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Prediction of Vertical Displacements in a
Subsiding Elastic Layer--A Model for
Subsidence in Karst Terrains

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Notation

a	half-width of opening
h	layer thickness
ε_{ij}	strains
σ_{ij}	stresses
v	vertical displacement
ν	Poisson's ratio
E	Young's modulus
ρ	density
g	acceleration due to gravity
x, y	horizontal and vertical coordinates
$\delta = h/a$	ratio of layer thickness to half-width of opening
α	constant
m_1, m_3	constants
$\eta = x/a$	ratio of x coordinate to half-width of opening
v_0, v_g	constants
J_n	Bessel function of order n

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Introduction

Subsidence over underground cavities is a common phenomenon, particularly

in karst terrains. At the ground surface, subsidence manifests itself by the

formation of a depression which may adversely affect adjacent manmade

structures. Models of subsidence have been presented by Hackett (1959, 1964),

Berry (1960, 1963, 1964a, b), Berry and Sales (1961, 1962) and Salamon (1963,

1964a, b, 1965). These and other subsidence models are reviewed by Voight and

Pariseau (1970). In many of these models, as well as in the model introduced

here, the subsiding region is assumed to be elastic. This is thought to give

a reasonable picture of displacements up to a state of incipient failure.

In what follows, the subsiding region is modeled as an infinitely long

elastic layer resting on a rigid base and deforming under its own weight into

an opening at its lower edge. An approximate solution for vertical

displacements on the ground surface and over the opening is found for the case

when the layer thickness is much greater than the width of the opening. This

solution, based on the superposition of solutions to two boundary-value

problems, differs from previous solutions in that the weight of the subsiding

layer is explicitly included. Because of the restricted form of the boundary

conditions, this solution is most appropriate for predicting subsidence up to

incipient failure in karst terrains.

The two boundary-value problems

Consider a homogeneous and isotropic elastic layer of thickness h, under its own weight, overlying an opening of width 2a (figure 1). The opening is considered to be of great length relative to its width (2a) so that the layer is in a state of plane strain. This general problem can be solved by the superposition of solutions to two separate problems.
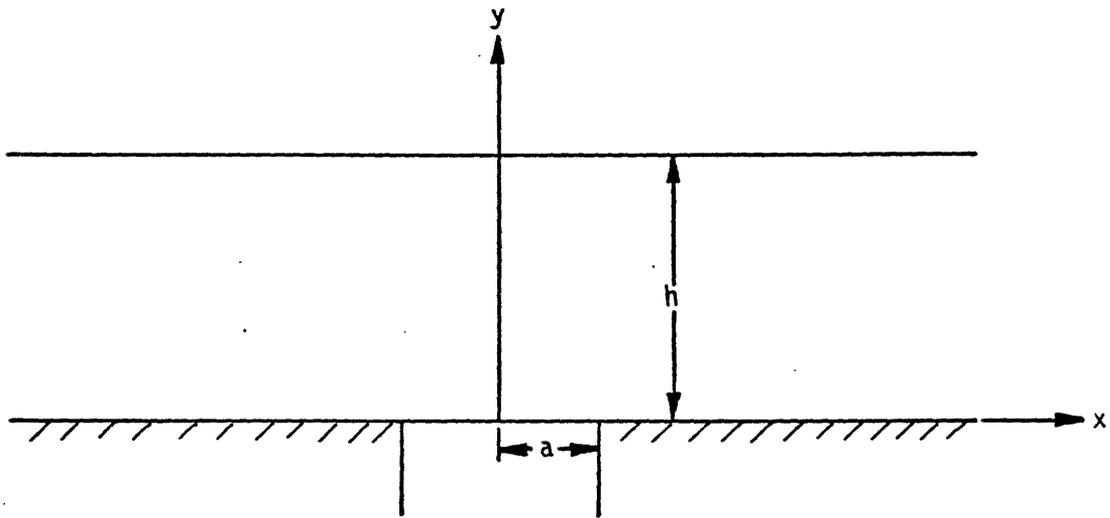
Figure 1.--Elastic layer of thickness h overlying an opening of width 2a.

The first problem is to determine the response of an infinitely wide layer of thickness h to gravity only. The stresses at $y = 0$ are

$$\sigma_{xy}(x,0) = v(x,0) = 0 \qquad\qquad x^2 \geq 0 \quad , \text{ and}$$

$$\sigma_{yy}(x,0) = -\rho gh \qquad\qquad x^2 \geq 0 \quad ,$$

and at $y = h$ are

$$\sigma_{yy}(x,h) = \sigma_{xy}(x,h) = 0 \qquad\qquad x^2 \geq 0$$

A solution for stresses which satisfies equilibrium and compatibility is

$$\sigma_{yy} = \rho g(y-h) \tag{1}$$

$$\sigma_{xy} = 0$$

$$\sigma_x = \sigma_z = \frac{\nu}{1-\nu}\, \rho g(y-h) \quad . \tag{2}$$

The layer is assumed to be constrained from lateral displacement so

$$\varepsilon_x = \varepsilon_z = 0 \quad , \qquad 0 \leq x^2 \leq \infty \text{ and } 0 \leq y \leq h$$

and

$$\varepsilon_y = \frac{(1+\nu)(1-2\nu)}{E(1-\nu)}\, \rho g(y-h) \quad ,$$

which leads to the vertical displacements, due to gravity only,

$$v = \frac{(1+\nu)(1-2\nu)}{E(1-\nu)}\, \rho g y(y/2-h) \tag{3}.$$

The second problem is the determination of the response of the layer to an opening of width 2a at its base. Directly over the opening, equilibrium requires that the total vertical stress vanish. Thus on $y = 0$,

$$\sigma_{yy}(x,0) = \begin{cases} \rho gh & x^2 < a^2 \\ 0 & x^2 > a^2 \end{cases} \quad ,$$

$$\sigma_{xy}(x,0) = 0 \qquad x^2 \geq 0 \quad , \text{ and}$$

$$v(x,0) = 0 \qquad x^2 > a^2 \quad ,$$

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and on $y = h$

$$\sigma_{xy}(x,h) = 0 \qquad x^2 \geq 0 \qquad , \text{ and}$$

$$\sigma_{yy}(x,h) = 0 \qquad x^2 \geq 0 \qquad .$$

A solution to this problem for the case $h/a \gg 1$ is given by Sneddon and Lowengrub (1969, p. 62-66).

Vertical displacements at the roof of the opening, that is at $y = 0$, as given by Sneddon and Lowengrub (1969), are

$$v(\eta,0) = -v_0[1-\eta^2]^{1/2}[1+\frac{m_1}{2\delta^2} + \frac{4m_1^2-m_3}{16\delta^4} - \frac{[1+2\eta^2]}{24\delta^4}m_3]+O(\delta^{-6}) \qquad 0<\eta<1 \qquad , (4)$$

where $\eta = x/a$, $v_0 = \frac{2[1-\nu^2]\rho gha}{E}$, $\delta = h/a$,

and the notation $O(\delta^{-6})$ means that terms of order $(a/h)^{-6}$ and smaller are omitted. Here $m_1 = 2.350$, and $m_3 = 9.962$.

Vertical displacements on the free surface, $y = h$, are given by Sneddon and Lowengrub (1969, eq. 2.11.7, p. 64) as

$$v(\eta,\delta) = - 2v_0 \int_0^\infty g(\xi) [\frac{\sinh\xi\delta + \xi\delta\cosh\xi\delta}{2\xi\delta + \sinh2\xi\delta}] \cos\xi\eta d\xi \qquad , \qquad (5)$$

where $g(\xi) = \int_0^1 \phi(t)J_0(\xi t)dt$ (6)

and

$$\phi(t) = t [1+\frac{m_1}{2\delta^2} + \frac{m_1^2}{4\delta^4} - \frac{(2t^2+1)m_3}{16\delta^4}] + O(\delta^{-6}) \qquad . \qquad (7)$$

Substitution of equation 7 in equation 6 and integrating leads to

$$g(\xi) = [1+\frac{m_1}{2\delta^2} + \frac{m_1^2}{4\delta^4} - \frac{3m_3}{16\delta^4}]\frac{1}{\xi} J_1(\xi)+\frac{m_3}{4\delta^4} \cdot \frac{J_2(\xi)}{\xi^2} + O(\delta^{-6}) \qquad , \qquad (8)$$

and thus,

$$v(\eta,\delta) = -2v_0 \left[\left[1 + \frac{m_1}{2\delta^2} + \frac{m_1^2}{4\delta^4} - \frac{3m_3}{16\delta^4} \right] \int_0^\infty \left[\frac{\sinh\xi\delta + \xi\delta\cosh\xi\delta}{2\xi\delta + \sinh2\xi\delta} \right] J_1(\xi)\cos\xi\eta \frac{d\xi}{\xi} \right.$$

$$\left. + \frac{m_3}{4\delta^4} \int_0^\infty \left[\frac{\sinh\xi\delta + \xi\delta\cosh\xi\delta}{2\xi\delta + \sinh2\xi\delta} \right] J_2(\xi)\cos\xi\eta \frac{d\xi}{\xi^2} \right] + O(\delta^{-6}) \quad . \tag{9}$$

Evaluation of vertical displacements on the free surface

The integrals in equation 9 can be found approximately by using Filon's quadrature (Sneddon, 1951, p. 519). Using this procedure, when $\delta \gg 1$, say 10 or greater, sufficient accuracy is obtained by excluding terms of order δ^{-4} and smaller. Vertical displacements of the free surface are then given by the approximate expression,

$$v(\eta,\delta) = -2v_0 \left[1 + \frac{m_1}{2\delta^2} \right] \int_0^\infty M(\xi\delta) J_1(\xi)\cos\xi\eta \frac{d\xi}{\xi} \quad , \tag{10}$$

where $M(\xi\delta) = \dfrac{\sinh\xi\delta + \xi\delta\cosh\xi\delta}{2\xi\delta + \sinh2\xi\delta}$.

An alternative to quadrature is to follow Sneddon (1951) and replace $M(\xi\delta)$ in equation 10 by the approximate expression,

$$M(\xi\delta) \simeq M_1(\xi\delta) = (1+\xi\delta)e^{-\xi\delta} - \tfrac{1}{2}[1+2\xi\delta]e^{-\alpha\xi\delta} \quad . \tag{11}$$

This approximation is best at large values of $\xi\delta$ where $M(\xi\delta) = M_1(\xi\delta) = (1+\xi\delta)e^{-\xi\delta}$, provided $\alpha > 1$. For smaller values of $\xi\delta$, the parameter α is chosen in such a way that the area under the curve $M_1(\xi\delta) - M(\xi\delta)$ is as small as possible. This condition is best satisfied when $\alpha = 1.85$.

Substituting equation 11 in the integral $\int_0^\infty M(\xi\delta)J_1(\xi)\cos\xi n\,\frac{d\xi}{\xi}$

gives the approximation,

$$\int_0^\infty M_1(\xi\delta)J_1(\xi)\cos\xi n\,\frac{d\xi}{\xi} = I_1 + \delta I_2 - \tfrac{1}{2}I_3 - \delta I_4 \quad ,$$

where

$$I_1 = \int_0^\infty e^{-\delta\xi}J_1(\xi)\cos\xi n\,\frac{d\xi}{\xi} \quad ,$$

$$I_2 = \int_0^\infty e^{-\delta\epsilon}J_1(\xi)\cos\xi n\,d\xi \quad ,$$

$$I_3 = \int_0^\infty e^{-\alpha\delta\xi}J_1(\xi)\cos\xi n\,\frac{d\xi}{\xi} \quad , \text{ and}$$

$$I_4 = \int_0^\infty e^{-\alpha\delta\xi}J_1(\xi)\cos\xi n\,d\xi \quad .$$

The integrals I_1, I_2, I_3, I_4, are found by methods described in Sneddon (1951). The results are

$$I_1 = \frac{1}{\sqrt{2}}[R_1^2+1+\delta^2-n^2]^{1/2} - \delta \quad ,$$

$$I_2 = 1 - \frac{1}{\sqrt{2}\,R_1^2}[\delta\,[R_1^2+1+\delta^2-n^2]^{1/2} + n[R_1^2-[1+\delta^2-n^2]]^{1/2}] \quad ,$$

$$I_3 = \frac{1}{\sqrt{2}}[R_2^2+1+\alpha^2\delta^2-n^2]^{1/2} - \alpha\delta \quad , \text{ and}$$

$$I_4 = 1 - \frac{1}{\sqrt{2}\,R_2^2}[\delta\,[R_2^2+1+\alpha^2\delta^2-n^2]^{1/2} + n[R_2^2-[1+\alpha^2\delta^2-n^2]]^{1/2}] \quad ,$$

where $R_1^2 = [\,[1+\delta^2-n^2]^2+4\delta^2 n^2]^{1/2}$ and $R_2^2 = [\,[1+\alpha^2\delta^2-n^2]^2+4\alpha^2\delta^2 n^2]^{1/2}$.

Vertical displacements on the free surface and over the opening

Displacements due to the weight of the layer deforming into the opening are now obtained by superposition of the displacements obtained in problems 1 and 2. For $y = 0$, this yields the vertical displacements

$$v(\eta,0) = - \frac{2[1-\nu^2]\rho gha}{E} \left[1+ \frac{m_1}{2\delta^2}\right] [1-\eta^2]^{1/2} \quad , \tag{12}$$

and for $y = h$, the vertical displacements

$$v(\eta,h) = - \frac{(1+\nu)(1-2\nu)}{2E(1-\nu)} \rho gh^2 - \frac{4[1-\nu^2]\rho gha}{E}\left[1+\frac{m_1}{2\delta^2}\right][I_1+\delta I_2 - \tfrac{1}{2} I_3 - \delta I_4] \quad . \tag{13}$$

We can now write equation 12 in the dimensionless form

$$\frac{v(\eta,0)}{v_0\left[1+ \frac{m_1}{2\delta^2}\right]} = - [1+\eta^2]^{1/2} \quad . \tag{14}$$

Since the first term on the right in equation 13 is a constant,

$$v_g = \frac{(1+\nu)(1-2\nu)\rho gh^2}{2E(1-\nu)} \quad , \text{ we have}$$

$$\frac{v(\eta,\delta)+v_g}{2 v_0 \left[1+\frac{m_1}{2\delta^2}\right]} = - [I_1+\delta I_2 - \tfrac{1}{2} I_3 - \delta I_4] \quad \tag{15}$$

for the vertical displacements on the free surface.

Vertical displacements on $y = h$ for $\delta = 10$ predicted by Filon's quadrature of equation 10 and by equation 15 are the same to 3 decimal places. Vertical displacements on $y = h$, predicted by equation 15 for three values of the ratio $\delta = h/a$, are shown in figure 2. Vertical displacements over the hole (on $y = 0$) predicted by equation 14 are shown in figure 3.
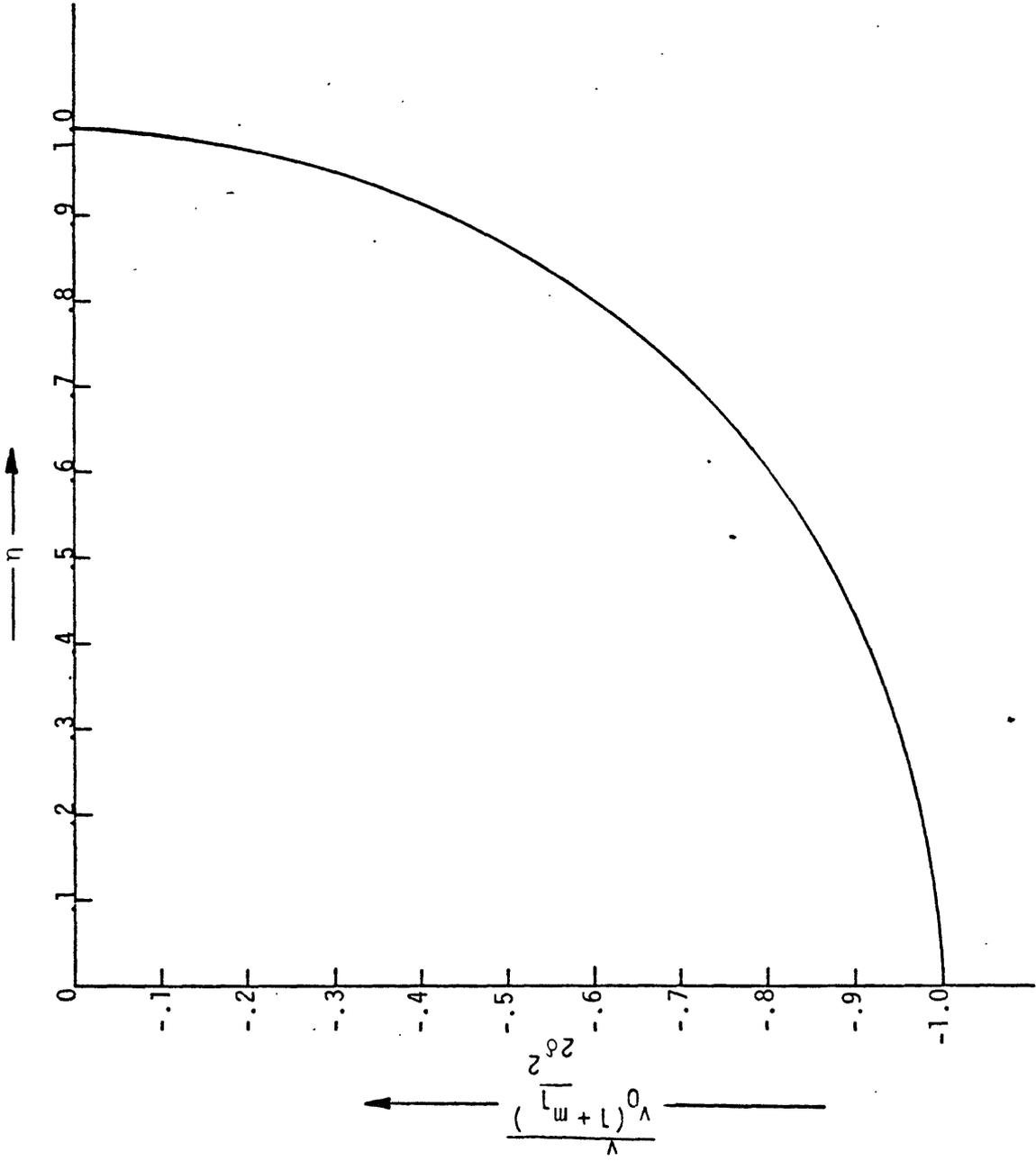
Figure 3.—Vertical displacements above the hole (at y = 0).

Discussion of resulting displacements

Vertical displacements on the free surface, shown in figure 2, are seen to decrease as h, the layer thickness, increases relative to the width of the opening, 2a. However, a broader trough of subsidence is formed as $\delta = h/a$ increases. Additionally, because $v_0 = \frac{2(1-\nu^2)}{E} \rho gha$ and $v_g = \frac{(1+\nu)(1-2\nu)}{2E(1-\nu)} \rho gh^2$, increasing Young's modulus, E, causes a decrease in free-surface vertical displacements. Conversely, increasing the density, ρ, and h or a causes an increase in vertical displacements at the surface. Also, increasing Poisson's ratio, ν, causes a decrease in v_0 and v_g and hence a decrease in vertical displacements at the ground surface. Note that the width of the subsidence trough is controlled only by the ratio h/a and is not affected by changing v_0 or v_g. Of equal interest is the formation of a ridge at the edge of the subsidence trough. This phenomenon is clearest in the case $\delta = h/a = 10$ (fig. 2). Vertical displacements over the hole, as shown in figure 3, will be increased as ρ, h, or a is increased and will be decreased as δ, E, or ν is increased.

Displacements over the hole, at y = 0, approximate actual field conditions as displacements for $x^2 > a^2$ will not vanish unless the subsiding layer overlies a very stiff base. This is also true for shear stresses at y = 0 when $x^2 > a^2$. The assumption that these stresses vanish implies that the layer is free to slide towards the hole as it subsides. Again this would be approximately true in the case of a relatively soft layer overlying a very stiff layer. Recognizing these restrictions, the present solution is best used as a model of subsidence in a karst terrain where a thick and relatively soft soil layer overlies solution cavities in much stiffer limestone or dolomite.

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