)\Theta - B_j \sin(j - 1)\Theta] \qquad (5\text{-}12)$$

$$k = (F_1{}^2 + F_2{}^2)^{1/2}k' \qquad (5\text{-}13)$$

where k is the scale factor on the map using coefficients A_j and B_j,

$$\partial k/\partial A_q = [(k')^2/k]q\rho^{q-1} \sum_{j=1}^{n} j\rho^{j-1}[A_j \cos(q - j)\Theta + B_j \sin(q - j)\Theta] \qquad (5\text{-}14)$$

$$\partial k/\partial B_q = [(k')^2/k]q\rho^{q-1} \sum_{j=1}^{n} j\rho^{j-1}[-A_j \sin (q - j)\Theta + B_j \cos (q - j)\Theta] \qquad (5\text{-}15)$$

where q is an integer identifying the particular coefficient which is being optimized by the differentiation. Equations (5-14) and (5-15) are separately calculated for each integer value of q from 1 to n in (5-14), and 2 to n in (5-15). It should be noted that (q - j) may be zero or negative at times, but the term is merely handled algebraically.

(III) Using the values calculated in equations (5-13) and (5-14) for each of the m points, and letting p identify the particular point, the following function is calculated separately for each value of q from 1 to n,

$$f(A_q) = 2 \sum_{p=1}^{m} (k_p - 1)\partial k_p/\partial A_q \qquad (5\text{-}16)$$

Likewise, using the results of equations (5-13) and (5-14), the following function is calculated separately for each value of q from 2 to n,

$$f(B_q) = 2 \sum_{p=1}^{m} (k_p - 1)\partial k_p/\partial B_q \qquad (5\text{-}17)$$

These two functions will be used in the iterative simultaneous equations (5-24) and (5-25). In addition, (5-24) and (5-25) also require the calculation of the derivatives of the above functions with respect to the various coefficient. This is done just below.

(IV) Calculated separately for each value of q from 1 to n as well as each value of l from 1 to n,

$$P = (1 - 1/k_p)(k')^2 qg\rho^{q+g-2} \qquad (5\text{-}18)$$

$$\partial f(A_q)/\partial A_g = 2 \sum_{p=1}^{m} [(\partial k_p/\partial A_g)(\partial k_p/\partial A_q)/k_p + P \cos (q - g)\Theta] \qquad (5\text{-}19)$$

Calculated separately for each value of q from 1 to n as well as each value of l from 2 to n,

$$\partial f(A_q)/\partial B_g = \partial f(B_g)/\partial A_q$$

$$= 2 \sum_{p=1}^{m} [(\partial k_p/\partial A_q)(\partial k_p/\partial B_g)/k_p + P \sin (q - g)\Theta] \quad (5\text{-}20)$$

Calculated separately for each value of q from 2 to n as well as each value of 1 from 2 to n,

$$\partial f(B_q)/B_g = 2 \sum_{p=1}^{m} [(\partial k_p/\partial B_q)(\partial k_p/\partial B_g)/k_p + P \cos (q - g)\Theta] \quad (5\text{-}21)$$

Actually,

$$\partial f(A_q)/\partial A_g = \partial f(A_g)/\partial A_q \quad (5\text{-}22)$$

and

$$\partial f(B_q)/\partial B_g = \partial f(B_g)/\partial B_q \quad (5\text{-}23)$$

so that some computations in (5-19) and (5-21) duplicate others and, depending on the programming, may be convenient to eliminate. In addition, second derivatives may be eliminated here, so that in equations (5-19) through (5-21), the terms with P and the division of the first term by k_p may all be eliminated.

(V) Using values calculated from equations (5-16) through (5-21), the following (2n - 1) simultaneous equations are solved by a Newton-Raphson iteration for ΔA_g and ΔB_g, which total (2n - 1) unknowns. The first n equations take the form

$$\sum_{g=1}^{n} (\partial f(A_q)/\partial A_g)\Delta A_g + \sum_{g=2}^{n} (\partial f(A_q)/\partial B_g)\Delta B_g = - f(A_q) \quad (5\text{-}24)$$

where q is given values of 1 to n for successive equations. The last (n - 1) equations take the form

$$\sum_{g=1}^{n} (\partial f(B_q)/\partial A_g)\Delta A_g + \sum_{g=2}^{n} (\partial f(B_q)/\partial B_g)\Delta B_g = - f(B_q) \quad (5\text{-}25)$$

where q is given values of 2 to n for successive equations. The matrix of coefficients of the Δ's is symmetric. The values of ΔA_1 to ΔA_n and ΔB_2 to ΔB_n found from these simultaneous equations (for which the solution is relatively standard and is not given here) are added to the previous trial values of coefficients, and the new coefficients are used in a repetition of equations (5-11) through (5-25), iterating until corrections are negligible.

The forward formulas for the GS50 or a similar projection

Once the coefficients are determined, they may be used to obtain x, y, and k for any points using, in order, equation (5-3) once for the map, then (5-4) through (5-8), followed by these equations:

$$x' = as \cos \chi \sin (\lambda - \lambda_0) \tag{5-26}$$

$$y' = as[\cos \chi_0 \sin \chi - \sin \chi_0 \cos \chi \cos (\lambda - \lambda_0)] \tag{5-27}$$

in which symbols are defined or determined just before equation (5-3) or by equations (5-3) through (5-8). Corrdinates x' and y' are converted to x and y with equations (4-65), and scale factor k may be determined as follows:

$$k = \left| \sum_{j=1}^{n} j(A_j + iB_j)(x' + iy')^{j-1} \right| k' \tag{5-28}$$

where the bars indicate the absolute value of the term enclosed, or $[F_2^2 + F_1^2]^{1/2}$, where $(F_2 + iF_1)$ equals the term enclosed.

If k is not desired, equations (5-8) and (5-28) may be omitted. Equations (4-65) and (5-28) are solved more efficiently by nesting rather than as a normal series, and even faster by application of the algorithm by Knuth (1969) which was previously described in an analogous application following equations (2-5) and (2-10). Here the algorithm is extended to include calculations of scale factor k:

Let

$$r = 2x';\ s' = (x')^2 + (y')^2;\ g_o = 0;\ g_f = A_f + iB_f;\ a_1 = g_n;$$

$$b_1 = g_{n-1};\ c_1 = ng_n;\ d_1 = (n - 1)g_{n-1};\ a_j = b_{j-1} + ra_{j-1};$$

$$b_j = g_{n-j} - s'a_{j-1};\ c_j = d_{j-1} + rc_{j-1};\ d_j = (n - j)g_{n-j} - s'c_{j-1}.$$

After j is given the value of successive integers from 2 to n for a_j and b_j and 2 to (n - 1) for c_j and d_j, then $x + iy = (x' + iy')\, a_n + b_n$; $F_2 + iF_1 = (x' + iy')\, c_{n-1} + d_{n-1}$, finally using equation (5-13) to find k.

<u>The inverse formulas</u>

For computing latitude ϕ and longitude λ from rectangular coordinates x and y, complex equation (4-65) may be inverted. While reversion of the series may be employed, the failure of the coefficients A_j and B_j to converge indicates a lengthy inverse series. An alternate solution is a standard Newton-Raphson iteration applied to complex variables:

$$\Delta(x' + iy') = -f(x' + iy')/f'(x' + iy') \tag{5-29}$$

where

$$f(x' + iy') = \sum_{j=1}^{n} (A_j + iB_j)(x' + iy')^j - (x + iy)/a \tag{5-30}$$

$$f'(x' + iy') = df(x' + iy')/d(x' + iy')$$

$$= \sum_{j=1}^{n} j(A_j + iB_j)(x' + iy')^{j-1} \tag{5-31}$$

using as the first trial $x' = x/a$ and $y' = y/a$.

The Knuth algorithm as used in forward computation may be directly applied to these similar equations to facilitate the iteration. Convergence is sufficient after three or four iterations to provide a final x' and y'.

Given ϕ_o, λ_o, a, n, A_1 to A_n, and B_1 to B_n ($B_1 = 0$), the values of ϕ and λ may be found from the final x' and y' noniteratively by the following inverses of equations (5-4) through (5-8), (5-26) and (5-27):

$$\rho = [(x')^2 + (y')^2]^{1/2} \tag{5-32}$$

$$z = 2 \arctan (\rho/2a) \tag{5-33}$$

$$\chi = \arcsin [\cos z \sin \chi_o + (y' \sin z \cos \chi_o/\rho)] \tag{5-34}$$

$$\phi = 2 \arctan \{\tan (\pi/4 + \chi/2)[(1 + e \sin \phi)/(1 - e \sin \phi)]^{e/2}\} - \pi/2 \tag{5-35}$$

$$\lambda = \lambda_o + \arctan_2 [x' \sin z/(\rho \cos \chi_o \cos z - y' \sin \chi_o \sin z)] \tag{5-36}$$

If $\rho = 0$, equations (5-34) and (5-36) are indeterminate, but $\chi = \chi_o$ and $\lambda = \lambda_o$. Equation (5-3) is used to find χ_o. Equation (5-35) involves iteration by successive substitution, using χ as the first trial ϕ on the right side of the equation, calculating ϕ on the left side, using the new value for ϕ on the right side, and so forth, until the change in ϕ is negligible.

Parameters for the GS50 Projection

For the 50-State map with ±2 percent maximum scale error in desired regions as described above, the parameters are shown below. To avoid the folding of the graticule and double plotting, if a computer is used for plotting, the plotted region should be limited to longitude 165°E. on the west, longitude 55°W. on the east, and latitudes 80° and 15°N. The rectangular coordinates should be limited to x = ±0.9a, and y = +0.7a, -0.5a with final cropping in some corner areas.

For the sphere:	For the Clarke 1866 ellipsoid:
n = 10	n = 10
ϕ_o = 45° N.lat.	ϕ_o = 45° N.lat.
λ_o = 120° W.long.	λ_o = 120° W.long.
A_1 = 0.9842990	A_1 = 0.9827497
A_2 = 0.0211642	A_2 = 0.0210669
A_3 = -0.1036018	A_3 = -0.1031415
A_4 = -0.0329095	A_4 = -0.0323337
A_5 = 0.0499471	A_5 = 0.0502303
A_6 = 0.0260460	A_6 = 0.0251805
A_7 = 0.0007388	A_7 = -0.0012315
A_8 = 0.0075848	A_8 = 0.0072202
A_9 = -0.0216473	A_9 = -0.0194029
A_{10} = -0.0225161	A_{10} = -0.0210072
B_1 = 0	B_1 = 0
B_2 = 0.0037608	B_2 = 0.0053804
B_3 = -0.0575102	B_3 = -0.0571664
B_4 = -0.0320119	B_4 = -0.0322847
B_5 = 0.1223335	B_5 = 0.1211983
B_6 = 0.0899805	B_6 = 0.0895678
B_7 = -0.1435792	B_7 = -0.1416121
B_8 = -0.1334108	B_8 = -0.1317091
B_9 = 0.0776645	B_9 = 0.0759677
B_{10} = 0.0853673	B_{10} = 0.0834037

The opposite sign for the two values of A_7 is correct. For the Clarke 1866 ellipsoid, e^2 = 0.006768658, a = 6378206.4 m. The resulting rectangular coordinates and scale factors are presented for a 15° graticule in table 5.

b. OTHER CONFORMAL TRANSFORMATIONS

The availability of the computer to perform repeated iterations of simultaneous equations may be applied to the selection of minimum-error parameters for conventional conformal projections, specifically the Transverse Mercator, Oblique Mercator, oblique Stereographic, and oblique conformal conic. The same principles may also be applied to many other projections. The following computations are designed to obtain, in the case of a conformal projection, the minimum value over-all of E, where

$$E = \sum_{j=1}^{m} P_j(k_j - 1)^2 \qquad (5\text{-}40)$$

with k_j as the scale factor at each of the m given points, and P_j as the weight assigned. This is slightly different from the natural logarithm (ln) function used by Tsinger (equation (4-32)), but it is more rigorous and does not make solution in these cases more difficult, as it would for the Tsinger formulas. There are two to four independent parameters involved in the four projections mentioned above. Each of them may be varied to obtain a minimum E.

For the Transverse Mercator, the longitude λ_o of the central meridian and the scale factor k_o along the central meridian may be varied. For the Oblique Mercator, latitude ϕ_p and longitude λ_p of the transformed pole of the projection, and scale factor k_o along the transformed equator or central line may be varied. For the oblique Stereographic, latitude ϕ_p and longitude λ_p of the projection center (the same as the transformed pole) and k_o at the projection center are suitable parameters. For the oblique conformal conic, the parameters chosen are ϕ_p and λ_p for the transformed pole, n for the cone constant, and k_o for the scale factor along the transformed parallel of latitude which would be tangent to the sphere for a tangent cone. In the latter case, the adjustment of k_o is equivalent to providing two standard transformed parallels, and is used partially because it is more consistent with the parameters used for the other projections. In addition, iterating the standard transformed parallels was not found to be practical.

The first three projections are limiting cases of the fourth, in that an oblique conformal conic with n=1 is an oblique Stereographic, but if n=0 the Oblique Mercator is obtained, and if n=0 and ϕ_p=0, the Transverse Mercator is formed. Therefore, equations may be derived for the

Table 5. Sample rectangular

[Note: y coordinate in parentheses below x coordinate; k
a = 1 unit; eccentricity is

Long. / Lat.	165°	180°	-165°	-150°	-135°
75°	-0.29450	-0.26954	-0.22462	-0.16629	-0.09888
	(0.68122)	(0.62252)	(0.57777)	(0.54832)	(0.53514)
	0.96940	0.93350	0.94680	0.98600	1.04351
60	-0.56708	-0.47652	-0.37432	-0.25945	-0.13450
	(0.55579)	(0.44931)	(0.36467)	(0.30448)	(0.26964)
	1.11056	1.03320	0.99720	0.98684	0.99638
45	-0.78438	-0.65970	-0.51358	-0.35313	-0.18060
	(0.40816)	(0.25882)	(0.14804)	(0.06723)	(0.01707)
	1.10999	1.01071	0.97599	0.96761	0.97461
30	-0.99437	-0.82970	-0.64556	-0.44699	-0.23176
	(0.18093)	(0.05909)	(-0.06996)	(-0.16831)	(-0.23587)
	0.92437	0.99489	0.98110	0.97955	1.01526
15	-1.26654	-0.99879	-0.77655	-0.54614	-0.30348
	(0.37724)	(-0.17525)	(-0.29355)	(-0.40445)	(-0.50718)
	5.35283	1.26758	1.02533	0.96750	1.17269

general case and simplified as needed for the limiting cases, if necessary resolving a few indeterminate equations. To determine the least-squares solution based on equation (5-40), only the projection equations leading to k are necessary.

Following the derivations, specific examples of the use of these minimum-error projections as applied to maps of North and South America and Alaska are given.

(1) Minimum-Error Oblique Conformal Conic Projection

Since the oblique conformal conic projection is the most general of the four projections discussed above, the derivations will first be applied to this projection, for which scale factor k may be expressed thus:

$$k = k_o \sin z_1 \tan^n(z/2)/\sin z \tan^n(z_1/2) \tag{5-41}$$

where

$$z_1 = \arccos n \tag{5-42}$$

$$z = \arccos [\sin \phi_p \sin \phi + \cos \phi_p \cos \phi \cos (\lambda - \lambda_p)] \tag{5-43}$$

coordinates for GS50 projection.

(scale factor) underlined. Equatorial radius of ellipsoid, based on Clarke 1866 ellipsoid.]

-120°	-105°	-90°	-75°	-60°	-45°
-0.02577	0.05019	0.12669	0.20199	0.27474	0.34149
(0.53917)	(0.56135)	(0.60290)	(0.66601)	(0.75568)	(0.88349)
1.11926	1.21874	1.35483	1.55468	1.87521	2.42649
-0.00285	0.13189	0.26642	0.39828	0.52663	0.66111
(0.26061)	(0.27713)	(0.31778)	(0.37908)	(0.45182)	(0.50581)
1.01989	1.05301	1.09134	1.12313	1.11492	1.17353
0.00000	0.18215	0.35975	0.52792	0.68068	0.78758
(0.00000)	(0.01665)	(0.06457)	(0.14091)	(0.24688)	(0.42634)
0.98441	0.99055	0.99558	0.99806	1.02418	1.44787
-0.00042	0.22878	0.44683	0.65324	0.83776	1.04409
(-0.26080)	(-0.23806)	(-0.17878)	(-0.08678)	(0.03834)	(0.00223)
1.02960	1.00249	0.99481	1.00384	0.87806	2.72764
0.00686	0.28360	0.53662	0.76638	1.12680	0.56142
(-0.54997)	(-0.49621)	(-0.43117)	(-0.31713)	(0.21682)	(1.25008)
1.24078	1.00240	1.10194	0.84755	2.88781	16.99865

and (ϕ_p, λ_p) are the latitude and longitude of the transformed pole, while (ϕ, λ) are the latitude and longitude of the point on the map. The constants k_o and n are defined in the preceding paragraphs.

To minimize E as defined in equation (5-40), this equation is differentiated with respect to each of the four parameters ϕ_p, λ_p, n, and k_o, using Σ in place of Σ with limits.

$$f(\phi_p) = \partial E/\partial \phi_p = 2\Sigma P_j(k_j - 1)\partial k_j/\partial \phi_p \tag{5-44}$$

$$f(\lambda_p) = \partial E/\partial \lambda_p = 2\Sigma P_j(k_j - 1)\partial k_j/\partial \lambda_p \tag{5-45}$$

$$f(n) = \partial E/\partial n = 2\Sigma P_j(k_j - 1)\partial k_j/\partial n \tag{5-46}$$

$$f(k_o) = \partial E/\partial k_o = 2\Sigma P_j(k_j - 1)\partial k_j/\partial k_o \tag{5-47}$$

To differentiate equation (5-41) for substitutions in equations (5-44) through (5-47), it is easier to rewrite it in terms of logarithms:

$$\ln k = \ln k_o + \ln \sin z_1 + n \ln \tan (z/2) - \ln \sin z - n \ln \tan (z_1/2) \tag{5-48}$$

The differential is as follows:

$$(dk)/k = (dk_o)/k_o + (\cos z_1/\sin z_1)dz_1 + [\ln \tan (z/2)]dn$$
$$+ (n/\sin z)dz - (\cos z/\sin z)dz - [\ln \tan (z_1/2)]dn$$
$$- (n/\sin z_1)dz_1 \quad (5\text{-}49)$$

Assembling terms for the respective partial derivatives, since z_1 is a function of n, and z is a function of ϕ_p and λ_p,

$$\partial k/\partial\phi_p = k[(n - \cos z)/\sin z]\partial z/\partial\phi_p \quad (5\text{-}50)$$

$$\partial k/\partial\lambda_p = k[(n - \cos z)/\sin z]\partial z/\partial\lambda_p \quad (5\text{-}51)$$

$$\partial k/\partial n = k \ln [\tan (z/2)/\tan (z_1/2)] + k[(n - \cos z_1)/\sin z_1]\partial z_1/\partial n$$
$$= k \ln [\tan (z/2)/\tan (z_1/2)] \quad (5\text{-}52)$$

since $\cos z_1 = n$ from equation (5-42),

$$\partial k/\partial k_o = k/k_o \quad (5\text{-}53)$$

For the additional partial derivatives required for equations (5-50) and (5-51), using equation (5-43),

$$\partial z/\partial\phi_p = - [\cos \phi_p \sin \phi - \sin \phi_p \cos \phi \cos (\lambda - \lambda_p)]/\sin z \quad (5\text{-}54)$$

$$\partial z/\partial\lambda_p = - \cos \phi_p \cos \phi \sin (\lambda - \lambda_p)/\sin z \quad (5\text{-}55)$$

To find parameters providing minimum E, equations (5-44) through (5-47) are iterated until each is sufficiently close to zero, using a Newton-Raphson iteration. For this, each of these four non-linear equations must be differentiated with respect to all four parameters, although several differentiations result in duplicate expressions, since

$$\partial f(\phi_p)/\partial\lambda_p = \partial^2E/\partial\phi_p\partial\lambda_p = \partial^2E/\partial\lambda_p\partial\phi_p = \partial f(\lambda_p)/\partial\phi_p \quad (5\text{-}56)$$

and so forth for others. There are two patterns of differentiation; for example,

$$\partial f(\phi_p)/\partial\phi_p = 2\ \Sigma P_j\ [(\partial k_j/\partial\phi_p)^2 + (k_j - 1)\partial^2k_j/\partial\phi_p^2] \quad (5\text{-}57)$$

$$\partial f(\phi_p)/\partial \lambda_p = 2\ \Sigma P_j\ [(\partial k_j/\partial \phi_p)(\partial k_j/\partial \lambda_p) + (k_j - 1)\partial^2 k_j/\partial \phi_p \partial \lambda_p] \qquad (5\text{-}58)$$

For simplicity of derivations and programming, the second derivatives of k_j may be ignored, since they are small, especially after multiplication by $(k_j - 1)$, when compared with the remainder of equations (5-57) and (5-58). After such simplification,

$$\partial f(\phi_p)/\partial \phi_p = 2\ \Sigma P_j (\partial k_j/\partial \phi_p)^2 \qquad (5\text{-}59)$$

$$\partial f(\phi_p)/\partial \lambda_p = \partial f(\lambda_p)/\partial \phi_p = 2\ \Sigma P_j (\partial k_j/\partial \phi_p)(\partial k_j/\partial \lambda_p) \qquad (5\text{-}60)$$

$$\partial f(\phi_p)/\partial n = \partial f(n)/\partial \phi_p = 2\ \Sigma P_j (\partial k_j/\partial \phi_p)(\partial k_j/\partial n) \qquad (5\text{-}61)$$

$$\partial f(\phi_p)/\partial k_o = \partial f(k_o)/\partial \phi_p = 2\ \Sigma P_j (\partial k_j/\partial \phi_p)(\partial k_j/\partial k_o) \qquad (5\text{-}62)$$

$$\partial f(\lambda_p)/\partial \lambda_p = 2\ \Sigma P_j (\partial k_j/\partial \lambda_p)^2 \qquad (5\text{-}63)$$

$$\partial f(\lambda_p)/\partial n = \partial f(n)/\partial \lambda_p = 2\ \Sigma P_j (\partial k_j/\partial \lambda_p)(\partial k_j/\partial n) \qquad (5\text{-}64)$$

and so forth. The iterations are performed by solving the following four simultaneous equations for the Δ values, or the increments in parameters, and then recalculating all terms using the new parameters, until all four changes are negligible:

$$\begin{bmatrix} \partial f(\phi_p)/\partial \phi_p & \partial f(\phi_p)/\partial \lambda_p & \partial f(\phi_p)/\partial n & \partial f(\phi_p)/\partial k_o \\ \partial f(\lambda_p)/\partial \phi_p & \partial f(\lambda_p)/\partial \lambda_p & \partial f(\lambda_p)/\partial n & \partial f(\lambda_p)/\partial k_o \\ \partial f(n)/\partial \phi_p & \partial f(n)/\partial \lambda_p & \partial f(n)/\partial n & \partial f(n)/\partial k_o \\ \partial f(k_o)/\partial \phi_p & \partial f(k_o)/\partial \lambda_p & \partial f(k_o)/\partial n & \partial f(k_o)/\partial k_o \end{bmatrix} \bullet \begin{bmatrix} \Delta\phi_p \\ \Delta\lambda_p \\ \Delta n \\ \Delta k_o \end{bmatrix} = \begin{bmatrix} -f(\phi_p) \\ -f(\lambda_p) \\ -f(n) \\ -f(k_o) \end{bmatrix} \qquad (5\text{-}65)$$

The solution involves standard matrix algebra, with multiplication of the inverse of the 4 x 4 matrix times the matrix to the right of the equal sign. It is necessary to start with appropriate initial estimates for each of the four parameters, and experience shows that the estimates do not always lead to convergence.

Of obvious importance is the selection of the m given points. The region represented by each point must be small enough to allow a reasonably accurate application of the least-squares principle to the entire region under consideration. The regional elements cannot be strips of land, such as those used in Tsinger's and Kavrayskiy's studies, since the parameter adjustments cause changes in the lines of constant scale.

To use the computed parameters for construction of the oblique conformal conic projection of the region of interest, the two transformed standard parallels may be of interest even though they are not needed in the calculation of rectangular coordinates, given n and k_o. Since the polar distances of these two parallels are the two values of z (polar distance) resulting in a k of 1.0 in equation (5-41), they may be found by rearranging that equation:

$$\sin z/\tan^n(z/2) = k_o \sin z_1/\tan^n(z_1/2) \tag{5-66}$$

By substituting from equation (5-42), this may also be written

$$\sin z/\tan^n(z/2) = k_o[(1 - n)^{(1-n)}(1 + n)^{(1+n)}]^{1/2} \tag{5-67}$$

In either case, the right side of the equation is a constant known from the parameters determined, and z is found from the left side by iteration. Letting

$$K = k_o[(1 - n)^{(1-n)}(1 + n)^{(1+n)}]^{1/2} \tag{5-68}$$

equation (5-67) is adapted to use a Newton-Raphson iteration:

$$f(z) = K - \sin z/\tan^n(z/2) \tag{5-69}$$

and

$$df(z)/dz = (n - \cos z)/\tan^n(z/2) \tag{5-70}$$

Then,

$$\Delta z = - f(z)/[df(z)/dz]$$

$$= - [K \tan^n(z/2) - \sin z]/(n - \cos z) \tag{5-71}$$

For the initial estimates of z, two angles respectively near 90° and near (but not at) 0° are suitable if n is positive, or with reversed signs if n is negative.

To determine rectangular coordinates using parameters as calculated, equation (5-43) may be used with the following formulas which are presented without derivation:

$$\lambda' = \arctan_2\{\cos\phi \sin(\lambda - \lambda_p)/[\cos\phi_p \sin\phi - \sin\phi_p \cos\phi \cos(\lambda - \lambda_p)]\} \tag{5-72}$$

$$\rho = R\, k_o \sin z_1 \tan^n(z/2)/(n \tan^n(z_1/2)) \quad (5\text{-}73)$$

$$\Theta = n\,\lambda' \quad (5\text{-}74)$$

$$x = -\rho \sin \Theta \quad (5\text{-}75)$$

$$y = -\rho \cos \Theta \quad (5\text{-}76)$$

where the X axis passes through the transformed pole perpendicular to the Y axis, which follows the meridian between the geographic pole and the transformed pole increasing southerly. The radius of the sphere at map scale is R, and (ρ,Θ) are polar coordinates; λ' is a transformed longitude, while $(90° - z)$ is a transformed latitude with respect to the transformed pole.

It may be desired to place the transformed pole at a latitude and longitude which are rounded off to the nearest 1°, 5°, or 10°. If so, n and k_o as calculated for the minimum-error pole will not produce quite as low an error as possible for the rounded-off pole. The need for an additional adjustment of n and k_o may be questioned, since the difference in root-mean-square error will usually be negligible, and is also a function of how well the m points were chosen and weighted in the first place. The values of n and k_o may be readjusted to the new pole, however, by a simple adaptation of Tsinger's non-iterative approach. Using equations (4-31) and (4-32) instead of (5-40), and adapting the format of equations (4-36) through (4-46) to equation (5-48), let

$$a = \ln k_o + \ln \sin z_1 - n \ln \tan (z_1/2) \quad (5\text{-}77)$$

$$b = \ln \tan (z/2) \quad (5\text{-}78)$$

$$c = \ln \sin z \quad (5\text{-}79)$$

where a is <u>not</u> a radius of the Earth here. Then equation (5-48) may be written

$$\ln k = a + nb - c \quad (5\text{-}80)$$

If (5-40) is changed as follows,

$$E = \Sigma P \ln^2 k \quad (5\text{-}81)$$

then

$$\partial E/\partial a = 2\ \Sigma P(a + nb - c) = 0 \tag{5-82}$$

$$\partial E/\partial n = 2\ \Sigma P(a + nb - c)b = 0 \tag{5-83}$$

Separating and transposing,

$$a\ \Sigma P + n\ \Sigma Pb = \Sigma Pc \tag{5-84}$$

$$a\ \Sigma Pb + n\ \Sigma Pb^2 = \Sigma Pbc \tag{5-85}$$

These equations can be readily solved for unknowns a and n, in the manner of equations (4-43) through (4-46). From (5-42) and (5-77), k_o can be found from a and n.

(2) Minimum-Error Oblique Mercator Projection

To apply the above formulas to the Oblique Mercator, n is made zero. In these equations no indeterminates result, but equation (5-65) reduces to three simultaneous equations in three unknowns ($\Delta\phi_p$, $\Delta\lambda_p$, k_o), after every term showing an n is omitted).

Equations (5-41), (5-50), and (5-51) simplify as follows, since z_1 = 90° from equation (5-42),

$$k = k_o/\sin z \tag{5-86}$$

$$\partial k/\partial\phi_p = -(k/\tan z)\ \partial z/\partial\phi_p \tag{5-87}$$

$$\partial k/\partial\lambda_p = -(k/\tan z)\ \partial z/\partial\lambda_p \tag{5-88}$$

Equations (5-43), and (5-53) through (5-55) are unchanged.

For rectangular coordinates of the final projection,

$$x = Rk_o\ \arctan_2\{[\tan\phi\cos\phi_p - \sin\phi_p\cos(\lambda - \lambda_p)]/[\sin(\lambda - \lambda_p)]\} \tag{5-89}$$

$$y = (1/2)Rk_o\ \ln[(1 + \cos z)/(1 - \cos z)] \tag{5-90}$$

Where the origin of the (x,y) axes occurs at $\phi = 0$ and $\lambda = \lambda_p + 90°$, with the x axis lying along the central line (z = 90°), x increasing easterly.

If the location of the pole is rounded off, k_o may be readjusted by using equations (5-77) through (5-85), but with n = 0. Thus, equations

(5-77), (5-80), and (5-82) become as follows, with equations (5-78) and (5-83) through (5-85) omitted:

$$a = \ln k_o \tag{5-91}$$

$$\ln k = a - c \tag{5-92}$$

$$a = \Sigma Pc/\Sigma P \tag{5-93}$$

where P is the weighting, and c is found from (5-79) and (5-43).

(3) Minimum-Error Oblique Stereographic Projection

To apply the oblique conformal conic projection to the oblique Stereographic, the cone constant n is set equal to 1. From equation (5-42), $z_1 = 0$, and equation (5-41) is indeterminate, but, using Stereographic projection formulas, (5-41) may be written as follows:

$$k = 2\,k_o/(1 + \cos z) \tag{5-94}$$

The rectangular coordinates may be found from the parameters as follows:

$$x = Rk \cos \phi \sin (\lambda - \lambda_p) \tag{5-95}$$

$$y = Rk\,[\cos \phi_p \sin \phi - \sin \phi_p \cos \phi \cos (\lambda - \lambda_p)] \tag{5-96}$$

where z is found from equation (5-43) and k from (5-94). The origin of the (x,y) axes is now at (ϕ_p,λ_p), and the y axis lies along the central meridian λ_p, with y increasing northerly. Equations (5-50) and (5-51) become the following:

$$\begin{aligned} \partial k/\partial\phi_p &= [k(1 - \cos z)/\sin z]\partial z/\partial\phi_p \\ &= -ky/(2Rk_o) \end{aligned} \tag{5-97}$$

$$\begin{aligned} \partial k/\partial\lambda_p &= [k(1 - \cos z)/\sin z]\partial z/\partial\lambda_p \\ &= -kx \cos \phi_p/(2Rk_o) \end{aligned} \tag{5-98}$$

Equations (5-43) and (5-53) through (5-55) are unchanged, and equation (5-65) becomes the same set of three simultaneous equations that it did for the Oblique Mercator. Now, however, (ϕ_p,λ_p) not only represent the transformed pole; they are also the center of the projection.

If the location of the pole is rounded off, k_o may be readjusted using equation (5-94) and the procedure in equations (5-77) through (5-85), with the following result:

$$\ln 2k_o = [\Sigma P \ln (1 + \cos z)]/\Sigma P \quad (5\text{-}99)$$

(4) Minimum-Error Transverse Mercator Projection

If the cone constant n is set equal to zero and the pole of the projection is placed on the Equator ($\phi_p = 0$), equations (5-86) of the Oblique Mercator and (5-53) apply without change. Normally parameters are described in terms of central meridian λ_o, rather than longitude λ_p of the pole, but $\lambda_o = \lambda_p \pm 90°$. Therefore $d\lambda_o = d\lambda_p$, so (5-88) may be rewritten with this substitution. Equations (5-43) and (5-55) then simplify, using -90° instead of ±90°, to

$$z = \arccos [\cos \phi \sin (\lambda - \lambda_o)] \quad (5\text{-}101)$$

$$\partial z/\partial \lambda_o = \cos \phi \cos (\lambda - \lambda_o)/\sin z \quad (5\text{-}102)$$

Equation (5-65) reduces to

$$\begin{bmatrix} \partial f(\lambda_o)/\partial \lambda_o & \partial f(\lambda_o)/\partial k_o \\ \partial f(k_o)/\partial \lambda_o & \partial f(k_o)/\partial k_o \end{bmatrix} \bullet \begin{bmatrix} \Delta\lambda_o \\ \Delta k_o \end{bmatrix} = \begin{bmatrix} -f(\lambda_o) \\ -f(k_o) \end{bmatrix} \quad (5\text{-}103)$$

for which solution by determinants is most straightforward:

$$\Delta\lambda_o = [f(k_o)\, \partial f(\lambda_o)/\partial k_o - f(\lambda_o)\, \partial f(k_o)/\partial k_o]/D \quad (5\text{-}104)$$

$$\Delta k_o = [f(\lambda_o)\, \partial f(\lambda_o)/\partial k_o - f(k_o)\, \partial f(\lambda_o)/\partial \lambda_o]/D \quad (5\text{-}105)$$

where

$$D = [\partial f(\lambda_o)/\partial \lambda_o][\partial f(k_o)/\partial k_o] - [\partial f(\lambda_o)/\partial k_o]^2 \quad (5\text{-}106)$$

For rectangular coordinates,

$$x = -Rk_o \ln \tan (z/2) \quad (5\text{-}107)$$

$$y = Rk_o \arctan_2 [\tan \phi/\cos (\lambda - \lambda_o)] \quad (5\text{-}108)$$

The origin of (x,y) axes is at $\phi = 0$ and $\lambda = \lambda_o$; the y axis coincides with the central meridian λ_o, y increasing northerly.

If λ_o is rounded off, k_o may be readjusted using equations (5-91), (5-93), (5-79), and (5-101).

For any of these projections, the RMSE r may be found thus:

$$r = [\Sigma P_j(k_j - 1)^2/\Sigma P_j]^{1/2} \tag{5-109}$$

(5) Minimum-Error Standard Conformal Map Projections for North and South America

The above computations were applied to map projections for North America, South America, both together, and just Alaska. Considering North America, for example, 162 points were used with weights proportional to the area centered on each point. Nearby coastal waters were included, as well as all water bodies essentially surrounded by North American land, such as Baffin Bay and the Caribbean Sea. The points are so distributed that regions do not exceed 5° x 5° of great-circle distance.

Table 6 lists the parameters, both as originally calculated, and as adjusted for rounded off latitude and (or) longitude of the central meridian or pole. It will be noted that the range of scale factors for North America is slightly higher for the oblique conformal conic than for the Oblique Mercator, even though the RMSE is lower, as it should be because of the greater flexibility of the former. This results from the geographic distribution of points for the region involved. If the computation were designed to find the minimum range of scale factors, the oblique conformal conic would have a lower range than the Oblique Mercator, although the RMSE should be higher.

The values of RMSE quantitatively show that for North America the oblique conformal conic (0.0245) is less distorted in overall scale than the Oblique Mercator (0.0326), which is considerably better than either the oblique Stereographic (0.0456) or Transverse Mercator (0.0487). These conclusions are based on using least squares and a particular pattern of points and weights. The pole for the minimum-error oblique conformal conic falls in mid-continent, and is therefore useless, since a discontinuity occurs (see figures 16 and 17). Starting iteration at a different point, a minimum RMSE of 0.0312 was found with the pole at 11.21° N., 156.42° W., and a cone constant of 0.4277. The various minimum-error projections of North America are shown in figures 14 through 18. The oblique conformal conic projections of figures 16 and 17 are identical, except that the cone is opened in a different orientation. Although the

Table 6.--Minimum-error conformal map projections for North and South America and Alaska

	North America	South America	North and South America	Alaska
No. of points used	162	107	263	177
Transverse Mercator:				
λ_o	-93.21°	-58.53°	-72.14°	-158.94°
k_o	0.9588	0.9743	0.9401	0.99169
RMSE	0.0487	0.0285	0.0694	0.01096
Rounded:				
λ_o	-95°	-60°	-70°	-160°
k_o	0.9617	0.9760	0.9464	0.99182
RMSE	0.0490	0.0291	0.0706	0.01104
Range	0.2503	0.1240	0.3156	0.04634
Oblique Mercator:				
ϕ_p	11.70°	11.68°	19.00°	-26.89°
λ_p	10.57°	29.38°	24.46°	-139.44°
k_o	0.9670	0.9731	0.9670	0.99661
RMSE	0.0326	0.0267	0.0354	0.00318
Rounded:				
ϕ_p	10°	10°	20°	-25°
λ_p	10°	30°	25°	-140°
k_o	0.9680	0.9743	0.9686	0.99554
RMSE	0.0332	0.0268	0.0356	0.00419
Range	0.1233	0.1038	0.1753	0.01501
Oblique Conformal Conic:				
ϕ_p	49.80°	-19.92°	22.48°	39.32°
λ_p	-99.45°	-64.55°	13.71°	-146.73°
n	0.9338	0.9566	0.1449	0.91414
k_o	0.9799	0.9868	0.9664	0.99723
RMSE	0.0245	0.0142	0.0342	0.00280

Table 6.--Minimum-error conformal map projections for North and South America and Alaska (cont'd.)

	North America	South America	North and South America	Alaska
		Oblique Conformal Conic: (Cont'd)		
Rounded:				
ϕ_p	10°[1]	-25°[2]	20°	40°
λ_p	-155°[1]	-75°[2]	15°	-145°
n	0.4241	0.9352	0.1146	0.91723
k_o	0.9695	0.9824	0.9681	0.99714
RMSE	0.0314	0.0201	0.0349	0.00286
Range	0.1339	0.0889	0.1743	0.01025
		Oblique Stereographic:		
ϕ_p	49.54°	-18.83°	16.37°	59.14°
λ_p	-97.63°	-62.92°	-91.40°	-158.66°
k_o	0.9432	0.9621	0.8262	0.99361
RMSE	0.0456	0.0277	0.1278	0.00543
Rounded:				
ϕ_p	50°	-20°	15°	60°
λ_p	-100°	-65°	-90°	-160°
k_o	0.9457	0.9618	0.8470	0.99376
RMSE	0.0459	0.0282	0.1309	0.00549
Range	0.2110	0.1134	0.6339	0.02297
		Conformal Transformation from Oblique Mercator:[3]		
No. of Coefficients:	20	20	20	12
min. k	0.9681	0.9774	0.9662	0.99707
RMSE	0.0170	0.0119	0.0200	0.00171
Range	0.0838	0.0512	0.1033	0.00765

Table 6.--Minimum-error conformal map projections for North and South America and Alaska (cont'd.)

Standard Maps Used for These Regions:

North America

Transverse Mercator:

λ_o	-100°
k_o	0.9260
RMSE	0.0603
Range	0.240

Bipolar Oblique Conic Conformal:

ϕ_p	45°
λ_p	-19.9933°
n	0.63056
k_o	0.9652
RMSE	0.0388
Range	0.177

South America

Bipolar Oblique Conic Conformal:

ϕ_p	-20°
λ_p	-110°
n	0.63056
k_o	0.9652
RMSE	0.0278
Range	0.1022

Alaska

Lambert Conformal Conic:[4]

ϕ_1	55°
ϕ_2	65°
RMSE	0.00539
Range	0.02097

Notes: All these computations are based on the Earth taken as a sphere.

λ_o = central meridian, degrees
ϕ_p = latitude of pole of projection, degrees
λ_p = longitude of pole of projection, degrees
k_o = central scale factor
n = cone constant
(ϕ_1, ϕ_2) = standard parallels
RMSE = root-mean-square error for all points used

Rounded means the pole or central meridian is rounded to nearest 5°, and other parameters are adjusted for minimum RMSE under these conditions.

Range = difference between maximum and minimum scale factors at given points (min. scale factor is k_o).

[1] this is rounded from another iteration leading to a pole at 11.2° latitude and -156.4° longitude, but which has an RMSE of 0.0312. The 49.8° pole has a lower RMSE, but it is on land. Moving the pole to nearby waters led to a 0.040 RMSE.

[2] moved and rounded to take pole off land, otherwise creating discontinuity in map.

[3] Starting with parameters of optimum Oblique Mercator (not rounded off), listed in this table.

[4] Albers Equal-Area Conic normally used with same standard parallels, Lambert Conformal Conic listed for consistency.

RMSE is lower for these than for the other alternates, the interruption is unacceptable in any direction. For other regions with different shapes, the relative preferability of projections changes, except that the oblique conformal conic would have to provide the lowest RMSE of the four,

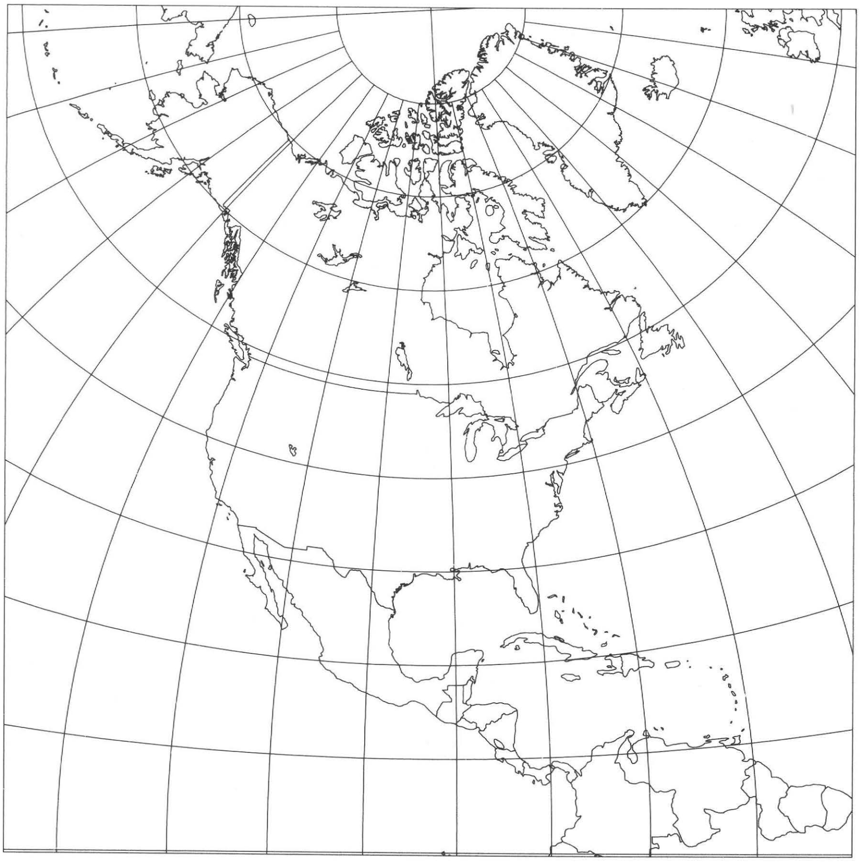

Figure 14.--Minimum-error Transverse Mercator projection of North America, with central meridian at longitude 93.21° W.

because of having more degrees of freedom. If regions were weighted in proportion to population or commercial income, the conclusions could be different for the same region. If conformal transformations with complex polynomial coefficients are used (figure 19) (see preceding and subsequent sections), a much lower RMSE is obtained.

The nine values of RMSE for North America in table 6 may further be compared with those calculated for two existing maps of North America issued by the U.S. Geological Survey. The parameters for these were based upon minimizing the range of scale factor variation and not least squares. For one map the base is the Bipolar Oblique Conic Conformal projection developed by Miller and Briesemeister of the American Geographical Society (Miller, 1941; Snyder, 1982, p. 111-121). The North American portion is almost entirely identical to an oblique conformal conic with transformed pole at latitude 45° N. and longitude 19.9933° W. The only portion varying slightly is in southern Central America and the Caribbean Sea, due to the bipolar concept. Assuming the single cone,

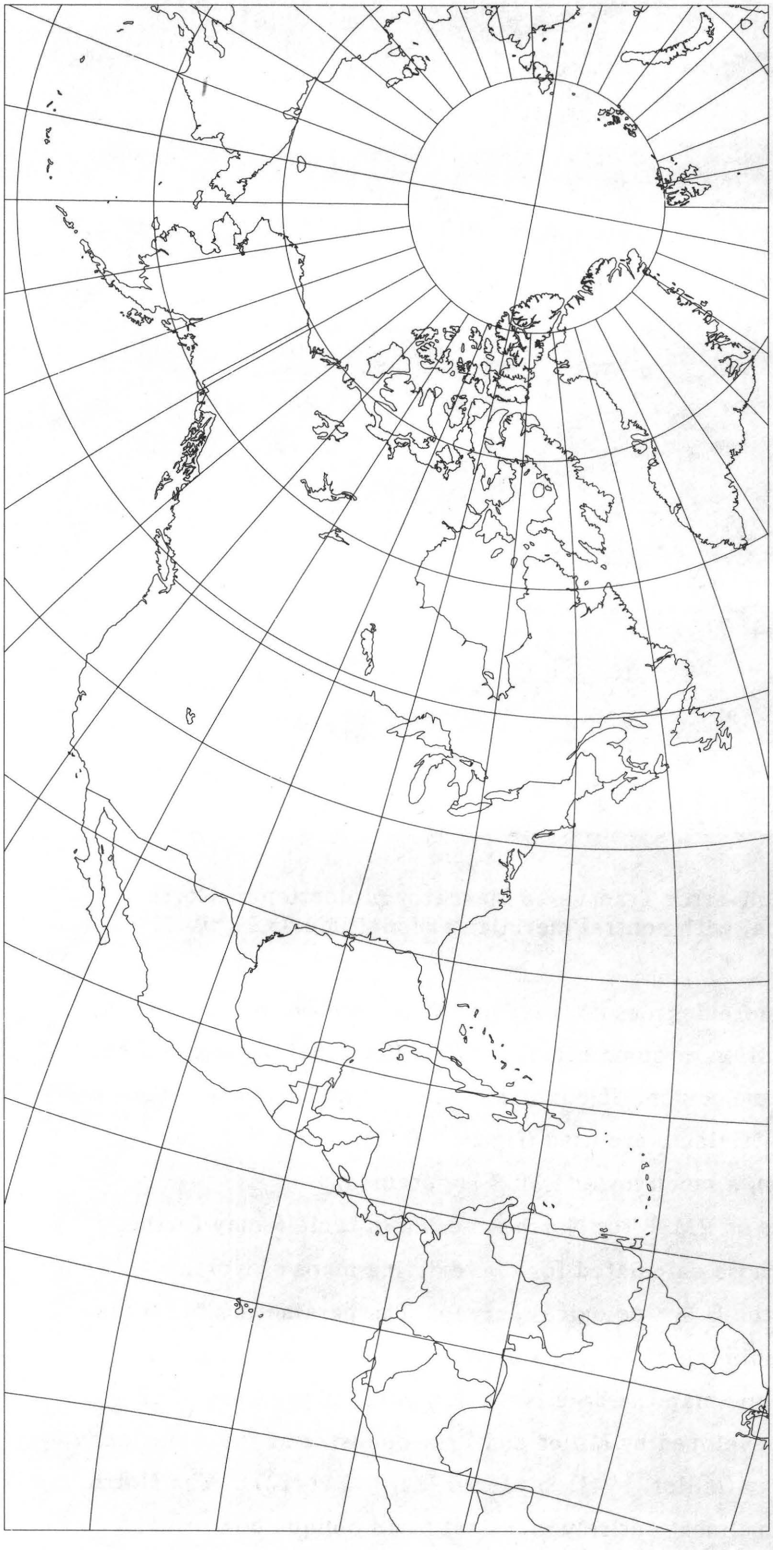

Figure 15.--Minimum-error Oblique Mercator projection of North America. The pole of projection is placed at latitude 11.70° N., longitude 10.57° E.

the RMSE for the same 162 points is 0.0388 as shown. A later USGS map of North America is based on the Transverse Mercator, with a central meridian of longitude 100° W. and central scale factor of 0.926, for which the RMSE is 0.0603.

(6) Minimum-Error Combinations of Standard Conformal Map Projection Bases and Complex Transformations

Another step in the optimization of map projection selection for a given region is to iterate not only the complex coefficients but also the latitude ϕ_p and longitude λ_p of the transformed pole of the oblique conformal conic base projection and the cone constant n simultaneously. It is not sufficient merely to determine the optimum base projection from the foregoing least-squares iteration, because the parameters are somewhat different if a complex transformation of a given degree is also imposed. It will be pointed out later, however, that the following derivations imply more versatility in parameter selection than actually exists, due to the difficulty in getting some combinations to iterate to convergence. Equations (5-3) through (5-36) use an oblique Stereographic base map with a fixed pole, but some of the equations apply to the more general problem.

The general equations (5-40), showing the basis for evaluating E, and (5-44) through (5-46), which differentiate E with respect to ϕ_ρ, λ_ρ, and n, are used here, as well as the following, which differentiate E with respect to the complex coefficients:

$$f(A_q) = \partial E/\partial A_q = 2 \sum_{j=1}^{m} P_j (k_j - 1)\partial k_j/\partial A_q \quad (5\text{-}110)$$

$$f(B_q) = \partial E/\partial B_q = 2 \sum_{j=1}^{m} P_j (k_j - 1)\partial k_j/\partial B_q \quad (5\text{-}111)$$

Equation (5-47), differentiating E with respect to the central scale coefficient k_o, may be omitted, since k_o is directly multiplied by coefficients A_q and B_q, and attempts to adjust it as well would be redundant. Therefore k_o is taken as a constant 1.

The equations for k as a function of the scale factor k' of the base projection and of the complex coefficients are the same as equations (5-11) through (5-13).

Figure 16.--Minimum-error oblique conformal conic projection of North America, cone open toward west. The pole of projection is placed at latitude 49.80° N., longitude 99.45° W., with a cone constant of 0.9338.

To obtain $\partial k_j/\partial\phi_p$ for use in equation (5-44), equation (5-13) is differentiated with respect to ϕ_p:

$$\partial k/\partial\phi_p = (1/2)(F_1^2 + F_2^2)^{-1/2}(2F_1\partial F_1/\partial\phi_p + 2F_2\partial F_2/\partial\phi_p)$$

$$+ (F_1^2 + F_2^2)^{1/2}\,\partial k'/\partial\phi_p$$

$$= [(k')^2/k](F_1\partial F_1/\partial\phi_p + F_2\partial F_2/\partial\phi_p) + (k/k')\partial k'/\partial\phi_p \qquad (5\text{-}112)$$

The value of $\partial k'/\partial\phi_p$ may be found from equations (5-50) for the oblique conformal conic, (5-87) for the Oblique Mercator, and (5-95) for the oblique Stereographic, in all of which k is the same term as k' above. For the first partial derivatives, however,

Figure 17.--Minimum-error oblique conformal conic projection of North America. Same parameters as figure 16, but cone open toward north-northeast. Neither arrangement is acceptable, although the root-mean-square error is lower for figures 16 and 17 than for 14, 15, and 18.

$$\partial F_1/\partial\phi_p = \Sigma j(j-1)\rho^{j-2}[A_j \sin(j-1)\Theta + B_j \cos(j-1)\Theta]$$
$$\partial\rho/\partial\phi_p + \Sigma j(j-1)\rho^{j-1}[A_j \cos(j-1)\Theta$$
$$- B_j \sin(j-1)\Theta]\partial\Theta/\partial\phi_p \qquad (5\text{-}113)$$

For the partial derivatives $\partial\rho/\partial\phi_p$ and $\partial\Theta/\partial\phi_p$ in equation (5-113), equations for ρ and Θ are needed. Although ρ and Θ for the Stereographic are given in (5-9) and (5-10), more general formulas, covering all base maps, are as follows:

$$\rho = [(x' - x_o')^2 + (y' - y_o')^2]^{1/2} \qquad (5\text{-}114)$$

$$\tan\Theta = (y' - y_o')/(x' - x_o') \qquad (5\text{-}114a)$$

Figure 18.--Minimum-error oblique Stereographic projection of North America, with the center of projection at latitude 49.54°N., longitude 97.63°W.

where (x',y') are rectangular coordinates of the base projection, and (x_o', y_o') are coordinates of an arbitrary origin roughly centered among the m points chosen for the least-squares iteration. The location of this origin does not affect the final RMSE of the coefficients, although it affects their values.

Differentiating ρ and Θ with respect to ϕ_p, using only equations (5-114) and (5-114a),

$$\partial\rho/\partial\phi_p = (1/2)[(x' - x_o')^2 + (y' - y_o')^2]^{-1/2}\,[2(x' - x_o')\partial x'/\partial\phi_p + 2(y' - y_o')\partial y'/\partial\phi_p]$$

$$= (\cos\Theta)\partial x'/\partial\phi_p + (\sin\Theta)\partial y'/\partial\phi_p \qquad (5\text{-}115)$$

$$(\sec^2\Theta)\partial\Theta/\partial\phi_p = [(x' - x_o')\partial y'/\partial\phi_p - (y' - y_o')\partial x'/\partial\phi_p]/(x' - x_o')^2$$

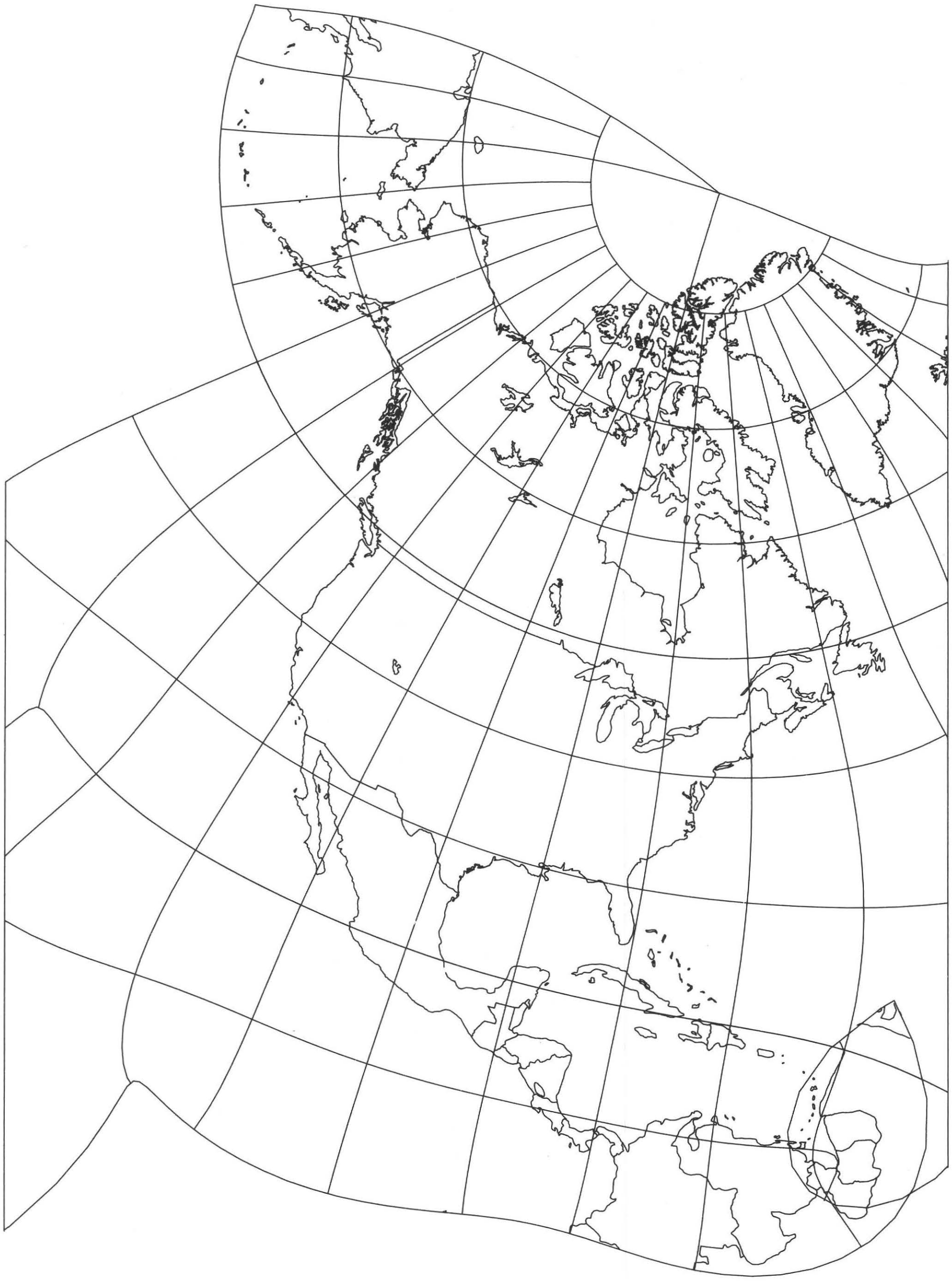

Figure 19.--Minimum-error complex conformal projection of North America, using 20 coefficients (10 pairs of complex numbers). Note the overlap in the lower right-hand corner. It is deliberately shown here, but would not be permissible on a finished map.

$$\partial\Theta/\partial\phi_p = (\cos^2\Theta/\rho \cos\Theta)\partial y'/\partial\phi_p - (\tan\Theta \cos^2\Theta/\rho \cos\Theta)\partial x'/\partial\phi_p$$

$$= (\cos\Theta/\rho)\partial y'/\partial\phi_p - (\sin\Theta/\rho)\partial x'/\partial\phi_p \qquad (5\text{-}116)$$

After equations (5-115) and (5-116) are substituted into (5-113), the latter simplifies to:

$$\partial F_1/\partial\phi_p = \Sigma j(j-1)\rho^{j-2}\{[A_j\partial x'/\partial\phi_p - B_j\partial y'/\partial\phi_p]\sin(j-2)\Theta + [A_j\partial y'/\partial\phi_p + B_j\partial x'/\partial\phi_p]\cos(j-2)\Theta\} \qquad (5\text{-}117)$$

Similarly differentiating (5-12), the equation for F_2, with respect to ϕ_p and substituting from equations (5-115) and (5-116),

$$\partial F_2/\partial\phi_p = \Sigma j(j-1)\rho^{j-2}\{[A_j\partial x'/\partial\phi_p - B_j\partial y'/\partial\phi_p]\cos(j-2)\Theta - [A_j\partial y'/\partial\phi_p + B_j\partial x'/\partial\phi_p]\sin(j-2)\Theta\} \qquad (5\text{-}118)$$

For the equivalent of equations (5-112), (5-117), and (5-118), but for λ_p or n instead of ϕ_p, it will be found that the new equations are identical except for substitution of λ_p, or n, respectively, in place of each usage of ϕ_p. This can be shown by differentiating (5-13) for k with respect to λ_p or n, and proceeding as before.

To determine $\partial x'/\partial\phi_p$, etc., it is convenient to use the projection equations for polar and rectangular coordinates of the base map. For the oblique conformal conic, in addition to equations (5-41) through (5-43), they include (5-72) through (5-76). In these equations ρ and Θ should be given primes to distinguish them from ρ and Θ of equations (5-114) through (5-118), while x and y are given primes to distinguish them from x and y of the complex transformation. Differentiating (5-75) and (5-76), which are standard equations for converting from polar to rectangular coordinates,

$$\partial x'/\partial\phi_p = -\rho' \cos\Theta' (\partial\Theta'/\partial\phi_p) - \sin\Theta' (\partial\rho'/\partial\phi_p) \qquad (5\text{-}119)$$

$$\partial y'/\partial\phi_p = \rho' \sin\Theta' (\partial\Theta'/\partial\phi_p) - \cos\Theta' (\partial\rho'/\partial\phi_p) \qquad (5\text{-}120)$$

These equations are identical to those for $\partial x'/\partial\lambda_p$, $\partial x'/\partial n$, etc., except for substitution of λ_p or n for ϕ_p. This does not apply to the following equations for differentials to use in (5-119) and (5-120). From (5-74) and (5-72),

$$\partial\theta'/\partial\phi_p = n\cos^2\lambda' \cos\phi \sin(\lambda - \lambda_p)\cos z/[\cos\phi_p \sin\phi - \sin\phi_p \cos\phi \cos(\lambda - \lambda_p)]^2 \tag{5-121}$$

Since $k' = n\rho'/R \sin z$, formed by combining (5-41) and (5-73), the following may be obtained from (5-73) and (5-43),

$$\partial\rho'/\partial\phi_p = -k'R[\cos\phi_p \sin\phi - \sin\phi_p \cos\phi \cos(\lambda - \lambda_p)]\sin z \tag{5-122}$$

Similarly,

$$\partial\theta'/\partial\lambda_p = n\cos\phi \cos^2\lambda'[\sin\phi_p \cos\phi - \cos\phi_p \sin\phi \cos(\lambda - \lambda_p)]/[\cos\phi_p \sin\phi - \sin\phi_p \cos\phi \cos(\lambda - \lambda_p)]^2 \tag{5-123}$$

$$\partial\rho'/\partial\lambda_p = -k'R\cos\phi_p \cos\phi \sin(\lambda - \lambda_p)/\sin z \tag{5-124}$$

$$\partial\theta'/\partial n = \lambda' \tag{5-125}$$

$$\partial\rho'/\partial n = \rho'[\ln\tan(z/2) - (1/2)\ln((1 - n)/(1 + n)) - 1/n] = \rho'\{(1/2)\ln[(1 - \cos z)(1 + n)/((1 + \cos z)(1 - n))] - 1/n\} \tag{5-126}$$

This provides all required terms for (5-112), whether for ϕ_p, λ_p, or n, except for the partial derivatives of k' with respect to the parameters, which are the same as (5-50) through (5-52), (5-54), and (5-55), replacing k with k'. Equation (5-52) may be rewritten

$$\partial k'/\partial n = (k'/2)\ln[(1 - \cos z)(1 + n)/((1 + \cos z)(1 - n))] \tag{5-127}$$

to reduce computation.

For uniformity, equations corresponding to (5-112), (5-113), and (5-117) may be developed for $\partial k/\partial A_j$ and $\partial k/\partial B_j$, instead of using (5-14) and (5-15): Equation (5-112) may be used intact except for substitution of A_j or B_j in place of ϕ_p, but the last term may be eliminated, since

$$\partial k'/\partial A_j = 0 \tag{5-128}$$

$$\partial k'/\partial B_j = 0 \tag{5-129}$$

The derivatives corresponding to (5-113) and (5-117), but with respect to A_j and B_j, are simpler. Differentiating (5-11) and (5-12) with respect to these coefficients,

$$\partial F_1/\partial A_j = j\rho^{j-1} \sin (j - 1)\Theta \tag{5-130}$$

$$\partial F_2/\partial A_j = j\rho^{j-1} \cos (j - 1)\Theta \tag{5-131}$$

$$\partial F_1/\partial B_j = j\rho^{j-1} \cos (j - 1)\Theta = \partial F_2/\partial A_j \tag{5-132}$$

$$\partial F_2/\partial B_j = -j\rho^{j-1} \sin (j - 1)\Theta = -\partial F_1/\partial A_j \tag{5-133}$$

To obtain optimum values of A_j, B_j, ϕ_p, λ_p, and n, matrices of the form (5-65) must be obtained, using equations (5-59) through (5-61), (5-63), and (5-64), without change. In addition to those shown in (5-65), partial derivatives involving A_j and B_j must be included.

Those involving only A_j and B_j are shown in (5-19) and (5-20), except that the second derivatives may be omitted, so that

$$\partial f(A_q)/\partial A_g = 2 \sum_{p=1}^{m} (\partial k_p/\partial A_q)(\partial k_p/\partial A_g) \tag{5-134}$$

$$\partial f(A_q)/\partial B_g = \partial f(B_g)/\partial A_q = 2 \sum_{p=1}^{m} (\partial k_p/\partial A_q)(\partial k_p/\partial B_g) \tag{5-135}$$

$$\partial f(B_q)/\partial B_g = 2 \sum_{p=1}^{m} (\partial k_p/\partial B_q)(\partial k_p/\partial N_g) \tag{5-136}$$

Those relating A_j and ϕ_p, etc., are comparable:

$$\partial f(A_j)/\partial \phi_p = \partial f(\phi_p)/\partial A_j = 2 \sum_{p=1}^{m} (\partial k_p/\partial A_j)(\partial k_p/\partial \phi_p) \tag{5-137}$$

and so forth. Equation (5-65) then becomes, using an example with A_1, A_2, B_2, ϕ_p, λ_p, and n, letting $B_1 = 0$,

$$\begin{bmatrix} \partial f(A_1)/\partial A_1 & \partial f(A_1)/\partial A_2 & \partial f(A_1)/\partial B_2 & \partial f(A_1)/\partial \phi_p & \partial f(A_1)/\partial \lambda_p & \partial f(A_1)/\partial n \\ \partial f(A_2)/\partial A_1 & \partial f(A_2)/\partial A_2 & \partial f(A_2)/\partial B_2 & \partial f(A_2)/\partial \phi_p & \partial f(A_2)/\partial \lambda_p & \partial f(A_2)/\partial n \\ \partial f(B_2)/\partial A_1 & \partial f(B_2)/\partial A_2 & \partial f(B_2)/\partial B_2 & \partial f(B_2)/\partial \phi_p & \partial f(B_2)/\partial \lambda_p & \partial f(B_2)/\partial n \\ \partial f(\phi_p)/\partial A_1 & \partial f(\phi_p)/\partial A_2 & \partial f(\phi_p)/\partial B_2 & \partial f(\phi_p)/\partial \phi_p & \partial f(\phi_p)/\partial \lambda_p & \partial f(\phi_p)/\partial n \\ \partial f(\lambda_p)/\partial A_1 & \partial f(\lambda_p)/\partial A_2 & \partial f(\lambda_p)/\partial B_2 & \partial f(\lambda_p)/\partial \phi_p & \partial f(\lambda_p)/\partial \lambda_p & \partial f(\lambda_p)/\partial n \\ \partial f(n)/\partial A_1 & \partial f(n)/\partial A_2 & \partial f(n)/\partial B_2 & \partial f(n)/\partial \phi_p & \partial f(n)/\partial \lambda_p & \partial f(n)/\partial n \end{bmatrix} \cdot \begin{bmatrix} \Delta A_1 \\ \Delta A_2 \\ \Delta B_2 \\ \Delta \phi_p \\ \Delta \lambda_p \\ \Delta n \end{bmatrix} = \begin{bmatrix} -f(A_1) \\ -f(A_2) \\ -f(B_2) \\ -f(\phi_p) \\ -f(\lambda_p) \\ -f(n) \end{bmatrix} \quad (5\text{-}138)$$

Its solution follows the pattern described for (5-65). Because of the relationships as expressed in (5-135) and (5-137), the square matrix is actually symmetric. In summary, the first matrix is composed of derivatives found from equations of the type (5-134) through (5-137) and (5-59) through (5-64). The terms for these equations are found from (5-112) and its similar variations for other variables, and in turn from (5-41) through (5-43), (5-50) through (5-55), replacing k with k', (5-11), (5-12), (5-117) through (5-120) and all their variations for other variables, (5-121) through (5-133), and (5-73) through (5-76).

The choice of initial estimates for parameters of ϕ_p, λ_p, and n is even more critical when the complex coefficients are involved, although in this case the initial estimates may again be 1 for A_1 and zero for other coefficients.

While the above appears to be an ideal way to find a minimum-error conformal map projection, it is not necessarily the case, even when the program properly iterates. Examples were given in table 6 and figures 16 and 17 in which the pole of the "optimum" oblique conformal conic projection occurs well inland in the region studied with the resulting unacceptable gore.

Therefore, one should incorporate into a general optimizing program the ability to hold one or more of the variables constant. This is done by striking out any row and column in which each element involves a differential of the variable to be eliminated. Thus, if the pole (ϕ_p, λ_p) is to be held constant, rows 4 and 5 and columns 4 and 5 of the first matrix in equation (5-138) and rows 4 and 5 of the second and third matrices are all deleted.

The cone constant can similarly be held constant, but if it is made 0 or 1, fixing the base projection as the Oblique Mercator or oblique Stereographic, respectively, some equations become indeterminate, in addition to those mentioned in the previous section. For the Oblique Mercator, equations (5-86), (5-89), and (5-90) apply, adding primes to x and y, and deleting k_o. Differentiating,

$$\partial x'/\partial\phi_p = -R \cos z \cos^2(x'/R)/\cos\phi \sin(\lambda - \lambda_p) \quad (5\text{-}139)$$

$$\partial y'/\partial\phi_p = R(k')^2[\cos\phi_p \sin\phi - \sin\phi_p \cos\phi \cos(\lambda - \lambda_p)] \quad (5\text{-}140)$$

$$\partial x'/\partial\lambda_p = R \cos^2(x'/R)[\cos\phi_p \tan\phi \cos(\lambda - \lambda_p) - \sin\phi_p]/\sin^2(\lambda - \lambda_p) \quad (5\text{-}141)$$

$$\partial y'/\partial\lambda_p = R(k')^2 \cos\phi_p \cos\phi \sin(\lambda - \lambda_p) \quad (5\text{-}142)$$

For other partials, equations (5-54), (5-55), (5-87), and (5-88) apply.

For the oblique Stereographic, equations (5-94) through (5-99) apply. After adding primes to x, y, and k and deleting k_o, differentiation leads to the following:

$$\partial x'/\partial\phi_p = -x' y'/2R \quad (5\text{-}143)$$

$$\partial y'/\partial\phi_p = -Rk' \cos z - (y')^2/2R \quad (5\text{-}144)$$

$$\partial x'/\partial\lambda_p = -Rk' \cos\phi \cos(\lambda - \lambda_p) - (x')^2 \cos\phi_p/2R \quad (5\text{-}145)$$

$$\partial y'/\partial\lambda_p = -x' \sin\phi_p - x' y' \cos\phi_p/2R \quad (5\text{-}146)$$

The equations for the GS50 (50-State) projection described earlier in this bulletin, although involving the Stereographic, do not include the above equations, since the position of the pole of projection (ϕ_p, λ_p) is held constant for GS50 computations.

In practice, the versatility in map projection optimization is not as great as the foregoing derivations might indicate. A number of attempts were made to iterate to convergence with a variety of constraints, using the 162 weighted points selected for establishing a map of North America. If the cone constant n is fixed at zero (the Oblique Mercator) and the pole and complex coefficients are allowed to seek optimum values, only first and second order complex coefficients could be made to iterate to convergence. With a cone constant of 0.5, only first- to third-order complex coefficients could be used, and with a cone constant of 1.0, only the first-order (which is equivalent to adjusting only k_o in earlier equations). If the pole is also fixed, the complex order can generally be increased indefinitely.

As a further refinement, the development of an optimum projection may be applied to the ellipsoid, since for moderately sized regions the scale variation is small enough to make ellipsoidal corrections worthwhile. In practice, the parameters obtained are close to those obtained using spherical formulas, so the spherical optimum may be obtained and then used in ellipsoidal formulas with almost negligible variation from the true ellipsoidal optimum. The additional complication is relatively minimal, since the change may involve merely a substitution of conformal latitude for geographic latitude, as was done for the GS50 projection. This is satisfactory, since the lines of constant scale are not following a particular parallel or meridian.

For the oblique conformal conic, using conformal latitude χ,

$$\begin{aligned} k' &= n\rho' \cos\chi\,(1 - e^2 \sin^2\phi)^{1/2}/(a \sin z \cos\phi) \\ &= nF[(1 - \cos z)/(1 + \cos z)]^{n/2} \cos\chi\,(1 - e^2\sin^2\phi)^{1/2}/ \\ &\quad (\sin z \cos\phi) \end{aligned} \tag{5-147}$$

where nF is a constant equal to the right side of equation (5-67) with k_o=1. Differentiation is eased by conversion to the logarithm form:

$$\begin{aligned} \ln k' &= \ln(nF) + (n/2)\ln(1 - \cos z) - (n/2)\ln(1 + \cos z) \\ &\quad - \ln\sin z + \ln[\cos\chi\,(1 - e^2\sin^2\phi)^{1/2}/\cos\phi] \end{aligned} \tag{5-148}$$

Differentiating with respect to χ_p, the conformal latitude of the pole,

$$\begin{aligned} (\partial k'/\partial\chi_p)/k' &= [(n/2)\sin z/(1 - \cos z) + (n/2)\sin z/(1 + \cos z) \\ &\quad - \cos z/\sin z]\,\partial z/\partial\chi_p \end{aligned} \tag{5-149}$$

$$\partial k'/\partial\chi_p = k'\,(n - \cos z)(\partial z/\partial\chi_p)/\sin z \tag{5-150}$$

Substituting χ and χ_p in place of ϕ and ϕ_p in equation (5-43) for z, the differential $\partial z/\partial\chi_p$ is the same as (5-54) with the same substitutions. Once χ_p is computed, using these formulas, it may be converted to ϕ_p using the inverse of equation (5-3). Thus, provided that k' is calculated according to (5-147) rather than (5-41), equation (5-150) is the same as (5-50) with the substitution of (χ,χ_p) for (ϕ,ϕ_p). Other equations for the spherical oblique conformal conic apply with the same substitution. For the ellipsoidal Oblique Mercator, the spherical equations also apply with the χ substitutions, except that the spherical k' is multiplied by $\cos\chi\,(1 - e^2\sin^2\phi)^{1/2}/\cos\phi$, and R is replaced by a.

For the ellipsoidal oblique Stereographic, equations (5-6) through (5-8), (5-26) and (5-27) for g, s, k', x', and y' apply without change, while (5-97), (5-98), and (5-143) through (5-146) become the following, respectively:

$$\partial k'/\partial\chi_p = -k'y'/(2Rk_o) \qquad (5\text{-}150a)$$

$$\partial k'/\partial\lambda_p = -k'x'\cos\chi_p/(2Rk_o) \qquad (5\text{-}150b)$$

$$\partial x'/\partial\chi_p = -x'y'/2a \qquad (5\text{-}150c)$$

$$\partial y'/\partial\chi_p = -as\cos z - (y')^2/2a \qquad (5\text{-}150d)$$

$$\partial x'/\partial\lambda_p = -as\cos\chi\cos(\lambda - \lambda_p) - (x')^2\cos\chi_p/2a \qquad (5\text{-}150e)$$

$$\partial y'/\partial\lambda_p = -x'\sin\chi_p - x'y'\cos\chi_p/2a \qquad (5\text{-}150f)$$

c. MINIMUM-ERROR PSEUDOCYLINDRICAL EQUAL-AREA PROJECTIONS

A type of map projection very different from the conformal projections evaluated for a given region is the pseudocylindrical. With straight, horizontal parallels of latitude, in the normal form, and curved meridians converging to a point or a line shorter than the Equator, it is impossible for a pseudocylindrical projection to be conformal. On the other hand, numerous equal-area pseudocylindricals have been developed, and there are infinite possibilities. Normally these projections are used for world maps.

It is probably less important to consider minimum-error pseudocylindrical projections of the world than to devise minimum-error regional maps--conformal, equal-area or otherwise. To indicate the application of least squares and the computer to the design of such a projection, however, some examples of the procedure will be shown.

It is known that normal equal-area pseudocylindricals, with either points or lines as poles, fit the following general equations:

$$x = R^2(\lambda - \lambda_0) \cos \phi/(dy/d\phi) \tag{5-151}$$

$$y = R\, f(\phi) \tag{5-152}$$

where (x,y) are rectangular coordinates with origin at the intersection of the Equator and the central meridian ($\lambda_0 = 0$), the X axis coincides with the Equator, x increases easterly, R is the radius of the sphere, (ϕ,λ) are latitude/longitude of a given point, and $f(\phi)$ is any function of ϕ.

(1) A Pointed-Polar Pseudocylindrical Projection

If the pole is to be shown as a point, $f(\phi)$ may be written as a polynomial as follows, omitting even powers of ϕ since there should be symmetry about the Equator,

$$y = R(A_1\phi + A_3\phi^3 + A_5\phi^5 + \ldots) \tag{5-153}$$

Then

$$dy/d\phi = R(A_1 + 3A_3\phi^2 + 5A_5\phi^4 + \ldots) \tag{5-154}$$

which may be substituted into equation (5-151). The basis for minimizing error may be taken as follows (another basis will be discussed later), adapting equation (4-21):

$$E = \sum_{j=1}^{m} [(h_j - 1)^2 + (k_j - 1)^2] \cos \phi_j \tag{5-155}$$

where h is the scale factor along the meridian and k is the scale factor along the parallel of latitude for each of the m given points. To compute h and k for a normal pseudocylindrical, since y is not a function of λ,

$$h = (1/R)[\partial x/\partial\phi)^2 + (dy/d\phi)^2]^{1/2} \tag{5-156}$$

$$k = (\partial x/\partial\lambda)/(R \cos \phi) \tag{5-157}$$

From (5-151), (5-153), and (5-154),

$$\partial x/\partial\phi = -R^3(\lambda - \lambda_o) \cos\phi\, (6A_3\phi + 20A_5\phi^3 + \ldots)/(dy/d\phi)^2$$

$$- R^2(\lambda - \lambda_o) \sin\phi/(dy/d\phi) \quad (5\text{-}158)$$

$$\partial x/\partial\lambda = R^2 \cos\phi/(dy/d\phi) \quad (5\text{-}159)$$

For convenience, let

$$F_1 = A_1 + 3A_3\phi^2 + 5A_5\phi^4 + \ldots \quad (5\text{-}160)$$

$$F_2 = 6A_3\phi + 20A_5\phi^3 + \ldots \quad (5\text{-}161)$$

Substituting into (5-158) and (5-159),

$$\partial x/\partial\phi = -R(\lambda - \lambda_o)[(F_2/F_1^{\,2}) \cos\phi + (1/F_1) \sin\phi] \quad (5\text{-}162)$$

$$\partial x/\partial\lambda = R(\cos\phi)/F_1 \quad (5\text{-}163)$$

Substituting into (5-156) and (5-157),

$$h = \{(\lambda - \lambda_o)^2[(F_2/F_1^{\,2}) \cos\phi + (1/F_1) \sin\phi]^2 + F_1^2\}^{1/2} \quad (5\text{-}164)$$

$$k = 1/F_1 \quad (5\text{-}165)$$

To minimize E, it is first differentiated with respect to each constant A_n, omitting the limits in writing Σ:

$$\partial E/\partial A_n = f(A_n) = 2\, \Sigma[(h_j - 1)(\partial h_j/\partial A_n) + (k_j - 1)(\partial k_j/\partial A_n)]$$

$$\cos\phi_j \quad (5\text{-}166)$$

From (5-156) and (5-157),

$$\partial h/\partial A_n = [(\partial x/\partial\phi)(\partial^2 x/\partial\phi\partial A_n) + (dy/d\phi)(\partial^2 y/\partial\phi\partial A_n)]/$$

$$\{R[(\partial x/\partial\phi)^2 + (dy/d\phi)^2]^{1/2}\} \quad (5\text{-}167)$$

$$\partial k/\partial A_n = (\partial^2 x/\partial\lambda\partial A_n)/(R \cos\phi) \quad (5\text{-}168)$$

From equations (5-154) and (5-160) through (5-163), omitting intermediate steps,

$$\partial^2x/\partial\phi\partial A_n = -[R(\lambda - \lambda_o)n/F_1^{\,2}]\{[(n-1)\phi^{n-2} - 2\phi^{n-1}(F_2/F_1)] \cos\phi - \phi^{n-1}\sin\phi\} \quad (5\text{-}169)$$

$$\partial^2x/\partial\lambda\partial A_n = -Rn\,\phi^{n-1}\cos\phi/F_1^{\,2} \quad (5\text{-}170)$$

$$\partial^2y/\partial\phi\partial A_n = Rn\,\phi^{n-1} \quad (5\text{-}171)$$

For the derivatives of equation (5-166) with respect to each constant A_n, using p as a subscript to indicate either a different or the same constant:

$$\partial f(A_n)/\partial A_p = 2\Sigma[(\partial h_j/\partial A_n)(\partial h_j/\partial A_p) + (\partial k_j/\partial A_n)(\partial k_j/\partial A_p) + (h_j - 1)(\partial^2 h_j/\partial A_n\partial A_p) + (k_j - 1)(\partial^2 k_j/\partial A_n\partial A_p)] \cos\phi_j$$

To simplify further derivations with minimal increase in iterations, these second derivatives may be ignored, so that

$$\partial f(A_n)/\partial A_p = 2\Sigma[(\partial h_j/\partial A_n)(\partial h_j/\partial A_p) + (\partial k_j/\partial A_n)(\partial k_j/\partial A_p)] \cos\phi_j \quad (5\text{-}172)$$

In order to iterate equation (5-166) to near zero for each value of n, to find the minimum, the following equations are placed in matrix form for solution by standard methods:

$$\begin{bmatrix} \partial f(A_1)/\partial A_1 & \partial f(A_1)/\partial A_3 & \partial f(A_1)/\partial A_5 \\ \partial f(A_3)/\partial A_1 & \partial f(A_3)/\partial A_3 & \partial f(A_3)/\partial A_5 \\ \partial f(A_5)/\partial A_1 & \partial f(A_5)/\partial A_3 & \partial f(A_5)/\partial A_5 \end{bmatrix} \bullet \begin{bmatrix} \Delta A_1 \\ \Delta A_3 \\ \Delta A_5 \end{bmatrix} = \begin{bmatrix} -f(A_1) \\ -f(A_3) \\ -f(A_5) \end{bmatrix} \quad (5\text{-}173)$$

using equations (5-154), (5-160) through (5-165), (5-169) through (5-171), (5-167), (5-168), (5-166) and (5-172) in order to develop the elements of the matrix, which, like (5-65), is symmetric. One may use initial estimates of 1 for A_1 and zero for other constants, and calculate points for say each 15° of both latitude and longitude for one-fourth of the sphere, bounded by the central meridian, the Equator, and one of the outer meridians. The weighting along the Equator and central meridian must then be reduced to avoid double weighting in order to make the quadrant representative of the Earth. Furthermore, computation for either pole leads to indeterminant expressions in some of the above equations, due to division by zero, so the latitude should be shifted to slightly less than 90° at that point for a northern quadrant.

(2) A Flat-Polar Pseudocylindrical Projection

Flat-polar pseudocylindricals, in which the poles are shown as straight lines shorter than the Equator, rather than as points, are probably used more than pointed-polar pseudocylindricals, since shape distortion can be less pronounced. Equations (5-151) and (5-152) apply also to flat-polar equal-area pseudocylindrical projections, but equations (5-153) and (5-154) are not satisfactory. Instead, it is desirable to use a derivative $(dy/d\phi)$ which will cancel the $\cos\phi$ term in the numerator of (5-151) to allow a non-zero x when $\phi = \pm 90°$. This is accomplished by multiplying the right side of equation (5-154) by $\cos\phi$ and using new coefficients:

$$dy/d\phi = R \cos\phi \, (B_1 + B_3\phi^2 + B_5\phi^4 + \ldots) \qquad (5\text{-}174)$$

Integration to obtain y may be obtained from standard tables of integrals:

$$\int \phi^n \cos\phi \, d\phi = \phi^{n-1}(\phi \sin\phi + n \cos\phi) - n(n-1)\int \phi^{n-2} \cos\phi \, d\phi \qquad (5\text{-}175)$$

Integrating for the first three coefficients of the series in (5-174),

$$y/R = B_1 \sin\phi + B_3(\phi^2 \sin\phi + 2\phi\cos\phi - 2\sin\phi) + B_5 [\phi^3(\phi\sin\phi + 4\cos\phi) - 12(\phi^2\sin\phi + 2\phi\cos\phi - 2\sin\phi)] + \ldots \qquad (5\text{-}176)$$

Equation (5-176) is not needed to determine the constants, but it is needed to compute rectangular coordinates for the final map projection. With derivations corresponding to those described earlier for the pointed-polar pseudocylindricals, let

$$F_1 = B_1 + B_3\phi^2 + B_5\phi^4 + \ldots \qquad (5\text{-}177)$$

$$F_2 = 2B_3\phi + 4B_5\phi^3 + \ldots \qquad (5\text{-}178)$$

Then

$$x = R(\lambda - \lambda_0)/F_1 \qquad (5\text{-}179)$$

$$dy/d\phi = RF_1 \cos\phi \qquad (5\text{-}180)$$

$$\partial x/\partial\phi = -R(\lambda - \lambda_0)F_2/F_1^2 \qquad (5\text{-}181)$$

$$\partial x/\partial\lambda = R/F_1 \tag{5-182}$$

$$\partial^2 y/\partial\phi\partial B_n = R\phi^{n-1} \cos\phi \tag{5-183}$$

$$\partial^2 x/\partial\phi\partial B_n = -[R(\lambda - \lambda_o)\phi^{n-1}/F_1^{\ 3}][F_1(n - 1)/\phi - 2F_2] \tag{5-184}$$

$$\partial^2 x/\partial\lambda\partial B_n = -R\phi^{n-1}/F_1^{\ 2} \tag{5-185}$$

These equations may be substituted into (5-167), (5-168), and (5-166), (5-172), and (5-173), using B in place of A, and equations (5-156) and (5-157) instead of (5-164) and (5-165), iterating the equations to convergence to obtain the coefficients. For the flat-polar version, however, k is infinite at the poles, so the range of latitude used in minimizing must be limited to less than the range ±90°, and the constants will vary with the range chosen.

(3) A Better Error Limitation

Instead of using equation (5-155) as a basis for minimizing error, it is better to use a and b in place of h and k, where a and b are the relative lengths of the semiaxes of Tissot's indicatrix, an ellipse representing a small circle on the Earth as portrayed on the map. The lengths a and b are made equal to the maximum and minimum scale factors at a given point, whereas h and k are not maximum and minimum unless the meridian intersects the parallel at a right angle. It should be noted that the intersection is perpendicular for the minimum-error conics discussed in an earlier section, while Airy's minimum-error azimuthal is based on minimizing (h' - 1) and (k' - 1), where h' and k' are measured in directions perpendicular to each other.

Without showing the derivation, the relationship between (h,k) and (a,b) for an equal-area map projection is as follows:

$$a = (a' + b')/2 \tag{5-187}$$

$$b = (a' - b')/2 \tag{5-188}$$

$$a' = (h^2 + k^2 + 2)^{1/2} \tag{5-189}$$

$$b' = (h^2 + k^2 - 2)^{1/2} \tag{5-190}$$

Adapting equation (5-155) and substituting from the above equations,

$$E = \Sigma[(a_j - 1)^2 + (b_j - 1)^2] \cos\phi_j \tag{5-191}$$

$$= a_j'(a_j' - 2) \cos \phi_j \tag{5-192}$$

To replace equation (5-166), using equations (5-192) and (5-189),

$$\partial E/\partial A_n = f(A_n) = 2\ \Sigma(a_j' - 1)(\partial a_j'/\partial A_n) \cos \phi_j \tag{5-193}$$

$$\partial a_j'/\partial A_n = (1/2)(h_j^{\ 2} + k_j^{\ 2} + 2)^{-1/2}(2h_j\partial h_j/\partial A_n + 2k_j\partial k_j/\partial A_n)$$

$$= (h_j\partial h_j/\partial A_n + k_j\partial k_j/\partial A_n)/a_j' \tag{5-194}$$

Substituting from (5-194) into (5-193),

$$f(A_n) = 2\ \Sigma(1 - 1/a_j')(h_j\partial h_j/\partial A_n + k_j\partial k_j/\partial A_n) \cos \phi_j \tag{5-195}$$

From (5-195) can be obtained the (a,b) equivalent of equation (5-172). In this case it is found that second derivatives cannot be ignored (the program will not iterate to convergence).

$$\partial f(A_n)/\partial A_p = 2\ \Sigma\{(\partial a_j'/\partial A_n)(\partial a_j'/\partial A_p)/a_j' + (1 - 1/a_j')[(\partial h_j/\partial A_n)$$

$$(\partial h_j/\partial A_p) + h_j\ \partial^2 h_j/\partial A_n\partial A_p + (\partial k_j/\partial A_n)(\partial k_j/\partial A_p)$$

$$+ k_j\partial^2 k_j/\partial A_n\partial A_p]\} \tag{5-196}$$

For the second derivatives, equations (5-167) and (5-168) are differentiated with respect to A_p,

$$\partial^2 h/\partial A_n\partial A_p = \{hR[(\partial^2 x/\partial\phi\partial A_n)(\partial^2 x/\partial\phi\partial A_p) + (\partial x/\partial\phi)(\partial^3 x/$$

$$\partial\phi\partial A_n\partial A_p) + (\partial^2 y/\partial\phi\partial A_n)(\partial^2 y/\partial\phi\partial A_p) + (dy/d\phi)$$

$$(\partial^3 y/\partial\phi\partial A_n\partial A_p)] - [(\partial x/\partial\phi)(\partial^2 x/\partial\phi\partial A_n) + (dy/d\phi)$$

$$(\partial^2 y/\partial\phi\partial A_n)][(\partial x/\partial\phi)\ (\partial^2 x/\partial\phi\partial A_p)$$

$$+ (dy/d\phi)(\partial^2 y/\partial\phi\partial A_p)]/hR\}h^2R^3 \tag{5-197}$$

$$\partial^2 k/\partial A_n\partial A_p = (\partial^3 x/\partial\lambda\partial A_n\partial A_p)/(R \cos \phi) \tag{5-198}$$

For the additional derivatives required in equations (5-197) and (5-198), equations (5-169) through (5-171) are differentiated to produce

$$\partial^3x/\partial\phi\partial A_n\partial A_p = 2R(\lambda - \lambda_o)np\,\phi^{n+p-2}\{[(n + p - 2)/\phi - 3F_2/F_1]\cos\phi - \sin\phi\}/F_1 \quad (5\text{-}199)$$

$$\partial^3x/\partial\lambda\partial A_n\partial A_p = 2Rnp\,\phi^{n+p-2}\cos\phi/F_1^3 \quad (5\text{-}200)$$

$$\partial^3y/\partial\phi\partial A_n\partial A_p = 0 \quad (5\text{-}201)$$

Using equations (5-195) and (5-196) instead of (5-166) and (5-172), then equations (5-154), (5-160) through (5-165), (5-169) through (5-171), (5-167), (5-168), (5-197) through (5-201), (5-189), (5-194) and (5-173) may be used with (5-195) and (5-196) to determine constants A_n for a minimum-error pointed-polar pseudocylindrical equal-area by iteration. The initial estimates of constants required trial and error before convergence would occur.

The solution is analogous for the flat-polar equivalent based on (a,b). That is, equations (5-177), (5-178), (5-180) through (5-185), (5-167), (5-168), (5-156), (5-157), (5-189), (5-201) through (5-203), (5-194) through (5-197), and (5-173), using B instead of A, may be employed in order for each iteration. For the additional derivatives, (5-201) is valid, but from (5-184) and (5-185),

$$\partial^3x/\partial\phi\partial A_n\partial A_p = 2R(\lambda - \lambda_o)\phi^{n+p}[(n + p)/\phi - 3F_2/F_1]/F_1^3 \quad (5\text{-}202)$$

$$\partial^3x/\partial\lambda\partial A_n\partial A_p = 2R\,\phi^{n+p}/F_1^3 \quad (5\text{-}203)$$

For a minimum-error pointed-polar pseudocylindrical projection based on equation (5-192), and fitting points at 5° intervals of latitude and longitude, the coefficients are found to be

$A_1 = 1.27326$

$A_3 = -0.04222$

$A_5 = -0.02930$

The RMSE or r is found to be 0.5749 from the following equivalent in principle to equation (5-109):

$$r = [E/\Sigma\cos\phi_j]^{1/2} \quad (5\text{-}204)$$

where E is found from (5-192).

Sample coordinates for the 180th meridian of this projection, for a sphere of radius 1, are as follows, using equations (5-151) through (5-154):

Latitude	x	y
90°	0.00000	1.55620
75	1.29863	1.45939
60	1.63934	1.24797
45	1.94968	0.97080
30	2.21642	0.65946
15	2.40096	0.33254
0	2.46737	0.00000

Symmetry exists about the Equator and central meridian, and meridians are equally spaced along each parallel of latitude. The projection is illustrated in figures 20 and 21, the latter including Tissot indicatrices (see also figure 9).

For a minimum-error flat-polar pseudocylindrical based on equation (5-192), but limited in given points to 5° intervals of latitude and longitude between latitudes 75° N. and S.,

$$B_1 = 1.24126$$

$$B_3 = 0.31970$$

$$B_5 = -0.00768$$

for which the RMSE is 0.4234.

The coordinates for the 180th meridian of this projection, for a sphere of radius 1, are as follows, using equations (5-151), (5-174), and (5-176):

Latitude	x	y
90°	1.58403	1.38700
75	1.77844	1.32443
60	1.98508	1.15840
45	2.18845	0.91976
30	2.36508	0.63463
15	2.48715	0.32313
0	2.53098	0.00000

Symmetry and meridian spacing are as described for the pointed-polar form. Each pole is thus a line 0.626 as long as the Equator on this projection. This projection resembles the pointed-polar form, in spite of the pole-line. In both cases, the central meridian varies appreciably from a length half that of the Equator, since the ratios are 0.631 and 0.548, respectively. This in turn resembles Behrmann's cylindrical equal-area projection (figures 6 and 9) in distortion near the Equator.

Because of the large remaining RMSE, the pseudocylindrical projections used are likely to remain those which have been developed in the past or which look satisfactory to the inventor and associates. Some, in fact, such as the Robinson projection, have been specifically chosen for the latter reason.

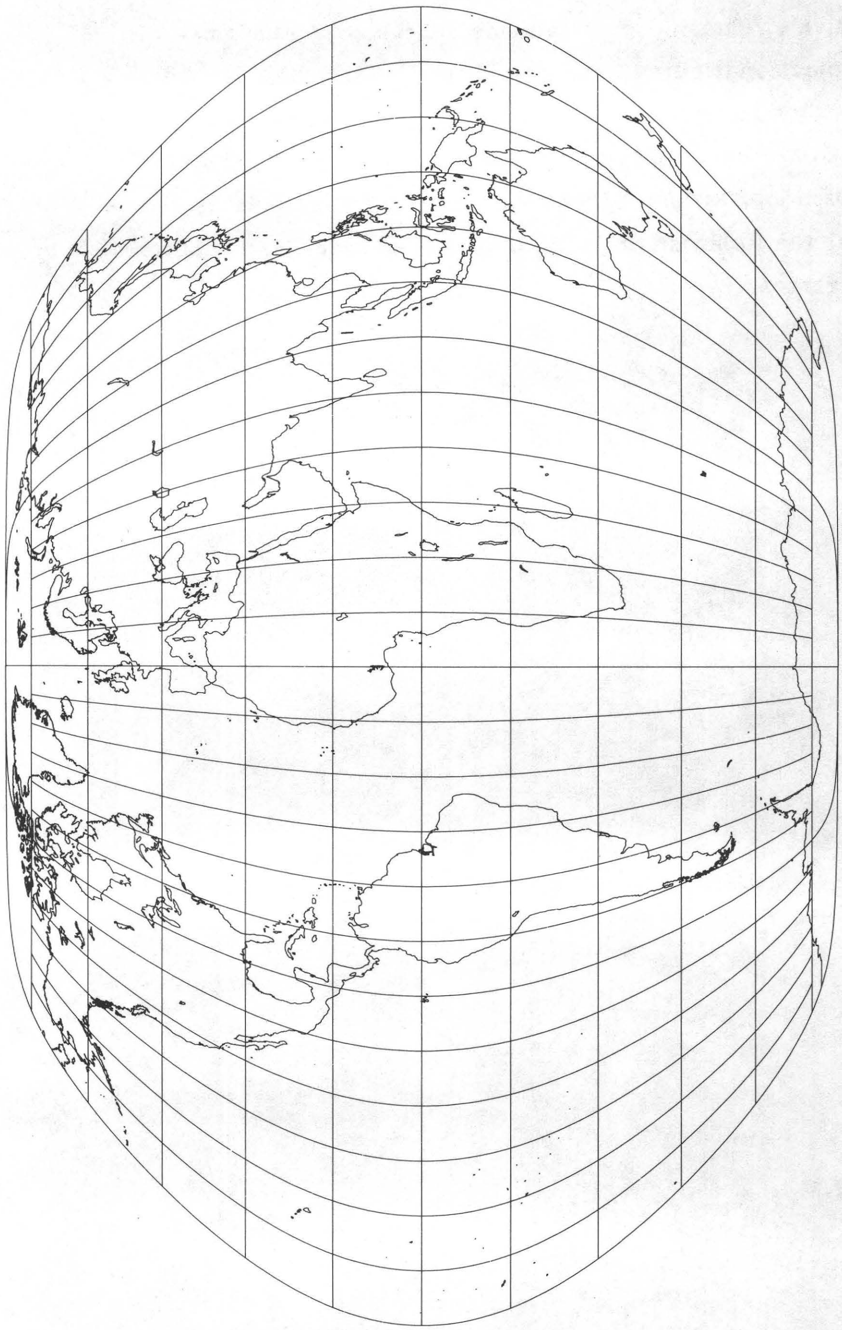

Figure 20.--Minimum-error pointed-polar pseudocylindrical equal-area projection.

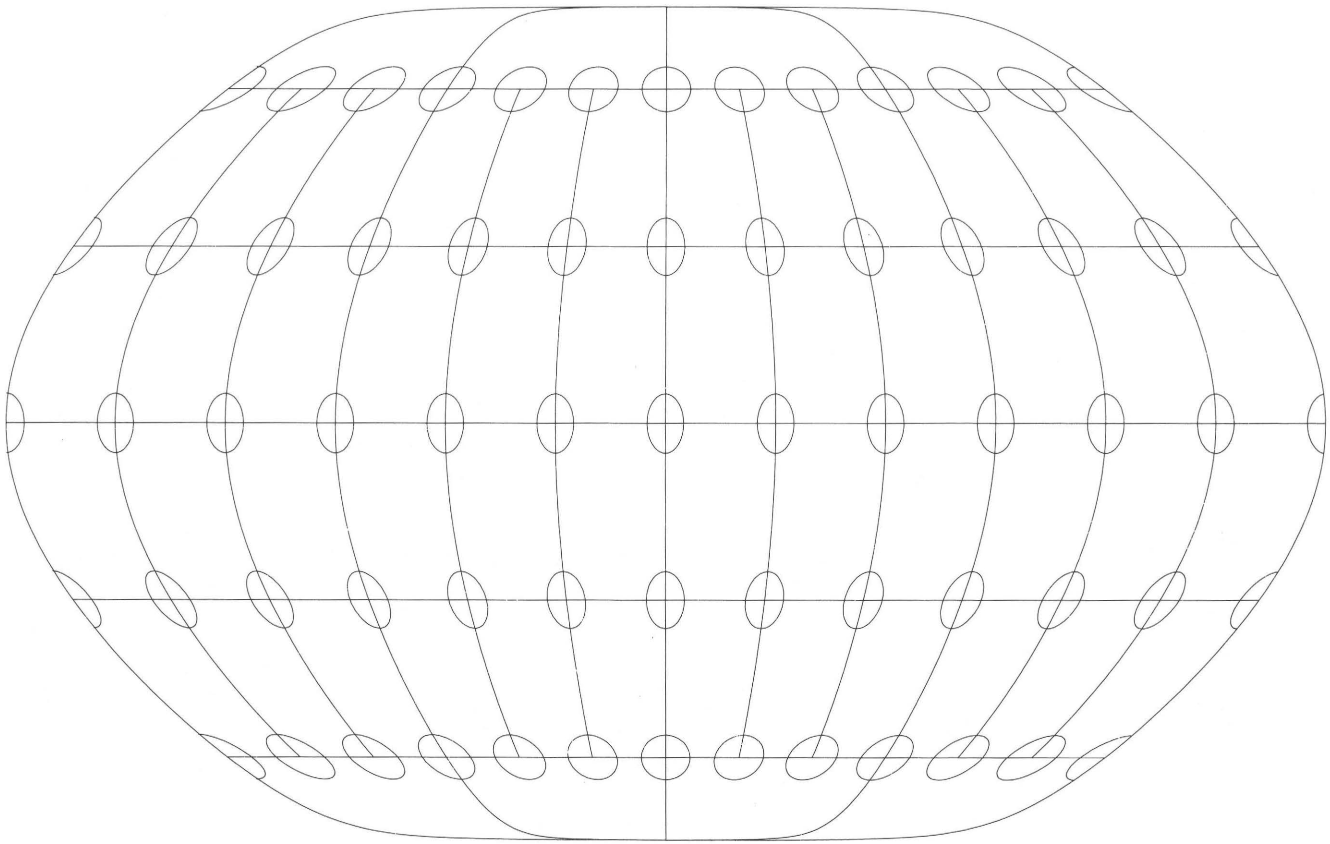

Figure 21.--Minimum-error pointed-polar pseudocylindrical equal-area projection, with Tissot indicatrices on a 30° graticule.

APPENDIX

The Appendix consists of sections 6 through 9, which list additional formulas and derivations omitted from the main text.

6. ADDITIONAL FORMULAS USED IN PROGRAM TO IDENTIFY UNMARKED MAP PROJECTIONS

(1) For the ellipsoidal form of the Albers Equal-Area Conic, spherical equation (3-58) is replaced by the following, which is indeterminate if $e = 0$:

$$q = (1 - e^2)\{\sin\phi/(1 - e^2\sin^2\phi) - (1/(2e)) \ln [(1 - e \sin\phi)/(1 + e \sin\phi)]\} \quad (6\text{-}1)$$

To find the standard parallels for the ellipsoid, substituting from (3-43), (3-57), and (3-62) into (3-64), then transposing,

$$\cos^2\phi_s - (C - nq_s)(1 - e^2\sin^2\phi_s) = 0 \quad (6\text{-}2)$$

where q_s is found from (6-1), using ϕ_s in place of ϕ. With the two values of ϕ_s calculated from (3-65) for the sphere as the first trial pair, equation (6-2) may be solved with a Newton-Raphson iteration: First equation (6-2) is rewritten with $f(\phi_s)$ in place of the zero. Differentiating the new equation with respect to ϕ_s, and simplifying,

$$f'(\phi_s) = -2 \sin\phi_s \cos\phi_s[1 - e^2(C - nq_s)] + 2n(1 - e^2) \cos\phi_s/ (1 - e^2\sin^2\phi_s) \quad (6\text{-}3)$$

From these equations, using one set of iterations for each of the two trial values of ϕ_s,

$$\Delta\phi_s = -f(\phi_s)/f'(\phi_s) \quad (6\text{-}4)$$

For the inverse computations, equations (3-45), (3-53), and (3-54) apply, but equation (3-66) is replaced by another Newton-Raphson iteration in which

$$\Delta\phi = (1 - e^2\sin^2\phi)^2[q/(1 - e^2) - \sin\phi/(1 - e^2\sin^2\phi) + (1/(2e)) \ln((1 - e\sin\phi)/(1 + e\sin\phi))]/(2\cos\phi) \quad (6\text{-}5)$$

where q is found from an inversion of (3-43) and (3-57):

$$q = (C - \rho^2 n^2/a^2)/n \quad (6\text{-}6)$$

and the first trial ϕ is the arcsine of (q/2), from an inversion of equation (3-58). Other equations given for the spherical form remain unchanged.

(2) For the Equidistant Conic for the ellipsoid (this readily simplifies for the sphere, if e = 0), the function in equation (3-43) is the following:

$$f(\phi) = G - M \quad (6\text{-}7)$$

where G is a constant,

$$M = E_0\phi - E_1\sin 2\phi + E_2\sin 4\phi \quad (6\text{-}8)$$

$$E_0 = 1 - e^2/4 - (e^4/64)(3 + 5e^2/4) \quad (6\text{-}9)$$

$$E_1 = 3e^2/8 + (3e^4/32)(1 + 15e^2/32) \quad (6\text{-}10)$$

$$E_2 = (15e^4/256)(1 + 3e^2/4) \quad (6\text{-}11)$$

and M is the distance along the meridian from the Equator to ϕ. Combining (3-43) and (6-7), and applying to points 1 and 2,

$$a = \rho_1/(G - M_1) = \rho_2(G - M_2) \quad (6\text{-}12)$$

Solving for G and eliminating a,

$$G = (\rho_2 M_1 - \rho_1 M_2)/(\rho_2 - \rho_1) \quad (6\text{-}13)$$

The value of G for points 1 and 2 is compared with G calculated for points 2 and 3 (replacing subscripts accordingly), using first the spherical form, and if unsatisfactory then the ellipsoidal form. The inverse follows the pattern for the Lambert and Albers, except that ϕ may be found from an iterative inverse of (6-8):

$$\phi = (M + E_1 \sin 2\phi - E_2 \sin 4\phi)/E_0 \tag{6-14}$$

(There is also an inverse series available, but it is not used in this program.) The value of M is in turn found from another transposition of (3-43) and (6-7):

$$M = G - \rho/a \tag{6-15}$$

For determining the standard parallels, substitution from (3-43), (6-7), and (3-62) into (3-64), followed by transposition and arrangement into a Newton-Raphson iteration, leads to the following:

$$f(\phi_s) = (G - M_s)n(1 - e^2\sin^2\phi_s)^{1/2} - \cos\phi_s \tag{6-16}$$

$$f'(\phi_s) = -n[(1 - e^2\sin^2\phi_s)^{1/2}(dM_s/d\phi_s) + e^2\sin\phi_s\cos\phi_s(G - M_s)/ (1 - e^2\sin^2\phi_s)^{1/2}] + \sin\phi_s \tag{6-17}$$

where $(dM_s/d\phi_s)$ is readily found from (6-8) (compare (6-73)), using subscript s on M and ϕ, and (6-4) is used to increment ϕ_s. This involves iteration for either sphere or ellipsoid, but initial values of 90° and -90° for the two standard parallels ϕ_s cover all cases, and lead to few iteration steps. The initial values are used in two separate iterative sequences until convergence of each.

(3) For the three polar azimuthal projections which are treated as limiting forms of the conics, no additional formulas are needed for this program. With a cone constant of 1 (north polar) or -1 (south polar), the formulas for the conics function without causing any indeterminate conditions, except for the scale at the pole of the Stereographic. Since this projection is conformal, true scale may be placed along any parallel on the polar Stereographic, but this parallel is indeterminate for a given unidentified map. Therefore the pole is the most suitable arbitrary latitude for true scale. Equation (3-50) is indeterminate for $\phi_s = \pm 90°$, but it is found that in this case

$$k' = [(1 + e)^{(1+e)}(1 - e)^{(1-e)}]^{1/2} \tag{6-18}$$

This value of k' may be used to find a from equations (3-46) and (3-49) for points j = 1, 2, or 3. Likewise, equation (3-48) is indeterminate for $\phi_s = \pm 90°$, but then k' = 1.

(4) For the polar Orthographic projection of the sphere,

$$f(\phi) = \pm \cos \phi \tag{6-19}$$

with the ± taking the sign of n. Thus, from (3-43),

$$a = \rho_j / \cos \phi_j \tag{6-20}$$

since the sign in (3-45) cancels the sign of n. After a is averaged for j = 1, 2, and 3, the projection is tested for fit using the forward formulas. If satisfactory, equations (3-53), (3-54), and (6-21) may be used for inverse calculations, where the ± takes the sign of n:

$$\phi = \pm \arccos (\rho/a) \tag{6-21}$$

(5) For the polar Gnomonic projection of the sphere, the description for the Orthographic applies, except that (6-19), (6-20), and (6-21) are replaced by the following, respectively, with the ± taking the sign of n.

$$f(\phi) = \cot \phi \tag{6-22}$$

$$a = \rho_j \tan \phi_j \tag{6-23}$$

$$\phi = \pm[\pi/2 - \arctan (\rho/a)] \tag{6-24}$$

(6) For the polar Vertical Perspective, the unknown point of perspective makes calculation more complicated than that for the Orthographic, but

$$\rho = -a(P - 1) \cos \phi/(\pm P + \sin \phi) \tag{6-25}$$

where P is the distance of the point of perspective from the center of the globe in radii, and the ± again takes the sign of n.

Since ρ is known from (3-20) for three values of ϕ, after calculation of (x_o, y_o), points 1 and 2 may be inserted into equation (6-25), one at a time, to produce two simultaneous equations which may be solved for P:

$$P = \pm(\rho_1 \sin \phi_1 \cos \phi_2 - \rho_2 \sin \phi_2 \cos \phi_1)/(\rho_1 \cos \phi_2 - \rho_2 \cos \phi_1) \tag{6-26}$$

where the ± is as above. Then P is calculated using points 1 and 3 in place of 1 and 2. The two values are averaged, and a is calculated from a direct inversion of equation (6-25). The parameters are used to test the projection. For inverse calculations, equation (3-53) is used for Θ and (3-54) for λ, while (6-27) and (6-28) are used for ϕ.

$$\phi = \pm(\pi/2 - z)$$

taking the sign of n (or of the pole). (6-27)

$$z = \arcsin \{[P - [1 - (\rho^2/a^2)(P + 1)/(P - 1)]^{1/2}]/[a(P - 1)/\rho + \rho/a(P - 1)]\} \quad (6\text{-}28)$$

If ρ as calculated from this ϕ in equation (6-25) is not the same as the ρ given, the arcsine is in the wrong quadrant, and z must be subtracted from π or 180°.

(7) For the spherical Transverse Mercator, expansion of the right side of equations (3-87) and (3-88) gives the following:

$$h(\phi,\lambda,\lambda_o) = (1/2)k \ln [(1 + \cos\phi \sin\lambda \cos\lambda_o - \cos\phi \cos\lambda \sin\lambda_o)/(1 - \cos\phi \sin\lambda \cos\lambda_o + \cos\phi \cos\lambda)] \quad (6\text{-}29)$$

$$g(\phi,\lambda,\lambda_o) = k \arctan_2[\sin\phi/(\cos\phi \cos\lambda \cos\lambda_o + \cos\phi \sin\lambda \sin\lambda_o)] \quad (6\text{-}30)$$

Using three-dimensional rectangular coordinates (the spherical versions of (3-120) through (3-123)) and other symbols for rotation, let

$$X = \cos\phi \cos\lambda \quad (6\text{-}31)$$

$$Y = \cos\phi \sin\lambda \quad (6\text{-}32)$$

$$Z = \sin\phi \quad (6\text{-}33)$$

$$A = X \cos\lambda_o + Y \sin\lambda_o \quad (6\text{-}34)$$

$$B = Y \cos\lambda_o - X \sin\lambda_o \quad (6\text{-}35)$$

Then, substituting in (6-29) and (6-30),

$$h(\phi,\lambda,\lambda_o) = (1/2)\, k \ln [(1 + B)/(1 - B)] \quad (6\text{-}36)$$

$$g(\phi,\lambda,\lambda_o) = k \arctan_2(Z/A) \tag{6-37}$$

Differentiating with respect to λ_o, then simplifying,

$$h' = -A/(1 - B^2) \tag{6-38}$$

$$g' = -ZB/(A^2 + Z^2) \tag{6-39}$$

These values, calculated for various points indicated by subscripts, may be substituted into (3-84) through (3-86).

The inverses of equations (3-80), (3-81), (6-36), and (6-37), for determining (ϕ,λ) from (x', y') and in turn from (x,y) using equations (3-39) and (3-40), are as follows:

$$F = e^{(x'/a)} \tag{6-40}$$

where e is the base of natural logarithms, 2.71828...

$$F_1 = (F - 1/F)/2 \tag{6-41}$$

$$F_2 = \cos(y'/a) \tag{6-42}$$

$$\phi = \pm\arcsin[(1 - F_2^{\,2})/(F_1^{\,2} + 1)]^{1/2} \tag{6-43}$$

taking the sign of y'.

$$\lambda = \arctan_2(F_1/F_2) + \lambda_o \tag{6-44}$$

(8) For the ellipsoidal Transverse Mercator, let

$$A' = (\lambda - \lambda_o)\cos\phi \tag{6-45}$$

$$(e')^2 = e^2/(1 - e^2) \tag{6-46}$$

$$C' = (e')^2\cos^2\phi \tag{6-47}$$

$$T = \tan^2\phi \tag{6-48}$$

The functions for equations (3-80) and (3-81) are as follows in the usual series form applying only to a 6° to 8° band of longitude:

letting $$N = 1/(1 - e^2\sin^2\phi)^{1/2} \tag{6-49}$$

$$h = N[A' + (1 - T + C')(A')^3/6 + (5 - 18T + T^2 + 72(C')^2 - 58(e')^2)(A')^5/120] \quad (6\text{-}50)$$

$$g = M + N \tan\phi\, [(A')^2/2 + (5 - T + 9C' + 4(C')^2)(A')^4/24 + (61 - 58T + T^2 + 600C' - 330(e')^2)(A')^6/720] \quad (6\text{-}51)$$

with M found from (6-8). Differentiation with respect to λ_o leads to

$$h' = -N \cos\phi\, [1 + (1 - T + C')(A')^2/2 + (5 - 18T + T^2 + 72(C')^2 - 58(e')^2)(A')^4/120] \quad (6\text{-}52)$$

$$g' = -N \sin\phi\, [A' + (5 - T + 9C' + 4(C')^2)(A')^3/6 + (61 - 58T + T^2 + 600C' - 330(e')^2)(A')^5/120] \quad (6\text{-}53)$$

For the inverse of (3-80), (3-81), (6-50), and (6-51), first a "footpoint latitude" ϕ_f may be found by successive substitution, using y'/a as the first trial ϕ_f:

$$\phi_f = (y'/a + E_1 \sin 2\phi_f - E_2 \sin 4\phi_f)/E_o \quad (6\text{-}54)$$

Using equations (6-45) through (6-49) with subscript f on ϕ, C', T, and N, and letting

$$D = x'\, N_f/a \quad (6\text{-}55)$$

then,

$$\phi = \phi_f - [\tan\phi_f/(N_f^2(1 - e^2))][D^2/2 - (5 + 3T_f + 10C_f' - 4(C_f')^2 - 9(e')^2)D^4/24 + (61 + 90T_f + 298C_f' + 45T_f^2 - 252(e')^2 - 3(C_f')^2)D^6/720] \quad (6\text{-}56)$$

$$\lambda = \lambda_o + [D - (1 + 2T_f + C_f')D^3/6 + (5 - 2C_f' + 28T_f - 3(C_f')^2 + 8(e')^2 + 24T_f^2)D^5/120]/\cos\phi_f \quad (6\text{-}57)$$

(9) For the ellipsoidal or spherical Polyconic, only the ellipsoidal formulas are used here, since they apply to the sphere without introducing problems if e = 0. For the functions in (3-80) and (3-81),

$$h = N \cot \phi \sin [(\lambda - \lambda_o) \sin \phi] \tag{6-58}$$

$$g = M + N \cot \phi \{1 - \cos [(\lambda - \lambda_o) \sin \phi]\} \tag{6-59}$$

obtaining M from (6-8) and N from (6-49).

Differentiating (6-58) and (6-59) with respect to λ_o,

$$h' = - N \cos \phi \cos [(\lambda - \lambda_o) \sin \phi] \tag{6-60}$$

$$g' = - N \cos \phi \sin [(\lambda - \lambda_o) \sin \phi] \tag{6-61}$$

Inverse formulas used for the Polyconic involve a Newton-Raphson iteration. Reusing some previous symbols, but obtaining N from (6-49), and M, E_o, E_1, and E_2 from (6-8) through (6-11),

$$A' = y'/a \tag{6-62}$$

$$B' = (x')^2/a^2 + (A')^2 \tag{6-63}$$

With $\phi = A'$ as the first trial ϕ,

$$C' = (\tan \phi)/N \tag{6-64}$$

$$M' = E_o - 2E_1 \cos 2\phi + 4E_2 \cos 4\phi \tag{6-65}$$

$$\Delta\phi = [2A'(C'M + 1) - 2M - (M^2 + B')C']/[e^2 \sin^2\phi\, (M^2 + B' - 2A'M)/ 2C' + 2(A' - M)(C'M' - 2/\sin 2\phi) - 2M'] \tag{6-66}$$

$$\lambda = [\arcsin (x'C'/a)]/\sin \phi + \lambda_o \tag{6-67}$$

For both forward and inverse equations, if $\phi = 0$ or $y' = 0$ the equations are indeterminate, but $h = \lambda - \lambda_o$, $g = 0$ (forward), and $\phi = 0$, $\lambda = x'/a + \lambda_o$ (inverse).

(10) For several oblique or equatorial azimuthal projections, namely those listed for equations (3-100) through (3-103), equations (3-84), (3-85), and (3-96) through (3-105) may be used. For the oblique Gnomonic, equation (3-117) uses similar functions.

Expanding and rewriting equations (3-98), (3-99), and (3-104) in terms of equations (6-31) through (6-35),

$$h = k'B \tag{6-68}$$

$$g = k'(Z \cos \phi_o - A \sin \phi_o) \tag{6-69}$$

$$\cos z = A \cos \phi_o + Z \sin \phi_o \tag{6-70}$$

Differentiating h and g with respect to λ_o and ϕ_o, after simplification and noting similarities between formulas for various azimuthal projections,

$$h'_{\lambda_o} = -Bk \sin \phi_o + (Z \cos \phi_o - A \sin \phi_o)\partial k'/\partial\lambda_o \tag{6-71}$$

$$g'_{\lambda_o} = Ak' + B\partial k'/\partial\lambda_o \tag{6-72}$$

$$h_{\phi_o} = B\partial k'/\partial\phi_o \tag{6-73}$$

$$g'_{\phi_o} = -k'\cos z + (Z \cos \phi_o - A \sin \phi_o)\partial k'/\partial\lambda_o \tag{6-74}$$

where $\partial k'/\partial\lambda_o = hf \cos \phi_o$ (6-75)

$$\partial k'/\partial\phi_o = gf \tag{6-76}$$

$f = 1/\cos z$ for Gnomonic (6-77)

$f = -1/(1 + \cos z)$ for Stereographic (6-78)

$f = 0$ for Orthographic (6-79)

$f = (z \cos z - \sin z)/(z \sin^2 z)$ for Azimuthal Equidistant (6-80)

$f = -1/[2(1 + \cos z)]$ for Lambert Azimuthal Equal-Area (6-81)

The above projections are tested one at a time, although the Gnomonic and Stereographic are handled differently from the rest, as described earlier. For the inverse formulas, another pattern of equations satisfactorily applies to all the non-polar azimuthals, including the Vertical Perspective. After x' and y' are found from (3-39) and (3-40) for a given point,

$$\rho = [(x')^2 + (y')^2]^{1/2} \tag{6-82}$$

$$R_1 = \rho/a \tag{6-83}$$

$$\phi = \arcsin [\cos z \sin \phi_o + (y' \sin z \cos \phi_o/\rho)] \tag{6-84}$$

but if $\rho = 0$, $\phi = \phi_o$.

$$B' = \cos z - \sin \phi_o \sin \phi \tag{6-85}$$

$$\lambda = \lambda_o + \arctan_2(x' \sin z \cos \phi_o/(B'\rho)) \tag{6-86}$$

but if B' = 0 and x' = 0 (or if $\rho = 0$) $\lambda = \lambda_o$.

For z,

$$z = \arcsin R_1 \qquad \text{for Orthographic} \qquad (6\text{-}87)$$

$$z = R_1 \qquad \text{for Azimuthal Equidistant} \qquad (6\text{-}88)$$

$$z = 2 \arcsin (R_1/2) \qquad \text{for Lambert Azimuthal Equal-Area} \qquad (6\text{-}89)$$

$$z = \arcsin \{[P - (1 - R_1^2(P + 1)/(P - 1))^{1/2}]/[(P - 1)/R_1 + R_1/(P - 1)]\} \qquad (6\text{-}90)$$

for the Vertical Perspective in which P is found from (3-129). If it is -1, the projection is Stereographic. For the Gnomonic inverse, (6-90) may be used with P = 0, simplified to

$$z = 2 \arctan (R_1/2) \qquad \text{for Stereographic} \qquad (6\text{-}91)$$

$$z = \arctan R_1 \qquad \text{for Gnomonic} \qquad (6\text{-}92)$$

7. DERIVATION OF MATRIX OPERATORS FOR LEAST-SQUARES COMPUTATION OF POLYNOMIAL COEFFICIENTS

In order to derive equations (2-14) through (2-17), the equation for least-squares error E is written for discrepancies in x-coordinates as follows (for y-coordinates, the derivation is analogous, as outlined after equation (7-19)):

$$E = \sum_{i=1}^{m} (x_i - x_i')^2 \qquad (7\text{-}1)$$

where x_i' are the coordinates for each of the m points based on the true analytic equations, and x_i are coordinates based on the polynomial series, equation (2-1), which is to be developed:

$$x = C_1 + C_2\lambda + C_3\phi + C_4\lambda^2 + C_5\lambda\phi + C_6\phi^2 + C_7\lambda^3 + C_8\lambda^2\phi + C_9\lambda\phi^2 + C_{10}\phi^3 + \ldots \qquad (2\text{-}1)$$

Differentiating equation (7-1) with respect to each coefficient, and setting equal to zero for minimum error, using Σ in place of Σ with limits,

$$\partial E/\partial C_1 = 2\Sigma(x_i - x_i')(1) = 0 \tag{7-2}$$

$$\partial E/\partial C_2 = 2\Sigma(x_i - x_i')(\lambda_i) = 0 \tag{7-3}$$

$$\partial E/\partial C_3 = 2\Sigma(x_i - x_i')(\phi_i) = 0 \tag{7-4}$$

$$\partial E/\partial C_4 = 2\Sigma(x_i - x_i')(\lambda_i^2) = 0 \tag{7-5}$$

and so forth. Simplifying and transposing equations (7-2) through (7-5), the equations become, respectively,

$$\Sigma x_i = \Sigma x_i' \tag{7-6}$$

$$\Sigma\lambda_i x_i = \Sigma\lambda_i x_i' \tag{7-7}$$

$$\Sigma\phi_i x_i = \Sigma\phi_i x_i' \tag{7-8}$$

$$\Sigma\lambda_i^2 x_i = \Sigma\lambda_i^2 x_i' \tag{7-9}$$

and so forth. Substituting from equation (2-1) into equations (7-6) through (7-9),

$$\Sigma C_1 + \Sigma C_2\lambda_i + \Sigma C_3\phi_i + \Sigma C_4\lambda_i^2 + \ldots = \Sigma x_i' \tag{7-10}$$

$$\Sigma C_1\lambda_i + \Sigma C_2\lambda_i^2 + \Sigma C_3\lambda_i\phi_i + \Sigma C_4\lambda_i^3 + \ldots = \Sigma\lambda_i x_i' \tag{7-11}$$

$$\Sigma C_1\phi_i + \Sigma C_2\lambda_i\phi_i + \Sigma C_3\phi_i^2 + \Sigma C_4\lambda_i^2\phi_i + \ldots = \Sigma\phi_i x_i' \tag{7-12}$$

$$\Sigma C_1\lambda_i^2 + \Sigma C_2\lambda_i^3 + \Sigma C_3\lambda_i^2\phi_i + \Sigma C_4\lambda_i^4 + \ldots = \Sigma\lambda_i^2 x_i' \tag{7-13}$$

and so forth. These equations may be rewritten in matrix form as follows:

$$\begin{bmatrix} m & \Sigma\lambda_i & \Sigma\phi_i & \Sigma\lambda_i^2 & \ldots \\ \Sigma\lambda_i & \Sigma\lambda_i^2 & \Sigma\lambda_i\phi_i & \Sigma\lambda_i^3 & \ldots \\ \Sigma\phi_i & \Sigma\lambda_i\phi_i & \Sigma\phi_i^2 & \Sigma\lambda_i^2\phi_i & \ldots \\ \Sigma\lambda_i^2 & \Sigma\lambda_i^3 & \Sigma\lambda_i^2\phi_i & \Sigma\lambda_i^4 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \bullet \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ \vdots \end{bmatrix} = \begin{bmatrix} \Sigma x_i' \\ \Sigma\lambda_i x_i' \\ \Sigma\phi_i x_i' \\ \Sigma\lambda_i^2 x_i' \\ \vdots \end{bmatrix} \tag{7-14}$$

The above derivation is given by Wu and Yang (1981). The following steps further simplify the operations. Calling the above matrix equation

$$Ma = b \tag{7-15}$$

the matrix M is equivalent to the following, which may be readily multiplied for proof:

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \cdots \lambda_m \\ \phi_1 & \phi_2 & \phi_3 & \phi_4 & \cdots \phi_m \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \cdots \lambda_m^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \bullet \begin{bmatrix} 1 & \lambda_1 & \phi_1 & \lambda_1^2 & \cdots \\ 1 & \lambda_2 & \phi_2 & \lambda_2^2 & \cdots \\ 1 & \lambda_3 & \phi_3 & \lambda_3^2 & \cdots \\ 1 & \lambda_4 & \phi_4 & \lambda_4^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \\ 1 & \lambda_m & \phi_m & \lambda_m^2 & \cdots \end{bmatrix} \tag{7-16}$$

Matrix b is similarly

$$b = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \cdots \lambda_m \\ \phi_1 & \phi_2 & \phi_3 & \phi_4 & \cdots \phi_m \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \cdots \lambda_m^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \bullet \begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \\ \vdots \end{bmatrix} \tag{7-17}$$

Identifying the right-hand matrix of equation (7-16) as A, it may be seen that one matrix in both (7-16) and (7-17) is the transpose A^T. Thus, equations (7-14) or (7-15) may be rewritten

$$[A^T A]a = A^T x' \tag{7-18}$$

where x' is the right-hand matrix of equation (7-17). To find a, equation (7-18) is inverted:

$$a = [A^T A]^{-1} A^T x' \tag{7-19}$$

From this, equations (2-14) through (2-16) are readily obtained, dropping the primes from x' in equation (7-17) for convenience, since the primes are used for other purposes in the main text. By substituting y, y', and C' for each x, x', and C, respectively, in equations (7-1) through (7-19), the solution is seen to be almost identical, yielding equations (2-14), (2-15), and (2-17).

8. DERIVATION OF FORMULAS FOR THE LEAST-SQUARES AFFINE LINEAR TRANSFORMATION OF ONE MAP TO FIT ANOTHER

In calculating equations (3-147) through (3-165), the error E in fitting m given coordinates (x_j,y_j) into m calculated coordinates (X_j,Y_j) is as follows:

$$E = \sum_{j=1}^{m} [(X_j - x_{aj})^2 + (Y_j - y_{aj})^2] \quad (8\text{-}1)$$

where the adjusted coordinates (x_{aj},y_{aj}) are obtained by the following linear-transformation formulas:

$$x_{aj} = a_1 x_j + a_2 y_j + a_3 \quad (3\text{-}164)$$

$$y_{aj} = a_4 x_j + a_5 y_j + a_6 \quad (3\text{-}165)$$

and a_1 through a_6 are the six constants to be determined. Setting the derivative of E in equation (8-1) with respect to each constant equal to zero, and using Σ in place of Σ with limits,

$$\partial E/\partial a_1 = -2\Sigma(X_j - a_1 x_j - a_2 y_j - a_3)x_j = 0 \quad (8\text{-}2)$$

$$\partial E/\partial a_2 = -2\Sigma(X_j - a_1 x_j - a_2 y_j - a_3)y_j = 0 \quad (8\text{-}3)$$

$$\partial E/\partial a_3 = -2\Sigma(X_j - a_1 x_j - a_2 y_j - a_3) = 0 \quad (8\text{-}4)$$

$$\partial E/\partial a_4 = -2\Sigma(Y_j - a_4 x_j - a_5 y_j - a_6)x_j = 0 \quad (8\text{-}5)$$

$$\partial E/\partial a_5 = -2\Sigma(Y_j - a_4 x_j - a_5 y_j - a_6)y_j = 0 \quad (8\text{-}6)$$

$$\partial E/\partial a_6 = -2\Sigma(Y_j - a_4 x_j - a_5 y_j - a_6) = 0 \quad (8\text{-}7)$$

Cancelling out the factors -2, expanding, and transposing, equations (8-2) through (8-4) may be written in matrix form:

$$\begin{bmatrix} \Sigma x_j^2 & \Sigma x_j y_j & \Sigma x_j \\ \Sigma x_j y_j & \Sigma y_j^2 & \Sigma y_j \\ \Sigma x_j & \Sigma y_j & m \end{bmatrix} \bullet \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \Sigma X_j x_j \\ \Sigma X_j y_j \\ \Sigma X_j \end{bmatrix} \quad (8\text{-}8)$$

To reduce computation, the m values of x_j and y_j are averaged:

$$\bar{x} = \Sigma x_j / m \quad (8\text{-}9)$$

$$\bar{y} = \Sigma y_j / m \quad (8\text{-}10)$$

and the averages are subtracted for each value of x_j and y_j in equation (8-8):

$$\begin{bmatrix} \Sigma(x_j - \bar{x})^2 & \Sigma(x_j - \bar{x})(y_j - \bar{y}) & 0 \\ \Sigma(x_j - \bar{x})(y_j - \bar{y}) & \Sigma(y_j - \bar{y})^2 & 0 \\ 0 & 0 & m \end{bmatrix} \bullet \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \Sigma X_j(x_j - \bar{x}) \\ \Sigma X_j(y_j - \bar{y}) \\ \Sigma X_j \end{bmatrix} \quad (8\text{-}11)$$

The solution for a_3 is seen to be direct:

$$a_3 = \Sigma X_j / m \quad (8\text{-}12)$$

The remainder of the matrices of (8-11) may be rewritten in second-order format. A 2 x 2 symmetric matrix may be inverted as follows:

$$\begin{bmatrix} C & B \\ B & A \end{bmatrix}^{-1} = \frac{1}{AC - B^2} \begin{bmatrix} A & -B \\ -B & C \end{bmatrix} \quad (8\text{-}13)$$

Therefore, inverting the second-order simplification of (8-11),

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} A/D & -B/D \\ -B/D & C/D \end{bmatrix} \bullet \begin{bmatrix} E \\ F \end{bmatrix} \quad (8\text{-}14)$$

where symbols are as defined in equations (3-154) through (3-159), and m = 9. These matrices expand to equations (3-148) and (3-149). Constant

a_3 must be adjusted further so that x and y do not have to be subtracted before x_j and y_j may be used in equation (3-164). This adjustment is found by writing equation (3-164) as follows:

$$
\begin{aligned}
x_{aj} &= a_1(x_j - \bar{x}) + a_2(y_j - \bar{y}) + a_3 \\
&= a_1 x_j + a_2 y_j + (a_3 - a_1\bar{x} - a_2\bar{y})
\end{aligned}
\tag{8-15}
$$

Thus a_3 as found from equation (8-12) must be reduced by $(a_1\bar{x} + a_2\bar{y})$ to give equation (3-150).

The corresponding transformations of equations (8-5) through (8-7) result in the following equivalent of (8-12) and (8-14) for constants a_4, a_5, and a_6:

$$a_6 = \Sigma Y_j/m \tag{8-16}$$

$$\begin{bmatrix} a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} A/D & -B/D \\ -B/D & C/D \end{bmatrix} \bullet \begin{bmatrix} G \\ H \end{bmatrix} \tag{8-17}$$

where symbols are as defined in equations (3-154) through (3-157), (3-160), and (3-161), and m = 9. Equations (3-151) and (3-152) may be found from equation (8-17), with an adjustment to a_6 like that made to a_3 to obtain (3-153) from (8-16).

9. DERIVATION OF FORMULAS FOR A LOW-ERROR CONFORMAL PROJECTION FOR THE 50 STATES

(see also section 5a; previously published in Snyder (1984))

In deriving the formulas for determining the coefficients in equation (4-65), using least squares, it is convenient to use deMoivre's theorem to rewrite that equation:

$$
\begin{aligned}
x + iy &= \sum_{j=1}^{n} (A_j + iB_j)\,\rho^j (\cos j\Theta + i \sin j\Theta) \\
&= \sum_{j=1}^{n} \rho^j [(A_j \cos j\Theta - B_j \sin j\Theta) + i(A_j \sin j\Theta + B_j \cos j\Theta)]
\end{aligned}
\tag{9-1}
$$

where ρ and Θ are polar coordinates corresponding to x' and y' as follows:

$$\rho = [(x')^2 + (y')^2]^{1/2} \tag{9-2}$$

$$\Theta = \arctan_2(y'/x') \tag{9-3}$$

using $\arctan_2$ as the equivalent of the Fortran ATAN2 function rather than ATAN. The real and imaginary portions of equation (9-1) may be readily separated:

$$x = \sum_{j=1}^{n} \rho^j (A_j \cos j\Theta - B_j \sin j\Theta) \tag{9-4}$$

$$y = \sum_{j=1}^{n} \rho^j (A_j \sin j\Theta + B_j \cos j\Theta) \tag{9-5}$$

For a conformal projection of the sphere, one form of the equation for the scale factor k is

$$k = [(\partial x/\partial\phi)^2 + (\partial y/\partial\phi)^2]^{1/2}/R \tag{8-6}$$

where R is the radius of the sphere at the nominal scale of the map, and ϕ is the latitude of the point.

This is applied to the ellipsoid by replacing R as follows:

$$k = [(\partial x/\partial\phi)^2 + (\partial y/\partial\phi)^2]^{1/2}(1 - e^2\sin^2\phi)^{3/2}/[a(1 - e^2)] \tag{9-7}$$

where a is the semimajor axis and e the eccentricity of the ellipsoid.

To solve for k, equations (9-2) and (9-3) may be differentiated and rearranged. Squaring (9-2), differentiating, and substituting from (9-3):

$$2\rho d\rho = 2x'\,dx' + 2y'\,dy'$$

$$= 2\rho\cos\Theta\,dx' + 2\rho\sin\Theta\,dy' \tag{9-8}$$

Transposing and differentiating (9-3), then multiplying through by $\cos\Theta$:

$$dy' = x'\sec^2\Theta\,d\Theta + \tan\Theta\,dx' \tag{9-9}$$

$$\cos\Theta\,dy' = \rho\,d\Theta + \sin\Theta\,dx' \tag{9-10}$$

For use in equations (9-13) and (9-14), the following may now be obtained:

$$\partial\rho/\partial\phi = (\partial\rho/\partial x')(\partial x'/\partial\phi) + (\partial\rho/\partial y')(\partial y'/\partial\phi)$$

$$= \cos\Theta\,(\partial x'/\partial\phi) + \sin\Theta\,(\partial y'/\partial\phi) \qquad (9\text{-}11)$$

$$\partial\Theta/\partial\phi = (\partial\Theta/\partial x')(\partial x'/\partial\phi) + (\partial\Theta/\partial y')(\partial y'/\partial\phi)$$

$$= (-\sin\Theta/\rho(\partial x'/\partial\phi) + (\cos\Theta/\rho)(\partial y'/\partial\phi) \qquad (9\text{-}12)$$

Differentiating equation (9-4) and substituting from (9-11) and (9-12),

$$\partial x/\partial\phi = \sum_{j=1}^{n} j\rho^{j-1}(A_j \cos j\Theta - B_j \sin j\Theta)\,\partial\rho/\partial\phi$$

$$+ \sum_{j=1}^{n} \rho^{j}(-jA_j \sin j\Theta - jB_j \cos j\Theta)\,\partial\Theta/\partial\phi$$

$$= \sum_{j=1}^{n} j\rho^{j-1}[A_j \cos(j-1)\Theta - B_j \sin(j-1)\Theta]\partial x'/\partial\phi$$

$$- \sum_{j=1}^{n} j\rho^{j-1}[A_j \sin(j-1)\Theta + B_j \cos(j-1)\Theta]\,\partial y'/\partial\phi \qquad (9\text{-}13)$$

Similarly from equations (9-5), (9-11), and (9-12),

$$\partial y/\partial\phi = \sum_{j=1}^{n} j\rho^{j-1}[A_j \sin(j-1)\Theta + B_j \cos(j-1)\Theta]\partial x'/\partial\phi$$

$$+ \sum_{j=1}^{n} j\rho^{j-1}[A_j \cos(j-1)\Theta - B_j \sin(j-1)\Theta]\,\partial y'/\partial\phi \qquad (9\text{-}14)$$

Substituting equations (9-13) and (9-14) into (9-7), equation (5-13) is obtained, where F_1 and F_2 are convenient terms identified by equations (5-11) and (5-12), and

$$k' = [(\partial x'/\partial\phi)^2 + (\partial y'/\partial\phi)^2]^{1/2}(1 - e^2\sin^2\phi)^{3/2}/[a(1-e)^2)] \qquad (9\text{-}15)$$

By comparison of equations (9-15) and (9-7), it is seen that k' is the scale factor on the initial conformal projection of equation (4-65). Since this was made the Oblique Stereographic projection for the GS50 projection, k' may be found directly from equation (5-8), for which the derivation is not shown.

Equations (5-11) through (5-13) may be combined in complex notation to provide (5-28), which may be readily expanded to (5-11) through (5-13) for proof.

To minimize scale error (k - 1), the least-squares principle states that, for the best fit to m points, and omitting the weighting shown in equation (5-40),

$$E = \sum_{p=1}^{m} (k_p - 1)^2 = \text{minimum} \tag{9-16}$$

For a given coefficient A_q or B_q, using subscripts q to distinguish from others, and assigning symbol f to the differential:

$$f(A_q) = \partial E/\partial A_q = 2\sum_{p=1}^{m} (k_p - 1)\, \partial k_p/\partial A_q \tag{9-17}$$

$$f(B_q) = \partial E/\partial B_q = 2\sum_{p=1}^{m} (k_p - 1)\, \partial k_p/\partial B_q \tag{9-18}$$

which are the same as equations (5-16) and (5-17). For a minimum E, equations (9-17) and (9-18) must eventually equal zero.

Differentiating equations (5-11) through (5-13) with respect to A_q,

$$\partial k/\partial A_q = (1/2)\, k' \,(F_1{}^2 + F_2{}^2)^{-1/2} (2F_1 \partial F_1/\partial A_q + 2F_2 \partial F_2/\partial A_q)$$

$$= (k')^2 (F_1 \partial F_1/\partial A_q + F_2 \partial F_2/\partial A_q)/k \tag{9-19}$$

$$\partial F_1/\partial A_q = q\rho^{q-1} \sin (q - 1)\theta \tag{9-20}$$

and

$$\partial F_2/\partial A_q = q\rho^{q-1} \cos (q - 1)\theta \tag{9-21}$$

Combining (9-15), (9-16), (5-28), and (9-19) through (9-21) yields equation (5-14). Similarly, differentiating (5-11) through (5-13) with respect to B_q, equation (5-15) is obtained.

Since equations (9-17) and (9-18) are nonlinear, Newton-Raphson iteration of simultaneous equations may be used for computation of coefficients. Coefficient B_1 merely rotates the map (changing other B coefficients); therefore, it is held at zero. There are then (2n - 1) coefficients to find, requiring (2n - 1) simultaneous equations. These equations take the forms of equations (5-24) and (5-25), with appropriate starting trial values of $A_1 = 1$ and zero for all other coefficients.

For equations (5-24) and (5-25), the following differentials are needed. From equation (9-17), treating g and q as equal or different subscripts,

$$\partial f(A_q)/\partial A_g = \partial^2 E/\partial A_q \partial A_g = 2 \sum_{p=1}^{m} [(\partial k_p/\partial A_g)(\partial k_p/\partial A_q) + (k_p - 1)(\partial^2 k_p/\partial A_q \partial A_g)] \tag{9-22}$$

Differentiating equation (5-14) with respect to A_g,

$$\partial^2 k/\partial A_q \partial A_g = -(1/k)[(\partial k/\partial A_q)(\partial k/\partial A_g) - (k')^2 \, qg\rho^{q+g-2} \cos (q - g)\Theta] \tag{9-23}$$

Substituting in (9-22), equations (5-18) and (5-19) are obtained. Differentiation of equation (9-17) with respect to B_g, (9-18) with respect to A_g, and (9-18) with respect to B_g, each involving (5-14) or (5-15), leads to (5-20) and (5-21).

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INDEX