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DRAWDOWN IN WELLS RESPONDING TO CYCLIC PUMPING

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Introduction

A report by Thompson (1928) on the ground-water supplies of the Atlantic City, N. J., region, presents data in the form of a graph showing, for a selected location, how water levels change in response to changes in the average daily pumping rate for part of the region. In describing the nature of the looped curves shown on the graph Thompson applied the term "hysteresis" and discussed possible explanations for their existence. However, Thompson was concerned over the fact that the "hysteresis" curves did not return to the origin but instead returned to points indicating progressive net lowering of water levels each year. A subsequent report on the Atlantic City region by Barksdale, Sundstrom, and Brunstein (1936) again presented the above-described graph, brought up to date, but in their discussion the authors were still concerned over what might be the full explanation for the evident progressive net lowering of water level each year.

The uncertainty connected with the explanation of the features exhibited by these graphs prompted the senior writer to devise and analyze a theoretical problem that would parallel the type of cyclic pumping phenomena already observed in the Atlantic City region. Part A of this Ground Water Note presents the results of the senior writer's study. These results were never published but have received some distribution in the Ground Water Branch in the form of a typed memorandum dated March 2, 1938. The "hysteresis" curves shown in figure 1 not only are of scientific interest but their general method of development is based on an assumed sequence of events that may be closely paralleled in a number of field areas where ground water is used for irrigation, or where seasonal demands on a public ground-water supply are of a definite cyclic nature.

In Part B of this Ground Water Note, R. H. Brown has prepared an expanded and slightly revised version of a "type" problem worked out by the senior writer, and originally published in the Water Resources Bulletin for May 1946 (p. 121-122). This study, which also concerns drawdowns resulting from cyclic pumping, was prompted by questions posed by W. E. Hale in connection with a field problem encountered in the Iowa ground-water investigations. Results of this study may also find application in analogous field situations.

Part A: "Hysteresis" Curves

Consider a well pumping at a fluctuating rate from an infinitely broad artesian aquifer. All water is therefore taken from storage. In order to compute the resulting drawdown in a neighboring well, assume that the well pumps in 3-month periods according to the following annual schedule: First quarter, 200 gallons per minute; second quarter, 400 gallons per minute; third quarter, 600 gallons per minute; fourth quarter, 400 gallons per minute.

The effects will be the same as if on January 1 one well begins pumping 200 gpm, on April 1 another such well begins pumping 200 gpm, on July 1 a third well begins at the same rate, on October 1 a well begins recharging 200 gpm, on January 1 of next year another well begins recharging 200 gpm, on April 1 another well begins discharging 200 gpm, and so on. The drawdown at any time will be the sum of the effects of all these hypothetical wells. Assume that the aquifer has a coefficient of transmissibility T = 20,000 and a coefficient of storage, S = 0.0003, and that an observation well is 2,000 feet from the pumped well.

The drawdown, in the observation well, due to any of these hypothetical wells, is

$$s = (114.6\ Q/T)\ W(u)$$

where $u = 1.87\ r^2S/Tt$

(See Ground Water Note No. 5 for nomenclature.)

In this case $u = (1.87 \times 4{,}000{,}000 \times 0.0003)/20{,}000 \times 91t$

$= 0.00125/t$

in which t is expressed in quarter years.

When u is small, as it is in this case where t is greater than 1 -- that is, after more than 3 months of pumping -- the following approximate expression is very nearly correct (the expression log u indicates the common logarithm to the base 10, and ln u indicates the natural logarithm to the base e):

$$W(u) = -0.577 - \ln u$$

$$= -0.577 - 2.3 \log u = 2.3\ (0.251 + \log u)$$

Whence,
$$s = -\frac{114.6 \times 200 \times 2.3}{20{,}000}\left(0.251 + \log \frac{0.00125}{t}\right)$$

$$s = -2.64\ (0.251 + 0.0969 - 3 - \log t)$$

$$= 2.64\ (2.6521 + \log t)$$

This is the drawdown in the observation well caused by any one of the hypothetical wells when t is expressed in quarter years for the time since the well began pumping. At the end of the first quarter, the above expression gives the total drawdown inasmuch as only one well has been pumping.

At the end of the second quarter the expression for the total drawdown is:

$$s_2 = 2.64\ (2.6521 + \log 2 + 2.6521 + \log 1)$$

$$= 2.64\ (2 \times 2.6521 + \log 2 + \log 1),$$

as one of the 200-gallon wells has pumped for two quarters and one for one quarter.

At the end of the third quarter and fourth quarter, respectively

$$s_3 = 2.64\ (3 \times 2.6521 + \log 3 + \log 2 + \log 1)$$

$$s_4 = 2.64\ (3 \times 2.6521 - 2.6521 + \log 4 + \log 3 + \log 2 - \log 1)$$

The minus signs result from the fact that the well introduced at the beginning of the fourth quarter was a recharge well.

Further,

$$s_5 = 2.64\ (3 \times 2.6521 - 2 \times 2.6521 + \log 5 + \log 4 + \log 3$$
$$- \log 2 - \log 1)$$

$$s_6 = 2.64\ (4 \times 2.6521 - 2 \times 2.6521 + \log 6 + \log 5 + \log 4$$
$$- \log 3 - \log 2 + \log 1)$$

$$= 2.64\ \left(2 \times 2.6521 + \log \frac{6 \times 5 \times 4 \times 1}{3 \times 2}\right)$$

The quantity inside the parentheses is seen to be made of log terms and multiples of the constant 2.6521. By inspection it will be seen that the constant terms for the respective quarters of any year are:

1st quarter	2.6521
2nd quarter	5.3042
3rd quarter	7.9563
4th quarter	5.3042

If n is the number of the quarter, the log terms for this quarter will be

$$\log \frac{n\,(n-1)\,(n-2)\,(n-5)\,(n-6)\,(n-9)\,\ldots}{(n-3)\,(n-4)\,(n-7)\,(n-8)\,\ldots}$$

The series continues until the last term entered is 1. Thus the following expressions may be set up. All final values for drawdown are to be multiplied by 2.64.

$s_1 = 2.65 + \log 1 = 2.65 + 0 = 2.65$

$s_2 = 5.30 + \log 2\cdot1 = 5.30 + 0.30 = 5.60$

$s_3 = 7.96 + \log 3\cdot2\cdot1 = 7.96 + 0.78 = 8.74$

$s_4 = 5.30 + \log (4\cdot3\cdot2)/1 = 5.30 + 1.38 = 6.68$

$s_5 = 2.65 + \log (5\cdot4\cdot3)/(2\cdot1) = 2.65 + 1.48 = 4.13$

$s_6 = 5.30 + \log (6\cdot5\cdot4\cdot1)/(3\cdot2) = 5.30 + 1.30 = 6.60$

$s_7 = 7.96 + \log (7\cdot6\cdot5\cdot2\cdot1)/(4\cdot3) = 7.96 + 1.54 = 9.50$

$s_8 = 5.30 + \log (8\cdot7\cdot6\cdot3\cdot2)/(5\cdot4\cdot1) = 5.30 + 2.00 = 7.30$

$s_9 = 2.65 + \log (9\cdot8\cdot7\cdot4\cdot3)/(6\cdot5\cdot2\cdot1) = 2.65 + 2.00 = 4.65$

$s_{10} = 5.30 + \log (10\cdot9\cdot8\cdot5\cdot4\cdot1)/(7\cdot6\cdot3\cdot2) = 5.30 + 1.76 = 7.06$

$s_{11} = 7.96 + \log (11\cdot10\cdot9\cdot6\cdot5\cdot2\cdot1)/(8\cdot7\cdot4\cdot3) = 7.96 + 1.95 = 9.91$

$s_{12} = 5.30 + \log (12\cdot11\cdot10\cdot7\cdot6\cdot3\cdot2)/(9\cdot8\cdot5\cdot4\cdot1) = 5.30 + 2.36 = 7.66$

$s_{13} = 2.65 + \log 214 = 2.65 + 2.33 = 4.98$

$s_{14} = 5.30 + \log 113.5 = 5.30 + 2.05 = 77.35$

$s_{15} = 7.96 + \log 167 = 7.96 + 2.20 = 10.16$

$s_{16} = 5.30 + \log 413 = 5.30 + 2.62 = 7.92$

$s_{17} = 2.65 + \log 369 = 2.65 + 2.57 = 5.22$

$s_{18} = 5.30 + \log 189 = 5.30 + 2.28 = 7.58$

$s_{19} = 7.96 + \log 269 = 7.96 + 2.43 = 10.39$

$s_{20} = 5.30 + \log 650 = 5.30 + 2.81 = 8.11$

$s_{21} = 2.65 + \log 566 = 2.65 + 2.75 = 5.40$

$s_{22} = 5.30 + \log 284 = 5.30 + 2.45 = 7.75$

$s_{23} = 7.96 + \log 395 = 7.96 + 2.60 = 10.56$

$s_{24} = 5.30 + \log 940 = 5.30 + 2.97 = 8.27$

$s_{25} = 2.65 + \log 809 = 2.65 + 2.91 = 5.56$

$s_{26} = 5.30 + \log 399 = 5.30 + 2.60 = 7.90$

$s_{27} = 7.96 + \log 545 = 7.96 + 2.74 = 10.70$

$s_{28} = 5.30 + \log 1280 = 5.30 + 3.11 = 8.41$

$s_{29} = 2.65 + \log 1092 = 2.65 + 3.04 = 5.69$

$s_{30} = 5.30 + \log 533 = 5.30 + 2.73 = 8.03$

$s_{31} = 7.96 + \log 720 = 7.96 + 2.86 = 10.82$

$s_{32} = 5.30 + \log 1675 = 5.30 + 3.22 = 8.52$

In the accompanying figure 1 these values, multiplied by 2.64, are plotted consecutively after the method of the Atlantic City reports. It will be noted that the curves are looped "hysteresis" curves, and that successive points for the same rate of pumping come closer and closer together. If natural discharge could be checked or recharge increased, the annual curves eventually would nearly coincide -- that is, the successive loops would be practically coincident, although the loops would remain. If water were being withdrawn from an outcrop area without replacement, the successive points for the same pumping rate would tend to become equally distant.

Part B: Drawdown in Pumped Well after Cyclic Pumping Periods

Consider a well pumped according to a certain regimen in virtually definite cycles, comprising a given pumping period followed by a given period of rest. The hydrograph for such a well would have the general appearance shown by figure 2, where the circled numbers 1, 2, and 3 identify the first, second and third cycles; where points a, b, and c indicate the beginning of the first, second and third pumping periods; and points d, e, and f designate the ends of the pumping periods, or the beginning of the rest or recovery periods, within the three cycles. Examination of figure 2 should suggest that if it is desired to write an expression for the drawdown in the pumped well at the end of, say, the third cycle, the expression can be built up by summing expressions for the individual drawdown effects that would be caused by turning on a pumped well at the times represented by points a, b, and c, and by turning on a recharging well at the times represented by points d, e, and f. Each well operates continuously, after it is turned on, and at the rate Q. All wells are at one and the same location. Thus an expression for the net drawdown in the real pumped well at the end of the third cycle may be stated as follows:

$$s_3 = s_{d_1} + s_{d_2} + s_{d_3} - s_{r_1} - s_{r_2} - s_{r_3} \qquad (1)$$

where the subscript and numbered d's refer to the discharging wells turned on during the first, second, and third cycles, respectively, and the subscript and numbered r's refer similarly to the recharging wells. Inasmuch as the total drawdown is being computed only at the cyclically pumped well (that is, the radius r in the basic Theis equation is small) the approximate form of the Theis (1935, p. 522 or 1952, p. 7) equation can be used as long as values of time are not small. Note further that if the component drawdown s_{d_1} is paired with the component negative drawdown

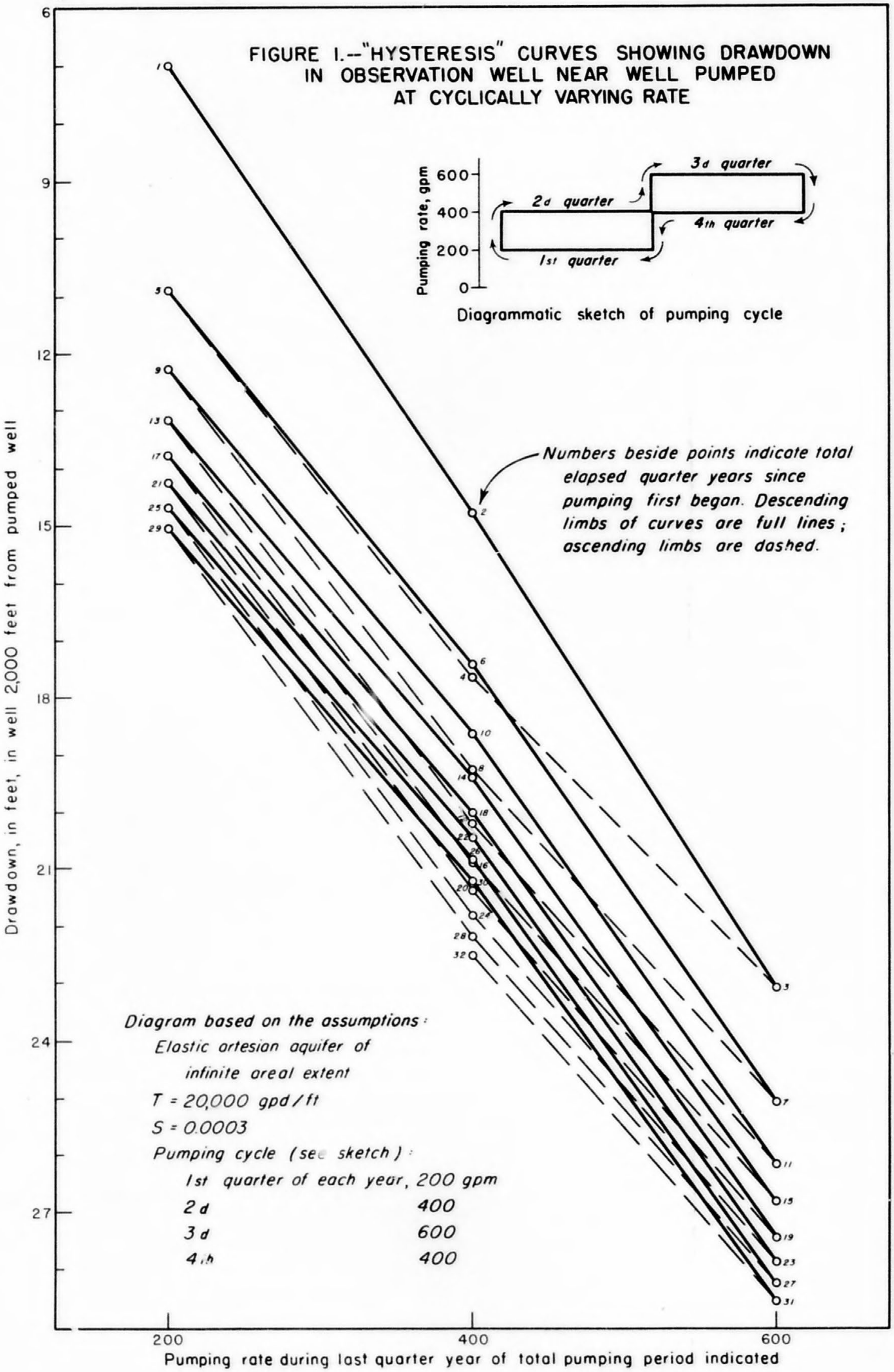

FIGURE 1.--"HYSTERESIS" CURVES SHOWING DRAWDOWN IN OBSERVATION WELL NEAR WELL PUMPED AT CYCLICALLY VARYING RATE

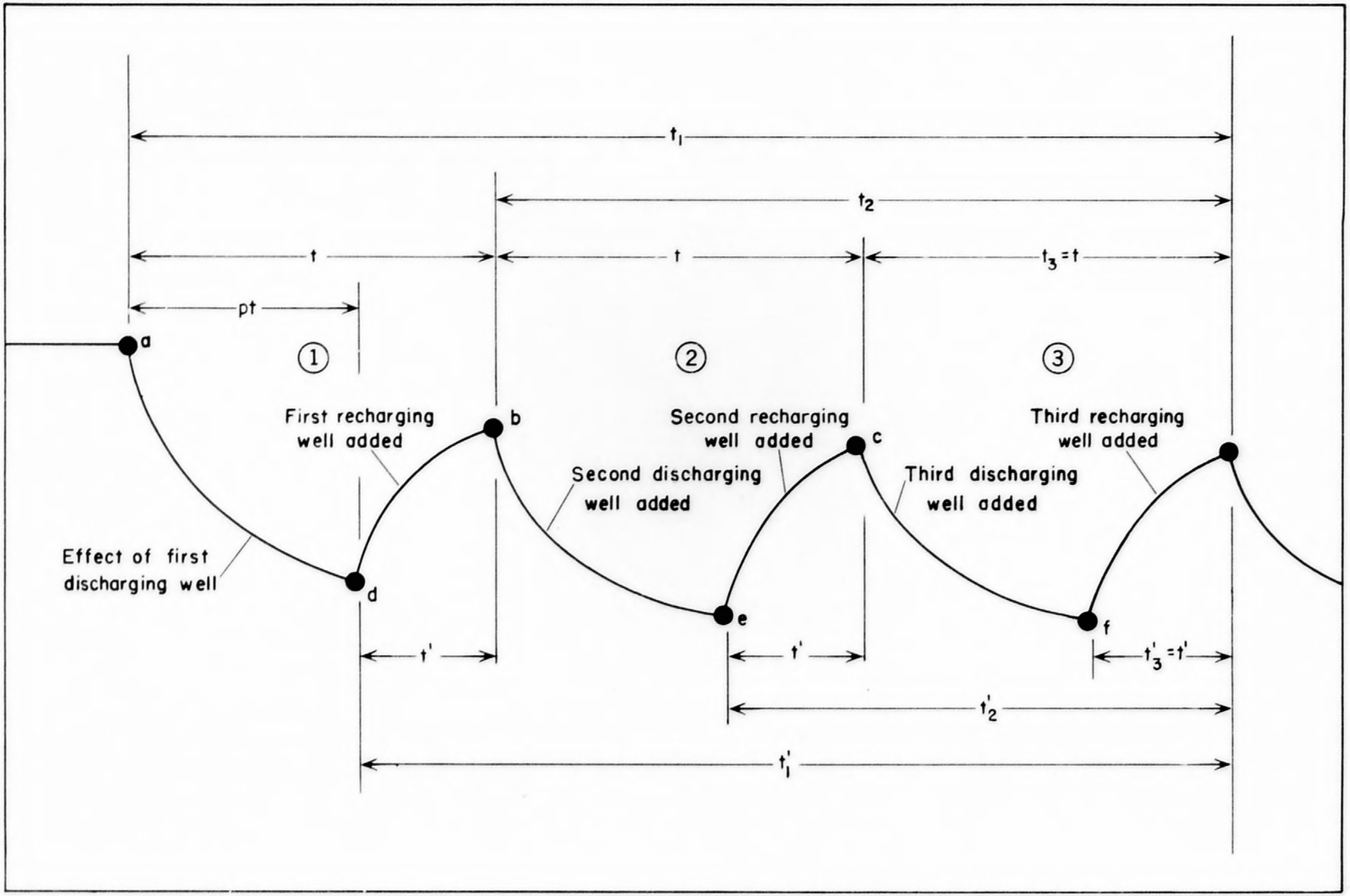

Figure 2 -- Hydrograph for cyclically pumped well showing nomenclature for time factors

s_{r_1} the expression for their algebraic sum is identical with the Theis (1935 or 1952) recovery formula. Thus

$$s_{d_1} - s_{r_1} = \frac{264Q}{T} \log \frac{t_1}{t'_1} \qquad (2)$$

Similarly

$$s_{d_2} - s_{r_2} = \frac{264Q}{T} \log \frac{t_2}{t'_2} \qquad (3)$$

and

$$s_{d_3} - s_{r_3} = \frac{264Q}{T} \log \frac{t_3}{t'_3} \qquad (4)$$

Equation (1), which evidently is the sum of equations (2), (3), and (4), can therefore be rewritten as

$$s_3 = \frac{264Q}{T} \log \frac{t_1 \cdot t_2 \cdot t_3}{t'_1 \quad t'_2 \quad t'_3} \qquad (5)$$

The form of equation (5) should demonstrate how simply the expression for the drawdown after any number of cycles may be written. Thus the drawdown in the pumped well just after completion of the n^{th} cycle and just before the beginning of the $(n + 1)$ pumping period is

$$s_n = \frac{264Q}{T} \log \frac{t_1 \cdot t_2 \cdot t_3 \cdot \ldots\ldots\ldots t_n}{t'_1 \quad t'_2 \quad t'_3 \quad \ldots\ldots\ldots t'_n} \qquad (6)$$

Looking at figure 2, again observe that each cycle spans a time interval t. Furthermore, within each cycle, the rest interval spans a time t' and pumping occurs over a time span pt, where p represents the pumping fraction of the cycle (that is, the ratio of pumping time to the period of the cycle, t). It should be clear, therefore, that each time factor appearing in equation (6) can be replaced by an equivalent factor written in terms of p and t.

Referring to figure 2, and the time factors for the <u>discharging</u> wells, evidently t_1 can be replaced by nt, and for the n^{th} cycle t_n is replaced simply by t. Note, however, that the new time factor for the last two cycles can be written as $2t$, for the last three cycles $3t$, and so on until all cycles are included by the factor nt already mentioned.

In similar fashion substitutions are made for the time factors related to the <u>recharging</u> wells. Thus t'_1 can be replaced by $(nt - pt)$, and for the n^{th} cycle t'_n is replaced by $(t - pt)$. Again note that the new time factor for the last two cycles can be written as $(2t - pt)$, for the last three cycles $(3t - pt)$, and so on until all cycles are included by the factor $(nt - pt)$ already mentioned. Now it is possible, therefore, to rewrite equation (6) as follows:

$$s_n = \frac{264Q}{T} \log \frac{nt \ \ldots\ldots\ldots\ldots\ldots\ldots \cdot 3t \cdot 2t \cdot t}{(nt - pt)\ldots\ldots\ldots(3t - pt)(2t - pt)(t - pt)} \qquad (7)$$

Cancelling the t in each term in the numerator with the t that can be factored from each term in the denominator, and reversing the order of writing the two infinite series there is left the relation

$$s_n = \frac{264Q}{T} \log \frac{1 \cdot 2 \cdot 3 \cdot \ldots\ldots\ldots n}{(1 - p)(2 - p)(3 - p) \ldots (n - p)} \tag{8}$$

Therefore, the absolute value of time is seen to have no significance and the only important factor is the number of cycles, whether those cycles be measured in minutes or days or years or any other time unit.

It will be helpful now to investigate the graphical tools that may aid in solving equation (8). Observe that the equation may also be written in the form

$$\frac{s_n T}{264Q} = \log \frac{1}{1 - p} + \log \frac{2}{2 - p} + \log \frac{3}{3 - p} + \ldots\ldots + \log \frac{n}{n - p} \tag{9}$$

Evidently the change in the factor $s_n T/264Q$, occurring during the n^{th} cycle, is represented by the last term of the series expression, that is, $\log \frac{n}{n - p}$. If a semilog plot is prepared, similar to figure 3, with values of $s_n T/264Q$ on the arithmetic scale and values of the number of cycles, n, on the logarithmic scale, the increment on the n-axis during the n^{th} cycle (that is, between points n - 1 and n) is log n - log (n-1), or $\log \frac{n}{n - 1}$. Therefore the limiting slope of this semilog plot -- that is, the slope across the n^{th} cycle -- is given by the change in the factor $s_n T/264Q$ divided by the change in log n. In mathematical terms, then, this limiting slope for the n^{th} cycle is

$$\frac{\Delta\left(\frac{s_n T}{264Q}\right)}{\Delta(\log n)} = \frac{\log \frac{n}{n - p}}{\log \frac{n}{n - 1}} = \frac{\log \left(1 + \frac{p}{n - p}\right)}{\log \left(1 + \frac{1}{n - 1}\right)}$$

$$= \frac{2.3 \left(\frac{p}{n - p}\right) - \frac{1}{2}\left(\frac{p}{n - p}\right)^2 + \frac{1}{3}\left(\frac{p}{n - p}\right)^3 - \ldots}{2.3 \left(\frac{1}{n - 1}\right) - \frac{1}{2}\left(\frac{1}{n - 1}\right)^2 + \frac{1}{3}\left(\frac{1}{n - 1}\right)^3 - \ldots} \tag{10}$$

The two series expansions for the two log terms are to be found in most comprehensive handbooks of chemistry, mathematics, or physics. (See, for example, Hodgman, 1952, p. 274.) The expansions will be found in terms of the natural logarithm, which is readily converted to the common logarithm by multiplying by 2.3.

If the two series expressions shown in equation (10) are multiplied by (n - 1) there results the fraction

$$\frac{\frac{(n-1)p}{n-p} - \frac{(n-1)}{2}\left(\frac{p}{n-p}\right)^2 + \left(\frac{n-1}{3}\right)\left(\frac{p}{n-p}\right)^3 - \cdots}{1 - \left(\frac{n-1}{2}\right)\left(\frac{1}{n-1}\right)^2 + \left(\frac{n-1}{3}\right)\left(\frac{1}{n-1}\right)^3 - \cdots}$$

It can now be seen that as n approaches infinity the value of this fraction approaches the quantity p. In other words, p is the limiting slope of the semilog plot previously described. Thus a means for handily resolving equation (8) begins to emerge.

Note that equation (8) can be rewritten in the form

$$\frac{s_n T}{264Q} = \log n! - \log(1-p) + \log(2-p) + \log(3-p) + \ldots + \log(n-p) \tag{11}$$

The right half of equation (11) can be evaluated for the end of any number of cycles, n, and for the value p, that pertain to the problem under study. For example, assume p = 0.75 and n = 100. Now evaluate equation (11) by looking up in tables (Hodgman, 1952, p. 225) the log of n factorial and subtracting from it log 0.25 + log 1.25 + log 2.25 .... log 99.25. The value so obtained is 2.060 and it might be labelled $\frac{sT}{264Q}_{100}$.

Computations of this nature have been made for selected values of n and p and the results are given in the following table. The results are also shown in plotted form as figure 3.

Values of $\frac{s_n T}{264Q}$ for selected values of n and p

Number of cycles = n

	10	20	30	40	50	60	70	80	90	100
p = 0.25	0.342	0.416	0.459	0.490	0.514	0.534	0.550	0.565	0.577	0.589
.50	.754	.902	.989	1.051	1.100	1.139	1.172	1.201	1.226	1.249
.75	1.313	1.537	1.669	1.762	1.834	1.894	1.946	1.987	2.025	2.060

Observe that within the limits of plotting accuracy each curve shown is a straight line over the log cycle of n values used (i.e., from n = 10 to n = 100). Furthermore, the slope of each of these curves is only a few tenths of 1 percent less than the limiting slope p.

Thus in the solution of practical field problems the following general rules of procedure may be stated:

When n is less than 10, solve equation (11) numerically, computing the factor $s_n T/264Q$ in the manner indicated by the example.

When n is more than 10 but less than 100, solve equation (11) graphically by use of figure 3. The family of curves shown in the figure can easily be expanded to include any other desired value of p merely by computing the factor $s_nT/264Q$ for n = 10 and n = 100, plotting the two computed values, and joining them by a straight line.

When n is greater than 100 compute, or pick off the semilog graph (figure 3), the factor $\frac{sT}{264Q}_{100}$ for the end of the 100th cycle. The desired factor, for the end of the n^{th} cycle, can then be computed using the relation

$$\frac{s_nT}{264Q} = \frac{sT}{264Q}_{100} + p \log \frac{n}{100} \tag{12}$$

Equation (12) is an approximation but again it will give values of the factor $s_nT/264Q$ that differ from (are more than) the true factors by only a few tenths of 1 percent. Equation (12) is of course based on the idea of extending the curves shown in figure 3 beyond the 100^{th} cycle at the slope p. If it is anticipated that a variety of factors $s_nT/264Q$ will have to be determined, it may be desirable to extend each curve in figure 3 at its slope, p, an appropriate distance beyond the n = 100 ordinate. The factors can then be picked off the graph, at the desired n values, without computation.

It is interesting to note that for the case in which p = 0.5 -- that is, the pumping and recovery periods are of equal length -- equation (9) can be simplified for a somewhat easier solution as follows:

$$\frac{s_nT}{264Q} = \log \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot \ldots\ldots\ldots n}{\frac{1}{2} \cdot 1\frac{1}{2} \cdot 2\frac{1}{2} \cdot 3\frac{1}{2} \cdot 4\frac{1}{2} \ldots\ldots (n - \frac{1}{2})}$$

$$= \log \frac{n!}{(\frac{1}{2})^n \; 1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot \ldots\ldots\ldots (2n - 1)}$$

$$= \log \frac{2^n \cdot n! \; (2 \cdot 4 \cdot 6 \cdot 8 \cdot \ldots\ldots\ldots 2n)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot \ldots\ldots\ldots 2n}$$

$$= \log \frac{2^n \cdot n! \cdot 2^n \; (1 \cdot 2 \cdot 3 \cdot \ldots\ldots n)}{(2n)!}$$

$$= \log \frac{2^{2n} \cdot n! \; (1 \cdot 2 \cdot 3 \cdot \ldots\ldots n)}{(2n)!} \tag{13}$$

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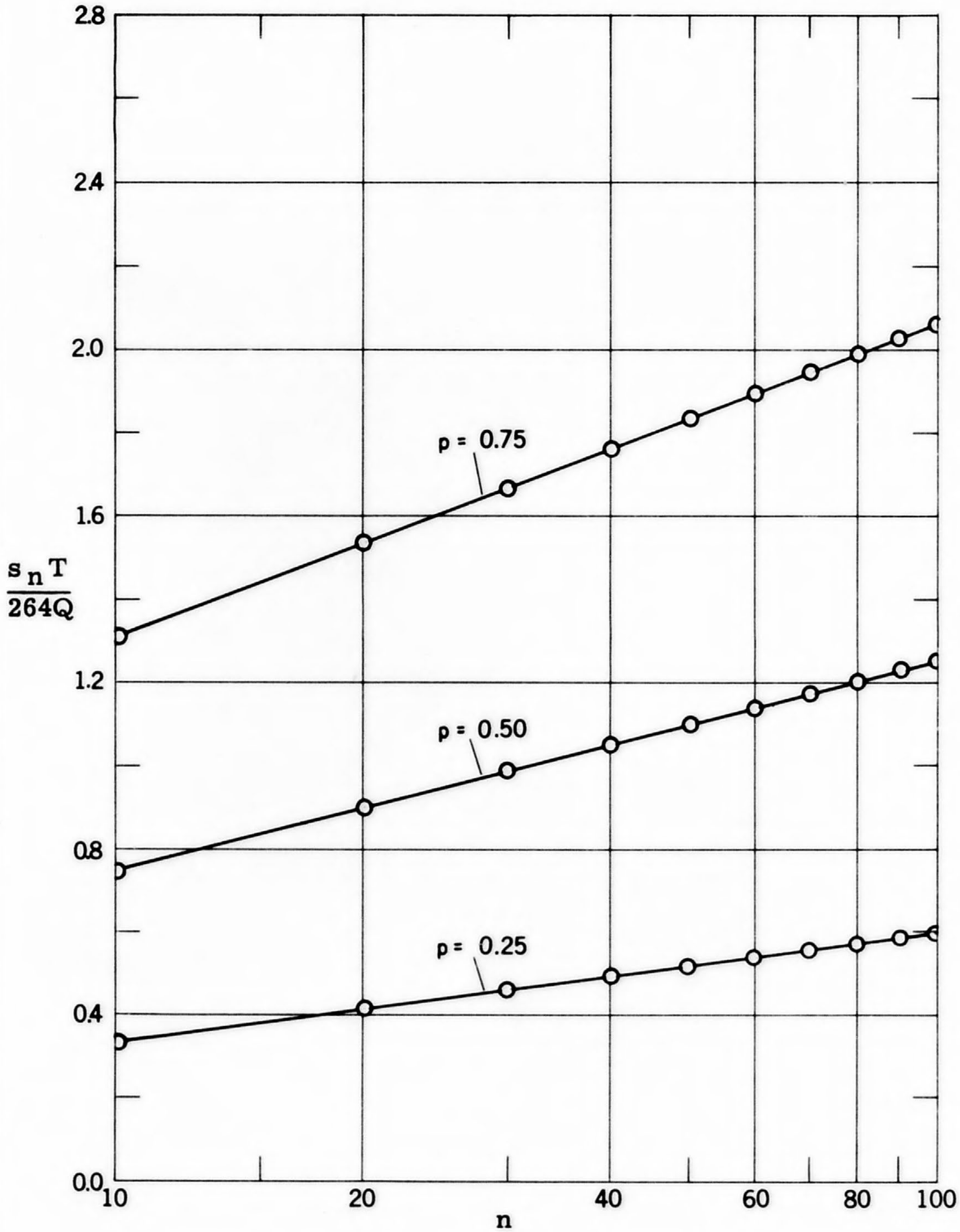

Figure 3. --- Semilog plot of factor $s_n T/264Q$ versus number of pumping cycles n for selected values of p