PROGRESS REPORT ON ANALOG MODEL CONSTRUCTION

ORANGE COUNTY CALIFORNIA

OPEN-FILE REPORT

UNITED STATES DEPARTMENT OF THE INTERIOR

GEOLOGICAL SURVEY

WATER RESOURCES DIVISION

PREPARED IN COOPERATION WITH

THE ORANGE COUNTY WATER DISTRICT

MENLO PARK, CALIFORNIA

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By

E. H. Cordes, J. R. Wall, and J. A. Moreland

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PROGRESS REPORT ON ANALOG MODEL CONSTRUCTION, ORANGE COUNTY, CALIFORNIA

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SUMMARY

An electrical analog model is a convenient device for integration of the cause and effect relations in a dynamic aquifer system.

The analogy between the flow of electrical charge in a conductor and the flow of fluid through saturated porous media is predicated on mathematical logic and an understanding of basic physical laws.

A first approximation of the Orange County ground-water system is simulated by an electrical analog model using the parameters, functions, and boundary conditions inferred from available hydrologic and geologic information.

The format for the data is prescribed by model use. As a tool, the model can provide the solution to a variety of hydrologic problems related to basin-management practices and engineering design.

INTRODUCTION

Purpose and Scope of the Project

The U.S. Geological Survey, in cooperation with the Orange County Water District, is constructing an electrical analog model of the Orange County ground-water basin. The analog model is expected to provide an understanding of the basin's hydrologic response through application of analogy to electrical-circuit theory.

The method of preparation of the data and progress in construction of the analog model are outlined in this report. Upon completion of the model, its use for solving specific hydrologic problems will be discussed in a subsequent report.

The basin-management program of the Orange County Water District includes, or will include: (1) Utilization of the ground-water basin for storage and regulation of water supplies, (2) installation of hydraulic barriers to arrest or prevent salt-water intrusion, (3) artificial recharge of the ground-water basin, (4) reclamation of sewage and waste water, and (5) augmentation of local and imported water supplies by desalination of ocean water.

The purpose of building the analog model is to provide a tool to analyze the response of the hydrologic system under various simulated conditions. The various choices of water management or development may then be evaluated on the basis of information that is pertinent and can be defined.

The most immediate need of the Water District, and therefore the initial application of the model, is for information on which to base the design of a hydraulic barrier in Santa Ana (Talbert) Gap and in any other Orange County coastal area where an immediate possibility of salt-water intrusion exists.

The model will be used in a continuing investigative program of the engineering and hydrologic problems involved in (1) determination of ground-water storage capacity and selection of recharge areas, (2) determination of water-storage requirements necessary to optimize surface deliveries of imported water, (3) selection of desirable pumping patterns and regulation of pumping, (4) sizing of a sewage reclamation plant, (5) design and operation of hydraulic barriers, and (6) quantitative determination of the subsurface flow of ground water across the Los Angeles-Orange County line.

The data available for initial construction and programing of the model include:

1. Annual water-level-contour maps for the period 1954-65.
2. Bimonthly water-level-contour maps for the period 1962-65.
3. Well-production records for the period 1954-65.
4. Well locations for approximately 2,700 metered wells.
5. Areas and volume of annual recharge for the period 1954-65.
6. Maps showing land use.
7. Maps showing annual precipitation and estimates of the rainfall-runoff relations in the bordering mountains.
8. Estimates of subsurface ground-water outflow to the ocean and inflow across the Los Angeles-Orange County line.
9. Data reports and interpretations of investigations in Alamitos, Sunset, Bolsa, and Santa Ana Gaps by the U.S. Geological Survey, California Department of Water Resources, and Los Angeles County Flood Control District.
10. Ground-water extraction data in Los Angeles County for the period 1950-65.
11. Lithologic logs of wells and other geologic information.

Additional information required for refinement of the model includes:

1. Delineation of the aquifers, the aquicludes, and their boundary conditions, which may require a program of deep test drilling.
2. Evaluation of aquifer transmissibilities, either from a program of field investigation or by a trial-and-error determination using the electrical analog model, or both.
3. Determination of the coefficients of storage in the "forebay" or water-table zone and the changes in storage in those parts of the semiconfined aquifer system that may be dewatered during periods of extensive pumping (fig. 2).
4. Determination of the changes of vertical permeability and storage so that a three-dimensional model can be constructed for those areas where engineering studies will require precision and detail.

The purpose of this report is to inform the Orange County Water District of the progress made in the preparation and construction of the electrical analog model and to explain the assumptions, methods, and techniques employed in the data analysis. The theoretical approach to the use of electrical and mathematical modeling is discussed in the appendix.

Location and Extent of the Area

The Orange County analog study area (fig. 1) is a part of the south-coastal basin of California. The south-coastal basin is composed of the Los Angeles County basin on the northwest and the Orange County basin on the southeast (fig. 2).

The study area is bounded on the east and south by the Puente Hills, Santa Ana Mountains, and San Joaquin Hills, which are made up of relatively impermeable, consolidated formations. On the coastal side, the study area is bounded by the Newport-Inglewood structural zone. This structural zone impedes ground-water movement along much of the coastal area, except locally where erosion and subsequent redeposition have placed permeable deposits in contact with the ocean.

Because hydrologic changes in the Los Angeles County basin affect ground-water conditions in the Orange County basin, the analog study area extends 6 to 8 miles northwestward into Los Angeles County.

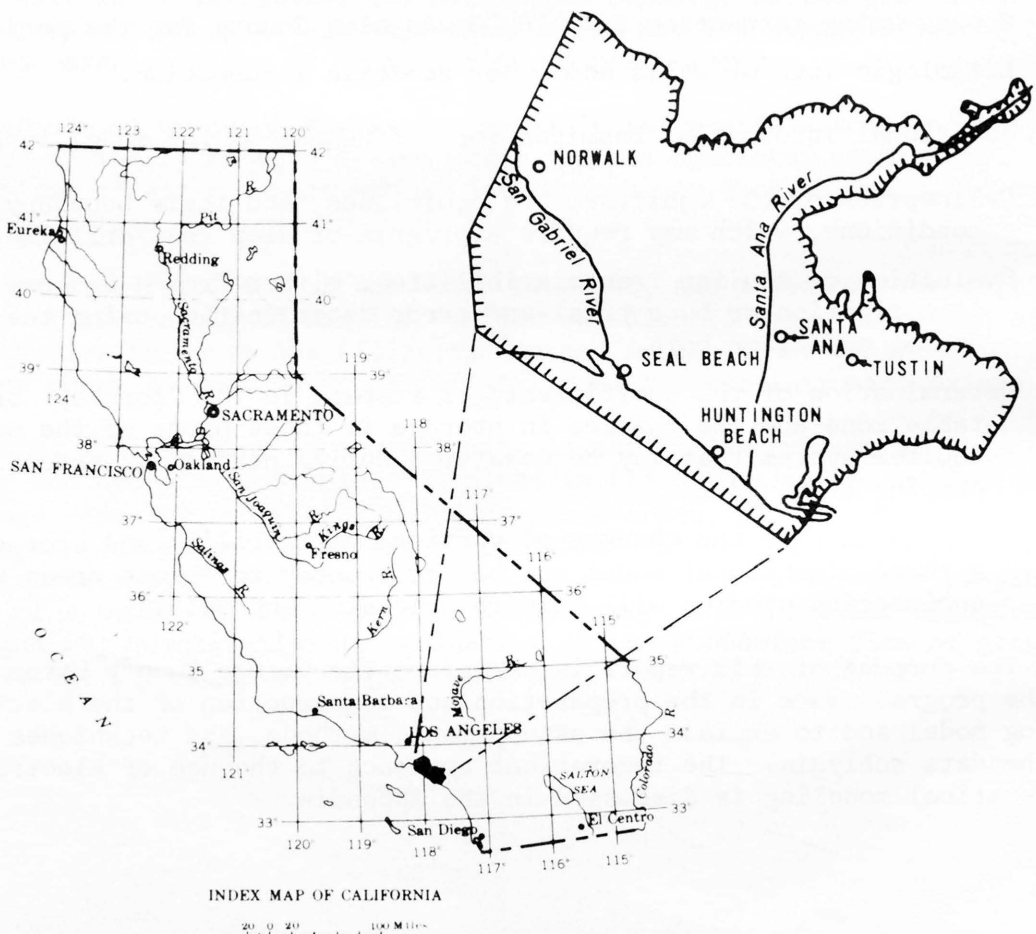

FIGURE 1.--Index map showing the location of the Orange County analog-model study.

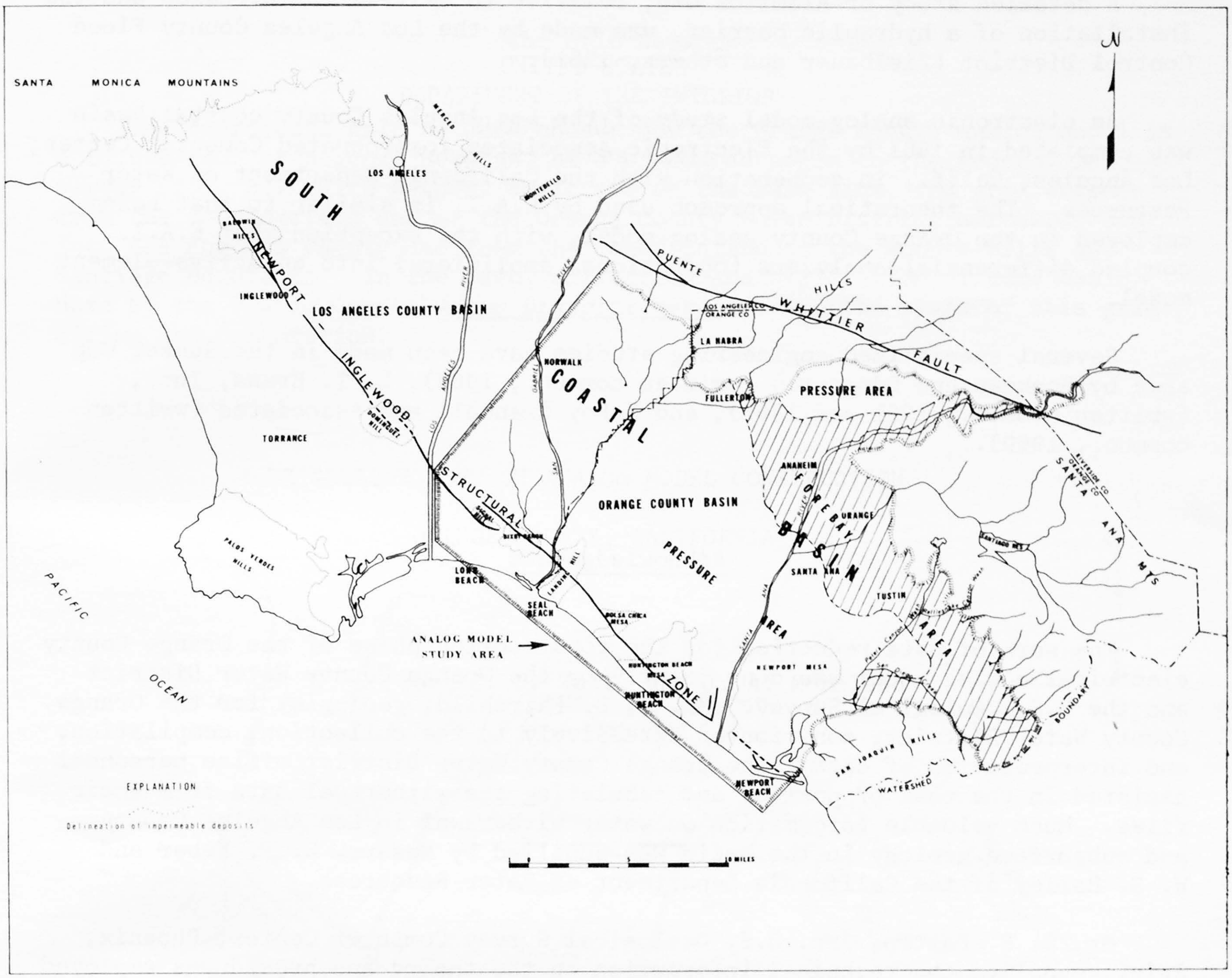

FIGURE 2.--Map of the south-coastal basin, California, and the Orange County analog-model study area.

Previous and Related Investigations

The first extensive ground-water investigation in Orange County was made by Mendenhall (1905). Subsequent studies have been made by the California Division of Public Works (Post, 1928; Eckis, 1934; and Gleason, 1947), the California Department of Water Resources (1959), and the U.S. Geological Survey (Piper, Garrett, and others, 1953; Poland, 1959; and Poland, Piper, and others, 1956).

The U.S. Geological Survey has released progress reports (Wall and Dutcher, 1965; and Wall, Cordes, and Moreland, 1966) on an investigation of the hazards of salt-water intrusion from inland waterways constructed in Sunset and Bolsa Gaps.

A detailed study of Alamitos Gap, relative to salt-water intrusion and the installation of a hydraulic barrier, was made by the Los Angeles County Flood Control District (Zielbauer and others, 1961).

An electronic analog-model study of the Los Angeles County coastal basin was completed in 1961 by the Electronic Associates Incorporated Computer Center, Los Angeles, Calif., in cooperation with the California Department of Water Resources. The theoretical approach used by E.A.I. is similar to that being employed in the Orange County analog model, with the exception that E.A.I. coupled differential analyzers (operational amplifiers) into an active-element model.

Several specialized engineering studies have been made in the Sunset Gap area by Bookman and Edmonston (written commun., 1963), L. T. Evans, Inc., (written commun., 1961 and 1962), and LeRoy Crandall and Associates (written commun., 1960).

Acknowledgments

The work of data reduction for the construction phase of the Orange County electrical analog model was done jointly by the Orange County Water District and the U.S. Geological Survey. Mr. J. B. Fairchild, geologist for the Orange County Water District, contributed extensively to the collection, compilation, and interpretation of data. The Orange County Water District office personnel assisted in the task of posting and tabulating the withdrawal data from their files. Much valuable information on water withdrawal in Los Angeles County and subsurface geology in the basin was supplied by Messrs. E. P. Weber and W. S. Harley of the California Department of Water Resources.

Mr. E. P. Patten, Jr., U.S. Geological Survey Computer Center, Phoenix, Ariz., supplied the technical information on the theory and procedures employed in analog model construction.

DATA PREPARATION AND CONSTRUCTION OF THE ORANGE COUNTY MODEL

The successful application of the theoretical concepts, as discussed in the appendix of this report, leading to analog model construction, involves several important considerations. Perhaps the most important consideration is selection of the proper scale or areal dimensions for the model. No single model will resolve all the problems that might be imposed upon it, so that the intended use of the model must serve as a guide for its construction. Also, it would be unrealistic to attempt to model the physical characteristics of the basin on a small scale if the basic data are widely scattered. The scale for the Orange County model was selected to be compatible with the density of points for which basic data were available and the detail of the solutions desired.

The Grid Network

The coordinate grid network chosen for the Orange County analog model is a 10,000-foot orthogonal network, at a 45-degree angle to the Federal land survey (fig. 3). With Newport Boulevard as the base line, the grid extends from the southeastern end of the Orange County Water District northwestward into Los Angeles County and from the coast inland to the Puente Hills and Santiago Mountains. In the past, collection and utilization of some basic data by the Orange County Water District were done on the basis of this grid sizing and orientation.

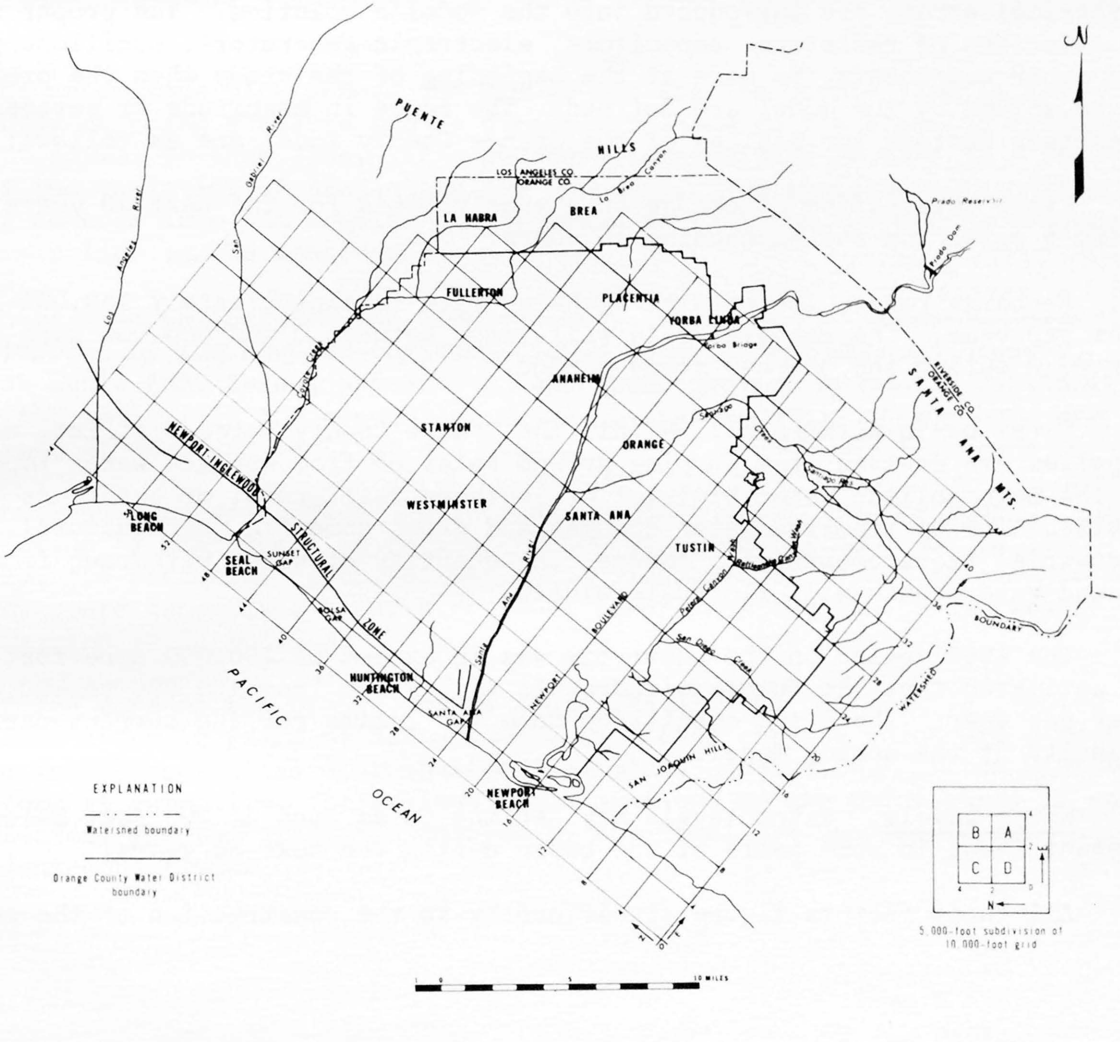

FIGURE 3.--The 10,000-foot grid pattern for analog model of Orange County, California.

For ground-water extraction data, the 10,000-foot grid was divided into 5,000-foot subdivisions (fig. 3 insert), labeled counterclockwise A, B, C, and D. A smaller nodal configuration of 2,500 feet was selected for construction of the resistance network of the electrical analog model. This smaller grid spacing results in a better approximation for conversion of continuous field parameters as lumped elements in the model. Each node is identified by the intersection of a north and east coordinate line having a designation (N/E).

Determination of Electrical Characteristics of Analog Components

The electrical analog model has three principal units: The function generators for excitation, a passive-element resistance-capacitance network for the conducting media, and the analyzing equipment for readout. The relative impedance of these three basic units is matched, so that no adverse electrical errors are introduced into the model's solution. The proper sizing or selection of resistors, capacitors, electronic generators, oscilloscopes, and other components, is made at the beginning of the study when the problems to be solved by the model are defined. The range in magnitude of several important factors for sealing of the Orange County model are as follows:

Time: Evaluation of change on a yearly basis for the next 20 years and every 5 years for the succeeding 10 years.

Recharge rate: The present recharge rate is approximately 150,000 acre-feet per year. The recharge rate will range between 0 and 300,000 acre-feet per year during the 30-year study period.

Total basin withdrawal: Within the Orange County Water District, water supplies are derived from pumping ground water or from surface water imported by the Metropolitan Water District of Southern California, or both. As the availability of imported water changes and gross domestic, agricultural, and industrial requirements also change, the quantity of water withdrawn from the ground-water basin will fluctuate widely.

The 1965 demand on the reservoir was in excess of 180,000 acre-feet. It is estimated that the future withdrawals will range from 150,000 to 400,000 acre-feet per year. These two quantities form the limits for the current-carrying capacity of the analog model.

Water levels: Water levels may decline to as much as 300 feet below the present level in some parts of the basin during the next 30 years.

All these factors figure significantly in the construction of the model.

Description of Boundary Conditions

A particular solution to the equation for a hydrologic-flow system (see appendix) is based upon a description of the boundaries of the fluid system and upon the physical conditions that are to be imposed upon these boundaries at the initial time.

The boundaries are not necessarily impermeable zones restraining fluid flow, but may be an irregular geometric surface where either the fluid velocity, the velocity potential, or a given function of both may be considered known.

For the Orange County model, the inland periphery of the ground-water basin is assumed to be an impermeable boundary along the foothills. This boundary is shown in figure 2. No significant water movement takes place across this boundary; the gradient is equal to zero for all time greater than zero.

The subsurface flow across the Orange-Los Angeles County line is a constantly changing variable and is directly related to the changing hydraulic gradient normal to that line. This boundary is permitted to vary as a function of the variation of head along the boundary. The resistance-capacitance network extends into Los Angeles County basin so that the varying flow across the county line can be considered.

Water movement between the Orange County basin and the ocean is assumed to be impeded by the Newport-Inglewood fault zone, except through breached zones at Santa Ana, Bolsa, and Sunset Gaps which provide hydraulic continuity across the fault. In the electrical model the water levels in the gaps are maintained at mean sea level at all times.

One important assumption and initial boundary condition for model construction is that the time rate of change of potential be equal to zero. Actually, the potential distribution in the fluid system is not measurable at a steady-state condition; thus, a residual error from the finite difference approximation is introduced.

In the case of the Orange County analog model, the residual is assumed to be zero after 10 years of historical water-level changes which are used to set the initial potential distribution for future prediction. If the historical record is extensive, the residual is diminished to the point where it can be neglected, and the predicted changes by the model are unaffected by antecedent conditions.

Storage and Transmissibility

The ratio of the storage coefficient (S) to the coefficient of transmissibility (T), which is in the basic differential equation of fluid flow (see appendix), is called the diffusivity coefficient and is a function of both the fluid and the porous media. In an equivalent electrical system the diffusivity or relating coefficient is the ratio of capacitance to conductance.

The capacitance and conductance used in the preliminary electrical analog model are based upon a first approximation of the storage and transmissibility of the Orange County ground-water system. Exact values of transmissibility and storage are not required for construction of the electrical analog model. Only relative change is built into the model network.

Storage: An artesian storage coefficient of 0.001 was assumed throughout the aquifer area under artesian pressure, and a storage coefficient of 0.2 was used throughout the outcrop or water-table area. Both storage values are estimates for the first trial run of the model and will be adjusted as necessary. Only a limited number of reliable aquifer-performance tests have been made in Orange County, and these were confined to wells that take water from the Talbert Formation. Throughout most of the basin no meaningful estimate of the storage coefficient is available.

Transmissibility: Transmissibility is defined as the rate of fluid flow, in gallons per day, through a saturated strip of aquifer 1 foot wide under unit hydraulic gradient. Transmissibility is related to permeability by the following expression:

$$T = Pm$$

where T is transmissibility, in gallons per day per foot
P is permeability, in gallons per day per square foot
m is saturated thickness of the aquifer, in feet.

If the aquifer is isotropic, its property of permeability from any point is equal in every direction; if the aquifer is homogeneous, the permeability is uniform throughout. As perfect homogeneity is nonexistent in nature, the effective permeability of the aquifer can only be approximated by an average permeability factor.

The following permeability values were assigned as a reasonable first approximation to the materials described by drillers' logs:

Lithologic unit	*Permeability ($gal/day/ft^2$)*
Clay----------------------------------	1
Silt----------------------------------	2-5
Fine sand-----------------------------	20-100
Medium sand---------------------------	100-300
Coarse sand---------------------------	300-500
Sand and gravel-----------------------	500-700
Gravel--------------------------------	700-1,000

These values, multiplied by the thickness of the formation, were used to compute the formation transmissibility.

Previous drilling in Orange County has provided some knowledge of the basin structure and lithology. A contour map, showing the elevation of the bottom of the aquifer system, was prepared from the available lithologic information and was used to define an effective aquifer thickness.

The summation of individual permeability values for the effective aquifer thickness determines the integrated coefficient of transmissibility for the aquifer. If the driller's log shows that the well did not penetrate the entire aquifer, an average permeability is computed for the interval penetrated, and other data are used to extrapolate the transmissibility of the full thickness of the aquifer.

Transmissibility values were calculated throughout the study area and used to construct a transmissibility-contour map. The transmissibility map, together with the storage information, is the basis for the first approximation of the resistance-capacitance electrical network in the analog model which represents the physical properties of the aquifers.

Recharge--Precipitation and Irrigation Return

The net inflow ($+W$) to the Orange County ground-water basin is a combination of artificial recharge from spreading operations, natural recharge from surface runoff, subsurface underflow, infiltration from precipitation, and irrigation return. The sources and quantities of this water were estimated by the Orange County Water District.

Six areas were considered in computing the recharge from precipitation and irrigation return.

Area 1: That part of the forebay region northwest of Newport Boulevard and east of the Yorba Bridge--39,504 acres.

Area 2: Tributary drainage to area 1--18,848 acres.

Area 3: All that part of the forebay region southeast of Newport Boulevard--23,425 acres.

Area 4: The surface drainage tributary to area 3--16,454 acres.

Area 5: The area along the river bottom in Santa Ana River Canyon between Yorba Bridge and Prado Dam--2,907 acres.

Area 6: All surface drainage tributary to area 5--27,621 acres.

The estimated quantity of recoverable recharge from precipitation and irrigation return was computed independently for each area.

Accretion to the ground-water basin from precipitation only was estimated by adjusting the recorded rainfall to reflect losses from soil moisture deficiency, surface runoff, and evapotranspiration. The provision for loss was incorporated in a net percolation factor determined for each area. The rainfall records from Orange County Flood Control District Station 26-A at Yorba Linda, Calif., were used to estimate the total annual recharge from deep penetration of precipitation.

Recharge from irrigation return was estimated on the basis of land usage and precipitation frequency. Again, a net percolation factor for each area was used to calculate the expected quantity of recharge to the ground-water basin. The natural recharge from both surface and subsurface flow in the Santa Ana River channel was estimated by use of a water budget of the surface drainage.

Artificial recharge from spreading of imported water along the Santa Ana River channel accounts for the major part of input to the system, and the quantity of this recharge has been determined by the Orange County Water District.

The sums of all these sources make up the yearly net recharge to the ground-water system. The artificial recharge into the electrical analog model was distributed proportionately between nodal points along the reaches of the Santa Ana River, Rattlesnake Canyon, and San Diego Creek channels (fig. 3). The remaining part of the input was distributed throughout the forebay area.

Discharge--Pumping and Subsurface Flow

Artificial withdrawal ($-W$), or net pumpage from the ground-water system, was evaluated from pumping records of the Orange County Water District. The production from each well was assigned to a quarter subdivision of the 10,000-foot grid, on the basis of geographic location. The centers of pumping for 1960 were assumed to be constant for the entire period of study.

To represent these extraction centers in the analog model, the pumpage for each well was assigned to a particular nodal point. Where possible the pumpage from adjacent subdivisions was combined and represented by a single nodal point. This minimized the number of function generators required to model the pumpage.

The distribution of pumping locations and the annual extraction rates in Los Angeles County were obtained from the records of the Water Master, California Department of Water Resources. About 80 percent of the withdrawal adjacent to the Orange County line was obtainable by sections and not individual wells. Because the exact well locations were unknown, it became a problem to assign the extractions from points on the Federal land survey to the proper location on the 10,000-foot grid used in the analog model. The two grids are oblique to each other; to simplify the transposition, a section of the Federal land survey is superimposed on several 10,000-foot grid units.

Production in Los Angeles County was assumed to be uniformly distributed within the sections of the Federal land survey. The quantity of this production assigned to a 10,000-foot grid in the analog model was estimated from the percentage of the Federal land section which lay within the boundary of the analog grid. For example, if 20 percent of a section, as determined by planimeter, was within a certain 10,000-foot grid unit, then 20 percent of the total production from the section was assigned to that unit. The remaining production in the section was distributed in a similar manner among the other 10,000-foot grid units containing parts of the same section until 100 percent of the production was accounted for. The total pumping production from each 10,000-foot grid unit in Los Angeles County was assigned to the node in the center of the grid unit.

Model Verification and Use

The electrical analog model is a convenient tool by which to solve a transient boundary-value problem whose physical performance can be described by a generalized partial differential equation. A discussion of the mathematical approach to electrical analog models is given in the appendix to this report.

The analog model will produce an exact solution for every set of boundary conditions used in its construction. This does not imply that the analog solution is correct for the hydrologic system. If the assumptions used to build the analog model are a reasonable facsimile of the hydrologic field conditions, then the analog model solution can be expected to approximate the response of the physical system and will be an exact solution for the assumptions or boundary conditions employed.

Calibration of the electrical model, which is called verification, is accomplished by matching the time rate of change of electrical potential in the model with the changes observed in the field. If the short-term and long-term water-level changes can be duplicated with the analog model for a reasonable set of known boundary conditions, then the model is considered verified and can be used to predict future change.

For preliminary verification of the Orange County analog model, the historical conditions began with known water levels in 1954. The change in water levels from 1954 to 1958 was taken as the short-term change and from 1954 to 1963 as the long-term change. Water-level-change maps were prepared for each of these periods to be compared with the voltage or potential change in the analog model for the same periods. The usual procedure for verification is to adjust the component values of the resistance-capacitance grid network until the magnitude of the electrical change is comparable to the water-level changes during the same time interval. The verified analog model is then ready to use for solving problems having a new set of conditions.

Construction Phase

The preceding paragraphs have briefly outlined the use of hydrologic data in construction of the analog model. The actual conversion of the hydrologic parameter into an electrical component or function was done by the U.S. Geological Survey Computer Center in Phoenix, Ariz. The first phase of model construction, the soldering of resistors and capacitors into an electrical network, is complete.

When the program for excitation by function generators that simulate water input is completed, the model is ready for verification. A preliminary run, using the assumption that the external boundaries are impermeable, will be made to verify the electrical continuity of the model and to evaluate the importance of the boundaries themselves.

Subsequent modification of the model to achieve optimum verification will consist of changing the resistance-capacitance element network, the boundary conditions, or the assumptions related to the hydrologic parameters incorporated in the electrical model.

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APPENDIX

THE THEORY OF ANALOG SIMULATION

The use of the principle of analogy is the process of attacking the unknown in terms of the known. It is one of man's oldest and most effective methods of solving problems. A simple and familiar type of analogy is the scale model in which every element of the original is reproduced at some smaller size. Other analogies, which are more complex and more difficult to visualize, are based upon the mathematical similarity between physical systems from entirely different disciplines.

One usually describes a dynamic system in terms of differential equations, or in terms of partial differential equations, if more than two variables are present. These equations describe the internal state of the physical system in relation to the forces or stresses that may be imposed upon it. Quite often the solving of a mathematical equation is a monumental effort. By using the principle of analogy, the mathematical laws governing two different systems can be related, and a given problem may be translated from one physical system, in which computations and solutions are difficult and expensive, into an analogous system in which a solution can readily be obtained.

If each element of a prototype system is replaced by a corresponding element in a model system and if all interactions between these elements are appropriately expressed and if the performance characteristics of the two systems are similar, the two systems are analogous. The chief advantage of this technique is that the scientist does not have to solve explicit mathematical equations, but can obtain a solution in the form of a direct readout from the model. It is necessary only that the analog be a true and valid model and that the applied forces and boundary conditions be known.

THE FLUID MODEL

By combining the equation of continuity and Darcy's law, a two-dimensional partial differential equation is derived; an equation that describes the dynamics of fluid flow in a confined aquifer:

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \frac{S}{T}\frac{\partial h}{\partial t} + \frac{W}{T}(x,y,t) \qquad (1)$$

where h = head of water, feet
x,y = space coordinates, feet
S = coefficient of storage of the aquifer (dimensionless)
T = transmissibility of the aquifer, gallons per day per each 1-foot wide strip of saturated aquifer under a unit hydraulic gradient
t = time, days
W = flux of a source and (or) sink, gallons per day per square foot.

This equation is valid only for two-dimensional laminar flow through homogeneous, isotropic, porous media.

Depending upon the data available, this equation can be used to compute the hydraulic constants (S,T) or to express either the changes in the head of water with time or the change in hydraulic gradient at any point in the system. An analytical solution to equation 1 is possible for fluid systems with simple geometric boundaries. If the geometry is complex, equation 1 cannot be solved by an analytical procedure.

FINITE DIFFERENCE APPROXIMATION

A numerical solution to equation 1 is easily found by the method of finite difference approximation. This is a mathematical convenience which permits the replacement of partial differential equations by a number of algebraic expressions which can be solved simultaneously.

A conformal coordinate grid (fig. A-1) is superposed on a continuously varying potential field, and attention is limited to the grid intersections or nodal points. If the heads h_1, h_2, h_3, h_4, and h_0 are assumed to exist at points 1, 2, 3, 4, and 0, the average potential gradient between points can be expressed as:

$$\left(\frac{\partial h}{\partial x}\right)_{1-0} \simeq \frac{h_1-h_0}{\Delta x} \qquad (2)$$

$$\left(\frac{\partial h}{\partial x}\right)_{0-3} \simeq \frac{h_0-h_3}{\Delta x} \qquad (3)$$

$$\left(\frac{\partial h}{\partial y}\right)_{4-0} \simeq \frac{h_4-h_0}{\Delta y} \qquad (4)$$

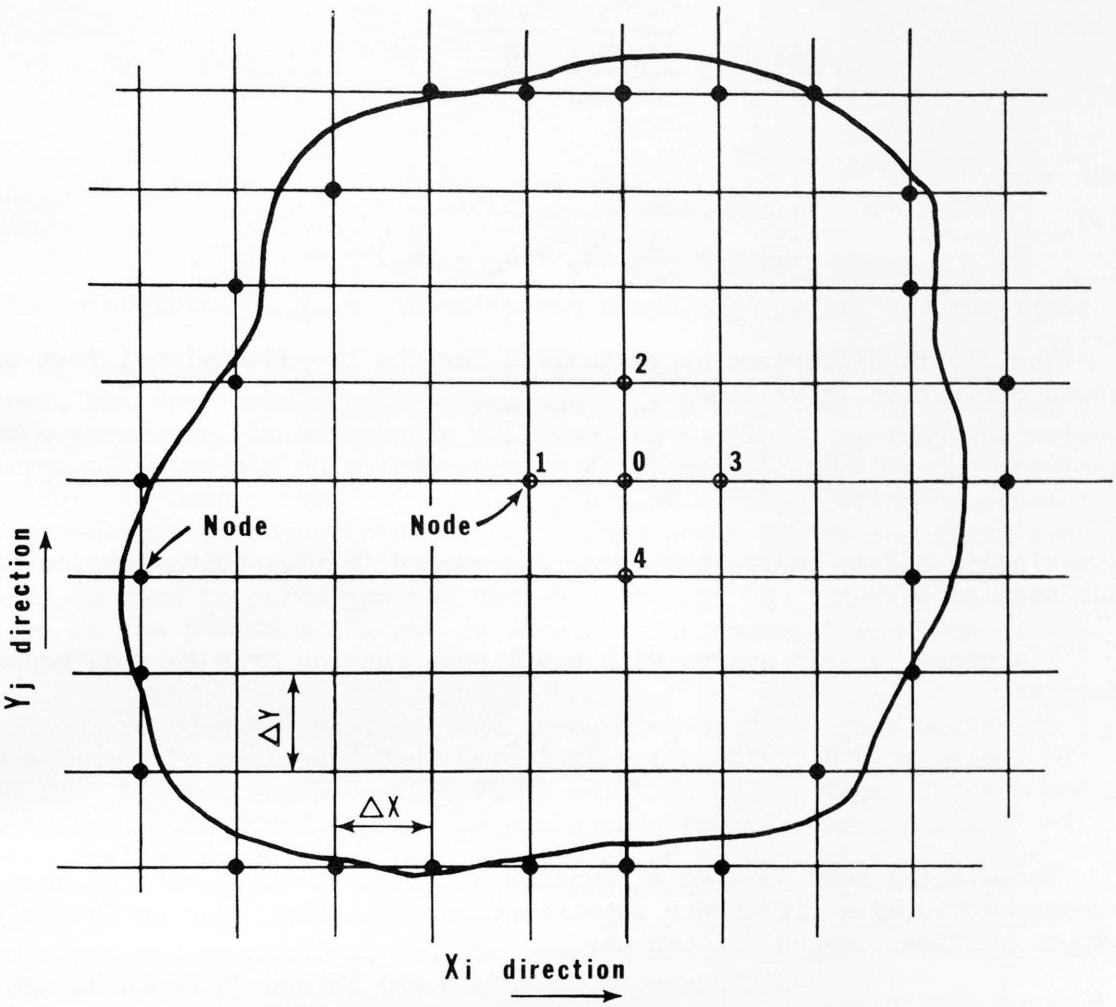
$$\left(\frac{\partial h}{\partial y}\right)_{0-2} \simeq \frac{h_0-h_2}{\Delta y} \qquad (5)$$

where the subscripts identify the effective nodes or coordinate points. The change in potential gradients or second derivatives can be written:

$$\left(\frac{\partial^2 h}{\partial x^2}\right)_0 \simeq \frac{\frac{h_1-h_0}{\Delta x} - \frac{h_0-h_3}{\Delta x}}{\Delta x}$$

such that

$$\left(\frac{\partial^2 h}{\partial x^2}\right)_0 \simeq \frac{1}{\Delta x^2}(h_1 + h_2 - 2h_0) \qquad (6)$$



After Stallman (1962)

The grid spacing is an arbitrary differential distance, ΔX and ΔY

The solid line is the external boundary of the potential distribution, dots indicate the nodes selected to approximate the irregular area, and circles are internal nodes

FIGURE A-1.--Finite grid and nomenclature used in numerical analysis.

and

$$\left(\frac{\partial^2 h}{\partial y^2}\right)_0 \simeq \frac{\frac{h_4-h_0}{\Delta y} - \frac{h_0-h_2}{\Delta y}}{\Delta y}$$

such that

$$\left(\frac{\partial^2 h}{\partial y^2}\right)_0 \simeq \frac{1}{\Delta y^2}(h_2 + h_4 - 2h_0) \tag{7}$$

The finite difference approximation for the two-dimensional form of the second derivative is written:

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \simeq \frac{1}{\Delta x^2}(h_1 + h_3 - 2h_0) + \frac{1}{\Delta y^2}(h_2 + h_4 - 2h_0) \tag{8}$$

By making Δx and Δy small, the error introduced in this approximation can be made negligible.

Therefore, a grid system with equal node spacing $\Delta x = \Delta y = a$, equation 8 becomes:

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \simeq \frac{h_1 + h_2 + h_3 + h_4 - 4h_0}{a^2} \tag{9}$$

Because the head is also a function of time, its time rate of change can be represented by a difference approximation. The first derivative, with respect to time, can be written as:

A forward difference,

$$\left(\frac{\partial h}{\partial t}\right)_0 \simeq \frac{h_{0,t} - h_{0,t+\Delta t}}{\Delta t}, \tag{10}$$

a backward difference,

$$\left(\frac{\partial h}{\partial t}\right)_0 \simeq \frac{h_{0,t} - h_{0,t-\Delta t}}{\Delta t}, \tag{11}$$

an average difference,

$$\left(\frac{\partial h}{\partial h}\right)_0 \simeq \frac{h_{0,t-\Delta t} - h_{0,t+\Delta t}}{2\Delta t}, \tag{12}$$

where Δt is small. The general form of the finite difference approximation for the governing partial differential equation of a ground-water flow system is written:

$$\frac{h_{(i+1,j,k)} + h_{(i-1,j,k)} + h_{(k,j+1,k)} + h_{(i,j-1,k)} - 4h_{(i,j,k)}}{a^2} \simeq$$

$$\frac{S}{T}\,\frac{h_{(i,j,k)} - h_{(i,j,k-\Delta t)}}{\Delta t} + \frac{W}{T}(x_i, y_j, t_k) \qquad (13)$$

where the subscripts, i, j, k, identify the coordinate points and the time.

Few of the parameters in a hydrologic system have discrete values or dimensions, but vary continuously throughout space and time. In modeling a continuous parameter, it is usually necessary to partition or lump the parameters into unit elements which represent a proportionate part of the physical system. This is accomplished by replacing a part of the continuous parameter with an assemblage of components having defined areal limits and fixed average magnitudes. The grid used for approximating the potential distribution (fig. A-1) is used to partition the transmissibility (T) and storage (S) parameters of the porous media and to identify the internal location of sources and (or) sinks.

In many ground-water systems, the properties of the porous media are unknown and must be evaluated by a trial-and-error solution (iteration) of equation 13. A simultaneous mathematical solution to equation 13 for a model containing hundreds of nodes would be a rigorous task, except, perhaps, with high-speed digital computers. With the aid of an electrical analog model, the solution becomes a matter of observing an image and making adjustments in the model until it duplicates historical data. The use of the electrical analog model is rapid and economical, and the results are often easier to comprehend than those obtained by use of its mathematical counterpart.

THE ANALOG MODEL

An electrical analog of a hydrologic system is a device in which the flow of electricity in a model is analogous to the flow of water in a hydrologic system. The electrical model is a network of resistors and capacitors that simulate the transmissibility and storage capacity of the aquifer system. The model functions by storing and resisting the flow of electricity in the same way that an aquifer stores and impedes the flow of water.

The space dimensions of the electrical model are directly proportional to the space dimensions of the hydrologic system. The geometric scaling of the hydrologic system is maintained in the electrical model by building the electrical conductive network on a scale map of the area.

In solving a problem, or in using the model to predict a condition in the future, electrical current is added or withdrawn from the model in a similar manner to recharge or extraction in the hydrologic prototype, and the resulting change is observed on an oscilloscope connected to the model network. The oscilloscope screen shows an image analogous to the hydrograph of the water-table fluctuations in the prototype area.

THE MATHEMATICAL BASIS OF ELECTRICAL ANALOGY

The analogy between electrical and fluid models is clearly shown by a comparison of the mathematical equations governing fluid flow through porous media and the flow of current through a resistance-capacitance conductor.

The equation that describes the flow of current through an electrical system is given by Karplus (1958) as:

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \simeq \frac{C}{\sigma}\frac{\partial v}{\partial t} + \frac{J}{\sigma}(x,y,t) \qquad (14)$$

where v = electrical potential, volts
x,y = space coordinates, feet
C = electrical capacitance, farads per cubic foot
σ = conductance of the media, mhos per foot
t = time, seconds
J = flux of electrical current through a unit thickness of the conducting media, amperes per cubic foot.

This equation states that the sum of the changes in potential gradient in the x and y directions is directly proportional to the sum of the rate of change of potential and the current density. This equation is similar in form to equation 1 which describes the flow of fluid through confined, isotropic, and homogeneous media.

The parallelism between equations 1 and 14 indicates that the cause-and-effect response of an electrical system is analogous to the response of a hydrologic system, provided the two systems are dimensionally equivalent.

The equivalence between electrical and hydrologic parameters can be derived by using basic mathematical expressions from the following laws (Bermes, 1960):

1. From Coulomb's law:

$$C = \frac{Q_e}{\Delta v \Delta x \Delta y \Delta z} \qquad (15)$$

where C = electrical capacitance, farads per cubic foot
Q_e = quantity of electrical charge, coulombs
Δv = differential change in electrical potential, volts
$\Delta(x,y,z)$ = differential length of the space coordinate, feet.

2. From Ohm's law:

$$\sigma = \frac{1}{\rho} = \frac{q_e}{\Delta v}\frac{\Delta x}{\Delta y \Delta z} \quad (16)$$

in x direction where σ = electrical conductance, mhos per foot
q_e = rate of flow of electrical current, amperes
Δv = differential change in electrical potential, volts
$\Delta(x,y,z)$ = differential change in space coordinates, feet
ρ = electrical resistance, ohm-feet.

3. From Theis' law:

$$S = \frac{Q_w}{\Delta h \Delta x \Delta y} \quad (17)$$

where Q_w = quantity of fluid, gallons
S = coefficient of storage, dimensionless
Δh = differential change of water head, feet
$\Delta(x,y)$ = differential length of space coordinates, feet.

4. From Darcy's law:

$$T = \frac{q_w}{\Delta h}\frac{\Delta x}{\Delta y} \quad (18)$$

in x direction where T = transmissibility of the aquifer, gallons per day per foot
q_w = rate of flow of water, gallons per day
Δh = differential change of water head, feet
$\Delta(x,y)$ = differential length of space coordinates, feet.

By substituting the equations from Coulomb's law and Ohm's law for like terms in the basic differential equation for an electrical model, the equation becomes:

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{1}{\Delta x^2}\left(\frac{Q_e}{q_e}\frac{\partial v}{\partial t_e}\right) + \frac{J}{q_e}\left(\frac{\Delta v \Delta y \Delta z}{\Delta x}\right) \quad (19)$$

where t_e = time, in seconds, in the electrical model.

By substituting the equations from Theis' law and Darcy's law for like terms in the basic differential equation for a hydrologic model, the equation becomes:

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \frac{1}{\Delta x^2}\left(\frac{Q_w}{q_w}\frac{\partial h}{\partial t_w}\right) + \frac{W}{q_w}\left(\frac{\Delta h \Delta y}{\Delta x}\right) \quad (20)$$

where t_w = time, in days, in the fluid model.

Because

$$W = \frac{q_w}{\Delta x \Delta y}$$

and

$$J = \frac{q_e}{\Delta x \Delta y \Delta z} ,$$

the preceding two equations may be simplified to the following forms:

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{1}{\Delta x^2}\left[\frac{Q_e}{q_e}\frac{\partial v}{\partial t_e} + \Delta v\right] \qquad (21)$$

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \frac{1}{\Delta x^2}\left[\frac{Q_w}{q_w}\frac{\partial h}{\partial t_w} + \Delta h\right] \qquad (22)$$

The final form of equations 21 and 22 shows that the head (h) in the fluid system is analogous to the voltage (v) in the electrical system, the quantity of fluid (Q_w) is analogous to the quantity of electrical charge (Q_e), the rate of liquid flow (q_w) is analogous to the rate of flow of current (q_e), and time (t_w) in the fluid model is analogous to time (t_e) in the electrical model. By introducing arbitrary scale factors, these terms can be related, such that:

$$K_1 = \frac{Q_w}{Q_e}$$

where K_1 has units of $\frac{\text{gallons}}{\text{coulomb}}$,

$$K_2 = \frac{h}{v}$$

where K_2 has units of $\frac{\text{feet of water}}{\text{volt}}$,

$$K_3 = \frac{q_w}{q_e}$$

where K_3 has units of $\frac{\text{gallons per day}}{\text{ampere}}$,

and

$$K_4 = \frac{t_w}{t_e}$$

where K_4 has units of $\frac{\text{days}}{\text{second}}$.

Theoretically, an infinite number of values can be assigned to the scaling factors (K_1, K_2, K_3, K_4) provided equations 21 and 22 are maintained. If proportional parameters from the electrical system are substituted in the counterpart differential equation 17, then:

$$K_2 \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} - \frac{\Delta v}{\Delta x^2} \right) = \frac{1}{\Delta x^2} \left(\frac{K_1 Q_e}{K_3 q_e} \frac{K_2 \partial v}{K_4 t_e} \right) \quad (23)$$

and

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} - \frac{\Delta v}{\Delta x^2} = \frac{1}{\Delta x^2} \left(\frac{Q_e}{q_e} \frac{\partial v}{\partial t_e} \right) \left(\frac{K_1}{K_3 K_4} \right) \quad (24)$$

Equation 24 is then equal to equation 21 provided

$$\frac{K_1}{K_3 K_4} = 1.$$

However, important limitations are imposed by the cost and the mechanics of electrical-circuit theory and narrow the selection of these constants.

INTERPRETATION

The solution to a problem using mathematical and electrical analog models may be subject to error. The reliability of the solution is dependent upon the reliability of the input data. Error may be inherent in the mathematical equations which describe the problem, or it may result from failure to satisfy some basic assumption.

The mathematical expression of a fluid-flow system assumes two-dimensional flow and an instantaneous release of water from storage. Both the coefficient of storage and the transmissibility of the aquifer are assumed to be constants, independent of time or changes in water level.

Because the Laplace equation was used to derive the finite difference approximation of equation 1, we can assume that

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0.$$

This corresponds to the steady-state case of equation 13:

$$h_1 + h_2 + h_3 + h_4 - 4h_0 \pm \frac{W}{T} a^2 = 0 \quad (25)$$

Generally the initial change in gradient is unknown, but we recognize that it is not zero; thus, equation 25 should be rewritten as:

$$h_1 + h_2 + h_3 + h_4 - 4h_0 \pm \frac{W}{T} a^2 = R_0, \qquad (26)$$

or in the electrical model:

$$v_1 + v_2 + v_3 + v_4 - 4v_0 \pm \frac{J}{\sigma} a^2 = R_0', \qquad (27)$$

where R_0' is the residual change in the electrical model.

With a known ratio of $\frac{J}{\sigma}$, the residual (R_0') can be eliminated at successive nodes in the electrical model by reducing the voltage by an amount $v_0 = \frac{R_0'}{4}$. A reduction of the residuals by this method is called iteration. If a gross change either larger or smaller than $v_0 = \frac{R_0'}{4}$ is applied rather than changes at each node, the adjustment procedure is known as the relaxation method (Ferris and others, 1962, p. 138-139).

Normally the time rate of change of the gradient is unknown at the beginning of a study period. After sufficient time, however, the initial change will decay to zero. Therefore, in the electrical analog model the residual is assumed to be zero at the beginning of the study period. This means that the maximum error in the time rate of head change is introduced at the beginning of stress when the residual is maximum, but the effect is continually diminished with time. In essence, the period of historical record defines the initial change in slope of the water table. Thus, subsequent changes will incorporate less and less of the residual error.

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