A Model of the Strain Response of

Barre Granite to Wetting and Drying

UNITED STATES DEPARTMENT OF THE INTERIOR

U.S. GEOLOGICAL SURVEY

A Model of the Strain Response of Barre Granite to Wetting and Drying

By

William Z. Savage

Open-File Report 79-768
1979

This report is preliminary and has not been edited or reviewed for conformity with U.S. Geological Survey standards or nomenclature.

CONTENTS

ILLUSTRATION

A MODEL OF THE STRAIN RESPONSE OF BARRE GRANITE TO WETTING AND DRYING

By

William Z. Savage

Introduction

Experiments on cores of Barre Granite suggest that wetting and drying cause strains in this rock. Terzaghi (1943) discussed a related problem: the consolidation of a clay layer in response to desiccation. However, his treatment is one dimensional and not applicable to the problem considered here. What follows is an attempt to model these strain responses using a special case of Biot's (1941) theory of three-dimensional consolidation.

Biot's theory of three-dimensional consolidation

Biot (1941) considered the stress-strain response of a porous material having the following properties: (1) isotropy, (2) reversibility of stress-strain relations under final equilibrium conditions, (3) linearity of stress-strain relations, (4) an incompressible pore fluid, (5) small strains, (6) and flow of water through the porous skeleton according to Darcy's law.

The first of Biot's constitutive (stress-strain) relations is,

$$e_{ij} = \frac{1}{2\mu}\left(\sigma_{ij} - \frac{\lambda}{3\lambda+2\mu}\delta_{ij}\sigma_{kk}\right) + \frac{p}{3H}\delta_{ij} \quad , \tag{1}$$

where e_{ij} represents strains, σ_{ij} represents stresses, μ and λ are, respectively, the shear modulus, and Lame's constants for the elastic skeleton. The pore pressure is represented by p and the coefficient 1/H is a measure of compressibility of the porous material for a given change in water pressure. As can be seen, equations (1) reduce to the usual elastic relations when the pore pressure, p, vanishes.

Because consolidation involves removal of pore water, an additional variable specifying the amount of pore fluid per unit volume must be defined. For a saturated material containing an incompressible fluid, this variable, θ, is equal to $\eta - \eta_o$ where η and η_o are porosities in the strained and unstrained states. Biot assumed the relation between water content (porosity), pore pressure, and mean stress $(\frac{\sigma_{ii}}{3})$ to be

$$\theta = p/R + \frac{\sigma_{ii}}{3H} \quad , \tag{2}$$

where 1/R measures the change in water content (porosity) for a given change in water pressure. This second constitutive relation (equation 2) predicts that an increase in pore-water pressure causes an increase in porosity, and that a decrease in mean stress (that is, increasing compression) will cause a decrease in porosity in a water-saturated body.

Constitutive equations (1) and (2) may be inverted for stress in terms of strain to give

$$\sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij} - \alpha \delta_{ij} p \quad , \tag{3}$$

where $\alpha = \frac{3\lambda + 2\mu}{3H} = \frac{K}{H}$ (K is the bulk modulus), and equation 2 may be written

$$\theta = \alpha e_{ii} + p/Q \quad , \tag{4}$$

where $1/Q = 1/R - {}^{\alpha}/H$.

The stresses given by equation (3) must satisfy equilibrium, or

$$\frac{\partial \sigma_{ij}}{\partial x_i} + \rho F_i = 0 \quad , \qquad (5)$$

where F_j represents the components of body force per unit mass at the point x_i. Substitution of equation (3) into (5) and utilization of the strain displacement relation $e_{ij} = 1/2 \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right]$ leads to the equations of equilibrium in terms of displacements,

$$(\lambda + \mu) \frac{\partial}{\partial x_j} \left(\frac{\partial u_i}{\partial x_i}\right) + \mu \frac{\partial}{\partial x_i} \left(\frac{\partial u_j}{\partial x_i}\right) - \alpha \frac{\partial p}{\partial x_j} + \rho F_j = 0 \quad . \qquad (6)$$

We see that the pressure gradients $\frac{\partial p}{\partial x_j}$ affect the displacements like a body force.

An additional relation is needed to describe the flow of pore fluid in response to changes in pore pressure. According to Darcy's law, the rate of flow of fluid, V_i, at a point, defined by the volume of fluid crossing a unit area per unit time, is proportional to the gradient of pore pressure at that point, or

$$V_i = -k \frac{\partial p}{\partial x_i} \quad , \qquad (7)$$

where k is a constant called the permeability.

Assuming the pore fluid to be incompressible, continuity requires the increase of fluid content per unit time in a volume of porous solid, or $\int_v \frac{\partial \theta}{\partial t}\, dv$, to equal the volume of fluid entering per unit time through the surface of the volume, or $-\int_s V_i \eta_i\, ds$, where η_i is an outward normal to s. We have, then

$$\int_v \frac{\partial \theta}{\partial t}\, dv = -\int_s V_i \eta_i\, ds$$

which, by the divergence theorem of Gauss, is

$$\int_v \frac{\partial \theta}{\partial t}\, dv = -\int_v \frac{\partial V_i}{\partial x_i}\, dv$$

or, finally, the statement of continuity:

$$\frac{\partial \theta}{\partial t} + \frac{\partial V_i}{\partial x_i} = 0 \quad . \tag{8}$$

Substituting equations (4) and (7) in (8) leads to

$$k \frac{\partial^2 p}{\partial x_i \partial x_i} = \alpha \frac{\partial e_{kk}}{\partial t} + 1/Q \frac{\partial p}{\partial t} \quad . \tag{9}$$

The four equations (6) and (9) in the four unknowns u_i and p constitute the basic equations in Biot's theory of consolidation. We will consider a particular solution to these equations for the case $F_j = 0$. Let the displacement be given by the gradient of a scalar, ϕ, or

$$u_i = \frac{\partial \phi}{\partial x_i} \quad . \tag{10}$$

Substituting equation (10) in (6) we find

$$(\lambda + 2\mu) \frac{\partial}{\partial x_j} \left[\frac{\partial^2 \phi}{\partial x_i \partial x_i}\right] - \alpha \frac{\partial p}{\partial x_j} = 0 \quad ,$$

which becomes

$$\frac{\partial^2 \phi}{\partial x_i \partial x_i} = \frac{\alpha}{\lambda+2\mu}{}_i \, p + C_o \quad , \tag{11}$$

when integrated with respect to x_j. The term C_o is a constant of integration.

Now, since

$$e_{kk} = \frac{\partial u_k}{\partial x_k} \quad ,$$

we have by equations (10) and (11)

$$e_{kk} = \frac{\partial^2 \phi}{\partial x_k \partial x_k} = \frac{\alpha p}{\lambda+2\mu} + C_o \quad . \tag{12}$$

Substituting equation (12) in equation (9), we find

$$k \frac{\partial^2 p}{\partial x_i \partial x_i} = \left[\frac{\alpha^2}{\lambda+2\mu} + 1/Q\right] \frac{\partial p}{\partial t}$$

or

$$\frac{\partial^2 p}{\partial x_i \partial x_i} = \frac{1}{C} \frac{\partial p}{\partial t} \quad , \tag{13}$$

where $1/C = 1/k \left[\frac{\alpha^2}{\lambda+2\mu} + 1/Q\right]$. Note that the pore pressure satisfies an equation of the same form as the heat conduction equation.

The coefficient $1/Q$ is a measure of the amount of water that can be forced into a porous material when the volume of the material is kept constant. In many rocks the compressibility of the mineral constituents is small; thus, $1/Q \quad 0$. Further, because of this small compressibility and because of the incompressibility of the water, changes in water content and volumetric changes are approximately equal. Such a condition is satisfied when $\alpha = 1$ and $1/Q = 0$ in equation (4). Then, since $1/Q = 1/R - \alpha/H$ and $\alpha = K/H$, we have for rocks with $\alpha = 1$ and $1/Q = 0$, $R=H=K$ and the coefficient of consolidation, $C = k(\lambda+2\mu)$.

Models of strain response to wetting and drying in rock cores

For modeling the strain response to wetting and drying in rock cores, we consider a long circular cylinder where the pore pressure is a function of only the radial coordinate, r, and time, t. The radial displacement is given by

$$u = \frac{\partial \phi}{\partial r} \quad , \qquad (14)$$

and equation (11) becomes, with $\alpha = 1$,

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} = \frac{1}{\lambda+2\mu} p + C_o \quad . \qquad (15)$$

Here p satisfies equation (13), or

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} = 1/C \; \frac{\partial p}{\partial t} \quad , \qquad (16)$$

with appropriate initial and boundary conditions and $C = k\,(\lambda+2\mu)$.

Equation (15) can be written as

$$\frac{1}{r}\frac{d}{dr}[r\frac{d\phi}{dr}] = \frac{1}{\lambda+2\mu} p + C_o$$

or

$$\frac{d}{dr}[r\frac{d\phi}{dr}] = \frac{1}{\lambda+2\mu} pr + C_o r \quad . \qquad (17)$$

Integrating equation (17), we find

$$u = \frac{d\phi}{dr} = \frac{1}{\lambda+2\mu}\frac{1}{r}\int^r prdr + c_1 r + \frac{c_2}{r}$$

where $c_1 = c_o/2$, or

$$u = \frac{(1+\nu)}{3K(1-\nu)}\frac{1}{r}\int^r prdr + c_1 r + \frac{c_2}{r} \quad . \qquad (18)$$

In equation (18) we used the identity

$$\lambda + 2\mu = \frac{3K(1-\nu)}{(1+\nu)} \quad ,$$

where K and ν are the bulk modulus and Poisson's ratio for the solid skeleton.

Equation (18) is completely analogous to the solution for displacements in the thermoelastic case (Timoshenko and Goodier, 1970, p. 444), except that temperature is replaced by pore pressure and the coefficient of thermal expansion is replaced by 1/3K. Because of this analogy, the expressions for displacements and stresses given in Timoshenko and Goodier for thermoelastic cylinders are applicable here.

For a long solid cylinder; i.e., a core, under no external loads and zero axial strain, we have, from equation 248 on page 445 of Timoshenko and Goodier (1970),

$$u = \frac{1+\nu}{1-\nu} \frac{1}{3K} \left[(1-2\nu) \frac{r}{a^2} \int_0^a prdr + \frac{1}{r} \int_0^r prdr \right] \quad , \qquad (19)$$

where a is the outer radius of the cylinder. For a long solid cylinder under zero axial force, from equation 252 on the same page,

$$u = \frac{1+\nu}{1-\nu} \frac{1}{3K} \left[\frac{1-3\nu}{1+\nu} \frac{r}{a^2} \int_0^a prdr + \frac{1}{r} \int_0^r prdr \right] \quad . \qquad (20)$$

Radial and hoop (circumferential) strains in both cases are given by $e_r = \frac{\partial u}{\partial r}$ and $e_\theta = u/r$. Note that equation 20 applies only at some distance from the ends of the cylinder.

Consider the hoop strains induced by drying a rock core in air. Since the core is under no external loads at the outset, the pore pressure is uniformly zero. As the core dries, water is evaporated from the surface, the rate of evaporation decreasing with increasing tension in the water. At some point, water in the surface pores is under tension such that the rate of evaporation becomes zero. As an approximation, we assume that the rate of evaporation is constant until the time when evaporation ceases, owing to surface tension in the water. The distribution of pore pressure during this process is given by a solution to the following boundary and initial value problem.

At $t=o$, $p=o$ for $o \leq r \leq a$ and for $o<t<t_o$, $V_e = k \frac{\partial p}{\partial r}$ on $r=a$ where V_e is the constant rate of evaporation. For $t_o<t<\infty$, $\frac{\partial p}{\partial r} = 0$ on $r=a$. The solution to equation (16) for $o<t<t_o$ (when $V_e = k \frac{\partial p}{\partial r} |_{r=a}$) as modified from Carslaw and Jaeger (1959, p. 203) is

$$p = \frac{V_e}{k}[\frac{2Ct}{a} + a[\frac{r^2}{2a^2} - 1/4 - 2 \sum_{n=1}^{\infty} [e^{-C\alpha_n^2 t/a^2}] \frac{J_o(r\alpha_n/a)}{\alpha_n^2 J_o(\alpha_n)}]] \quad , \qquad (21)$$

where α_n are positive roots of $J_1(\alpha) = 0$. For $t>t_o$ when $\frac{\partial p}{\partial r} |_{r=a} = 0$,

$$p = \frac{V_e}{k} [\frac{2Ct_o}{a} - 2 a \sum_{n=1}^{\infty} [e^{-C\alpha_n^2 t/a^2} - e^{\frac{-C\alpha_n^2 (t-t_o)^2}{a^2}}] \frac{J_o(r\alpha_n/a)}{\alpha_n^2 J_o(\alpha_n)}] \quad . \qquad (22)$$

From equation (22) we see that after evaporation from the surface stops and as $t \to \infty$, $p \to \frac{2Ct_o V_e}{ka}$, and the pore pressure becomes constant throughout the cylinder. Evaporation removes water from the cylinder, V_e, and hence the pore pressure is negative. The final result of the process is then a cylinder subjected to a constant negative pore pressure.

Insertion of equations (21) and (22) into (19) and (20) leads to expressions for radial displacements occurring during the desiccation process. Radial and hoop strains are obtained using $e_r = \frac{\partial u}{\partial r}$ and $e_\theta = u/r$. Predicted hoop strains at the surface of a long cylinder which is free from axial force are of most interest. They are given by

$$e_\theta = \frac{2V_e(\lambda+2\mu)t}{3Ka} \quad , \qquad (23)$$

for $o \leq t \leq t_o$ and by

$$e_\theta = \frac{2V_e(\lambda+2\mu)t_o}{3Ka} \quad , \qquad (24)$$

for $t>t_o$.

During desiccation, V_e is negative and the surface hoop strains decrease linearly from zero (equation 23) until the time, t_o, when V_e vanishes. After this time the surface hoop strains are constant. The constant surface hoop strain for $t>t_o$ reflects the fact that the pore pressure becomes constant throughout the cylinder after evaporation from the surface ceases.

Let us assume that after desiccation a rock core undergoes wetting. The desiccated core is initially under uniform negative pore pressure. When the core is wetted, water is drawn into the rock in response to this negative pore pressure and the core swells. Quantitatively, this process is described by the following boundary value problem. At $t = o$,

$p = p_o = \frac{2CV_e t_o}{ka}$ for $o \leq r \leq a$ and for $o < t < t_e$ $p = 0$ on $r = a$. Here t_e represents a time when surface evaporation causes the desiccation process to begin again. The solution to equation (16) for this case, as modified from Carslaw and Jaeger (1959, p. 199), is

$$p = \frac{2p_o}{a} \sum_{n=1}^{\infty} e^{-C\alpha_n^2 t} \frac{J_o(r\alpha_n)}{\alpha_n J_1(a\alpha_n)} , \tag{25}$$

where the α_n are positive roots of $J_o(a\alpha_n) = 0$.

From equation (25) we see that as time increases the pore pressure approaches zero. A state of zero pore pressure would be obtained if the core were immersed during the entire process, but if the core were simply wetted and then air dried, the desiccation process would eventually predominate.

Insertion of equation (25) in equations (19) and (20) gives expressions for the radial displacements occurring during wetting. Again, the hoop strain at the surface of a cylinder free from axial force is of most interest. It is given by

$$e_\theta = \frac{4p_o}{3K} \sum_{n=1}^{\infty} e^{\frac{-C\alpha_n^2 t}{a^2 \alpha_n^2}} , \qquad (26)$$

where $p_o = \dfrac{2CV_e t_o}{ka}$.

At the beginning of the wetting cycle the hoop strain resulting from desiccation is compressive. From equation (26) we see that, as water is drawn in, the hoop strain increases exponentially with time until an asymptotic value of zero strain is reached. This is consistent with the fact (equation 25) that the pore pressure throughout the cylinder approaches zero as time increases. However, if the core is wetted and then air dried, swelling occurs, but desiccation eventually causes the hoop strain to become more compressive.

Application to experiments on Barre Granite

Laboratory experiments on the strain response of Barre Granite cores to wetting and drying cycles are currently being carried out in U.S. Geological Survey laboratories. The 7.62-cm-radius cores which have been stored under constant temperature and humidity conditions for more than 1 year have strain gages attached to the ends and cylindrical surfaces. The strain gages are connected to a 50-channel Vishay continuous recorder to give strain versus time readings for various intervals.

A typical example of the hoop strain response of a granite core wetted for more than 1 hour with tap water and then dried under constant temperature and humidity conditions is shown in figure 1. The hoop strain shown is measured away from the ends of the core. The strains shown in figure 1 are relative to the value at the start of the experiment.

From figure 1 we infer that water flows into the core for the first 60 hours. Note that an asymptotic value of strain is reached, indicating that the wetting process goes to completion. The evaporation rate appears to remain constant from 60 to 350 hours. After 350 hours, evaporation apparently ceases and the strain becomes constant.

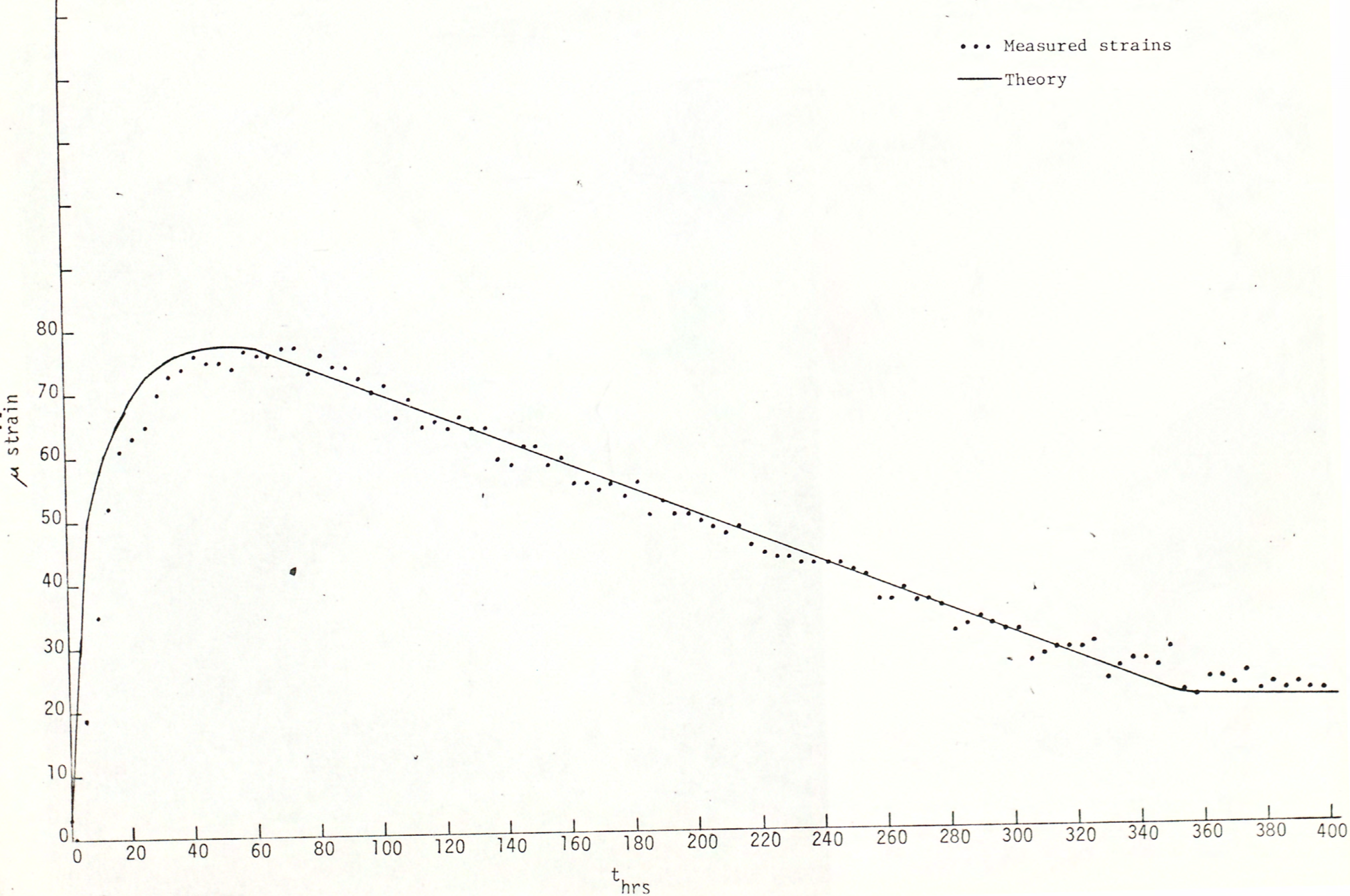

Figure 1.--Hoop strain versus time after wetting for 7.62-cm-radius core of Barre Granite.

The solid line in figure 1 represents the application of the theory described above. The theoretical curve shown in figure 1 has been adjusted to start at zero strain. The wetting cycle strains (0-60 hours) are described by equation (26) with $P_o \simeq 100$ atm, $K=4.5X10^{11}$ dyne/cm^2 and a=7.62 cm. The constant $C=k(\lambda+2\mu)$ is given by the product of k=0.65 nanodarcy and $\lambda+2\mu=4.25X10^{11}$ dyne/cm^2. The values of K and $\lambda+2\mu$ are from Thill, Bur, and Steckley (1973). The permeability is three orders of magnitude smaller than that measured on Barre Granite by Mesri, Adachi, and Ullrich (1976). However, their permeability tests were made with a high fluid pressure differential. The value 0.65 nanodarcy is closer to permeabilities of Westerly Granite measured by Brace, Walsh, and Frangos (1968) with a small pressure differential.

The drying cycle strains beginning at 60 hours are described by equations (23) and (24). We can estimate the constant rate of evaporation, V_e, from equation (24). Taking t_o to be 290 hours and the final hoop strain to be $22X10^{-6}$ we solve for V_e and find it to be $2.56X10^{-10}$ cm/s over a surface area of 1 cm^2. It is interesting to note that Thill, Bur, and Steckley (1973) found that during desiccation the water content in Barre Granite stabilizes after approximately 16-25 days. Here I estimate that evaporation ceases with a consequent stabilization of water content after approximately 12 days.

Figure 1 shows that the final strain is $22X10^{-6}$ greater than the initial strain. This could represent a permanent set or may reflect changes in rates of evaporation, V_e (equation 24), between the initial and final desiccation processes.

Finally, the negative pore pressure resulting from desiccation (equation 22) can be considered a capillary pressure, since a capillary pressure is a measure of the tendency of a porous medium to draw in a wetting fluid phase (Bear, 1972). The initial value of 100 atm is an order of magnitude higher than that resulting from desiccation of oil sands (Bear, 1972). However, the permeability of the granite is several orders of magnitude smaller than that of the oil sands, thus causing larger capillary tensions in the granite (Bear, 1972).

Conclusions and recommendations

The theory presented here is only a first attempt at modeling the strain response to wetting and drying in Barre Granite. To test this theory further, refinements in experimental procedure are necessary. For example, cores should be initially desiccated under vacuum conditions. During wetting the cores should be immersed in distilled, de-aired water and the strain monitored until it reaches a steady state. The subsequent drying cycle should be carried out in a vacuum until final steady strains are reached. Similar procedures are described in Thill, Bur, and Steckley (1973) and procedures for determination of capillary pressures are given in Bear (1972). However, determination of capillary pressures may be complicated by the granites' low permeability. In any case, further work and refinement of experimental procedures should help resolve some of the questions brought out above.

References

Bear, J., 1972, Dynamics of fluids in porous media: New York, American Elsevier, 764 p.

Biot, M. A., 1941, General theory of three-dimensional consolidation: Journal of Applied Physics, v. 12, p. 155-164.

Brace, W. F., Walsh, J. B., and Frangos, W. T., 1968, Permeability of granite under high pressure: Journal of Geophysical Research, v. 73, no. 6, p. 2225-2236.

Carslaw, H. S., and Jaeger, J. C., 1959, Conduction of heat in solids (2d ed.): Oxford, Clarendon Press, 510 p.

Mesri, G., Adachi, K., and Ullrich, C. R., 1976, Pore-pressure response in rock to undrained change in all-round stress: Geotechnique, v. 26, no. 2, p. 317-330.

Terzaghi, Karl, 1943, Theoretical soil mechanics: New York, John Wiley, 510 p.

Thill, R. E., Bur, T. R., and Steckley, R. C., 1973, Velocity anisotropy in dry and saturated rock spheres and its relation to rock fabric: International Journal of Rock Mechanics and Mining Science, v. 10, no. 6, p. 535-557.

Timoshenko, S. P., and Goodier, J. N., 1970, Theory of elasticity (3d ed.): New York, McGraw-Hill, 567 p.