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Derivation of Linearized Constitutive Equations for Plane-Strain of an Elastic-Plastic (Strain Hardening) Material

by

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This report is preliminary and has not been edited or reviewed for conformity with Geological Survey standards and nomenclature.

Introduction

The purpose of this report is to derive several of the basic equations for a paper on folding and folding of idealized rock (Johnson, 1979). That paper is based on constitutive equations for an ideal, strain-hardening material defined by Hill (1950, p. 30) and generalized to include Coulomb behavior and dilatancy by Rudnicki and Rice (1975). The most attractive feature of the theory is that folding and faulting, which are intimately related in nature, are different responses of the same material to different boundary conditions. In the paper by Johnson (1979) it is shown that single layers of sedimentary rocks are unlikely to fold, rather they will fault, because of low contrasts in elasticity and strength properties of sedimentary rocks such as layers of dolomite, limestone, sandstone, or siltstone in media of shale. Further, multilayers of these same rocks will fault rather than fold if contacts are bonded, but they will fold readily if contacts between layers are frictionless, or have low yield strengths, for example due to high pore-water pressure. These conclusions are based on solutions of the equations presented in the following pages, using experimental stress-strain curves for various rock types.

Results

First, I will define some symbols (Malvern, 1969; Fung, 1965). Let σ be Cauchy stress (Fung, 1965, p. 439). Let x_i be a Cartesian coordinate of a point in a body of the current state, u_i be a component of the displacement field, and v_i be a component of the velocity field associated with the coordinates (Euler description of flow). Then D_{ij} is the rate of deformation tensor,

$$D_{ij} = (1/2)[(\partial v_j/\partial x_i)+(\partial v_i/\partial x_j)] \qquad (1)$$

Ω_{ij} is the rate of spin tensor,

$$\Omega_{ij} = (1/2)[(\partial v_j/\partial x_i)-(\partial v_i/\partial x_j)]; \; i,j=1,2 \qquad (2)$$

and <u>infinitesimal</u> Green strains are

$$\varepsilon_{ij} = (1/2)[(\partial u_j/\partial x_i)+(\partial u_i/\partial x_j)] \tag{3}$$

Now let us derive the basic equations for incompressible, isotropic, elastic, strain-hardening plastic materials. We shall assume that both the elastic and the plastic strains are infinitesimal so that they can be superimposed; otherwise the analysis becomes much more complicated (Lee, 1969; Wang, 1973). The elastic strains are

$$\varepsilon_{ij}^{(e)} = \sigma'_{ij}/2G \tag{4a}$$

where the deviatoric stresses are

$$\sigma'_{ij} = \sigma_{ij} - (1/3)\sigma_{kk}\delta_{ij} \tag{4b}$$

and G is the elasticity shear modulus.

Also, the material is incompressible, so that

$$\varepsilon_{kk} = \varepsilon_{kk}^{(e)} = 0$$

The total strains are

$$\varepsilon_{ij} = \varepsilon_{ij}^{(p)} + \sigma'_{ij}/2G \tag{4c}$$

where $\varepsilon_{ij}^{(p)}$ is the plastic strain.

Most of the remainder of this paper will be devoted to the derivation of an expression for the plastic strain in eq. (4c). We first formulate the response of the material in terms of rates of stress and rates of deformation. Then we integrate the equations with respect to time in order to derive stress-strain relations.

We assume that the plastic material is a von Mises plastic, the yield condition for which is

$$J_2 \leq k^2 \tag{5a}$$

where

J_2 is the second invariant of the deviatoric stresses,

$$J_2 = (1/2)\ \sigma'_{ij}\ \sigma'_{ij} \tag{5b}$$

and <u>k</u> is the shear strength of the material (Kachanov, 1974). The material we are considering strain hardens (or softens), however, so that <u>k</u> is a function of the total plastic strain. If the material hardens isotropically,

$$k = \sqrt{J_2} = H[\int \sqrt{I_2}^{(p)} dt] \tag{6a}$$

where H[ ] is a function of the logarithmic plastic strain through the invariant

$$I_2^{(p)} = 2D_{ij}^{(p)} D_{ij}^{(p)} \tag{6b}$$

and <u>t</u> is time. Eq. (6a) apparently is equivalent to a relationship defined by Hill (1950, p. 30), who should be consulted for further details.

We shall now derive the following relation between rate of plastic deformation and the function <u>H</u>:

$$D_{ij}^{(p)} dt = [\sigma'_{ij}/(\sqrt{J_2} h)]\ d(\sqrt{J_2}) \tag{7a}$$

where <u>h</u> is the <u>hardening</u> <u>modulus</u>,

$$h = d(\sqrt{J_2})/(\sqrt{I_2}^{(p)} dt) \equiv dH/(\sqrt{I_2}^{(p)} dt) \tag{7b}$$

Two basic equations are needed in addition to those already presented: First, according to the Saint Venant-von Mises theory of plasticity (Kachanov, 1974, p. 51), an increment of plastic deformation is proportional to the deviatoric stress,

$$(D_{ij}^{(p)} dt/\sigma'_{ij}) = (D_{xx}^{(p)} dt/\sigma'_{xx}) = (D_{xz}^{(p)} dt/\sigma_{xz}) = d\lambda \qquad (8)$$

where λ is some parameter. Second, the increment of plastic work, $\underline{dW}^{(p)}$, is defined as

$$dW^{(p)} = \sigma'_{ij} D_{ij}^{(p)} dt \qquad (9)$$

because the mean stress does no work.

Using eq. (8), we can write eq. (9) in terms of the invariants, J_2 and $I_2^{(p)}$. For example, eliminating $D_{ij}^{(p)}dt$ in eq. (9) with eq. (8),

$$dW^{(p)} = \sigma'_{ij}\sigma'_{ij} \; d\lambda = 2J_2 d\lambda$$

Similarly, we can eliminate σ'_{ij} in eq. (9) with eq . (8) and show that

$$4J_2(d\lambda)^2 = I_2^{(p)}(dt)^2$$

or

$$d\lambda = (1/2)\sqrt{(I_2^{(p)}/J_2)} \; dt \qquad (10)$$

Equating (8) and (10),

$$D_{ij}^{(p)} = (1/2)\sigma'_{ij} \sqrt{(I_2^{(p)}/J_2)}$$

Finally, eliminating $I_2^{(p)}$ with eq. (7b) we derive eq. (7a).

Performing the differentiation of $\sqrt{J_2}$ indicated in eq. (7a),

$$d\sqrt{J_2} = (1/2)\sigma'_{k\ell} \, d\sigma'_{k\ell}/\sqrt{J_2} = (1/2)\sigma'_{k\ell} \, d\sigma_{k\ell}/\sqrt{J_2}$$

we derive

$$D_{ij}^{(p)} = [\sigma'_{ij}\sigma_{k\ell}/(4hJ_2)](d\sigma_{k\ell}/dt) \tag{11}$$

The remaining problem is to provide a proper form for the time derivative, $d\sigma_{k\ell}/dt$. The rate of deformation tensor, $\underline{D}_{ij}$, is a well-behaved measure of rate of deformation; it vanishes if the body is subjected solely to rigid-body spin. We require a measure of stress rate that is similarly insensitive to spin of the body; otherwise, eq. (11) would be nonsense. The measure of the rate of change of stress usually employed is the Jaumann rate, the rationale of which is explained clearly by Fung (1965). The Jaumann stress rate is defined as

$$\overset{\nabla}{\sigma}_{ij} = D\sigma_{ij}/Dt - \sigma_{ip}\Omega_{pj} - \sigma_{jp}\Omega_{pi} \tag{12}$$

where D is the material time derivative and Ω_{ij} is the spin tensor defined in eq. (12).

For plane deformation, which we assume in folding analyses, the Jaumann stress rates are

$$\overset{\nabla}{\sigma}_{xx} = D\sigma_{xx}/Dt + 2\sigma_{xz}\Omega_{xz} \tag{13a}$$

$$\overset{\nabla}{\sigma}_{zz} = D\sigma_{zz}/Dt - 2\sigma_{xz}\Omega_{xz} \tag{13b}$$

$$\overset{\nabla}{\sigma}_{xz} = D\sigma_{xz}/Dt + (\sigma_{zz} - \sigma_{xx})\Omega_{xz} \tag{13c}$$

where

$$D\ /Dt = \partial\ /\partial t + \dot{u}\partial\ /\partial x + \dot{w}\partial\ /\partial z \tag{13d}$$

in which $\underline{\dot{u}}$ and $\underline{\dot{w}}$ are components of velocity in the $\underline{x}$- and $\underline{z}$- directions, respectively.

Now we can write eq. (11) in the final form,

$$D_{ij}^{(p)} = [\sigma'_{ij}\ \sigma'_{k\ell}/(4hJ_2)]\overset{\nabla}{\sigma}_{k\ell} \tag{14}$$

which is equivalent to a special case of relations derived by Rudnicki and Rice (1975, p. 10). Further, if we take appropriate time derivatives of eq. (4c), we can write the expression for the total rate of deformation of the strain-hardening material. From eqs. (4c) and (14),

$$D_{ij} = D_{ij}^{(p)} + \overset{\nabla}{\sigma}_{ij}/2G$$

so that

$$D_{ij} = (\overset{\nabla}{\sigma}{}'_{ij}/2G) + [\sigma'_{ij}\sigma'_{k\ell}/(4hJ_2)]\overset{\nabla}{\sigma}_{k\ell} \qquad (15)$$

This completes the derivation of the basic equation for the elastic, strain-hardening (or softening) plastic material.

In order to solve problems in folding theory we linearize eq. (15) by considering a mean flow, $\bar{D}_{ij}$, and a perturbing flow, $\tilde{D}_{ij}$ and $\tilde{\Omega}_{ij}$, so that,

$$D_{ij} = \bar{D}_{ij} + \tilde{D}_{ij} \qquad (16a)$$

$$\Omega_{ij} = \tilde{\Omega}_{ij} \qquad (16b)$$

Similarly,

$$\sigma_{ij} = \bar{\sigma}_{ij} + \tilde{\sigma}_{ij} \qquad (16c)$$

The mean flow and associated stresses are homogeneous, so that they are independent of position. Further, the perturbing flow and associated perturbing stresses are small excursions from the mean state. They are in general inhomogeneous. In folding, for example, the mean flow is the uniform shortening of a layer and the perturbing flow is the flow associated with the growth of folds in the layer. We shall substitute eqs. (16) into eq. (15), maintaining terms to <u>first order</u> in the perturbing stresses and deformations. To first order,

$$J_2 \simeq \bar{J}_2 + \tilde{J}_2$$

where

$$\bar{J}_2 = (1/2)\ \bar{\sigma}'_{ij}\bar{\sigma}'_{ij} \tag{17a}$$

$$\tilde{J}_2 = \bar{\sigma}'_{ij}\tilde{\sigma}'_{ij} \tag{17b}$$

If the mean flow involves zero shear, $\bar{\sigma}_{xz} = 0 = \bar{D}_{xz}$,

$$h \simeq \bar{h} + \tilde{h} \tag{17c}$$

where

$$\bar{h} = [dH/\sqrt{\bar{I}_2}^{(p)} dt]\ ;\ \text{at constant}\ \bar{I}_2^{(p)} \tag{17d}$$

$$\tilde{h} = [dh/\sqrt{\bar{I}_2}^{(p)} dt) 2\bar{D}^{(p)}_{xx}\ \tilde{D}^{(p)}_{xx} dt/\sqrt{\bar{I}_2}^{(p)}];\ \text{at constant}\ \bar{I}_2^{(p)} \tag{17e}$$

$$\sqrt{(\bar{I}_2^{(p)})} = 2|\bar{D}^{(p)}_{xx}| \tag{17f}$$

In Johnson (1979) the bar over the h defined in eq. (17d) has been dropped for simplicity.

Two forms of linearized equations can be derived now, depending upon what assumptions are made concerning the state of mean stress. If the mean stress is time dependent, the expressions for the perturbed Jaumann stress rates can be derived from:

$$2\tilde{D}_{xx} = (\tilde{\sigma}^{\nabla}_{xx}{}'/G_N) - (2\tilde{h}\bar{\sigma}^{\nabla}_{xx}{}'/\bar{h}^2) \tag{18a}$$

$$D_{zz} = -D_{xx};\quad \sigma^{\nabla}_{zz}{}' = -\sigma^{\nabla}_{xx}{}' \tag{18b}$$

$$2\tilde{D}_{xz} = (\tilde{\sigma}^{\nabla}_{xz}/G) + (\bar{\sigma}'_{xx}\bar{\sigma}^{\nabla}_{xx}{}'\tilde{\sigma}_{xz}/\bar{h}\bar{J}_2) \tag{18c}$$

where

$$G_N = [(1/G) + (1/\bar{h})] = G\bar{h}/(G+\bar{h}) = \bar{h}_{tan} \tag{18d}$$

and the expressions for the mean Jaumann stress rates are

$$\bar{\sigma}^{\nabla}_{xx}{}' = -\bar{\sigma}^{\nabla}_{zz}{}' = 2G_N\bar{D}_{xx};\quad \bar{\sigma}^{\nabla}_{xz}{}' = 0 \tag{18e}$$

These equations are somewhat difficult to deal with because they contain a mixture of perturbed stress and stress rates and strain and strain rates. The second term in eq. (18a) contains $\underline{\tilde{h}}$, which, according to eq. (17e), contains an increment of perturbed plastic strain, $\underline{\tilde{D}^{(p)}_{xx}dt} = \tilde{\varepsilon}^{(p)}_{xx}$, and the second term in eq. (18c) contains the perturbed shear stress.

We can derive a second form for the linearized equations by assuming that the mean stresses are time independent. This form of equations is used in Johnson (1979) to solve <u>buckling</u> problems; problems where there may be two equilibrium forms of a layer, a uniformly shortened layer and a folded layer. These equations are exact to first order, yet they avoid the complications associated with the solution of eqs. (18).

Thus, we assume that

$$\bar{\sigma}^{\nabla\prime}_{ij} = 0 = \bar{D}_{ij}$$

and in this case eq. (15) becomes, to first order,

$$\tilde{D}_{ij} = \tilde{\sigma}^{\nabla\prime}_{ij}/2G + (\sigma'_{ij}\sigma'_{k\ell}/4\bar{h}J_2)\tilde{\sigma}^{\nabla}_{k\ell}$$

so that, for example,

$$\tilde{D}_{xx} = (1/2G_N)\sigma^{\nabla\prime}_{xx}$$

where $\underline{G}_N$ is defined in eq. (18d).

Further, the strains are infinitesimal, so that

$$\tilde{\varepsilon}_{xx} = \tilde{D}_{xx}t$$

$$\tilde{\sigma}_{xx} = \tilde{\sigma}^{\nabla}_{xx}t$$

and

$$\tilde{\sigma}'_{xx} = 2G_N\tilde{\varepsilon}_{zz} \qquad (19a)$$

Similarly,

$$\tilde{\sigma}'_{zz} = 2G_N\tilde{\varepsilon}_{zz} \qquad (19b)$$

For shearing,

$$\tilde{D}_{xz} = \overset{\nabla}{\tilde{\sigma}}_{xz}/2G + (1/4\bar{h}J_2)\tilde{\sigma}_{xz}[(\bar{\sigma}'_{xx} + \tilde{\sigma}'_{xx})(\overset{\nabla}{\tilde{\sigma}}_{xx} - \overset{\nabla}{\tilde{\sigma}}_{zz})]$$

so that, to first order,

$$\overset{\nabla}{\tilde{\sigma}}_{xz} = 2G\tilde{D}_{xz}$$

Using the definition of the Jaumann stress rates in eqs. (13),

$$\partial\tilde{\sigma}_{xz}/\partial t = 2G\tilde{D}_{xz} + (\bar{\sigma}_{xx} - \bar{\sigma}_{zz})\tilde{\Omega}_{xz}$$

and

$$\tilde{\sigma}_{xz} = 2G\tilde{\varepsilon}_{xz} + (\bar{\sigma}_{xx} - \bar{\sigma}_{zz})\tilde{\omega}_{xz} \tag{19c}$$

where $\tilde{\omega}_{xz}$ is the rotation,

$$\tilde{\omega}_{xz} = (1/2)[(\partial w/\partial x) - (\partial u/\partial z)] \tag{19d}$$

in which $\underline{u}$ and $\underline{w}$ are displacements in the $\underline{x}$- and $\underline{z}$-directions, respectively. This completes the derivation of the basic linearized equations used to solve folding problems.

Now I will briefly explain how one extracts the parameter $\underline{h}$ from experimental data. The parameter is the slope of the stress-strain curve for pure shear. According to eq. (7b),

$$h = d\sqrt{J_2}/\sqrt{I_2}^{(p)}dt$$

For pure shear,

$$J_2 = (1/2)\sigma'_{ij}\sigma'_{ij} = \sigma^2_{xz} = \tau^2$$

Thus,

$$\sqrt{J_2} = |\tau|.$$

Also,

$$I_2^{(p)} = 2D_{ij}^{(p)}D_{ij}^{(p)} = 4(\dot{\varepsilon}_{xz}^{(p)})^2 = (\dot{\gamma}^{(p)})^2$$

$$\gamma \equiv 2\varepsilon_{xz}$$

so that

$$\sqrt{I_2}^{(p)}dt = |d\gamma^{(p)}|$$

and

$$h = d\tau/d\gamma^{(p)} \tag{20}$$

as shown in Fig. 4C of Johnson (1979).

For <u>triaxial loading</u>, where σ_a is axial stress, σ_c is confining pressure and ε_a is axial strain,

$$\sqrt{J_2} = \sqrt{(1/3)}|\sigma_a - \sigma_c|$$

$$\sqrt{I_2}^{(p)} = \sqrt{3}|\dot{\varepsilon}_a{}^{(p)}|$$

so that

$$h = (1/3)d(\sigma_a - \sigma_c)/d\varepsilon_a \tag{21}$$

Expressions (20) and (21) are valid if plastic strains are infinitesimal. If the strains are finite, the strain is replaced by the logarithmic strain, as explained by Hill (1950, p. 31).

Finally, I want to show that eqs. (19) reduce to the expressions for an incompressible, Mooney material if both the mean and perturbing strains are infinitesimal. I must assume that the reader is familiar with notation presented elsewhere (Johnson, 1977, p. 340) in order to keep the discussion brief. According to the analysis of buckling of an elastic material (Johnson, 1977, p. 347), the perturbing stresses, $\underline{s}$, are

$$s_{xx}' = 2G(\partial u/\partial x)\lambda_X^2$$

$$s'_{zz} = 2G(\partial w/\partial z)\lambda_Z^2$$

$$s_{xz} = G[(\partial w/\partial x)\lambda_X^2 + (\partial u/\partial z)\lambda_Z^2]$$

where

$$\lambda_X = 1 + \partial U/\partial X;\ \lambda_Z = 1 + \partial W/\partial Z$$

are the finite strains and U and W are displacements associated with the mean deformation. If normal strains are infinitesimal, $(\partial u/\partial x)(\partial U/\partial X) \simeq 0$, for example, and if rotation can be moderate, $(\partial w/\partial x)(\partial U/\partial X) \neq 0$, for example. Thus, to the order of accuracy indicated above,

$$s'_{xx} = 2G\partial u/\partial x \tag{22a}$$

$$s'_{zz} = 2G\partial w/\partial x \tag{22b}$$

$$s_{xz} = G(\partial w/\partial x + \partial u/\partial x) + 2G(\partial U/\partial X)(\partial w/\partial x - \partial u/\partial z)$$

because $\partial U/\partial X = -\partial W/\partial Z$.

Using our notation for mean stress,

$$\bar{\sigma}_{xx} - \bar{\sigma}_{zz} = 4G\partial U/\partial X$$

so that

$$s_{xz} = G(\partial w/\partial x + \partial u/\partial z) + (\bar{\sigma}_{xx} - \bar{\sigma}_{zz})\tilde{\omega}_{xz} \tag{22c}$$

Equations (22) are equivalent to eqs. (19) if the hardening modulus, h, approaches infinity. Thus, eqs. (19) are exact to first order if both the mean strains and the perturbing strains are infinitesimal. This qualification will allow us to apply the analysis to folding of sedimentary rocks, where typical strains at peak stresses are in the range 10^{-4} to 10^{-2}.

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