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AN EQUATION FOR POTENTIAL DISTRIBUTION ABOUT A WELL BEING BAILED

By

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A brief ground-water investigation made at the Navajo Ordnance Depot at Bellemont, Ariz., during parts of 1949 and 1950 included recommendations for the drilling of what later proved to be a 1,650-foot test well. In the course of the study it became evident that the only practical opportunity for estimating the transmissibility of the deep-seated water-bearing zone would hinge upon the successful analysis of data collected during and after the testing of the proposed well by bailing. Responding to a request to study the problem, H. E. Skibitzke in an unpublished paper dating back to 1950 developed and described the analysis presented herewith as a ground-water note. Subsequently M. I. Rorabaugh examined and discussed Skibitzke's analysis, offering an alternative development of the same solution which also is presented herewith.

The described analysis affords a most useful means of gleaning, from data which commonly may be overlooked, a preliminary appraisal of aquifer transmissibility. Skibitzke's derivation of equation

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(6) obviously supports equation (18) in part 1 of Ground-Water Note 28; furthermore, these two equations are identical with equation (1) in Ground-Water Note 26. It may be observed that there is no basic difference between the analyses of the effects of either instantaneously injecting a "slug" of water into, or removing a slug from, a well.

Introduction

Where the expense of installing a test pump on a new well is substantial, a significant saving may be made by using the bailer of the drill rig as the pump, provided that the aquifer is of low transmissibility. The test can be made as a recovery test to determine the coefficients of permeability and transmissibility. Discharging-well tests, in which drawdown varies as a function of time, generally are analyzed using the nonequilibrium equation developed by Theis (1935). This equation was derived through analogy with the conduction of heat in solids and through suitable modification of a solution that Theis credits to H. S. Carslaw.

One of the assumptions inherent in the derivation of Theis' nonequilibrium equation is that the rate of well discharge, Q, is steady and constant. Obviously, if a bailer is used as a pump Q will not be steady but will be discontinuous or intermittent. Therefore the development of a new equation is required, and recourse may be had to solutions available in the theory of heat flow or to modification of Theis' equation.

Derivation from a Selected Heat-Flow Equation

An approximate fundamental differential equation of hydrodynamics, describing the unsteady-state flow of an incompressible fluid in a compressible porous medium, has been given by Muskat (1937, p. 133) and Jacob (1950, p. 333) in the general form

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = \frac{S'}{P}\frac{\partial h}{\partial t} \quad \ldots\ldots\ldots \quad (1)$$

where h = a head or potential function (the sum of the gravity potential and the pressure potential), as represented by the piezometric surface, of the incompressible fluid

S' = storage coefficient of the water-bearing material

P = coefficient of permeability of the water-bearing material

t = time

x, y, z = coordinate distances from the origin of coordinates, which later in this paper will be taken at the center of the well to be bailed (the axis of the well being along the z-coordinate axis)

For an instantaneous point source, of strength Q'/S', at the coordinate origin, Carslaw and Jaeger (1947, p. 216-217) show that a particular solution of equation (1) is

$$h = \frac{Q'/S'}{8\left[\pi(P/S')t\right]^{3/2}}\, e^{-(x^2 + y^2 + z^2)/4(P/S')t} \quad \ldots \quad (2)$$

where h is now the change in head attributable to the instantaneous point source, Q' is the volume of water removed instantaneously, at time t = 0, from point x = y = z = 0, and S' is as previously defined. Equation (2) expresses the head distribution resulting from the assumed conditions listed above in a homogeneous and isotropic medium of infinite thickness and extent. Inasmuch as it is the solution for a point source, whereas a pumped or bailed well would approximate a (vertical) line source, equation 2 can be used to find the head distribution in the vicinity of a bailed well by integrating or summing up all the effects of the point sources from $-\infty$ to $+\infty$ along the z axis. Over an infinitesimal length dz along the z-axis the source strength is $(Q'/S')dz$. Therefore,

$$h = \int_{-\infty}^{\infty} \frac{Q'/S'}{8\left[\pi(P/S')t\right]^{3/2}} e^{-(x^2 + y^2 + z^2)/4(P/S')t} \, dz \; . \quad (3)$$

Because x, y, and t are not variables of integration it is convenient to let $r^2 = x^2 + y^2$; to substitute this relation in equation (3); and to rewrite the equation in the following form

$$h = \frac{Q'/S'}{8\left[\pi(P/S')t\right]^{3/2}} e^{-r^2/4(P/S')t} \int_{-\infty}^{\infty} e^{-z^2/4(P/S')t} dz \quad (4)$$

The required integration can be obtained by noting the similarity of the integral in equation (4) to the error function (erf) of x which is written

$$\text{erf } x = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-\alpha^2} \, d\alpha$$

For this well-known function

$$\text{erf } (-x) = -\text{erf } (x)$$

which means that the curve obtained by plotting the function in the x-y plane is symmetrical about the y-axis. Thus it follows that the function to be integrated in equation (4) is symmetrical about the axis $z = 0$, and hence the integration between the limits $(-\infty)$ and (∞) is simply double the result of integrating from (0) to (∞). Using this relation, and a table of integrals (Peirce, 1929, p. 63, item 492) it follows that

$$\int_{-\infty}^{\infty} e^{-z^2/4(P/S')t} dz = 2\int_{0}^{\infty} e^{-(S'/4Pt)z^2} dz = 2\left[\pi(P/S')t\right]^{1/2}$$

Substituting this result in equation (4) there follows

$$h = \frac{(Q'/S')(2)\left[\pi(P/S')t\right]^{1/2}}{8\left[\pi(P/S')t\right]^{3/2}} e^{-r^2/4(P/S')t}$$

or

$$h = \frac{Q'/S'}{4\left[\pi(P/S')t\right]} e^{-r^2/4(P/S')t} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (5)$$

This solution gives the radial distribution of head changes in the vicinity of an infinite line source in terms of an instantaneous point source of strength Q'/S'. However, in the practical application of this solution to the problem of a fully penetrating bailed well it is more convenient to have the strength in terms of the quantity of water, q, removed by the bailer, and the conventionally defined storage coefficient, S, of an aquifer of finite thickness, m.

From equation (5) it is evident that the head distribution is independent of z, and therefore the flow characteristics of the system would not be changed by confining a segment of the aquifer between two impermeable strata in x-y planes. The terms Q', S', and P in equation (5) may then be replaced by equivalent terms incorporating the thickness and hydrologic properties of the aquifer.

By definition

$$Q' = \frac{q}{m}, \; S' = \frac{S}{m}, \text{ and } P = \frac{T}{m}$$

where T is the coefficient of transmissibility of the aquifer. Substituting these ratios for Q', S', and P in equation (5) and adopting the more commonly used symbol, s', in place of h, there results the final equation needed

$$s' = \frac{q}{4\pi Tt} e^{-r^2 S/4Tt} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

where s' = residual drawdown of the piezometric surface
q = volume of water removed from the well in one bailer cycle
r = distance from center of bailed well to point at which drawdown is observed

and the remaining terms are as previously defined.

Note in equation (6) that as r becomes small and as t becomes large the term $r^2S/4Tt$ approaches zero. When this occurs the value of $e^{-r^2S/4Tt}$ approaches unity. Thus in and near the bailed well, when r is small in comparison with the extent of the aquifer and when t is large, equation (6) may be written in the simplified form

$$s' = \frac{q}{4\pi Tt} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

Derivation from the Theis Recovery Equation

Most of the material presented in this section is credited to M. I. Rorabaugh, who independently derived equation (7) from the Theis (1935, p. 522, eq. 7) recovery formula. In nondimensional form the recovery formula is written as

$$s' = \frac{Q}{4\pi T} \log_e \frac{t}{t'} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (8)$$

where t = time elapsed since well discharge began
t' = time elapsed since well discharge stopped

It should be remembered that in the derivation of equation (8) the term Q is a rate of discharge, and it is specified that the times t and t' must be large.

Consider a short pumping period of length Δt during which a quantity (volume) of water, q, is removed from the well. In the ensuing recovery period apply equation (8) to an observation of residual drawdown in the pumped well, made at a time, t_n, which represents the elapsed time from the midpoint of the Δt pumping interval. It should now be evident that the following identities may be written and substituted in equation (8)

$$Q = \frac{q}{\Delta t}$$

$$t = t_n + \frac{\Delta t}{2}$$

and

$$t' = t_n - \frac{\Delta t}{2}$$

Performing the indicated substitutions equation (8) now becomes

$$s' = \frac{q/\Delta t}{4\pi T} \log_e \frac{t_n + \Delta t/2}{t_n - \Delta t/2}$$

or

$$s' = \frac{q/\Delta t}{4\pi T} \log_e \frac{(2t_n/\Delta t) + 1}{(2t_n/\Delta t) - 1} \quad \ldots\ldots\ldots \quad (9)$$

The log term in equation (9) may be expanded into a series form by referring to an appropriate mathematical handbook. Thus it is found that

$$\log_e \frac{n+1}{n-1} = 2\left[\frac{1}{n} + \frac{1}{3n^3} + \frac{1}{5n^5} + \cdots\right] \quad \ldots\ldots\ldots \quad (10)$$

In this problem evidently $n = \frac{2t_n}{\Delta t}$ which, when substituted in the foregoing series, will yield

$$\log_e \frac{(2t_n/\Delta t) + 1}{(2t_n/\Delta t) - 1} = 2\left[\frac{\Delta t}{2t_n} + \frac{\overline{\Delta t}^{\,3}}{3(2t_n)^3} + \frac{\overline{\Delta t}^{\,5}}{5(2t_n)^5} + \cdots\right] \quad \ldots \quad (11)$$

For a well that is bailed the pumping time, Δt, per bailer cycle, will be very small relative to the time t_n. Therefore, in the bracketed part of the series shown in equation (11) each term beyond the first will be so small that it may be neglected, leaving as the equivalent of the log term merely $\frac{\Delta t}{t_n}$. Substituting this equivalent in equation (9), it follows that

$$s' = \frac{q/\Delta t}{4\pi T}\left(\frac{\Delta t}{t_n}\right)$$

or

$$s' = \frac{q}{4\pi T t_n} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (12)$$

which is identical to equation (7)

Recovery Analysis After Repeated Bailing

Equation (7) specifies the residual drawdown, in or near a bailed well, for some time t in the recovery period, after the removal of only one bailer of water. If after *n* bailer cycles a single observation of residual drawdown is made it is obviously specified by the following equation

$$s' = \frac{q_1}{4\pi T t_1} + \frac{q_2}{4\pi T t_2} + \frac{q_3}{4\pi T t_3} + \cdots + \frac{q_n}{4\pi T t_n}$$

$$= \frac{1}{4\pi T}\left[\frac{q_1}{t_1} + \frac{q_2}{t_2} + \frac{q_3}{t_3} + \cdots + \frac{q_n}{t_n}\right] \quad . \; . \; . \; . \; . \; . \quad (13)$$

where the subscripts identify each bailer cycle in chronological order. Each time factor represents the interval from the instant of occurrence of the indicated bailer cycle to the instant at which s' was observed. It is likely that the volume of water removed in each bailer cycle can be assumed to be constant. Therefore equation (13) may be further simplified to read

$$s' = \frac{q}{4\pi T}\left[\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \cdots + \frac{1}{t_n}\right] \quad . \; . \; . \; . \qquad (14)$$

or, in abbreviated notation

$$s' = \frac{q}{4\pi T} \sum_{i=1}^{n} \frac{1}{t_i} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (15)$$

By expressing s' in feet, q in gallons, and t_i in days, T will be given in units commonly used by the Geological Survey--that is, gallons per day per foot.

Conclusion

This paper could have begun by writing equation (6) directly from Carslaw's solution (Theis, 1935, p. 520). However, some of the steps and integration procedures involved in the derivation of equation (6) from the more general or fundamental "point-source" solution were considered worthy of documentation to the extent shown.

The development of equations (7) and (15) involves most of the assumptions inherent in the Theis recovery formula, particularly the stipulation that r be small and t large. Rorabaugh (personal communication) has observed that the new equations developed herein will be most useful when the bailing is "hit-or-miss" and a reasonable average pumping rate cannot be ascertained. When bailing is at a fairly uniform rate the strictly cyclic effects might well be dissipated before satisfaction of the requirement that time be large. In this latter event the Theis method would be preferred because all the recovery data are more easily studied graphically. It will be evident that use of the analytical method described in this paper, particularly when the number of bailer cycles is large, will require much computation for each observed residual drawdown in the recovery period.

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