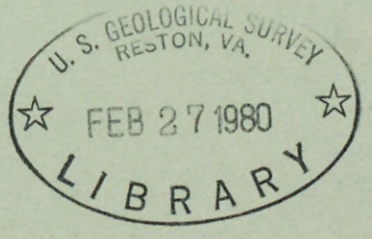
UNITED STATES
DEPARTMENT OF THE INTERIOR
GEOLOGICAL SURVEY

A PRACTICAL LAGRANGIAN TRANSPORT MODEL

By Harvey E. Jobson

U.S. GEOLOGICAL SURVEY
OPEN-FILE REPORT 80-206



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NSTL Station, Mississippi

1980

UNITED STATES DEPARTMENT OF THE INTERIOR

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CONTENTS

ILLUSTRATIONS

A Practical Lagrangian Transport Model

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ABSTRACT

An unconditionally stable and practical transport model for use in upland streams and rivers has been developed and verified. Basing the model on the Lagrangian, rather than the Eulerian, reference frame greatly reduces the numerical problems associated with solving the advective terms of the convective-diffusion equation. The model contains almost no numerical dispersion, is conceptually simple, and is relatively easy to code. Model results closely simulated dye concentrations measured in the Chattahoochee River near Atlanta, Ga. under highly unsteady flow conditions.

INTRODUCTION

The simulation of the transport of pollutants in rivers has received wide attention. Historically, the problem has been approached by applying the conservation principle in the Eulerian sense. Although an enormous amount of literature exists that describes various numerical techniques for solving the resulting convective-diffusion equation, all practical solution schemes contain an undesirably large amount of numerical dispersion. Most of the dispersion results from deficiencies in simulating the advective term of the equation.

If the conservation principle is applied in the Lagrangian sense, the troublesome advection term does not appear so the mathematical difficulties of numerically solving the equation are small. Although straightforward, transforming the Lagrangian solution back to a fixed grid coordinate system is somewhat involved.

The purpose of this paper is to present a practical, one-dimensional, transport model which is based on the Lagrangian reference frame and to verify the model's accuracy by comparison with field data and a solution obtained by an existing Eulerian model of the implicit finite-difference type.

In addition to accuracy and unconditional stability, the Lagrangian solution scheme offers several significant advantages. It can be used to assess the contribution of each surface exchange process to the accuracy of a temperature model as well as to determine rate coefficients for surface exchange (Jobson, 1977). Jobson and others (1978) used observed temperature records in conjunction with a continuous measure of traveltimes, which are a direct result of the Lagrangian solution scheme, to estimate the natural temperature of a thermally altered river. Fischer (1972) found a Lagrangian scheme to be quite effective for simulating transport in the Bolinas Lagoon, Calif. A final advantage of Lagrangian scheme is its mathematical simplicity and its physical understandability.

After the theoretical basis for the model has been outlined, it will be verified by use of 4 days of dye dilution data obtained on a 27.8-km reach of the Chattahoochee River near Atlanta, Ga. The reach starts at Buford Dam, where hydroelectric units are used for peak power generation, so the river flow rate is highly unsteady. Dye concentrations are simulated by both the Lagrangian and an Eulerian model, and the lack of numerical dispersion of the Lagrangian model is vividly illustrated.

THEORETICAL DEVELOPMENT

In the Lagrangian framework, one conceptually follows an individual fluid parcel while keeping track of all factors which tend to change its concentration. The Lagrangian transport model is unique because it does not use a gradient-type despersive term and it can more accurately simulate the transport than the Eulerian dispersive model does. For example, the Lagrangian model gives an exact solution to the plug flow problem, provided the velocity field is precisely defined. Because of numerical dispersion, Eulerian solution schemes are generally very poor at simulating the plug flow problem.

In order to develop a solution approach, it is well to begin with some review of the fundamentals of mixing. Consider a localized flow element in the river segment of length Δx and area da as illustrated in figure 1. The flow segment is moving downstream with a velocity U, the cross-sectional mean stream velocity. Neglecting molecular diffusion, the continuity of mass for a conservative substance in the element requires

$$\frac{\partial(c\ da\ \Delta x)}{\partial t} = c(u - U)da - \{c(u - U) + \frac{\partial[c(u - U)]}{\partial x}\Delta x\}\ da$$

or

$$\frac{\partial c}{\partial t} + \frac{\partial[c(u - U)]}{\partial x} = 0 \qquad (1)$$

in which u and c are local instantaneous values of velocity and concentration while t is time.

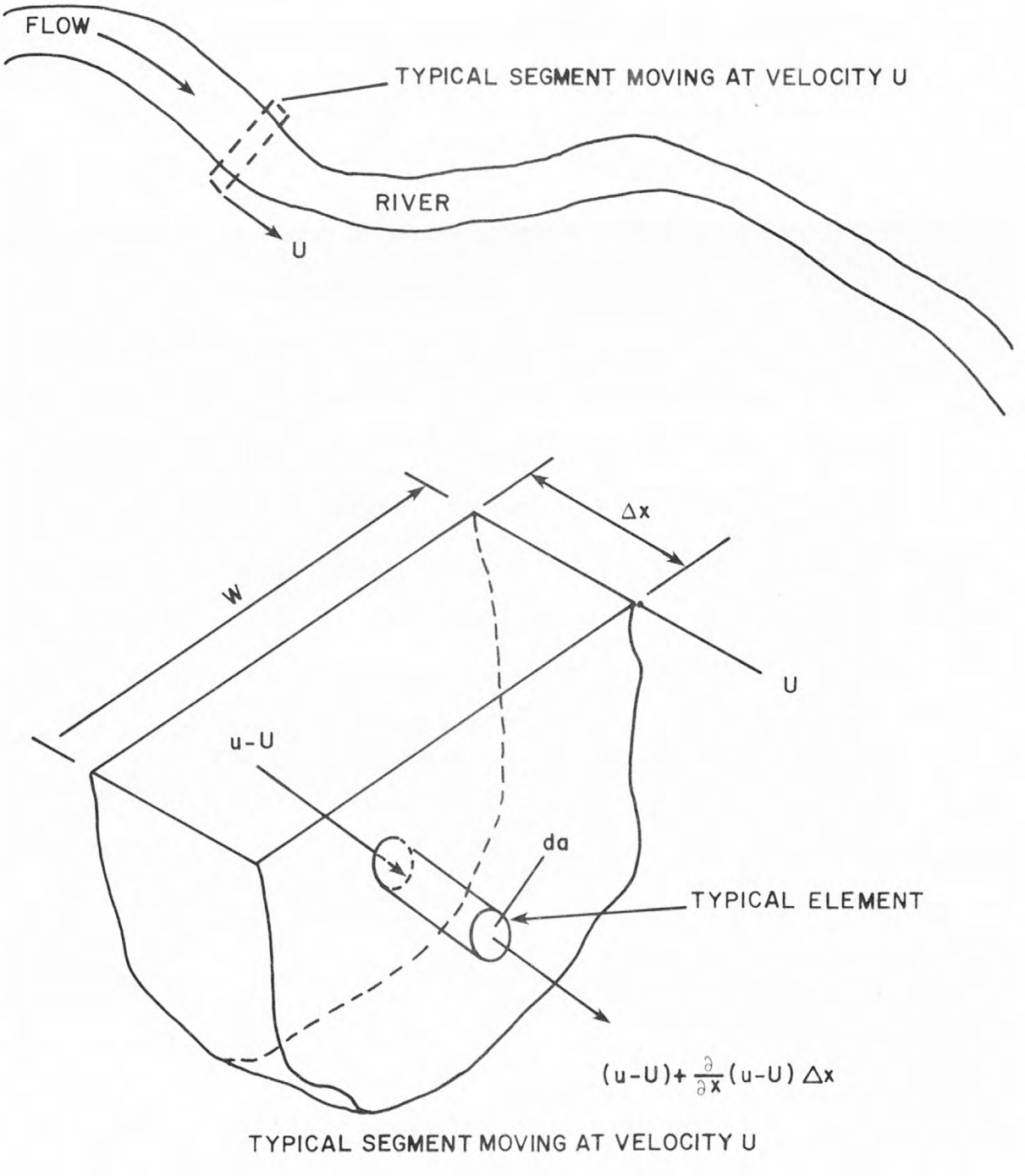

Figure 1.--Typical flow element in a typical segment of a river.

Along the same lines used by Reynolds in dealing with turbulent fluctuations, let's define the local instantaneous value of u and c as

$$u = U + u_a + u' \qquad (2)$$

$$c = C + c_a + c'$$

in which U and C are cross sectional mean values which have also been averaged over a time which is long with-respect-to turbulent fluctuations, u_a and c_a are local values (in the cross-section) of the time mean values of u and c, and u' and c' are instantaneous deviations of the value of u or c from $U + u_a$ or $C + c_a$. By definition then, the time averaged value of u' or c' is zero. Also by definition the cross-sectional mean value of u_a and c_a is zero.

Replacing the instantaneous values of u and c in equation 1 with their equivalent values from 2 and expanding 1 obtains

$$\frac{\partial C}{\partial t} + \frac{\partial c_a}{\partial t} + \frac{\partial c'}{\partial t} + \frac{\partial (u_a C)}{\partial x} + \frac{\partial (u_a c_a)}{\partial x} + \frac{\partial (u_a c')}{\partial x} + \frac{\partial (u'C)}{\partial x} + \frac{\partial (u' c_a)}{\partial x}$$

$$+ \frac{\partial (u'c')}{\partial x} = 0 \qquad (3)$$

Now average each term in equation 3 with respect to time (short relative to normal variation); it is seen that all terms involving only one primed term are zero by definition so equation 3 reduces to

$$\frac{\partial C}{\partial t} + \frac{\partial c_a}{\partial t} + \frac{\partial (u_a C)}{\partial x} + \frac{\partial (u_a c_a)}{\partial x} + \frac{\partial \overline{u'c'}}{\partial x} = 0 \qquad (4)$$

in which the overbar indicates a time averaged value. Next, integrating each term in equation 4 over the entire cross-sectional area, A, and dividing by the cross sectional area (i.e., averaging over the cross section) all terms involving a single term with a subscript a are identically zero so equation 4 reduces to

$$\frac{\partial C}{\partial t} + \frac{1}{A} \frac{\partial}{\partial x} \int_A (u_a c_a + \overline{u'c'}) da = 0 \qquad (5)$$

using braces < > to indicate a cross sectional averaged value, equation 5 can be written

$$\frac{\partial C}{\partial t} + \frac{\partial \langle u_a c_a + \overline{u'c'} \rangle}{\partial x} = 0 \qquad (6)$$

Equation 6 is an expression of Taylor's (1954) theory of one-dimensional dispersion. The term $\langle u_a c_a \rangle$ designates the effect of differential convection while the term $\langle \overline{u'c'} \rangle$ designates the turbulent diffusion effect. According to Fischer (1966), who applied Taylor's theory to natural channels, the $\langle u_a c_a \rangle$ term far outweighs the $\langle \overline{u'c'} \rangle$ term. Based on many experimental observations, equation 6 can be simplified to

$$\frac{\partial c}{\partial t} + \frac{\partial \langle u_a c_a \rangle}{\partial x} = 0 \qquad (7)$$

According to the convective dispersion theory of Taylor and Fischer, the quantity $\langle u_a c_a \rangle$ is usually expressed as a diffusive term

$$\langle u_a c_a \rangle = -D_x \frac{\partial c}{\partial x} \qquad (8)$$

in which D_x is the dispersion coefficient. Equation 8 has been verified as the correct form of equation for the asymptotic solution in a steady uniform channel. It can be shown theoretically that $D_x = 5.9\ U_* Y$ for flow in an infinitely wide open channel with a logarithmic velocity profile (Elder, 1959). Here U_* and Y are the shear velocity and depth, respectively. Fischer (1973) summarized measurements in both the laboratory and natural channels. The reported values ranged from $D_x = 5.8\ U_* Y$ to $D_x = 7500\ U_* Y$. With this much variation it is difficult to predict an appropriate value of D_x for use in a given flow condition. Much of the uncertainty about the "real" dispersion coefficient is academic when using a conventional finite-difference Eulerian transport model because numerical dispersion (smearing because of limitations of the model) is often equal to or greater than the actual dispersion observed in a natural stream.

In view of the difficulties involved in the use of equation 8, the proposed Lagrangian model uses an alternative approach to evaluate the term $\langle u_a c_a \rangle$. This approach is much more straightforward in its physical interpretation. Recall the definition

$$\langle u_a c_a \rangle \equiv \frac{1}{A} \int_A u_a c_a \, da \tag{9}$$

where

$$u_a = u - U$$

$$c_a = c - C$$

since the turbulent components (u' and c') are ignored. That is, u_a is the local deviation from the cross-sectional mean velocity. Equation 9 is an expression for the net flux of material by the convective velocity u_a through a vertical plane which is moving downstream at velocity U. Consider figure 2 which illustrates an infinitely wide river with the finite segments (parcels) separated by a moving plane, the dashed lines. Assuming the concentration to be well mixed in each segment, equation 9 can be evaluated directly from the velocity distribution. At the boundary between segments k and $k - 1$ the local concentration, c, can be approximated by C_{k-1} in the region where $u_a > 0$ and by C_k in the region where $u_a < 0$. The cross-sectional average concentration (in segment k) is C_k. It then follows from equation 9

$$\langle u_a c_a \rangle = \frac{1}{A} \int_{A^+} u_a (C_{k-1} - C_k) \, da$$

where C_k is the uniform concentration in segment k and A^+ indicates integration over the portion of the area where the local velocity is greater than the cross-sectional mean ($u_a > 0$). Since C_{k-1} and C_k are assumed independent of position in a cross section, they can be moved

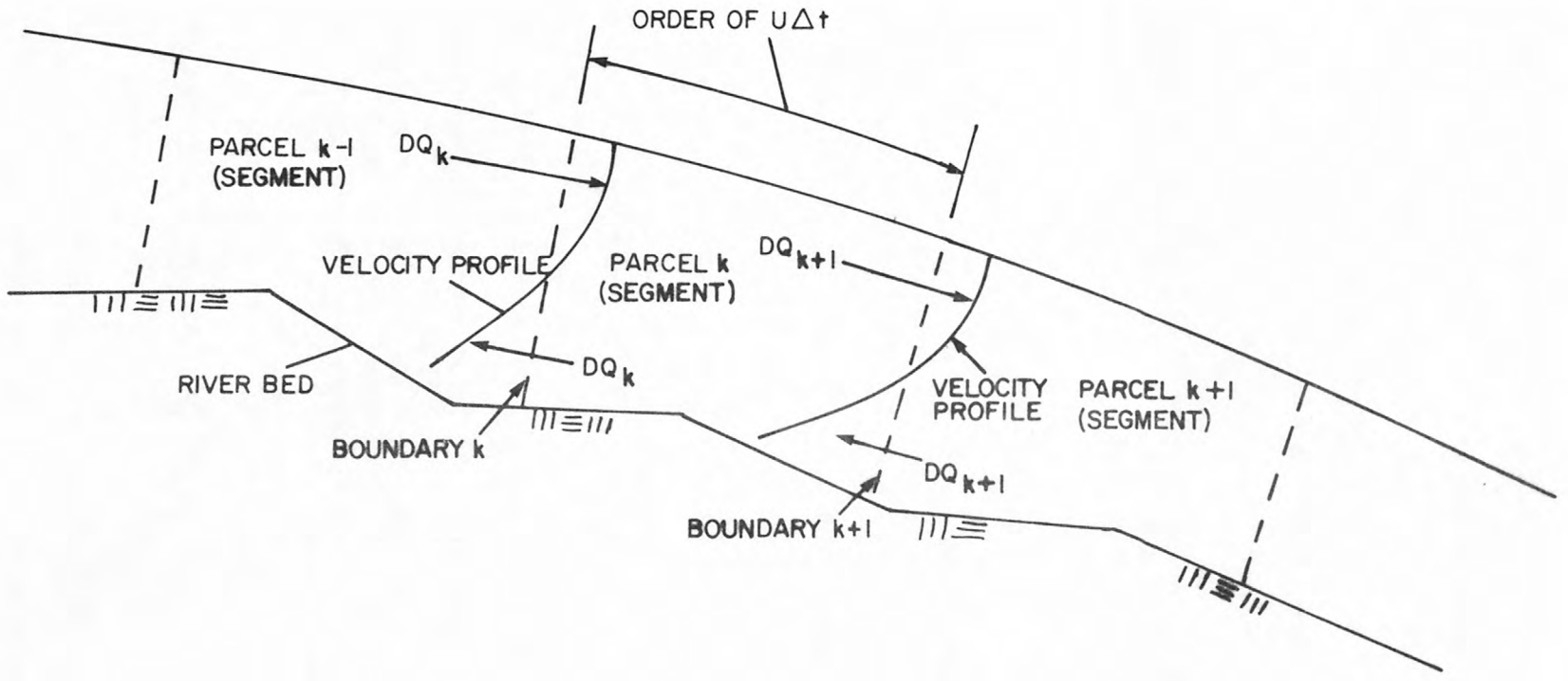

Figure 2.--Schematic of a longitudinal section of a river with fluid parcels delineated to illustrate the dispersive process.

outside of the integral leaving

$$\langle u_a c_a \rangle = \frac{C_{k-1} - C_k}{A} \int_{A^+} u_a da \qquad (10)$$

Similarly, at the boundary between segments k and $k + 1$,

$$\langle u_a c_a \rangle = \frac{C_{k+1} - C_k}{A} \int_{A^-} u_a da \qquad (11)$$

where A^- indicates integration over the portion of the area where $u_a < 0$.

By observing that continuity of water at the boundary must be maintained, it is seen that

$$\int_{A^+} u_a da + \int_{A^-} u_a da = 0.$$

Let's define a mixing coefficient, DQ_k, at the boundary between segments k and $k + 1$ as

$$DQ_k \equiv \int_{A^+} u_a da = 1/2 \int_A |u - U| da \qquad (12)$$

so that the second term of equation 7 is approximated by

$$\frac{\partial \langle u_a c_a \rangle}{\partial x} = \frac{DQ_k (C_k - C_{k+1}) - DQ_{k-1} (C_{k-1} - C_k)}{A \Delta x} \qquad (13)$$

in which Δx is the distance between the segment boundaries which is also approximately equal to $U\Delta t$ and $A\Delta x$ is recognized as the segment volume. Of course, equation 13 could be derived just as easily from a mass balance and DQ as defined in figure 2. But notice only 2 assumptions have been required: first $\overline{u'c'} = 0$,

and second that the concentration in each segment is well mixed. The first assumption is well founded, but the accuracy of the second depends on the time step size. Ideally for equation 13 to be most accurate the value of the time step must be some fixed percentage of the time required for complete mixing in the cross section. It has been found empirically that equation 13 has an optimum accuracy when the time step size is equal to $5D_x/U^2$. Equation 13 will underestimate the true dispersion for small time steps and overestimate it for large time steps.

Equation 13 can also become unstable for highly unsteady flow where adjacent segment volumes of very unequal size exist in the river. This instability is easily overcome by limiting the value of *DQ*.

The rate of flow of water between segments *(DQ)* is determined from the distribution of velocity in the cross section and mean velocity as shown by equation 12. The value of *DQ* will remain a constant percentage of the discharge, *Q*, provided the velocity distribution in the cross section remains similar. For example, if the lateral variation in velocity were zero and the vertical variation were logarithmic, the ratio of *DQ/Q* would be only a function of the ratio of the mean velocity to the shear velocity. The ratio of *DQ/Q* would increase from 0.092 to 0.184 as the ratio of the mean to shear velocity decreased from 10.0 to 5.0. Lateral variations in velocity increase the value of *DQ/Q*. At high flow conditions in the Chattahoochee River for example, inclusion of lateral velocity variations increased the value of *DQ* by 75, 25, 28, and 48 percent over values estimated from the vertical velocity variation alone at the Highway 20, Littles Ferry, Highway 120, and Highway 141 Bridges, respectively.

It is worthwhile to compare equation 13 to an explicit finite-difference formulation of equation 8, namely,

$$\frac{\partial \langle u_a c_a \rangle}{\partial x} = -\frac{\partial}{\partial x}\left(D_x \frac{\partial c}{\partial x}\right) = \frac{D_{xk}\,(C_k - C_{k+1}) - D_{xk-1}(C_{k-1} - C_k)}{\Delta x^2} \qquad (14)$$

Equation 14 is identical to equation 13 if the value of $D_x/\Delta x^2$ is replaced by $DQ/A\Delta x$. On the other hand, the value of D_x is very difficult to estimate, especially for natural channels, while the computation of *DQ* is straightforward and requires only stream gaging data. This is an important advantage to the Lagrangian approach.

Returning to equation 7 and integrating gives

$$C = C_O - \int_O^{TT} \frac{\partial \langle u_a c_a \rangle d\tau}{\partial x} \qquad (15)$$

where C_O is the concentration of the parcel (segment) at time zero and C is the concentration after a time lapse of TT. The time lapse, or traveltime, TT, can be considered either as a single time step in the model or as the total time for the parcel to pass through the system. Equation 15 is quite amenable to numerical analysis because of its integral rather than differential form. On the other hand, in order to construct a solution net over space and time, equation 15 must be solved for a series of fluid parcels, and the locations of the parcels must be continually tracked.

The location of a particular water parcel (or moving water segment) can be determined at any time from

$$x = x_O + \int_O^{TT} U d\tau \qquad (16)$$

where x and x_O are the locations of the fluid parcel at the time TT and at time zero, respectively, and U is the mean flow velocity. As in equation 15, the traveltime TT could be considered as the time step in the model, in which case x_O must be the location of the parcel at the beginning of the time step. Of course, the value of U is evaluated at the location of the parcel.

The traditional Eulerian solution scheme for transport problems involves discretizing the river reach into a number of subreaches separated by grid points. The initial conditions for the solution net are known concentrations of water parcels at each grid point. New concentrations of different water parcels at the same grid point are then computed for some later time, Δt. One boundary condition and a finite-difference approximation of the convective diffusion equation are used to compute the new concentrations. The new concentrations are then considered as initial or old values and the process is repeated as many times as desired.

In the Lagrangian approach, the Eulerian grid must be retained because it is the reference upon which the depth, area, and velocity of flow are defined. The solution scheme starts at time zero with a known concentration of a water parcel at each grid point just as in the Eulerian scheme. In the Lagrangian approach, however, new concentrations are computed by use of equation 15 for the same water parcels, which, of course, are no longer at the same location after a time lapse of Δt. The new location of each water parcel is determined by equation 16. The boundary condition is incorporated by adding a new parcel to the upstream boundary at each time step.

The basic bookkeeping necessary to keep track of all the information about individual fluid parcels is accomplished by setting up Parcel Characteristic arrays. The parcels are numbered in the order of their location, or time of entry to the reach, with parcel number 1 occupying the upstream boundary and parcel *NS* occupying the downstream boundary. In general, the total number of parcels in the system, *NS*, will be different for each time step. A minimum of three parcel characteristics (location, volume, and concentration) must be tracked but it is usually desirable to follow more. For example, Jobson and others (1978) kept track of 13 characteristics in a temperature model. The characteristics and identifying index of each parcel are updated during each time step of the model.

A copy of the program deck, 177 statements, may be obtained by contacting the author (U.S. Geological Survey, Building 2101, NSTL Station, MS 39529) if further detail is desired.

MODEL VERIFICATION

The U.S. Geological Survey undertook an intensive study of the Chattahoochee River near Atlanta, Ga. in the spring of 1975. The study plan included the development and verification of unsteady flow and transport models. The Lagrangian model is verified with data collected during this intensive study on a 27.9-km reach extending from Buford Dam, about 65 km northeast of Atlanta, to the Highway 141 Bridge. The reach has four small tributaries and an overall slope of 0.00036. Its roughly rectangular cross section ranges in width from 45 to 65 m.

The U.S. Army Corps of Engineers controlled the river flow below Buford Dam during the verification period, March 21-24, 1976. They maintained a steady flow of 15.3 m^3/s until 0700 hours on Monday morning, March 22. At 0700 hours a release rate of 113 m^3/s was started and maintained for 15 hours (fig. 3). At 2200 hours the release rate was returned 15.3 m^3/s. At 0700 hours on Tuesday they increased the release rate to 226 m^3/s and maintained it for 5 hours before returning to 15.3 m^3/s for the duration of the study. As reported previously (Keefer and Jobson, 1978), the flow was modeled using a finite-difference solution to the one-dimensional continuity and momentum equations for gradually varied flow. Figure 3 shows a comparison of modeled and observed stages at Highway 20, Littles Ferry, Highway 120, and Highway 141 which are 3.69, 13.25, 20.65, and 27.88 km downstream of the dam, respectively. The river reach was subdivided by use of 48 unequally spaced grid points and the flow model was operated with a time step of 5 minutes. Results of the flow model (velocity, area, top width, and tributary inflow) were stored on magnetic disks.

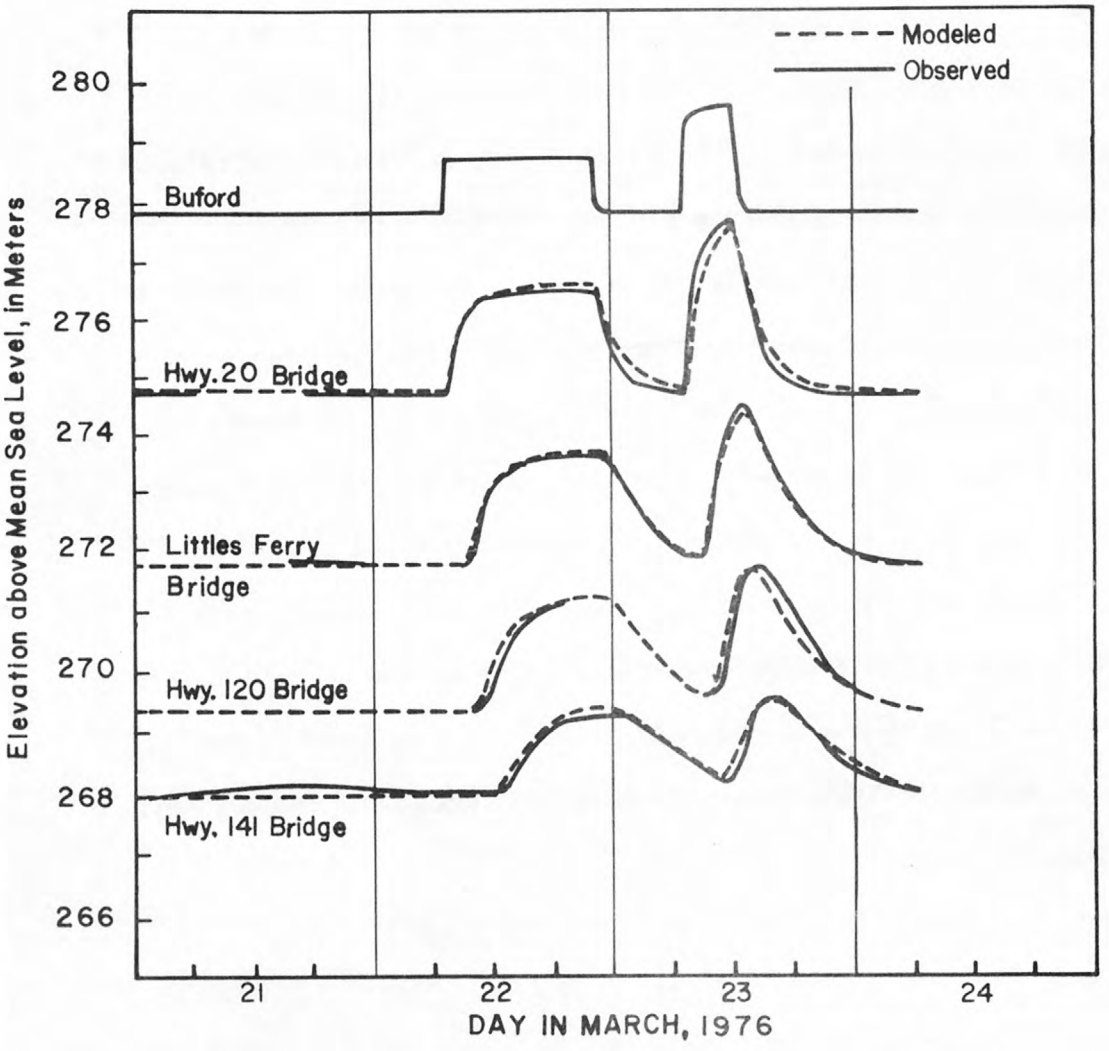

Figure 3.--Comparison of modeled and observed stages on the Chattahoochee River.

Starting at 1100 hours on March 21, rhodamine-WT dye was injected below Buford Dam from a point near the center of the river. The dye was injected at a constant rate for the duration of the study, and dip samples were collected from the Littles Ferry and Highway 141 bridges. Figure 4 contains a plot of the observed concentrations as well as values computed with the Lagrangian model. The Lagrangian model was operated with a 30-minute time step so it averaged six values from the flow model during each time step and for each grid point. The traveltime through the reach varied from a maximum of 23.4 hours to a minimum of 8.0 hours. In general, the results of the Lagrangian model appear to be excellent. The root-mean-square (RMS) error between the modeled and observed dye concentrations was 0.75 mg/l at Littles Ferry during the 58 hours when concentration was non-zero. The RMS error at Highway 141 was 0.86 mg/l during the 49 hours of non-zero concentration.

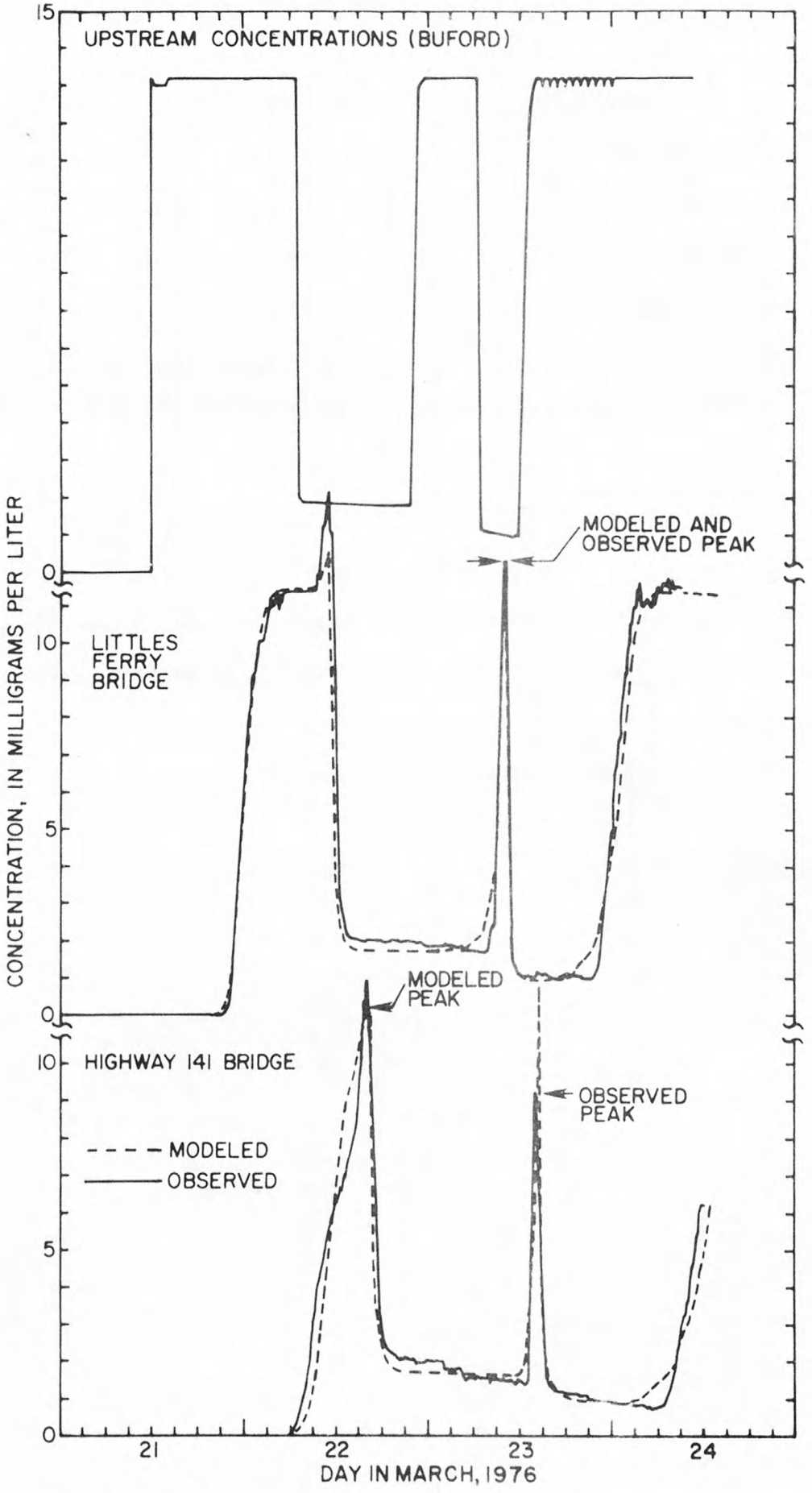

Figure 4.--Comparison of observed dye concentration to values predicted by use of the Lagrangian transport model.

For purposes of comparison, results obtained by use of an implicit finite-difference solution to the Eulerian dispersion equation similar to that presented by Stone and Brian (1963) are shown in figure 5. These results, which have been presented previously (Keefer and Jobson, 1978), were obtained by assuming the dispersion coefficient, D_x, to be equal to zero. In other words, all of the apparent mixing shown in figure 5 results from numerical dispersion over which the modeler has little or no control. By some fortuitous coincidence, as is often the case, the numerical dispersion in the Eulerian finite-difference scheme approximates the actual dispersive process, and the agreement in figure 5 is also good. The RMS errors are 0.88 and 0.81, respectively, at Littles Ferry and Highway 141. Numerical dispersion appears to overly "smear" the results in figure 5 in several places and to reduce the peak concentrations during the low flow between the high flow pulses by more than it should.

The results shown in both figures 4 and 5 are very good and in terms of the root-mean-square errors one would have to say the Lagrangian and Eulerian models are of about equal accuracy. Visually, the results shown in figure 4 appear superior, especially at Littles Ferry.

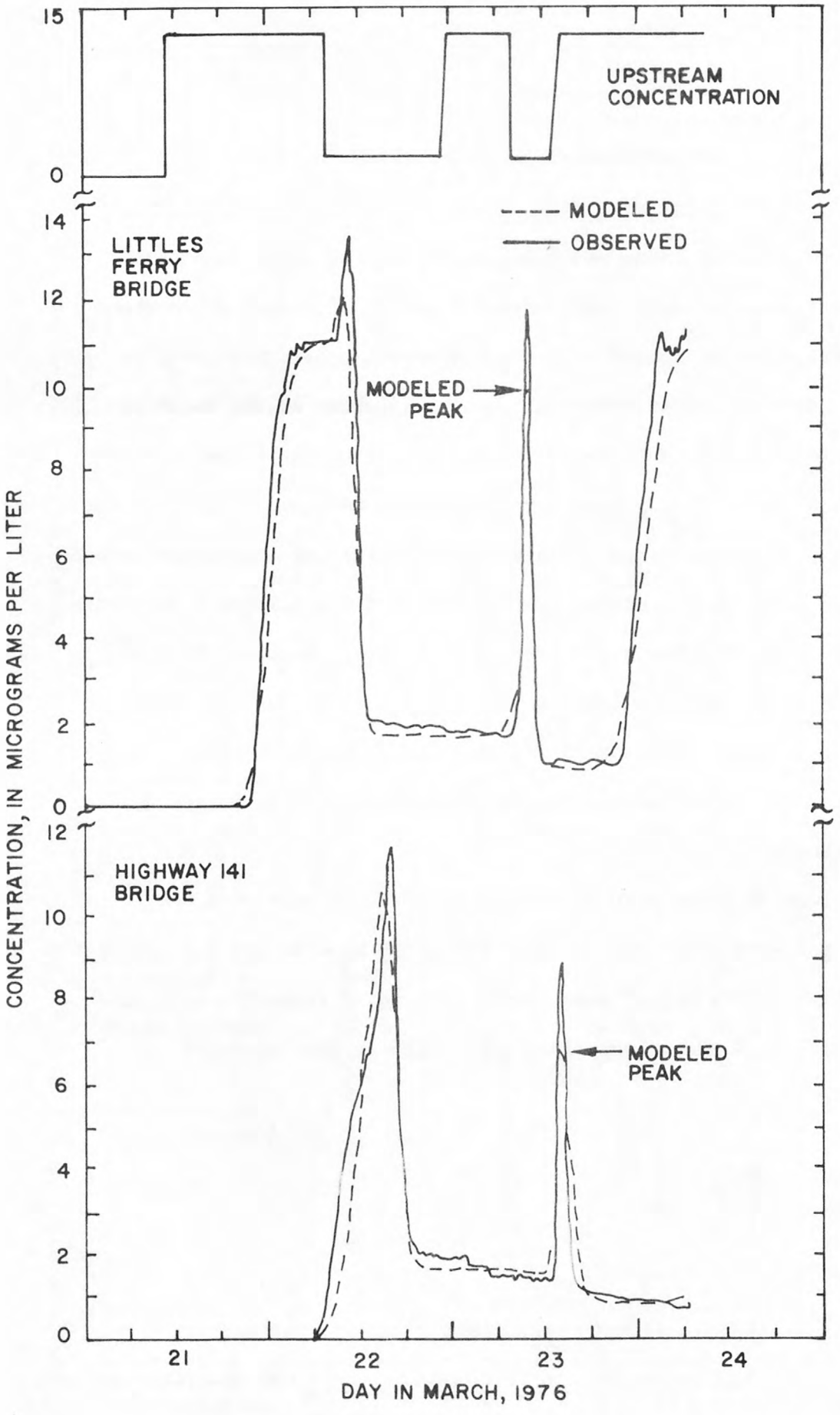

Figure 5.--Comparison of observed dye concentration to values predicted by use of an implicit finite difference model with zero dispersion.

In order to illustrate that equation 13 does indeed model the actual mixing process and that the Lagrangian model contains virtually no numerical dispersion (except in the dispersive term itself) the model was rerun for plug flow assumption, $DQ = 0$. These results are shown in figure 6. The RMS "error" in figure 6 is 1.15 at both Littles Ferry and Highway 141. The difference between the modeled results in figure 4 and 6 is the mixing process as modeled by equation 13. Equation 13 is not perfect because it ignores lateral variations in concentration. In fact, it appears to do a rather poor job on the arrival of the first dye front at Highway 141 and on rapidly falling stage (Littles Ferry at 0600, and 2100 hours on March 23 and Highway 141 at 0600 hours on the 24th). The difference in the computed curves of figures 5 and 6 is a direct measure of the numerical dispersion of the Eulerian model since both are calculated for plug flow conditions ($D_x = 0$). Some errors in the flow model certainly exist (fig. 3), and these errors will influence the accuracy of the transport model. For example, the observed peak discharge at Highway 141 which occurred at about 1500 hours on March 23 was 20 percent larger than the modeled value (Keefer and Jobson, 1978, p. 641) while the observed concentration peak here was about 20 percent lower than the value modeled by the Lagrangian scheme (fig. 4). The nearly simultaneous arrival of the dye front and flood wave at Highway 141 about noon on March 22 may help explain the poor fit of the transport model at this time. The second dye front and flood wave arrived at Highway 141 about noon on March 23. This and the fact that fluid segments of very unequal size existed in the river at this time are believed to account for the poor match of the peak concentration at Highway 141 about noon on March 23 by the Lagrangian model.

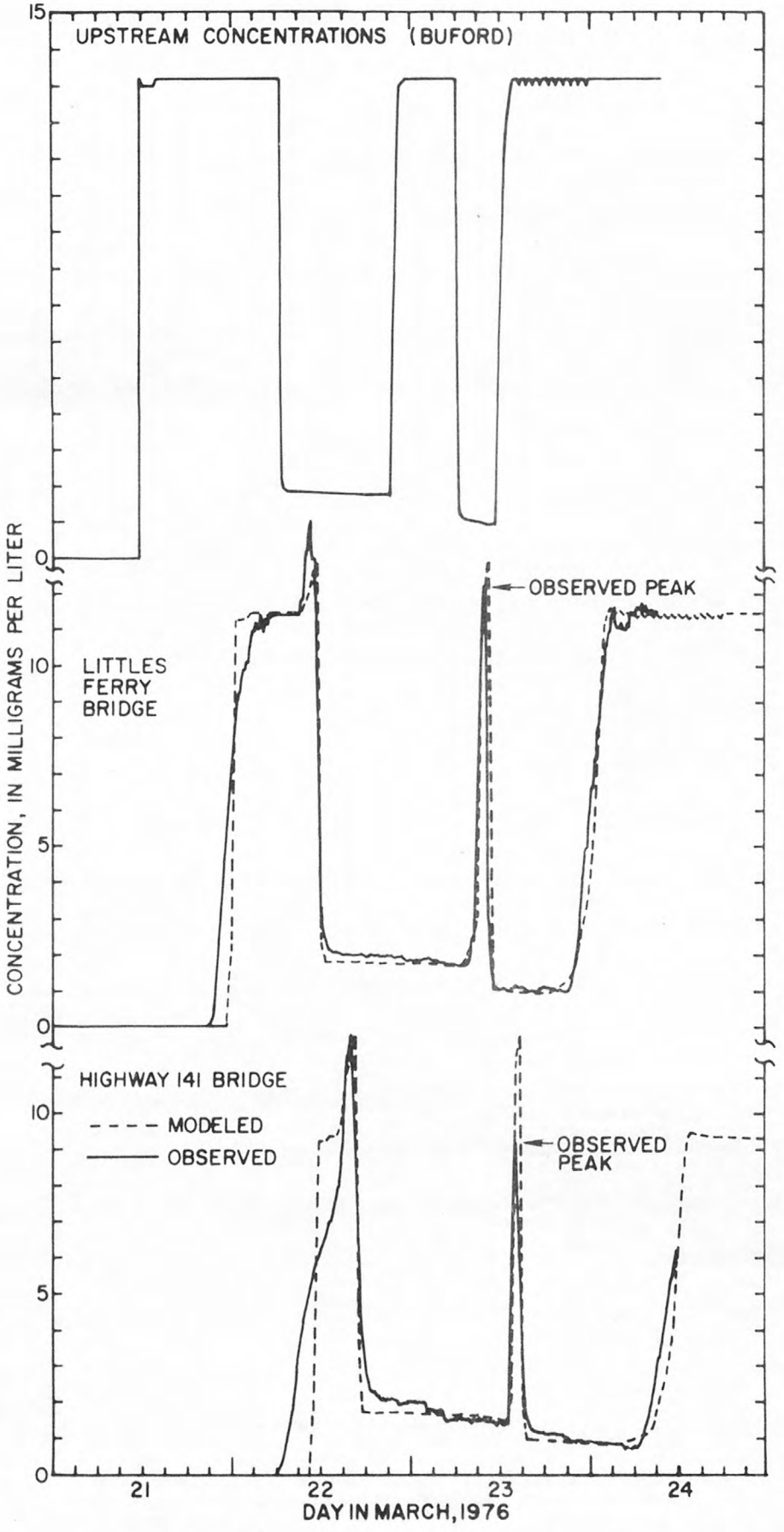

Figure 6.--Comparison of observed dye concentration to values predicted by use of the Lagrangian model with plug flow assumptions, $DQ = 0$.

In summary, it is believed that the proposed Lagrangian transport model is basically superior to the finite-difference model and that equation 13, while not perfect, provides a reasonable approximation of the mixing process. The Lagrangian model offers significant advantages; it can be unconditionally stable for any time step size; the coding is relatively simple and straightforward; the magnitude of the mixing coefficient, DQ, can be easily assessed from stream gaging data; and the conceptual model is easily visualized in the physical sense so that program modification is easily accomplished.

To further illustrate the nondispersive nature of the Lagrangian model, it was run assuming $DQ = 0$, Δt = 30 minutes and steady low-flow conditions like those on March 21. The upstream concentration was assumed to vary as a square wave (period 12 hours, amplitude 13.18), and the predicted concentrations are shown in figure 7. The slight angles at the corners occur because the concentration at specific Eulerian grid points is determined by linear interpolation between the nearest Lagrangian parcels. The concentration changed from zero to 13.18 at time zero so it is easily seen that the low-flow traveltimes are slightly over 12 hours and slightly under 24 hours to Littles Ferry and Highway 141, respectively. The reduction in amplitude is the result of tributary dilution. The Eulerian finite difference model was also run under these conditions with the dispersion coefficient set equal to zero. These results are also shown in figure 7. All dispersion in figure 7 is the result of numerical dispersion.

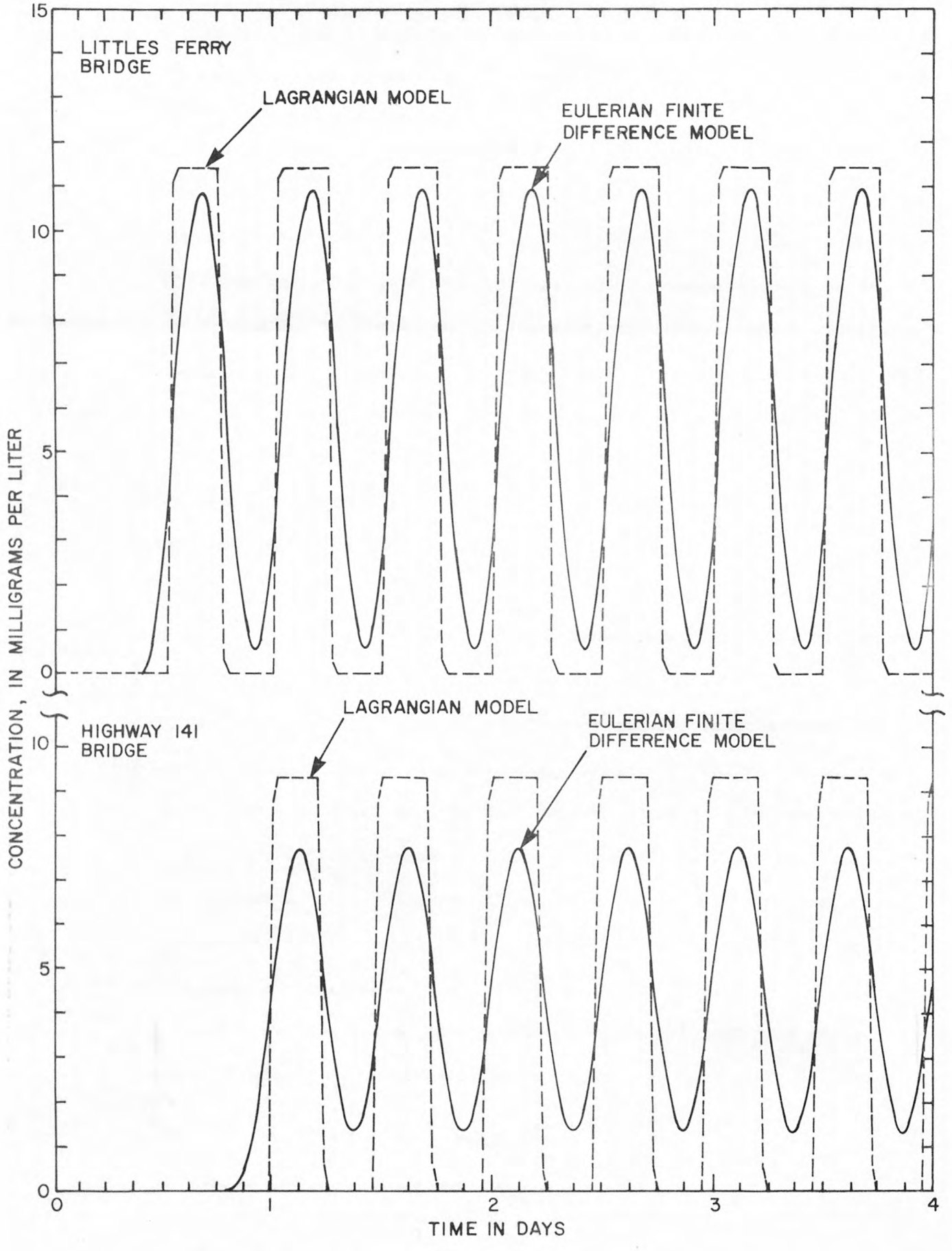

Figure 7.--Modeled response of the Chattahoochee River to a square wave input. Steady low flow and zero dispersion.

The cost of running the Lagrangian model is not large. For example, the results shown in figures 4, 6, and 7 were obtained on an IBM 360-91[1/] machine with CPU times of 4.66, 5.23, and 2.56 seconds, respectively. The model contained the coding necessary to handle nonconservative substances, tracked eight parcel characteristics, created instructions for two computer-drawn plots, and except for figure 7, averaged six values of each hydraulic variable for each of the 48 grid points and time steps.

[1/] The use of the brand name in this report is for identification purposes only and does not imply endorsement by the U.S. Geological Survey.

SUMMARY AND CONCLUSIONS

The simulation of transport in open channels has long been approached from the Eulerian frame of reference. Approaching the problem from a Lagrangian point of view is much simpler conceptually and offers several significant advantages. Among those are:

1. Numerical dispersion can be practically eliminated.

2. A continuous record of traveltimes, which are very useful in the interpretation of results, is a direct by-product.

3. The mixing coefficient DQ is easily evaluated in comparison with the difficulty of estimating the dispersion coefficient, D_x.

4. The contribution of each physical process such as tributary inflow, mixing, or surface exchange on the predicted concentration can be easily tabulated. This information is extremely useful in calibrating models.

An unconditionally stable and practical solution algorithm for solving the one-dimensional transport problem from the Lagrangian reference frame has been developed. The algorithm has been verified by comparison of modeled and observed dye concentrations in the Chattahoochee River under highly transient conditions.

NOTATION

A	cross sectional area of river.
c	local instantaneous concentration.
c_a	deviation of local time averaged velocity from cross sectional mean.
c'	deviation of local instantaneous velocity from local time averaged value.
C	cross sectional average concentration.
C_k	concentration of parcel k.
C_0	concentration at time zero.
da	differential area.
DQ_k	flow rate of water from one segment (parcel) to the next.
D_x	longitudinal dispersion coefficient.
k	index number of a Lagrangian parcel.
Q	discharge.
t	time.
TT	traveltime of the parcel.
u	local instantaneous velocity of the water.
u_a	deviation of local time averaged local velocity from cross sectional mean.
u'	deviation of local instantaneous velocity from local time averaged value.
U	cross sectional mean velocity of the water.
U_*	shear velocity.

W	width of river.
x	longitudinal distance from origin.
x_o	location of a water parcel before the time step.
Y	river depth.
Δx	distance between parcel boundaries.
Δt	time step size.

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