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# DRAWDOWN PATTERNS FOR TWO-WELL SYSTEMS AS APPLIED TO STRAIGHT-LINE-BOUNDARY CONDITIONS

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## INTRODUCTION

Application of the Theis nonequilibrium formula has been presented (Stallman, 1952) to broaden the scope of its use to situations where the pumped formation is adjacent to another formation of a vastly different transmissibility, and the boundary between the formations approximates a straight line. The most common examples of this situation might be (1) the aquifer that is hydraulically connected with a surface stream and (2) the permeable material of a buried valley abutting the relatively impermeable walls of the valley. In either example, the application is made by the use of the image-well theory. In the first example, the condition is imposed that no drawdown occurs along the boundary; in the second example, the condition is imposed that no water flows across the boundary. These conditions are simulated by assuming that the aquifer is infinite and an image well is the same distance from the boundary as the pumping well but on the opposite side of the boundary. In the first example, the image well would operate at the same time and rate as the pumping well but would be a recharging well instead of a discharging well. In the second example, the image well would operate in the same fashion as the pumping well. In this manner, the problem is converted from that involving boundary conditions as such to that of two-well systems in infinite aquifers.

The purpose of this paper is to extend the solution given by Stallman and provide a more convenient method of computing drawdowns in the vicinity of a pumping well at future times with permeable or impermeable boundary conditions, and of computing times of approximate stabilization of selected net drawdown contours ($u < 0.02$) for a pumping well near a recharge boundary.

### Case of Pumped Well near Recharge Boundary

The observed net drawdown at any point is given by the following equation (Stallman, 1952, p. 2).

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$$s_o = s_p \pm s_i = \frac{114.6\ Q}{T} \left[W(u)_p \pm W(u)_i\right] \qquad (1)$$

where
- $s_o$ = observed net drawdown at any point, in feet
- $s_p$ = drawdown produced by the pumping well, in feet
- $s_i$ = drawdown (+) or recovery (-) produced by the image well, in feet
- $Q$ = discharge rate of the real well, in gallons per minute
- $T$ = coefficient of transmissibility of the aquifer, in gallons per day per foot
- $W(u)_p$ = value of $W(u)$ for the pumping well
- $W(u)_i$ = value of $W(u)$ for the image well
- $W(u)$ = $-0.5772 - \log_e u + u - u^2/2 \cdot 2! + u^3/3 \cdot 3! - u^4/4 \cdot 4! + \ldots$
- $u$ = $1.87r^2S/Tt$
- $S$ = coefficient of storage of the aquifer, dimensionless
- $r_p$ = distance from the pumping well to point of observation, in feet
- $r_i$ = distance from image well to point of observation, in feet
- $t$ = time since pumping began, in days

If the boundary is of the recharge type, then

$$s_o = s_p - s_i = \frac{114.6\ Q}{T} \left[W(u)_p - W(u)_i\right] \qquad (2)$$

If two observation points are selected such that $s_{o_1} = s_{o_2}$, then

$$\frac{114.6\ Q}{T} \left[W(u)_{p_1} - W(u)_{i_1}\right] = \frac{114.6\ Q}{T} \left[W(u)_{p_2} - W(u)_{i_2}\right]$$

or

$$W(u)_{p_1} - W(u)_{i_1} = W(u)_{p_2} - W(u)_{i_2} \qquad (2a)$$

When $t$ becomes sufficiently large so that $u$ is equal to or less than 0.02, the value of $W(u)$ can be approximated as $(-0.5772 - \log_e u)$ and equation 2a rewritten in the form

$$-0.5772 - \log_e u_{p_1} + 0.5772 + \log_e u_{i_1} = -0.5772 - \log_e u_{p_2} + 0.5772 + \log_e u_{i_2}$$

$$\log_e u_{i_1} - \log_e u_{p_1} = \log_e u_{i_2} - \log_e u_{p_2}$$

or

$$\frac{u_{i_1}}{u_{p_1}} = \frac{u_{i_2}}{u_{p_2}} \qquad (3)$$

If $t$ is the same for both observation points, the only variable in $u$ is $r^2$ and

$$\frac{r_{i_1}^2}{r_{p_1}^2} = \frac{r_{i_2}^2}{r_{p_2}^2}$$

or
$$\frac{r_{i_1}}{r_{p_1}} = \frac{r_{i_2}}{r_{p_2}} \tag{4}$$

A line connecting points of equal net drawdown is a circle (Muskat, 1937, p. 175-177) whose center would be on an extension of a line joining the image and pumping wells and on the landward side of the pumping well, that is the side of the pumping well away from the boundary. Such a circle can easily be located by determining the two points at which it intersects the line through the image and pumping wells. For convenience, pick one of these points of intersection anywhere between the pumping well and the boundary, on the line through the image and pumping wells and determine the ratio of $r_i/r_p$ which is constant for all points on a circle of equal net drawdown. From the geometry of the system, it is obvious that the other point of intersection, on the landward side of the pumping well, is at the maximum distance from both the image and pumping wells $[r_{i(max)}, r_{p(max)}]$. Therefore,

$$\frac{r_{i(max)}}{r_{p(max)}} = \frac{r_i}{r_p}$$

and

$$r_{i(max)} = r_{p(max)} + r_{ip} \tag{5}$$

where $r_{ip}$ is the distance from the pumping well to the image well. Solving the two preceding equations for $r_{p(max)}$ in terms of the known ratio of $r_i/r_p$ and the known distance $r_{ip}$, there follows

$$r_{p(max)} = \frac{r_{ip}}{(r_i/r_p) - 1} \tag{6}$$

The distance $r_{p(max)}$ fixes the point of intersection of the circle of equal net drawdown on the landward side of the pumping well, and the point first assumed for computing $r_i/r_p$ is the point of intersection on the boundary side. Therefore, one-half the distance between the two points of intersection locates the center of the circle and is the radius of the circle.

Furthermore, it is obvious from the geometry of the system that the point of intersection on the boundary side of the pumping well is at the minimum distance from both the image and pumping wells $[r_{i(min)}, r_{p(min)}]$. Therefore,

$$\frac{r_{i(min)}}{r_{p(min)}} = \frac{r_i}{r_p}$$

and

$$r_{i(min)} = r_{ip} - r_{p(min)} \tag{7}$$

whence

$$r_{p(min)} = \frac{r_{ip}}{(r_i/r_p) + 1} \tag{8}$$

Because the value of u varies with the square of the distance (r), if the time necessary for u to become as small as 0.02 is determined for the greatest distance, $r_{i(max)}$, on a given contour circle of net drawdown, then u will be less than 0.02 for any other point on that circle for that same time. In other words, all the points on the circle may be considered to have reached approximate stabilization at the time necessary for u at $r_{i(max)}$ to equal 0.02.

For diagrammatic presentation, it is generally more desirable to determine lines or contours of net drawdown of specified magnitude and uniform interval, as shown in figure 1. By letting u equal 0.02 for the distance $r_{i(max)}$ of each contour, it is possible to compute the buildup due to the recharge image well. If the buildup is added to the specified magnitude of net drawdown, the result is the drawdown effect of the pumping well. The ratio of $u_i/u_p$ may then be determined, which would also be equal to the ratio of $r_i^2/r_p^2$. Equations 5, 6, 7, and 8 may then be used to compute the maximum and minimum values of $r_i$ and $r_p$. Finally, the time necessary for u to equal 0.02 for the distance $r_{i(max)}$ may be determined.

Example 1

The results of certain ground-water investigations have identified an artesian aquifer (values of T and S are $1 \times 10^5$ gpd/ft and $1 \times 10^{-4}$ respectively) cut by a perennial stream. Test drilling has demonstrated the feasibility of constructing a supply well, capable of being pumped continuously at 1,000 gpm, at a location 1,000 feet from the stream. It is desired to determine, and show by means of a contour map, the net drawdowns that will ultimately occur in the vicinity of the supply well, and to compute the elapsed pumping time required for each net drawdown contour to become stabilized in position as a circle.

The analysis will evidently require use of a recharging image well placed across the stream, and at a distance of 2,000 feet, from the supply well. Let it be assumed that it is desired to draw the contour circles representing net drawdowns of 1, 2, 3, 4, and 5 feet. The following sample computations indicate the method of determining the location of the 5-foot net drawdown contour circle.

Let $u_i$ for the distance $r_{i(max)}$ of this contour equal 0.02. Therefore:

$$s_o = 5'; \; u_i = 0.02, \; W(u)_i = 3.35; \; T = 1 \times 10^5 \text{ gpd/ft}$$

$$S = 1 \times 10^{-4}; \; r_{ip} = 2000'; \; Q = 1000 \text{ gpm}$$

$$s_i = \frac{114.6\,Q}{T} W(u)_i = \frac{114.6 \times 1000 \times 3.35}{1 \times 10^5} = 3.84'$$

$$s_p = s_o + s_i = 5.00 + 3.84 = 8.84'$$

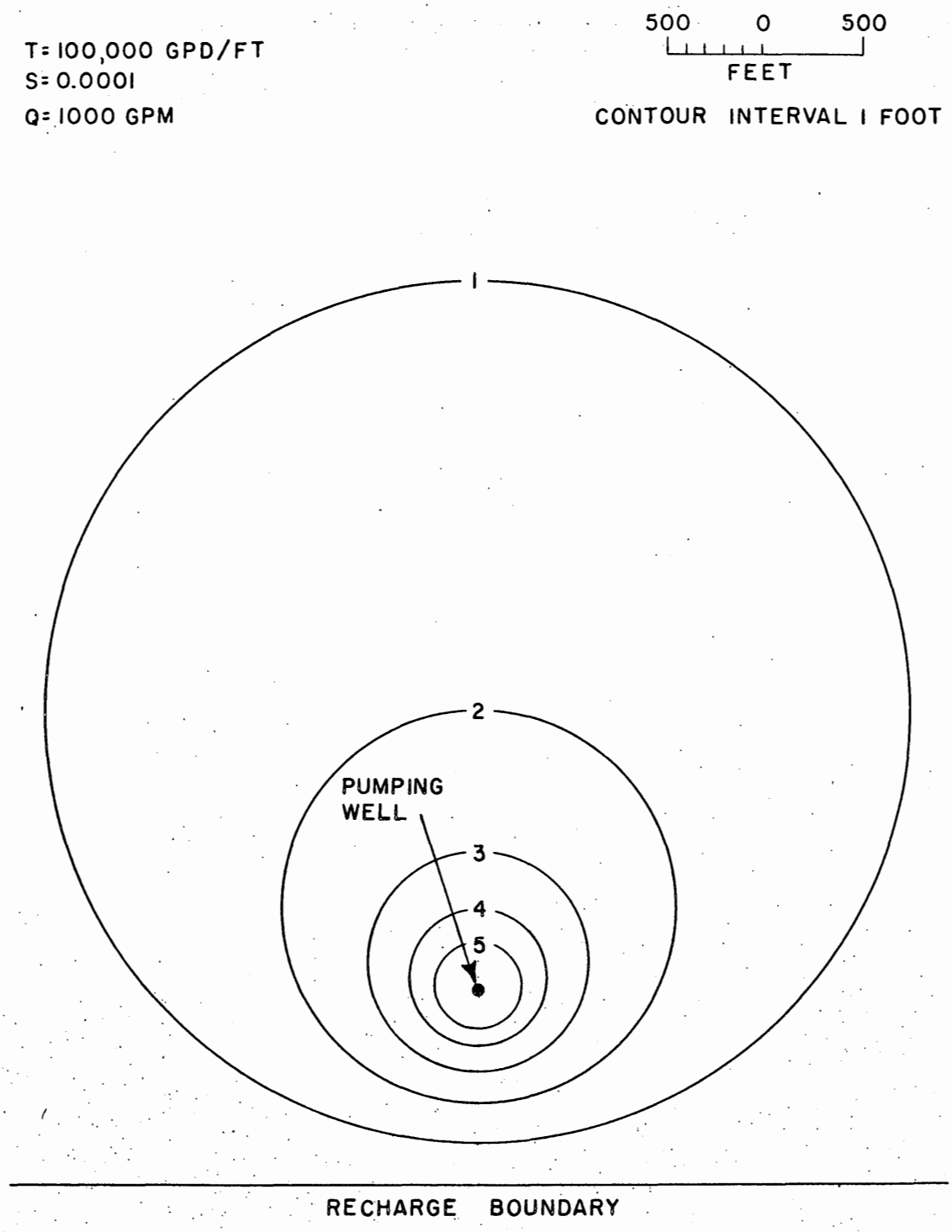


Figure 1.--Equilibrium positions of net drawdown contours for a discharging well near a recharge boundary

$$W(u)_p = \frac{T \times s_p}{114.6 \times Q} = \frac{1 \times 10^5 \times 8.84}{114.6 \times 1000} = 7.71 \qquad \therefore u_p = 2.5 \times 10^{-4}$$

$$\frac{u_i}{u_p} = \frac{2 \times 10^{-2}}{2.5 \times 10^{-4}} = 80 = \frac{r_i^2}{r_p^2} \qquad \therefore \frac{r_i}{r_p} = 8.94$$

$$r_{p(max)} = \frac{r_{i_p}}{(r_i/r_p) - 1} = \frac{2000}{8.94 - 1} = 252'$$

$$r_{i(max)} = r_{i_p} + r_{p(max)} = 2000 + 252 = 2252'$$

$$r_{p(min)} = \frac{r_{ip}}{(r_i/r_p) + 1} = \frac{2000}{8.94 + 1} = 201'$$

$$r_{i(min)} = r_{i_p} - r_{p(min)} = 2000 - 201 = 1799'$$

$$\text{Radius of Circle} = \frac{r_{i(max)} - r_{i(min)}}{2} = \frac{2252 - 1799}{2} = 226'$$

Distance of circle center from image well = $r_{i(min)}$ + Radius

$$= 1799 + 226 = 2025'$$

$$t = \frac{1.87\, r_{i(max)}^2\, S}{T\, u_i} = \frac{(1.87)(2252)^2(1 \times 10^{-4})}{(1 \times 10^5)\,(2 \times 10^{-2})} = 0.475 \text{ day}$$

Similar computations were made for the other net drawdown contour circles to determine their radii, locate their centers, and determine their times of approximate equilibrium. The results are given in table 1, and the contours are drawn as shown in figure 1.

Table 1.--Data used in preparation of Figure 1

| $s_o$ | $r_{p(max)}$ | $r_{i(max)}$ | $r_{p(min)}$ | $r_{i(min)}$ | Radius of Circle | Circle Center from image well | t |
|---|---|---|---|---|---|---|---|
| 1 | 3640 | 5640 | 785 | 1215 | 2212 | 3427 | 2.97 |
| 2 | 1440 | 3440 | 590 | 1410 | 1015 | 2425 | 1.11 |
| 3 | 719 | 2719 | 419 | 1581 | 569 | 2150 | .69 |
| 4 | 419 | 2419 | 296 | 1704 | 358 | 2062 | .55 |
| 5 | 252 | 2252 | 201 | 1799 | 226 | 2025 | .48 |

## Case of Pumped Well near Impermeable Boundary

With impermeable boundary conditions, the modified Theis equation for net drawdown at any point is

$$s_o = s_p + s_i = \frac{114.6\ Q}{T} \left[W(u)_p + W(u)_i\right] \tag{9}$$

Here again, selecting two observation points such that $s_{o_1} = s_{o_2}$, it follows that

$$W(u)_{p_1} + W(u)_{i_1} = W(u)_{p_2} + W(u)_{i_2}$$

When u is equal to or less than 0.02, the approximate value of W(u) is:

$$W(u) = -0.5772 - \log_e u$$

Therefore

$$-0.5772 - \log_e u_{p_1} - 0.5772 - \log_e u_{i_1} = -0.5772 - \log_e u_{p_2} - 0.5772 - \log_e u_{i_2}$$

or

$$\log_e u_{p_1} + \log_e u_{i_1} = \log_e u_{p_2} + \log_e u_{i_2} \tag{10}$$

$$u_{p_1}\ u_{i_1} = u_{p_2}\ u_{i_2}$$

If t is the same for both observation points

$$r_{p_1}\ r_{i_1} = r_{p_2}\ r_{i_2} \tag{11}$$

Net drawdown contours about a well near an impermeable boundary are not circles, and therefore their positions at a given time cannot be completely fixed by determining their points of intersection with the line through the pumping and image wells. The contours nearest the pumping well are oval-shaped; those farther away from the pumping well become progressively more elongated toward the boundary until the last contour that closes, before the boundary is reached, is distinctly teardrop-shaped. More distant contours do not close upon themselves but intersect the boundary at right angles. However, all contours are symmetrical about the line through the pumping and image wells.

To obtain a net drawdown contour for a particular elapsed pumping time, various values of $r_p$ must be assumed for substitution in equation 11 so that corresponding values of $r_i$ may be computed. These values of $r_p$ and $r_i$ are then plotted to obtain points on the required net drawdown contour.

As an aid in assuming proper values of $r_p$, the points of intersection of the net drawdown contour with the line through the pumping and image wells are determined. From the geometry of the system, it is obvious that the point of intersection on the landward side of the pumping well will be at a minimum distance from the pumping well but at a maximum distance from the image well; the point on the boundary side of the pumping well (if present) will be at a maximum distance from the pumping well but at a minimum distance from the image well. Therefore, it may be stated that

$$r_{i(max)}\ r_{p(min)} = r_{i(min)}\ r_{p(max)} = r_i\ r_p \tag{12}$$

$$r_{i(max)} = r_{ip} + r_{p(min)} \tag{13}$$

$$r_{i(min)} = r_{ip} - r_{p(max)} \tag{14}$$

Therefore,

$$r_{p(min)}^2 + r_{p(min)}\ r_{ip} - r_i\ r_p = 0 \tag{15}$$

$$r_{p(max)}^2 - r_{p(max)}\ r_{ip} + r_i\ r_p = 0 \tag{16}$$

Equations 15 and 16 permit the determination of the points of intersection of the net drawdown contour and the line through the pumping and image wells.

However, as in the case of the recharge boundary, net drawdown contours of a specified magnitude and uniform interval are generally desirable for use in diagrams. To obtain such contours, it is necessary to determine the product of $u_i$ and $u_p$ for each contour. When $u_i$ and $u_p$ are less than 0.02, equation 9 may be written as follows:

$$s_o = \frac{114.6\ Q}{T}\left[-2\ (0.5772) - \log_e u_i\ u_p\right] \tag{17}$$

$$\log_e u_i\ u_p = -\left(\frac{s_o T}{114.6\ Q} + 1.154\right) \tag{18}$$

By substituting proper values for $s_o$, Q, and T in equation 18, the product of $u_i u_p$ may be directly computed. Using the relation

$$u_i u_p = \left(\frac{K r_i r_p}{t}\right)^2 \quad \text{where } K = \frac{1.87\ S}{T} \tag{19}$$

it follows that

$$r_i r_p = \frac{t}{K}\sqrt{u_i u_p} \tag{20}$$

Equation 20 thus permits computation of the product of $r_i r_p$ which may be used in equations 15 and 16 to determine the points of intersection of the desired net drawdown contour and the line through the pumping and image wells. When these two points are located other points on the same

net drawdown contour may be located by assuming intermediate values of $r_p$ and computing corresponding values of $r_i$, using equation 11.

Example 2

Assume the same field situation described in example 1 except that the artesian aquifer is cut off by an impermeable barrier. It is again desired to determine, and show by means of a contour map, the net drawdowns that will occur in the vicinity of the supply well, assuming this time that the pumping rate of 1,000 gpm has been maintained for 20 days. Evidently the analysis now requires use of a discharging image well placed across the impermeable boundary, and at a distance of 2,000 feet, from the supply well.

The sample computations that follow were made to determine the position of the 20-foot net drawdown contour.

$$s_o = 20'; \; T = 1 \times 10^5 \text{ gpd/ft}; \; S = 1 \times 10^{-4}; \; t = 20 \text{ days};$$

$$r_{ip} = 2000'; \; Q = 1000 \text{ gpm}$$

$$\log_e u_i u_p = -\left(\frac{s_o T}{114.6\,Q} + 1.154\right) = -\left(\frac{20 \times 1 \times 10^5}{114.6 \times 1000} + 1.154\right) = -18.60$$

$$\therefore \; u_i u_p = 8.35 \times 10^{-9} \text{ and } \sqrt{u_i u_p} = 9.14 \times 10^{-5}; \; K = \frac{1.87S}{T} = 1.87 \times 10^{-9}$$

$$r_i r_p = \frac{t}{K}\sqrt{u_i u_p} = \frac{20 \times 9.14 \times 10^{-5}}{1.87 \times 10^{-9}} = 9.78 \times 10^5$$

$$r_{p(min)}^2 + r_{p(min)}\, r_{ip} - r_i r_p = 0$$

$$r_{p(min)}^2 + 2000\, r_{p(min)} - 9.78 \times 10^5 = 0; \; r_{p(min)} = 405'$$

$$r_{i(max)} = r_{ip} + r_{p(min)} = 2000 + 405 = 2405'$$

$$r_{p(max)}^2 - r_{p(max)}\, r_{ip} + r_i r_p = 0$$

$$r_{p(max)}^2 - 2000\, r_{p(max)} + 9.78 \times 10^5 = 0; \; r_{p(max)} = 850'$$

$$r_{i(min)} = r_{ip} - r_{p(max)} = 2000 - 850 = 1150'$$

The above computations determine the points of intersection of the 20-foot net drawdown contour with the line through the pumping and image wells. They also provide the limits to be used in assuming values of $r_p$ for determining corresponding values of $r_i$.

Computations to determine the positions of the other net drawdown contours indicate that they do not close upon themselves but intersect the boundary. For these contours, after $r_{p(min)}$ and $r_{i(max)}$ are determined, values for $r_p$ are assumed directly, and the corresponding values of $r_i$ are computed. The results are given in table 2, and the contours are drawn as shown in figure 2.

It must be remembered that, if an impermeable boundary is present, the net drawdown contours do not represent conditions at equilibrium but conditions after a definite period of discharge. The position of the net drawdown contour will change with time. In this respect, it is possible to prepare a diagram that would show such changes. The product $u_i u_p$ remains the same for a specified contour. By substituting various various values for t in equation 20, corresponding values for the product $r_i r_p$ may be determined. The positions of the contour at the different times are then found by using the procedures previously described.

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Theis, C. V., 1935, The relation between the lowering of the piezometric surface and the rate and duration of discharge of a well using ground-water storage: Am. Geophys. Union Trans., pt. 2, p. 519-524. (See also U. S. Geol. Survey Ground Water Note No. 5, August 1952.)

Table 2.--Data used in preparation of figure 2

| $s_o = 15'$ | | $s_o = 16'$ | | $s_o = 17'$ | | $s_o = 18'$ | | $s_o = 19'$ | | $s_o = 20'$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $r_{p(min)} = 2105$ | | $r_{p(min)} = 1570$ | | $r_{p(min)} = 1150$ | | $r_{p(min)} = 828$ | | $r_{p(min)} = 585$ | | $r_{p(min)} = 405$ | |
| $r_{i(max)} = 4105$ | | $r_{i(max)} = 3570$ | | $r_{i(max)} = 3150$ | | $r_{i(max)} = 2828$ | | $r_{i(max)} = 2585$ | | $r_{i(max)} = 2405$ | |
| $r_p$ | $r_i$ | $r_p$ | $r_i$ | $r_p$ | $r_i$ | $r_p$ | $r_i$ | $r_p$ | $r_i$ | $r_p$ | $r_i$ |
| 2200 | 3930 | 1600 | 3510 | 1200 | 3020 | 850 | 2750 | 600 | 2520 | 440 | 2220 |
| 2300 | 3760 | 1680 | 3340 | 1270 | 2850 | 900 | 2600 | 650 | 2320 | 480 | 2040 |
| 2400 | 3600 | 1760 | 3190 | 1340 | 2700 | 950 | 2460 | 700 | 2160 | 520 | 1880 |
| 2500 | 3460 | 1840 | 3050 | 1410 | 2570 | 1000 | 2340 | 750 | 2010 | 560 | 1750 |
| 2600 | 3330 | 1920 | 2930 | 1480 | 2450 | 1050 | 2230 | 800 | 1890 | 600 | 1630 |
| 2700 | 3200 | 2000 | 2810 | 1550 | 2340 | 1100 | 2130 | 850 | 1780 | 640 | 1530 |
| 2800 | 3090 | 2080 | 2700 | 1620 | 2240 | 1150 | 2030 | 900 | 1680 | 680 | 1440 |
| 2900 | 2980 | 2160 | 2600 | 1690 | 2140 | 1200 | 1950 | 950 | 1590 | 720 | 1360 |
| | | 2240 | 2510 | 1760 | 2060 | 1250 | 1870 | 1000 | 1510 | 760 | 1290 |
| | | 2320 | 2420 | 1830 | 1980 | 1300 | 1800 | 1050 | 1440 | 800 | 1220 |
| | | | | 1900 | 1910 | 1350 | 1730 | 1100 | 1370 | 850 | 1150 |
| | | | | | | 1400 | 1670 | 1150 | 1310 | | |
| | | | | | | 1450 | 1610 | 1200 | 1260 | | |
| | | | | | | 1500 | 1560 | | | | |

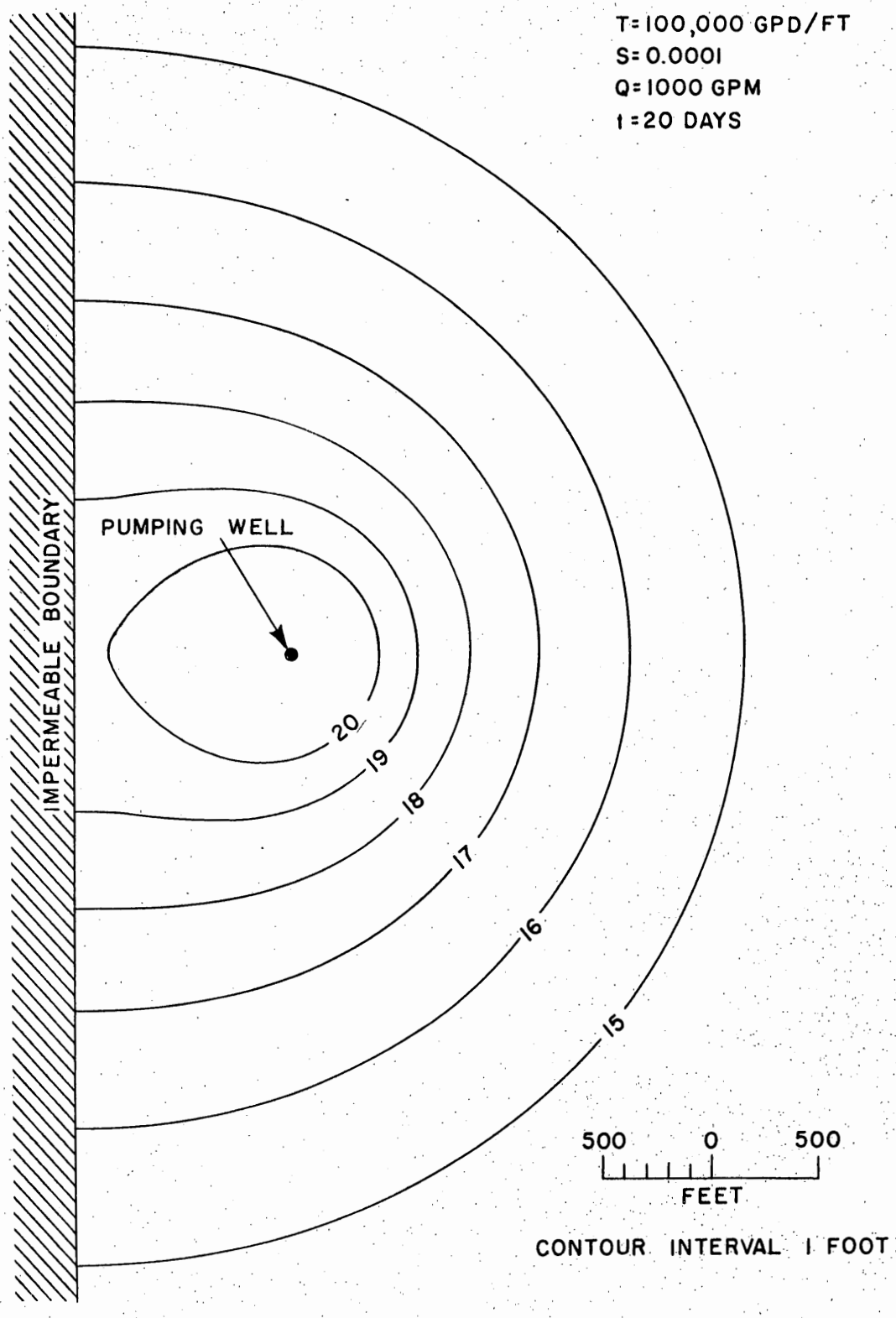


Figure 2.--Positions of net drawdown contours for a discharging well near an impermeable boundary

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