UNITED STATES DEPARTMENT OF THE INTERIOR

GEOLOGICAL SURVEY

HEWLETT PACKARD 9845 COMPUTER PROGRAMS TO COMPUTE HANKEL TRANSFORMS USING CONVOLUTION

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Open-File Report 80-806

1980

Hewlett Packard (1) 9845 computer programs to compute Hankel transforms using convolution.

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INTRODUCTION

Digital filters used to evaluate Hankel(J0,J1) transforms have been published by Anderson (1975). In his publication Anderson presents Fortran IV programs that compute Hankel transforms by direct and lagged convolution. In some applications the latter technique can result in a considerable savings in time. These programs represent some of the fastest methods for computing Hankel transforms while giving an accuracy of 3 to 5 digits. In general the Hankel transform of order n has the form

$$\int_0^\infty GH(G)J_n(BG)dG, \quad B>0, \qquad (1)$$

where, for the purposes of this report, GH(G) is the kernel function, B is the transform argument, and J is the Bessel function of the first kind.

(1) Citation of particular manufacturers and model numbers does not constitute endorsement by the U.S. Geological Survey.

Applications using Hankel transforms can be found in many electrical and electromagnetic problems. In DC resistivity transforms of order zero are used to calculate the potential at a point due to a point source. By combining point potentials it is possible to compute many arrays of interest to geophysicists, specifically, Wenner and dipole-dipole arrays. In the same vein, Hankel transforms of order one can be used to compute the electric field at a point due to a point source. This computation has direct application in computing Schlumberger soundings. In electromagnetics, Hankel transforms of orders zero and one can be found in the computation of mutual coupling over a two-layered earth (Frischnecht,1967).

The purpose of this report is to present Anderson's Hankel transform routines converted to HP enhanced Basic for use on a Hewlett Packard 9845B computer. This report is meant to be used by those familiar with Anderson's Hankel transform routines. If the reader has not already done so, he should read the Anderson publication. Limits on the application of convolution for computing Hankel transforms and specific criteria for kernel functions is discussed. Examples of calling programs, kernel functions and test results are also given.

THE HANKEL TRANSFORM FUNCTIONS

The Basic programs for four Hankel transform functions are given in Appendix A. The names of the functions and a brief description of their purpose follows:

Rhank0 performs a Hankel transform of order zero using direct convolution.

Rlagh0 performs a Hankel transform of order zero using lagged convolution.

Rhank1 performs a Hankel transform of order zero using direct convolution.

Rlagh1 performs a Hankel transform of order one using lagged convolution.

The lagged-convolution functions should be used to compute Hankel transforms for values of the Hankel transform argument (B) that are equally spaced logarithmically at the filter spacing (.2).

All the Hankel transform functions have the same parameter list. The parameters are described below:

X The natural logarithm (LOG(B)) of the transform argument. For the lagged-convolution transforms this parameter must decrease by .2 in the log domain.

Tol The relative error is about 100 times "Tol". Therefore if a relative error of about .001 is desired then "Tol" should be equal to .00001

L The number of filter weights used for the convolution. This value is returned to the calling program.

New For direct convolution (Rhank0, Rhank1) "New" is set unequal to one for the first call. With "New" unequal to one the filter weights are read into the common area. On subsequent calls "New" should be set equal to

one. This will use the weights already in the common area and save computer time by not re-reading the weights. For lagged convolution (Rlagh0, Rlagh1) "New" should be set unequal to one for the first call with a given set of B values, using a single kernel function. Again, on subsequent calls with "New" equal to one the stored filter weights are used. In addition, on subsequent calls only one additional kernel function value will need to be calculated.

These functions use a common area (designated by COM statements) to store the convolution filter weights and, in the case of the lagged-convolution functions, kernel functions and other arrays. For kernels requiring parameters other than the kernel function integration spacing (supplied by the Hankel transform function), additional parameters will have to be passed in common. This will require editing all the COM statements in an appropriate manner (see examples below). If J0 and J1 Hankel transforms are to be used in the same program or if a regular convolution function is to be used with a lagged-convolution function, as in the examples below, the COM statements will have to be edited.

In all the Hankel transform functions the user is required to supply the coded kernel function. For J0 and J1 transforms the function must be named Kern10 and Kern11 respectively. The name convention can be changed by editing the appropriate lines in the Hankel transform functions. As stated in Anderson (1975 page 16), the kernel function must monotonically decrease as the argument increases, and it can have no singularities in the range of

integration. In the examples below the kernel functions are coded to calculate the kernels for two integrals in one kernel function. This technique can be expanded to any number of kernels.

EXAMPLES

Tests were run comparing convolution derived Hankel transform values to known solutions for these integrals. The following two integrals and their analytic solutions were used to test the Hankel transform functions of order zero:

$$\int_0^\infty e^{-ax} J0(bx)dx = 1/(a^2+b^2)^{1/2}, \qquad (2)$$

$$\int_0^\infty x e^{-ax^2} J0(bx)dx = \exp(-b^2/4a)/2a. \qquad (3)$$

The following Basic test program and kernel function will calculate the convolved Hankel transform (Rhank0), the lagged convolution (Rlagh0), and the analytic solution for equations (2) and (3). The output of this program is given in Appendix B.

```
10    OPTION BASE 1
20    ! THE FOLLOWING COM STATEMENT WILL BE DIFFERENT FOR
30    ! DIFFERENT APPLICATIONS
31    ! Dum is the array of filter weights
32    ! Dum1 is the array of kernel function evaluations
33    ! Dum2 is an array used to determine where in Dum1 we should be
34    ! D1 & D2 are parameters used by Rlagh0
35    ! A & F are parameters used by Kern10
40    COM Dum(193),Dum1(193),Dum2(193),D1,D2,A,F
50    DIM A1(6)
60    DATA .1,.5, 1,5,10,50
70    MAT READ A1
80    PRINTER IS 0
90    Tol=1E-7
100   N=11
110   F=1
120 Cont:  Xmax=LOG(4)
130   PRINT USING L;F+1
140 L: IMAGE /,30X,"EQUATION ",DD
150   PRINT USING L1
160 L1:  IMAGE /, 8X,"A",12X,"B",12X,"EXACT        RHANK0        RLAGH0
    L1  L2",/
170   FOR J=1 TO 6
180   A=A1(J)
190   New=0
200   FOR I=N TO 1 STEP -1
201   ! calculate the log of the transform argument decreasing by .2
210   B1=Xmax-.2*(N-I)
220   B=EXP(B1)
230   An1=FNAns(B)
231   !
232   ! Rhank0 and Rlagh0 can be found in Appendix A
233   !
240   An2=FNRhank0(B1,Tol,N1,New)
250   An3=FNRlagh0(B1,Tol,N2,New)
260   New=1
270   PRINT USING L2;A;B;An1;An2;An3;N1;N2
280 L2:  IMAGE 1X,M.7DE,1X,M.7DE,1X,M.7DE,1X,M.7DE,1X,M.7DE,1X,3D,1X,3D
290   NEXT I
300   NEXT J
310   IF F=2 THEN 1540
311   !
312   ! Do next integral
320   F=2
330   GOTO Cont
340   PRINTER IS 16
350   STOP
351   !
352   ! Kern10 is the coded kernel function
353   !
360   DEF FNKern10(X)
370   OPTION BASE 1
380   COM Dum(193),Dum1(193),Dum2(193),D1,D2,A,F
381   !
382   ! IF F=1 compute the first kernel
383   !    F=2 compute the second kernel
384   !
390   ON F GOTO L1,L2
400 L1: C=X*EXP(-A*X*X)
410   RETURN C
```

```
420 L2:  C=EXP(-A*X)
430    RETURN C
440    FNEND
441    !
442    ! Ans computes the analytical solution to equations 2 & 3
443    !
450    DEF FNAns(B)
460    OPTION BASE 1
470    COM D1(193),D2(193),D3(193),D4,D5,A,F
480    ON F GOTO L1,L2
490 L1:  An=EXP(-B*B/(4*A))/(2*A)
500    RETURN An
510 L2:  An=1/SQR(A*A+B*B)
520    RETURN An
530    FNEND
```

The following integrals and their analytic solutions were used to test the Hankel transforms of order one:

$$\int_0^\infty x e^{-ax} J1(bx)dx = b/(a^2+b^2)^{3/2}, \qquad (4)$$

$$\int_0^\infty x^2 e^{-ax^2} J1(bx)dx = b\exp(-b^2/4a)/(2a)^2. \qquad (5)$$

The following Basic test program and kernel function will calculate the convolved Hankel transform (Rhank1), the lagged convolution (Rlagh1), and the analytic solution for equations 4 and 5. The output of this program is given in Appendix C.

```
10     OPTION BASE 1
11     ! THE FOLLOWING COM STATEMENT WILL BE DIFFERENT FOR
12     ! DIFFERENT APPLICATIONS
13     ! Dum is the array of filter weights
14     ! Dum1 is the array of kernel function evaluations
15     ! Dum2 is an array that tells use where in Dum1 we are
16     ! D1 & D2 are parameters used by Rlagh1
17     ! A & F are parameters used by Kern11
20     COM Dum(236),Dum1(236),Dum2(236),D1,D2,A,F
30     DIM A1(6)
40     DATA .1,.5, 1,5,10,50
```

```
50    MAT READ A1
60    PRINTER IS 0
70    Tol=1E-7
80    N=11
90    F=1
100 Cont:   Xmax=LOG(4)
101   PRINT USING L;F
102 L:  IMAGE /,30X,"EQUATION ",DD
110   PRINT USING L1
120 L1:   IMAGE /, 8X,"A",12X,"B",12X,"EXACT        RHANK1         RLAGH1
    L1  L2",/
130   FOR J=1 TO 6
140   A=A1(J)
150   New=0
160   FOR I=N TO 1 STEP -1
161   !
162   ! calculate the log of the transform argument decreasing by .2
163   !
170   B1=Xmax-.2*(N-I)
180   B=EXP(B1)
190   An1=FNAns(B)
191   !
192   ! Rhank1 & Rlagh1 can be found in Appendix A
193   !
200   An2=FNRhank1(B1,Tol,N1,New)
210   An3=FNRlagh1(B1,Tol,N2,New)
220   New=1
230   PRINT USING L2;A;B;An1;An2;An3;N1;N2
240 L2:   IMAGE 1X,M.7DE,1X,M.7DE,1X,M.7DE,1X,M.7DE,1X,M.7DE,1X,3D,1X,3D
250   NEXT I
260   NEXT J
270   IF F=2 THEN 300
271   !
272   ! Do next integral
280   F=2
290   GOTO Cont
300   PRINTER IS 16
310   STOP
311   !
312   ! Kernl1 is the coded kernel function
313   !
320   DEF FNKernl1(X)
330   OPTION BASE 1
340   COM Dum(236),Dum1(236),Dum2(236),D1,D2,A,F
350   ON F GOTO L1,L2
360 L1:  C=X*EXP(-A*X)
370   RETURN C
380 L2:   C=X*X*EXP(-A*X*X)
390   RETURN C
400   FNEND
401   !
402   ! Ans calculates the analytical solutions for equations 4 & 5
403   1
410   DEF FNAns(B)
420   OPTION BASE 1
430   COM D1(236),D2(236),D3(236),D4,D5,A,F
440   ON F GOTO L1,L2
450 L1:   An=B/(A*A+B*B)^1.5
460   RETURN An
470 L2:   An=B*EXP(-B*B/(4*A))/(2*A)^2
```

```
480   RETURN An
490   FNEND
```

TEST RESULTS

Appendix B and C give tabular listings of the output of the test programs (listed above). The headings are described below:

A	A parameter used in the kernel function.
B	The transform argument.
EXACT	The value of the analytic solution of the indicated integral.
Rhank0(1)	The Hankel transform value using Rhank0(1).
Rlagh0(1)	The Hankel transform value using Rlagh0(1)
L1	The number of filter weights used by Rhank0(1)
L2	The number of filter weights used by Rlagh0(1)

In all the test cases "Tol" was .0000001 and "B" was decreased by .2 in the log domain.

In some instances, such as when "A" is .1 and "B" is 2.718 the convolved Hankel transforms may change sign. These values are still small and are considered to be correct (See Anderson, 1975, p. 22). Rows marked with an asterisk are test results that can be found in Anderson (1975) Appendix 2.

SUMMARY

The Hankel transform functions, given as Basic programs, store filter weights by means of COM statements. The COM statements are also used to pass parameters, other than the transform argument, to the kernel function. These statements require editing for user applications involving nontrivial kernel functions. The kernel function must be coded by the user and must be named Kernl0 for zero order transforms and Kernl1 for transforms of order one. For lagged transforms the transform argument must decrease by .2 in the log domain.

REFERENCES

Anderson, W. L., 1975, Improved digital filters for evaluating Fourier and Hankel transform integrals: U.S. Geol. Survey Report USGS-GD-75-012 available from Natl. Tech. Inf. Service 5285 Port Royal Road, Springfield, Va 22161 as NTIS Report PB-242-800/inc, 223p.

Frischnecht, F. C., 1967, Fields about an oscillating magnetic dipole over a two layered earth and applications to ground and airborne electomagnetic surveys: Quart. Colo. School of Mines, v. 62, no 1, 326 p.

APPENDIX A

RHANK0

```
10    DEF FNRhank0(X,Tol,L,New)
20    !
30    ! HANKEL TRANSFORM OF ORDER ZERO  USING
40    ! CONVOULUTION.  CODED BY ROBERT J. BISDORF
50    ! FROM ORIGINAL PROGRAM WRITTEN IN FORTRAN BY
60    ! WALTER ANDERSON.
70    !
80    OPTION BASE 1
90    COM Wt(193)
100   DATA 5.8565723E-08,7.1143477E-11,-7.8395565E-11,8.7489547E-11
110   DATA -8.9007811E-11,9.8790055E-11,-9.8675347E-11,1.1118797E-10
120   DATA -1.0893474E-10,1.2543400E-10,-1.1979399E-10,1.4200767E-10
130   DATA -1.3106341E-10,1.6153229E-10,-1.4238602E-10,1.8486236E-10
140   DATA -1.5315381E-10,2.1319755E-10,-1.6238115E-10,2.4824144E-10
150   DATA -1.6850378E-10,2.9243813E-10,-1.6909302E-10,3.4934366E-10
160   DATA -1.6043759E-10,4.2417082E-10,-1.3690001E-10,5.2458440E-10
170   DATA -8.9946096E-11,6.6188220E-10,-6.6964033E-12,8.5276151E-10
180   DATA 1.3222770E-10,1.1219600E-09,3.5591442E-10,1.5061956E-09
190   DATA 7.0795382E-10,2.0600379E-09,1.2535947E-09,2.8646623E-09
200   DATA 2.0904225E-09,4.0409101E-09,3.3642886E-09,5.7687700E-09
210   DATA 5.2930786E-09,8.3164338E-09,8.2021809E-09,1.2083635E-08
220   DATA 1.2577400E-08,1.7666303E-08,1.9143895E-08,2.5953011E-08
230   DATA 2.8983953E-08,3.8268851E-08,4.3712685E-08,5.6590075E-08
240   DATA 6.5740136E-08,8.3864288E-08,9.8662323E-08,1.2448811E-07
250   DATA 1.4784461E-07,1.8501974E-07,2.2129198E-07,2.7527203E-07
260   DATA 3.3094739E-07,4.0974828E-07,4.9462868E-07,6.1030809E-07
270   DATA 7.3891802E-07,9.0939667E-07,1.1034727E-06,1.3554600E-06
280   DATA 1.6474556E-06,2.0207696E-06,2.4591294E-06,3.0131400E-06
290   DATA 3.6701680E-06,4.4934101E-06,5.4770076E-06,6.7015208E-06
300   DATA 8.1726989E-06,9.9954201E-06,1.2194425E-05,1.4909101E-05
310   DATA 1.8194388E-05,2.2239184E-05,2.7145562E-05,3.3174088E-05
320   DATA 4.0499452E-05,4.9486730E-05,6.0421440E-05,7.3822001E-05
330   DATA 9.0141902E-05,1.1012552E-04,1.3448017E-04,1.6428337E-04
340   DATA 2.0062570E-04,2.4507680E-04,2.9930366E-04,3.6560582E-04
350   DATA 4.4651421E-04,5.4541300E-04,6.6612648E-04,8.1365181E-04
360   DATA 9.9374786E-04,1.2138120E-03,1.4824945E-03,1.8107657E-03
370   DATA 2.2115938E-03,2.7012675E-03,3.2991969E-03,4.0295817E-03
380   DATA 4.9214244E-03,6.0106700E-03,7.3405529E-03,8.9643708E-03
390   DATA 1.0946310E-02,1.3365017E-02,1.6314985E-02,1.9910907E-02
400   DATA 2.4289325E-02,2.9612896E-02,3.6070402E-02,4.3876936E-02
410   DATA 5.3264829E-02,6.4465091E-02,7.7664144E-02,9.2918324E-02
420   DATA 1.1000121E-01,1.2811102E-01,1.4543025E-01,1.5832248E-01
430   DATA 1.6049224E-01,1.4170064E-01,8.8788108E-02,-1.1330934E-02
440   DATA -1.5331864E-01,-2.9094670E-01,-2.9084655E-01,-2.9708834E-02
450   DATA 3.9009601E-01,1.7999785E-01,-4.1858139E-01,1.5317216E-01
460   DATA 6.5184953E-02,-1.0751806E-01,7.8429567E-02,-4.6019124E-02
470   DATA 2.5309571E-02,-1.3904823E-02,7.8187120E-03,-4.5190369E-03
480   DATA 2.6729062E-03,-1.6073718E-03,9.7715622E-04,-5.9804407E-04
490   DATA 3.6749320E-04,-2.2635296E-04,1.3960805E-04,-8.6172618E-05
500   DATA 5.3212947E-05,-3.2867888E-05,2.0304203E-05,-1.2543926E-05
510   DATA 7.7499633E-06,-4.7882430E-06,2.9584108E-06,-1.8278645E-06
520   DATA 1.1293571E-06,-6.9778174E-07,4.3113019E-07,-2.6637753E-07
```

```
530    DATA 1.6458373E-07,-1.0168954E-07,6.2829807E-08,-3.8819969E-08
540    DATA 2.3985272E-08,-1.4819520E-08,9.1563774E-09,-5.6573541E-09
550    DATA 3.4954514E-09,-2.1597005E-09,1.3343946E-09,-8.2447148E-10
560    DATA 5.0941033E-10,-3.1474631E-10,1.9447072E-10,-1.2015685E-10
570    DATA 7.4241055E-11,-4.5871468E-11,2.8343095E-11,-1.7513137E-11
580    DATA 6.9049613E-12
590    IF New=1 THEN 610
600    MAT READ Wt
610    E=1.22140275816
620    Er=.81873075308
630    A=EXP(-X-26.304557)
640    Cmax=0
650    L=18
660    Y1=A*1.31201480803E11
670    Y=Y1
680    FOR I=129 TO 146
690    Y=Y*E
700    GOSUB A1
710    R1=R1+C
720    Cmax=MAX(ABS(C),Cmax)
730    NEXT I
740    IF Cmax=0 THEN L60
750    Cmax=Tol*Cmax
760    FOR I=147 TO 193
770    Y=Y*E
780    GOSUB A1
790    R1=R1+C
800    L=L+1
810    IF ABS(C)<=Cmax THEN L30
820    NEXT I
830 L30:  Y=Y1*E
840    FOR J=1 TO 128
850    Y=Y*Er
860    I=129-J
870    GOSUB A1
880    R1=R1+C
890    L=L+1
900    IF ABS(C)<=Cmax THEN Return
910    NEXT J
920    GOTO Return
930 L60:  Y=A
940    FOR I=1 TO 128
950    Y=Y*E
960    GOSUB A1
970    R1=R1+C
980    L=L+1
990    IF C=0 THEN L80
1000   NEXT I
1010 L80: Y=A*7.08966799402E16
1020   FOR J=1 TO 47
1030   Y=Y*Er
1040   I=194-J
1050   GOSUB A1
```

```
10     DEF FNRlagh0(X,Tol,L,New)
20     !
30     ! HANKEL TRANSFORM OF ORDER ZERO  USING
40     ! LAGGED CONVOULUTION.  CODED BY ROBERT J. BISDORF
50     ! FROM ORIGINAL PROGRAM WRITTEN IN FORTRAN BY
60     ! WALTER ANDERSON.
70     !
80     OPTION BASE 1
90     COM Wt(193),Save(193),Key(193),X0,Lag
100    DATA 5.8565723E-08,7.1143477E-11,-7.8395565E-11,8.7489547E-11
110    DATA -8.9007811E-11,9.8790955E-11,-9.8675347E-11,1.1118797E-10
120    DATA -1.0893474E-10,1.2543400E-10,-1.1979399E-10,1.4200767E-10
130    DATA -1.3106341E-10,1.6153229E-10,-1.4238602E-10,1.8486236E-10
140    DATA -1.5315381E-10,2.1319755E-10,-1.6238115E-10,2.4824144E-10
150    DATA -1.6850378E-10,2.9243813E-10,-1.6909302E-10,3.4934366E-10
160    DATA -1.6043759E-10,4.2417082E-10,-1.3690001E-10,5.2458440E-10
170    DATA -8.9946096E-11,6.6188220E-10,-6.6964033E-12,8.5276151E-10
180    DATA 1.3222770E-10,1.1219600E-09,3.5591442E-10,1.5061956E-09
190    DATA 7.0795382E-10,2.0600379E-09,1.2535947E-09,2.8646623E-09
200    DATA 2.0904225E-09,4.0409101E-09,3.3642886E-09,5.7687700E-09
210    DATA 5.2930786E-09,8.3164338E-09,8.2021809E-09,1.2083635E-08
220    DATA 1.2577400E-08,1.7666303E-08,1.9143895E-08,2.5953011E-08
230    DATA 2.8983953E-08,3.8268851E-08,4.3712685E-08,5.6590075E-08
240    DATA 6.5740136E-08,8.3864288E-08,9.8662323E-08,1.2448811E-07
250    DATA 1.4784461E-07,1.8501974E-07,2.2129198E-07,2.7527203E-07
260    DATA 3.3094739E-07,4.0974828E-07,4.9462868E-07,6.1030809E-07
270    DATA 7.3891802E-07,9.0939667E-07,1.1034727E-06,1.3554600E-06
280    DATA 1.6474556E-06,2.0207696E-06,2.4591294E-06,3.0131400E-06
290    DATA 3.6701680E-06,4.4934101E-06,5.4770076E-06,6.7015208E-06
300    DATA 8.1726989E-06,9.9954201E-06,1.2194425E-05,1.4909101E-05
310    DATA 1.8194388E-05,2.2239184E-05,2.7145562E-05,3.3174088E-05
320    DATA 4.0499452E-05,4.9486730E-05,6.0421440E-05,7.3822001E-05
330    DATA 9.0141902E-05,1.1012552E-04,1.3448017E-04,1.6428337E-04
340    DATA 2.0062570E-04,2.4507680E-04,2.9930366E-04,3.6560582E-04
350    DATA 4.4651421E-04,5.4541300E-04,6.6612648E-04,8.1365181E-04
360    DATA 9.9374786E-04,1.2138120E-03,1.4824945E-03,1.8107657E-03
370    DATA 2.2115938E-03,2.7012675E-03,3.2991969E-03,4.0295817E-03
380    DATA 4.9214244E-03,6.0106700E-03,7.3405529E-03,8.9643708E-03
390    DATA 1.0946310E-02,1.3365017E-02,1.6314985E-02,1.9910907E-02
400    DATA 2.4289325E-02,2.9612896E-02,3.6070402E-02,4.3876936E-02
410    DATA 5.3264829E-02,6.4465091E-02,7.7664144E-02,9.2918324E-02
420    DATA 1.1000121E-01,1.2811102E-01,1.4543025E-01,1.5832248E-01
430    DATA 1.6049224E-01,1.4170064E-01,8.8788108E-02,-1.1330934E-02
440    DATA -1.5331864E-01,-2.9094670E-01,-2.9084655E-01,-2.9708834E-02
450    DATA 3.9009601E-01,1.7999785E-01,-4.1858139E-01,1.5317216E-01
460    DATA 6.5184953E-02,-1.0751806E-01,7.8429567E-02,-4.6019124E-02
470    DATA 2.5309571E-02,-1.3904823E-02,7.8187120E-03,-4.5190369E-03
480    DATA 2.6729062E-03,-1.6073718E-03,9.7715622E-04,-5.9804407E-04
490    DATA 3.6749320E-04,-2.2635296E-04,1.3960805E-04,-8.6172618E-05
500    DATA 5.3212947E-05,-3.2867888E-05,2.0304203E-05,-1.2543926E-05
510    DATA 7.7499633E-06,-4.7882430E-06,2.9584108E-06,-1.8278645E-06
520    DATA 1.1293571E-06,-6.9778174E-07,4.3113019E-07,-2.6637753E-07
530    DATA 1.6458373E-07,-1.0168954E-07,6.2829807E-08,-3.8819969E-08
540    DATA 2.3985272E-08,-1.4819520E-08,9.1563774E-09,-5.6573541E-09
550    DATA 3.4954514E-09,-2.1597005E-09,1.3343946E-09,-8.2447148E-10
```

```
560    DATA 5.0941033E-10,-3.1474631E-10,1.9447072E-10,-1.2015685E-10
570    DATA 7.4241055E-11,-4.5871468E-11,2.8343095E-11,-1.7513137E-11
580    DATA 6.9049613E-12
590    IF New=1 THEN L30
600    MAT READ Wt
610    Lag=-1
620    X0=-X-26.30455704
630    MAT Key=(0)
640 L30:  Lag=Lag+1
650    R1=0
660    Cmax=0
670    L=0
680    M=1
690    I=129
700    GOTO L200
710 L110:  Cmax=MAX(ABS(C),Cmax)
720    I=I+1
730    IF I<147 THEN GOTO L200
740    IF Cmax=0 THEN L150
750    Cmax=Cmax*Tol
760    M=2
770    I=128
780    GOTO L200
790 L120:  IF ABS(C)<=Cmax THEN L130
800    I=I-1
810    IF I>0 THEN L200
820 L130:  M=3
830    I=147
840    GOTO L200
850 L140:  IF ABS(C)<=Cmax THEN L190
860    I=I+1
870    IF I<194 THEN L200
880    GOTO L190
890 L150:  M=4
900    I=1
910    GOTO L200
920 L160:  IF C=0 THEN L170
930    I=I+1
940    IF L<129 THEN L200
950 L170:  M=5
960    I=193
970    GOTO L200
980 L180:   IF C=0 THEN L190
990    I=I-1
1000   IF I>146 THEN L200
1010 L190: RETURN R1/EXP(X)
1020 L200: Look=I+Lag
1030   Iq=INT(Look/194)
1040   Ir=Look MOD 194
1050   IF Ir=0 THEN Ir=1
1060   Iroll=Iq*193
1070   IF Key(Ir)<=Iroll THEN L220
1080 L210: C=Save(Ir)*Wt(I)
1090   R1=R1+C
1100   L=L+1
1110   ON M GOTO L110,L120,L140,L160,L180
1120 L220: Key(Ir)=Iroll+Ir
1130   Z=EXP(X0+Look*.2)
1140   Save(Ir)=FNKern10(Z)
1150   GOTO L210
```

```
1160  FNEND
```

RHANK1

```
10    DEF FNRhank1(X,Tol,L,New)
20    !
30    !   HANKEL TRANSFORM OF ORDER ONE USING
40    !   CONVOLUTION. CODED BY ROBERT J. BISDORF
50    !   FROM ORIGINAL PROGRAM WRITTEN IN FORTRAN
60    !   BY WALTER ANDERSON.
70    !
80    OPTION BASE 1
90    COM Wt(236)
100   E=1.22140275816
110   Er=.81873075307
120   DATA -8.8863805E-10,1.1293811E-09,-1.2050872E-09,1.2696232E-09
130   DATA -1.3223909E-09,1.3642393E-09,-1.3969439E-09,1.4225941E-09
140   DATA -1.4427475E-09,1.4580582E-09,-1.4682563E-09,1.4732179E-09
150   DATA -1.4735606E-09,1.4719870E-09,-1.4727091E-09,1.4828225E-09
160   DATA -1.5102619E-09,1.5667752E-09,-1.6634522E-09,1.8172900E-09
170   DATA -2.0412753E-09,2.3595230E-09,-2.7861077E-09,3.3592871E-09
180   DATA -4.0940172E-09,5.0571015E-09,-6.2604109E-09,7.8269461E-09
190   DATA -9.7514701E-09,1.2267639E-08,-1.5312389E-08,1.9339924E-08
200   DATA -2.4126297E-08,3.0576829E-08,-3.8060204E-08,4.8423732E-08
210   DATA -6.0051116E-08,7.6787475E-08,-9.4700993E-08,1.2192844E-07
220   DATA -1.4918997E-07,1.9392737E-07,-2.3464786E-07,3.0911127E-07
230   DATA -3.6815394E-07,4.9413800E-07,-5.7554168E-07,7.9301529E-07
240   DATA -8.9502818E-07,1.2794292E-06,-1.3811469E-06,2.0789668E-06
250   DATA -2.1069398E-06,3.4103188E-06,-3.1584463E-06,5.6639045E-06
260   DATA -4.6059955E-06,9.5561672E-06,-6.4142855E-06,1.6440205E-05
270   DATA -8.2010619E-06,2.8945217E-05,-8.6348466E-06,5.2317398E-05
280   DATA -3.9915035E-06,9.7273612E-05,1.5220520E-05,1.8614373E-04
290   DATA 7.2023760E-05,3.6620099E-04,2.2062958E-04,7.3874539E-04
300   DATA 5.8623480E-04,1.5226779E-03,1.4538718E-03,3.1930365E-03
310   DATA 3.4640868E-03,6.7790882E-03,8.0328420E-03,1.4484339E-02
320   DATA 1.8201316E-02,3.0866143E-02,4.0106549E-02,6.4527872E-02
330   DATA 8.4285526E-02,1.2773175E-01,1.6020907E-01,2.1948043E-01
340   DATA 2.3636305E-01,2.4895051E-01,1.2586300E-01,-5.1060445E-02
350   DATA -3.4376222E-01,-2.9042175E-01,1.1564736E-01,4.9253231E-01
360   DATA -4.6748595E-01,1.5280945E-01,3.3348541E-02,-8.2485252E-02
370   DATA 7.9740630E-02,-6.6934498E-02,5.5150465E-02,-4.5868721E-02
380   DATA 3.8651958E-02,-3.2935834E-02,2.8303994E-02,-2.4475127E-02
390   DATA 2.1259541E-02,-1.8526278E-02,1.6182037E-02,-1.4158101E-02
400   DATA 1.2402225E-02,-1.0873526E-02,9.5392016E-03,-8.3723743E-03
410   DATA 7.3506490E-03,-6.4551136E-03,5.6696335E-03,-4.9803353E-03
420   DATA 4.3752213E-03,-3.8438703E-03,3.3772023E-03,-2.9672872E-03
430   DATA 2.6071877E-03,-2.2908274E-03,2.0128794E-03,-1.7686706E-03
440   DATA 1.5540998E-03,-1.3655666E-03,1.1999089E-03,-1.0543497E-03
450   DATA 9.2644973E-04,-8.1406593E-04,7.1531559E-04,-6.2854459E-04
460   DATA 5.5229955E-04,-4.8530352E-04,4.2643446E-04,-3.7470650E-04
470   DATA 3.2925334E-04,-2.8931382E-04,2.5421910E-04,-2.2338147E-04
480   DATA 1.9628455E-04,-1.7247455E-04,1.5155278E-04,-1.3316889E-04
490   DATA 1.1701502E-04,-1.0282066E-04,9.0348135E-05,-7.9388568E-05
500   DATA 6.9758436E-05,-6.1296474E-05,5.3860978E-05,-4.7327436E-05
510   DATA 4.1586435E-05,-3.6541840E-05,3.2109174E-05,-2.8214208E-05
520   DATA 2.4791718E-05,-2.1784390E-05,1.9141864E-05,-1.6819888E-05
530   DATA 1.4779578E-05,-1.2986765E-05,1.1411426E-05,-1.0027182E-05
540   DATA 8.8108499E-06,-7.7420630E-06,6.8029235E-06,-5.9777053E-06
```

```
550   DATA 5.2525892E-06,-4.6154325E-06,4.0555653E-06,-3.5636118E-06
560   DATA 3.1313335E-06,-2.7514911E-06,2.4177236E-06,-2.1244417E-06
570   DATA 1.8667342E-06,-1.6402859E-06,1.4413051E-06,-1.2664597E-06
580   DATA 1.1128220E-06,-9.7781908E-07,8.5919028E-07,-7.5494920E-07
590   DATA 6.6335060E-07,-5.8286113E-07,5.1213358E-07,-4.4998431E-07
600   DATA 3.9537334E-07,-3.4738689E-07,3.0522189E-07,-2.6817250E-07
610   DATA 2.3561831E-07,-2.0701397E-07,1.8188012E-07,-1.5979545E-07
620   DATA 1.4038968E-07,-1.2333746E-07,1.0835294E-07,-9.5185048E-08
630   DATA 8.3613184E-08,-7.3443411E-08,6.4505118E-08,-5.6648167E-08
640   DATA 4.9740428E-08,-4.3665572E-08,3.8321109E-08,-3.3616717E-08
650   DATA 2.9472836E-08,-2.5819439E-08,2.2594957E-08,-1.9745353E-08
660   DATA 1.7223359E-08,-1.4987869E-08,1.3003472E-08,-1.1240058E-08
670   DATA 9.6723739E-09,-8.2794392E-09,7.0438407E-09,-5.9509676E-09
680   DATA 4.9882405E-09,-4.1443813E-09,3.4088114E-09,-2.7712762E-09
690   DATA 2.2217311E-09,-1.7504755E-09,1.3485207E-09,-1.0080937E-09
700   DATA 7.2300885E-10,-4.8860666E-10,3.0121413E-10,-9.1649798E-11
710   IF New=1 THEN 730
720   MAT READ Wt
730   A=EXP(-X-17)
740   R1=0
750   Cmax=0
760   L=13
770   Y1=A*2.41549527535E7
780   Y=Y1
790   FOR I=86 TO 98
800   Y=Y*E
810   GOSUB A1
820   R1=R1+C
830   Cmax=MAX(ABS(C),Cmax)
840   NEXT I
850   IF Cmax=0 THEN L60
860   Cmax=Tol*Cmax
870   FOR I=99 TO 236
880   Y=Y*E
890   GOSUB A1
900   R1=R1+C
910   L=L+1
920   IF ABS(C)<=Cmax THEN L30
930   NEXT I
940 L30:  Y=Y1*E
950   FOR I=85 TO 1 STEP -1
960   Y=Y*Er
970   GOSUB A1
980   R1=R1+C
990   L=L+1
1000  IF ABS(C)<=Cmax THEN Return
1010  NEXT I
1020  GOTO Return
1030 L60: Y=A
1040  FOR I=1 TO 85
1050  Y=Y*E
1060  GOSUB A1
1070  R1=R1+C
1080  L=L+1
1090  IF C=0 THEN L80
1100  NEXT I
1110 L80: Y=A*3.8508663159SE20
1120  FOR I=236 TO 99 STEP -1
1130  Y=Y*Er
1140  GOSUB A1
```

```
1150  R1=R1+C
1160  L=L+1
1170  IF C=0 THEN Return
1180  NEXT I
1190 Return:  RETURN R1/EXP(X)
1200 A1:  C=FNKern11(Y)*Wt(I)
1210  RETURN
1220  FNEND
```

RLAGH1

```
10    DEF FNRlagh1(X,Tol,L,New)
20    !
30    ! HANKEL TRANSFORM OF ORDER ONE USING
40    ! LAG CONVOULUTION.  CODED BY ROBERT J. BISDORF
50    ! FROM ORIGINAL PROGRAM WRITTEN IN FORTRAN BY
60    ! WALTER ANDERSON.
70    !
80    OPTION BASE 1
90    COM Wt(236),Save(236),Key(236),X0,Lag
100   DATA -8.8863805E-10,1.1293811E-09,-1.2050872E-09,1.2696232E-09
110   DATA -1.3223909E-09,1.3642393E-09,-1.3969439E-09,1.4225941E-09
120   DATA -1.4427475E-09,1.4580582E-09,-1.4682563E-09,1.4732179E-09
130   DATA -1.4735606E-09,1.4719870E-09,-1.4727091E-09,1.4828225E-09
140   DATA -1.5102619E-09,1.5667752E-09,-1.6634522E-09,1.8172900E-09
150   DATA -2.0412753E-09,2.3595230E-09,-2.7861077E-09,3.3592871E-09
160   DATA -4.0940172E-09,5.0571015E-09,-6.2604109E-09,7.8269461E-09
170   DATA -9.7514701E-09,1.2267639E-08,-1.5312389E-08,1.9339924E-08
180   DATA -2.4126297E-08,3.0576829E-08,-3.8060204E-08,4.8423732E-08
190   DATA -6.0051116E-08,7.6787475E-08,-9.4700993E-08,1.2192844E-07
200   DATA -1.4918997E-07,1.9392737E-07,-2.3464786E-07,3.0911127E-07
210   DATA -3.6815394E-07,4.9413800E-07,-5.7554168E-07,7.9301529E-07
220   DATA -8.9502818E-07,1.2794292E-06,-1.3811469E-06,2.0789668E-06
230   DATA -2.1069398E-06,3.4103188E-06,-3.1584463E-06,5.6639045E-06
240   DATA -4.6059955E-06,9.5561672E-06,-6.4142855E-06,1.6440205E-05
250   DATA -8.2010619E-06,2.8945217E-05,-8.6348466E-06,5.2317398E-05
260   DATA -3.9915035E-06,9.7273612E-05,1.5220520E-05,1.8614373E-04
270   DATA 7.2023760E-05,3.6620099E-04,2.2062958E-04,7.3874539E-04
280   DATA 5.8623480E-04,1.5226779E-03,1.4538718E-03,3.1930365E-03
290   DATA 3.4640868E-03,6.7790882E-03,8.0328420E-03,1.4484339E-02
300   DATA 1.8201316E-02,3.0866143E-02,4.0106549E-02,6.4527872E-02
310   DATA 8.4285526E-02,1.2773175E-01,1.6020907E-01,2.1948043E-01
320   DATA 2.3636305E-01,2.4895051E-01,1.2586300E-01,-5.1060445E-02
330   DATA -3.4376222E-01,-2.9042175E-01,1.1564736E-01,4.9253231E-01
340   DATA -4.6748595E-01,1.5280945E-01,3.3348541E-02,-8.2485252E-02
350   DATA 7.9740630E-02,-6.6934498E-02,5.5150465E-02,-4.5868721E-02
360   DATA 3.8651958E-02,-3.2935834E-02,2.8303994E-02,-2.4475127E-02
370   DATA 2.1259541E-02,-1.8526278E-02,1.6182037E-02,-1.4158101E-02
380   DATA 1.2402225E-02,-1.0873526E-02,9.5392016E-03,-8.3723743E-03
390   DATA 7.3506490E-03,-6.4551136E-03,5.6696335E-03,-4.9803353E-03
400   DATA 4.3752213E-03,-3.8438703E-03,3.3772023E-03,-2.9672872E-03
410   DATA 2.6071877E-03,-2.2908274E-03,2.0128794E-03,-1.7686706E-03
420   DATA 1.5540998E-03,-1.3655666E-03,1.1999089E-03,-1.0543497E-03
430   DATA 9.2644973E-04,-8.1406593E-04,7.1531559E-04,-6.2854459E-04
440   DATA 5.5229955E-04,-4.8530352E-04,4.2643446E-04,-3.7470650E-04
450   DATA 3.2925334E-04,-2.8931382E-04,2.5421910E-04,-2.2338147E-04
460   DATA 1.9628455E-04,-1.7247455E-04,1.5155278E-04,-1.3316889E-04
470   DATA 1.1701502E-04,-1.0282066E-04,9.0348135E-05,-7.9388568E-05
```

```
480    DATA 6.9758436E-05,-6.1296474E-05,5.3860978E-05,-4.7327436E-05
490    DATA 4.1586435E-05,-3.6541840E-05,3.2109174E-05,-2.8214208E-05
500    DATA 2.4791718E-05,-2.1784390E-05,1.9141864E-05,-1.6819888E-05
510    DATA 1.4779578E-05,-1.2986765E-05,1.1411426E-05,-1.0027182E-05
520    DATA 8.8108499E-06,-7.7420630E-06,6.8029235E-06,-5.9777053E-06
530    DATA 5.2525892E-06,-4.6154325E-06,4.0555653E-06,-3.5636118E-06
540    DATA 3.1313335E-06,-2.7514911E-06,2.4177236E-06,-2.1244417E-06
550    DATA 1.8667342E-06,-1.6402859E-06,1.4413051E-06,-1.2664597E-06
560    DATA 1.1128220E-06,-9.7781908E-07,8.5919028E-07,-7.5494920E-07
570    DATA 6.6335060E-07,-5.8286113E-07,5.1213358E-07,-4.4998431E-07
580    DATA 3.9537334E-07,-3.4738689E-07,3.0522189E-07,-2.6817250E-07
590    DATA 2.3561831E-07,-2.0701397E-07,1.8188012E-07,-1.5979545E-07
600    DATA 1.4038968E-07,-1.2333746E-07,1.0835294E-07,-9.5185048E-08
610    DATA 8.3613184E-08,-7.3443411E-08,6.4505118E-08,-5.6648167E-08
620    DATA 4.9740428E-08,-4.3665572E-08,3.8321109E-08,-3.3616717E-08
630    DATA 2.9472836E-08,-2.5819439E-08,2.2594957E-08,-1.9745353E-08
640    DATA 1.7223359E-08,-1.4987869E-08,1.3003472E-08,-1.1240058E-08
650    DATA 9.6723739E-09,-8.2794392E-09,7.0438407E-09,-5.9509676E-09
660    DATA 4.9882405E-09,-4.1443813E-09,3.4088114E-09,-2.7712762E-09
670    DATA 2.2217311E-09,-1.7504755E-09,1.3485207E-09,-1.0080937E-09
680    DATA 7.2300885E-10,-4.8860666E-10,3.0121413E-10,-9.1649798E-11
690    IF New=1 THEN L30
700    MAT READ Wt
710    Lag=-1
720    X0=-X-17
730    MAT Key=(0)
740 L30:  Lag=Lag+1
750    R1=0
760    Cmax=0
770    L=0
780    M=1
790    I=86
800    GOTO L200
810 L110:  Cmax=MAX(ABS(C),Cmax)
820    I=I+1
830    IF I<99 THEN GOTO L200
840    IF Cmax=0 THEN L150
850    Cmax=Cmax*Tol
860    M=2
870    I=85
880    GOTO L200
890 L120:  IF ABS(C)<=Cmax THEN L130
900    I=I-1
910    IF I>0 THEN L200
920 L130:  M=3
930    I=99
940    GOTO L200
950 L140:  IF ABS(C)<=Cmax THEN L190
960    I=I+1
970    IF I<237 THEN L200
980    GOTO L190
990 L150:  M=4
1000   I=1
1010   GOTO L200
1020 L160: IF C=0 THEN L170
1030   I=I+1
1040   IF L<86 THEN L200
1050 L170: M=5
1060   I=236
1070   GOTO L200
```

```
1080 L180:  IF C=0 THEN L190
1090  I=I-1
1100  IF I>98 THEN L200
1110 L190: RETURN R1/EXP(X)
1120 L200: Look=I+Lag
1130  Iq=INT(Look/237)
1140  Ir=Look MOD 237
1150  IF Ir=0 THEN Ir=1
1160  Iroll=Iq*236
1170  IF Key(Ir)<=Iroll THEN L220
1180 L210: C=Save(Ir)*Wt(I)
1190  R1=R1+C
1200  L=L+1
1210  ON M GOTO L110,L120,L140,L160,L180
1220 L220: Key(Ir)=Iroll+Ir
1230  Z=EXP(X0+Look*.2)
1240  Save(Ir)=FNKern1(Z)
1250  GOTO L210
1260  FNEND
```

APPENDIX B

EQUATION 2

	A	B	EXACT	RHANK0	RLAGH0	L1	L2
*	.1	.4000000E+01	.2124177E-16	-.4940951E-06	-.4940979E-06	55	55
	.1	.3274923E+01	.1133230E-10	.1045882E-05	.1045890E-05	55	55
	.1	.2681280E+01	.7822155E-07	-.2819013E-05	-.2819006E-05	55	55
	.1	.2195247E+01	.2928806E-04	.4615496E-04	.4615494E-04	55	55
	.1	.1797316E+01	.1554778E-02	.1391187E-02	.1391187E-02	55	55
	.1	.1471518E+01	.2228206E-01	.2222885E-01	.2222884E-01	55	55
	.1	.1204777E+01	.1327510E+00	.1332464E+00	.1332464E+00	55	55
	.1	.9863879E+00	.4391279E+00	.4393603E+00	.4393603E+00	57	57
	.1	.8075861E+00	.9791698E+00	.9788740E+00	.9788739E+00	58	58
	.1	.6611956E+00	.1676132E+01	.1675747E+01	.1675746E+01	58	58
	.1	.5413411E+00	.2403227E+01	.2403018E+01	.2403018E+01	59	59
	.1	.4432126E+00	.3059784E+01	.3059768E+01	.3059768E+01	59	59
	.5	.4000000E+01	.3354626E-03	.3016334E-03	.3016332E-03	55	55
	.5	.3274923E+01	.4688885E-02	.4680832E-02	.4680830E-02	55	55
	.5	.2681280E+01	.2747080E-01	.2757095E-01	.2757094E-01	55	55
	.5	.2195247E+01	.8985539E-01	.8989900E-01	.8989899E-01	57	57
	.5	.1797316E+01	.1988564E+00	.1987958E+00	.1987958E+00	58	58
	.5	.1471518E+01	.3386859E+00	.3386091E+00	.3386091E+00	58	58
	.5	.1204777E+01	.4839645E+00	.4839239E+00	.4839239E+00	59	59
	.5	.9863879E+00	.6147863E+00	.6147838E+00	.6147838E+00	59	59
	.5	.8075861E+00	.7217347E+00	.7217530E+00	.7217530E+00	60	60
	.5	.6611956E+00	.8036513E+00	.8036759E+00	.8036759E+00	61	61
	.5	.5413411E+00	.8637040E+00	.8637266E+00	.8637266E+00	61	61
	.5	.4432126E+00	.9064506E+00	.9064682E+00	.9064682E+00	62	62
	1.0	.4000000E+01	.9157819E-02	.9198552E-02	.9198549E-02	55	55
	1.0	.3274923E+01	.3423772E-01	.3427466E-01	.3427465E-01	56	56
	1.0	.2681280E+01	.8287159E-01	.8285138E-01	.8285138E-01	58	58
	1.0	.2195247E+01	.1498794E+00	.1498395E+00	.1498395E+00	58	58
	1.0	.1797316E+01	.2229666E+00	.2229405E+00	.2229405E+00	58	58
	1.0	.1471518E+01	.2909836E+00	.2909781E+00	.2909781E+00	59	59
	1.0	.1204777E+01	.3478378E+00	.3478450E+00	.3478450E+00	60	60
	1.0	.9863879E+00	.3920416E+00	.3920535E+00	.3920535E+00	60	60
	1.0	.8075861E+00	.4247749E+00	.4247866E+00	.4247866E+00	61	61
	1.0	.6611956E+00	.4482330E+00	.4482425E+00	.4482425E+00	62	62
	1.0	.5413411E+00	.4646784E+00	.4646852E+00	.4646852E+00	63	63
	1.0	.4432126E+00	.4760385E+00	.4760429E+00	.4760430E+00	65	65
	5.0	.4000000E+01	.4493290E-01	.4492778E-01	.4492778E-01	58	58
	5.0	.3274923E+01	.5849341E-01	.5849238E-01	.5849238E-01	59	59
	5.0	.2681280E+01	.6980510E-01	.6980658E-01	.6980658E-01	60	60
	5.0	.2195247E+01	.7858767E-01	.7859007E-01	.7859008E-01	60	60
	5.0	.1797316E+01	.8508519E-01	.8508753E-01	.8508753E-01	61	61
	5.0	.1471518E+01	.8973869E-01	.8974058E-01	.8974058E-01	62	62
	5.0	.1204777E+01	.9299966E-01	.9300101E-01	.9300101E-01	63	63
	5.0	.9863879E+00	.9525163E-01	.9525251E-01	.9525251E-01	65	65
	5.0	.8075861E+00	.9679162E-01	.9679212E-01	.9679212E-01	67	67
	5.0	.6611956E+00	.9783782E-01	.9783804E-01	.9783804E-01	70	70
	5.0	.5413411E+00	.9854543E-01	.9854545E-01	.9854546E-01	76	76
	5.0	.4432126E+00	.9902262E-01	.9902250E-01	.9902251E-01	83	83
	10.0	.4000000E+01	.3351600E-01	.3351649E-01	.3351649E-01	60	60
	10.0	.3274923E+01	.3824049E-01	.3824163E-01	.3824164E-01	60	60
	10.0	.2681280E+01	.4177472E-01	.4177593E-01	.4177593E-01	61	61

10.0	.2195247E+01	.4432484E-01	.4432586E-01	.4432586E-01	62	62
10.0	.1797316E+01	.4612082E-01	.4612156E-01	.4612156E-01	63	63
10.0	.1471518E+01	.4736525E-01	.4736575E-01	.4736575E-01	64	64
10.0	.1204777E+01	.4821817E-01	.4821846E-01	.4821846E-01	66	66
10.0	.9863879E+00	.4879847E-01	.4879861E-01	.4879862E-01	69	69
10.0	.8075861E+00	.4919137E-01	.4919140E-01	.4919140E-01	74	74
10.0	.6611956E+00	.4945650E-01	.4945646E-01	.4945646E-01	81	81
10.0	.5413411E+00	.4963503E-01	.4963493E-01	.4963493E-01	91	91
10.0	.4432126E+00	.4975506E-01	.4975492E-01	.4975492E-01	107	107
50.0	.4000000E+01	.9231163E-02	.9231312E-02	.9231312E-02	63	63
50.0	.3274923E+01	.9477869E-02	.9477967E-02	.9477968E-02	64	64
50.0	.2681280E+01	.9646921E-02	.9646979E-02	.9646979E-02	66	66
50.0	.2195247E+01	.9761924E-02	.9761953E-02	.9761953E-02	69	69
50.0	.1797316E+01	.9839780E-02	.9839787E-02	.9839787E-02	74	74
50.0	.1471518E+01	.9892316E-02	.9892307E-02	.9892307E-02	81	81
50.0	.1204777E+01	.9927688E-02	.9927669E-02	.9927669E-02	91	91
50.0	.9863879E+00	.9951470E-02	.9951443E-02	.9951444E-02	107	107
50.0	.8075861E+00	.9967443E-02	.9967412E-02	.9967412E-02	137	137
50.0	.6611956E+00	.9978165E-02	.9978130E-02	.9978130E-02	147	147
50.0	.5413411E+00	.9985358E-02	.9985321E-02	.9985321E-02	147	147
50.0	.4432126E+00	.9990183E-02	.9990144E-02	.9990145E-02	147	147

EQUATION 3

	A	B	EXACT	RHANK0	RLAGH0	L1	L2
*	.1	.4000000E+01	.2499219E+00	.2499219E+00	.2499219E+00	109	109
	.1	.3274923E+01	.3052084E+00	.3052084E+00	.3052084E+00	108	108
	.1	.2681280E+01	.3726971E+00	.3726970E+00	.3726970E+00	107	107
	.1	.2195247E+01	.4550578E+00	.4550578E+00	.4550578E+00	107	107
	.1	.1797316E+01	.5555260E+00	.5555260E+00	.5555260E+00	107	107
	.1	.1471518E+01	.6780067E+00	.6780067E+00	.6780067E+00	107	107
	.1	.1204777E+01	.8271847E+00	.8271848E+00	.8271848E+00	106	106
	.1	.9863879E+00	.1008630E+01	.1008630E+01	.1008630E+01	105	105
	.1	.8075861E+00	.1228873E+01	.1228873E+01	.1228873E+01	106	106
	.1	.6611956E+00	.1495406E+01	.1495406E+01	.1495406E+01	106	106
	.1	.5413411E+00	.1816531E+01	.1816532E+01	.1816532E+01	105	105
	.1	.4432126E+00	.2200928E+01	.2200929E+01	.2200929E+01	104	104
	.5	.4000000E+01	.2480695E+00	.2480696E+00	.2480696E+00	106	106
	.5	.3274923E+01	.3018529E+00	.3018530E+00	.3018530E+00	106	106
	.5	.2681280E+01	.3666359E+00	.3666362E+00	.3666362E+00	105	105
	.5	.2195247E+01	.4441547E+00	.4441551E+00	.4441551E+00	104	104
	.5	.1797316E+01	.5360298E+00	.5360302E+00	.5360302E+00	105	105
	.5	.1471518E+01	.6434409E+00	.6434413E+00	.6434413E+00	105	105
	.5	.1204777E+01	.7666297E+00	.7666295E+00	.7666295E+00	104	104
	.5	.9863879E+00	.9042608E+00	.9042589E+00	.9042589E+00	103	103
	.5	.8075861E+00	.1052809E+01	.1052805E+01	.1052805E+01	102	102
	.5	.6611956E+00	.1206327E+01	.1206319E+01	.1206319E+01	103	103
	.5	.5413411E+00	.1357000E+01	.1356992E+01	.1356992E+01	103	103
	.5	.4432126E+00	.1496648E+01	.1496644E+01	.1496644E+01	103	103
	1.0	.4000000E+01	.2425356E+00	.2425358E+00	.2425358E+00	106	106
	1.0	.3274923E+01	.2920393E+00	.2920396E+00	.2920396E+00	105	105
	1.0	.2681280E+01	.3494440E+00	.3494441E+00	.3494441E+00	104	104
	1.0	.2195247E+01	.4145451E+00	.4145447E+00	.4145447E+00	103	103
	1.0	.1797316E+01	.4861969E+00	.4861953E+00	.4861953E+00	103	103
	1.0	.1471518E+01	.5620671E+00	.5620640E+00	.5620640E+00	104	104
	1.0	.1204777E+01	.6386827E+00	.6386785E+00	.6386786E+00	103	103
	1.0	.9863879E+00	.7119357E+00	.7119322E+00	.7119322E+00	103	103
	1.0	.8075861E+00	.7779816E+00	.7779811E+00	.7779812E+00	105	105

1.0	.6611956E+00	.8341506E+00	.8341542E+00	.8341542E+00	105 105
1.0	.5413411E+00	.8794118E+00	.8794184E+00	.8794184E+00	105 105
1.0	.4432126E+00	.9142290E+00	.9142365E+00	.9142365E+00	107 107
5.0	.4000000E+01	.1561738E+00	.1561737E+00	.1561737E+00	105 105
5.0	.3274923E+01	.1673066E+00	.1673074E+00	.1673074E+00	105 105
5.0	.2681280E+01	.1762562E+00	.1762576E+00	.1762576E+00	105 105
5.0	.2195247E+01	.1831271E+00	.1831286E+00	.1831286E+00	107 107
5.0	.1797316E+01	.1882097E+00	.1882110E+00	.1882110E+00	109 109
5.0	.1471518E+01	.1918634E+00	.1918644E+00	.1918644E+00	109 109
5.0	.1204777E+01	.1944352E+00	.1944358E+00	.1944358E+00	111 111
5.0	.9863879E+00	.1962182E+00	.1962184E+00	.1962184E+00	113 113
5.0	.8075861E+00	.1974412E+00	.1974411E+00	.1974411E+00	115 115
5.0	.6611956E+00	.1982739E+00	.1982736E+00	.1982736E+00	117 117
5.0	.5413411E+00	.1988380E+00	.1988375E+00	.1988375E+00	117 117
5.0	.4432126E+00	.1992189E+00	.1992182E+00	.1992183E+00	117 117
10.0	.4000000E+01	.9284767E-01	.9284838E-01	.9284839E-01	107 107
10.0	.3274923E+01	.9503354E-01	.9503411E-01	.9503412E-01	109 109
10.0	.2681280E+01	.9658826E-01	.9658863E-01	.9658864E-01	111 111
10.0	.2195247E+01	.9767418E-01	.9767436E-01	.9767436E-01	113 113
10.0	.1797316E+01	.9842294E-01	.9842295E-01	.9842295E-01	113 113
10.0	.1471518E+01	.9893459E-01	.9893448E-01	.9893448E-01	115 115
10.0	.1204777E+01	.9928206E-01	.9928186E-01	.9928186E-01	117 117
10.0	.9863879E+00	.9951704E-01	.9951677E-01	.9951677E-01	117 117
10.0	.8075861E+00	.9967549E-01	.9967524E-01	.9967525E-01	147 147
10.0	.6611956E+00	.9978212E-01	.9978186E-01	.9978187E-01	147 147
10.0	.5413411E+00	.9985380E-01	.9985353E-01	.9985354E-01	147 147
10.0	.4432126E+00	.9990193E-01	.9990167E-01	.9990167E-01	147 147
50.0	.4000000E+01	.1993631E-01	.1993626E-01	.1993626E-01	147 147
50.0	.3274923E+01	.1995724E-01	.1995718E-01	.1995719E-01	147 147
50.0	.2681280E+01	.1997130E-01	.1997125E-01	.1997125E-01	147 147
50.0	.2195247E+01	.1998075E-01	.1998070E-01	.1998070E-01	147 147
50.0	.1797316E+01	.1998709E-01	.1998704E-01	.1998704E-01	147 147
50.0	.1471518E+01	.1999134E-01	.1999130E-01	.1999130E-01	147 147
50.0	.1204777E+01	.1999420E-01	.1999416E-01	.1999416E-01	147 147
50.0	.9863879E+00	.1999611E-01	.1999609E-01	.1999609E-01	147 147
50.0	.8075861E+00	.1999739E-01	.1999738E-01	.1999738E-01	147 147
50.0	.6611956E+00	.1999825E-01	.1999826E-01	.1999826E-01	147 147
50.0	.5413411E+00	.1999883E-01	.1999885E-01	.1999885E-01	147 147
50.0	.4432126E+00	.1999921E-01	.1999926E-01	.1999926E-01	147 147

APPENDIX C

EQUATION 4

	A	B	EXACT	RHANK1	RLAGH1	L1	L2
*	.1	.4000000E+01	.6244145E-01	.6244139E-01	.6244139E-01	54	54
	.1	.3274923E+01	.9310879E-01	.9310870E-01	.9310870E-01	53	53
	.1	.2681280E+01	.1388066E+00	.1388065E+00	.1388065E+00	52	52
	.1	.2195247E+01	.2068631E+00	.2068629E+00	.2068629E+00	51	51
	.1	.1797316E+01	.3081326E+00	.3081323E+00	.3081323E+00	50	50
	.1	.1471518E+01	.4586353E+00	.4586348E+00	.4586348E+00	49	49
	.1	.1204777E+01	.6818896E+00	.6818889E+00	.6818889E+00	48	48
	.1	.9863879E+00	.1012146E+01	.1012145E+01	.1012145E+01	47	47
	.1	.8075861E+00	.1498683E+01	.1498681E+01	.1498681E+01	46	46
	.1	.6611956E+00	.2211093E+01	.2211091E+01	.2211091E+01	45	45
	.1	.5413411E+00	.3244884E+01	.3244880E+01	.3244880E+01	44	44
	.1	.4432126E+00	.4725301E+01	.4725300E+01	.4725300E+01	49	49
	.5	.4000000E+01	.6106325E-01	.6106319E-01	.6106319E-01	46	46
	.5	.3274923E+01	.9007145E-01	.9007135E-01	.9007135E-01	45	45
	.5	.2681280E+01	.1321440E+00	.1321438E+00	.1321438E+00	44	44
	.5	.2195247E+01	.1923473E+00	.1923473E+00	.1923473E+00	49	49
	.5	.1797316E+01	.2768159E+00	.2768159E+00	.2768159E+00	48	48
	.5	.1471518E+01	.3920049E+00	.3920049E+00	.3920049E+00	49	49
	.5	.1204777E+01	.5428297E+00	.5428296E+00	.5428296E+00	48	48
	.5	.9863879E+00	.7293379E+00	.7293379E+00	.7293379E+00	48	48
	.5	.8075861E+00	.9424059E+00	.9424059E+00	.9424059E+00	49	49
	.5	.6611956E+00	.1160712E+01	.1160712E+01	.1160712E+01	48	48
	.5	.5413411E+00	.1352730E+01	.1352730E+01	.1352730E+01	48	48
	.5	.4432126E+00	.1485837E+01	.1485837E+01	.1485837E+01	47	47
	1.0	.4000000E+01	.5706721E-01	.5706720E-01	.5706720E-01	49	49
	1.0	.3274923E+01	.8156899E-01	.8156899E-01	.8156899E-01	48	48
	1.0	.2681280E+01	.1144129E+00	.1144129E+00	.1144129E+00	49	49
	1.0	.2195247E+01	.1563863E+00	.1563863E+00	.1563863E+00	48	48
	1.0	.1797316E+01	.2065669E+00	.2065669E+00	.2065669E+00	49	49
	1.0	.1471518E+01	.2612943E+00	.2612943E+00	.2612943E+00	48	48
	1.0	.1204777E+01	.3138789E+00	.3138789E+00	.3138789E+00	47	47
	1.0	.9863879E+00	.3559344E+00	.3559344E+00	.3559344E+00	48	48
	1.0	.8075861E+00	.3802741E+00	.3802741E+00	.3802741E+00	47	47
	1.0	.6611956E+00	.3837632E+00	.3837632E+00	.3837632E+00	49	49
	1.0	.5413411E+00	.3681696E+00	.3681696E+00	.3681696E+00	49	49
	1.0	.4432126E+00	.3386703E+00	.3386703E+00	.3386703E+00	51	51
	5.0	.4000000E+01	.1523646E-01	.1523646E-01	.1523646E-01	47	47
	5.0	.3274923E+01	.1533700E-01	.1533700E-01	.1533700E-01	49	49
	5.0	.2681280E+01	.1468168E-01	.1468168E-01	.1468168E-01	49	49
	5.0	.2195247E+01	.1348160E-01	.1348160E-01	.1348160E-01	51	51
	5.0	.1797316E+01	.1198257E-01	.1198257E-01	.1198257E-01	52	52
	5.0	.1471518E+01	.1039303E-01	.1039303E-01	.1039303E-01	54	54
	5.0	.1204777E+01	.8855875E-02	.8855875E-02	.8855875E-02	56	56
	5.0	.9863879E+00	.7451874E-02	.7451874E-02	.7451874E-02	59	59
	5.0	.8075861E+00	.6215871E-02	.6215871E-02	.6215871E-02	62	62
	5.0	.6611956E+00	.5153787E-02	.5153787E-02	.5153787E-02	66	66
	5.0	.5413411E+00	.4255682E-02	.4255682E-02	.4255682E-02	71	71
	5.0	.4432126E+00	.3504317E-02	.3504317E-02	.3504317E-02	77	77
	10.0	.4000000E+01	.3201644E-02	.3201644E-02	.3201644E-02	51	51
	10.0	.3274923E+01	.2810812E-02	.2810812E-02	.2810812E-02	53	53
	10.0	.2681280E+01	.2416102E-02	.2416102E-02	.2416102E-02	55	55

10.0	.2195247E+01	.2045609E-02	.2045609E-02	.2045609E-02	57	57
10.0	.1797316E+01	.1713615E-02	.1713615E-02	.1713615E-02	60	60
10.0	.1471518E+01	.1424984E-02	.1424984E-02	.1424984E-02	64	64
10.0	.1204777E+01	.1179014E-02	.1179014E-02	.1179014E-02	68	68
10.0	.9863879E+00	.9721652E-03	.9721652E-03	.9721652E-03	73	73
10.0	.8075861E+00	.7997494E-03	.7997494E-03	.7997494E-03	80	80
10.0	.6611956E+00	.6568832E-03	.6568832E-03	.6568832E-03	96	96
10.0	.5413411E+00	.5389702E-03	.5389702E-03	.5389702E-03	99	99
10.0	.4432126E+00	.4419099E-03	.4419099E-03	.4419099E-03	99	99
50.0	.4000000E+01	.3169524E-04	.3169524E-04	.3169524E-04	81	81
50.0	.3274923E+01	.2603169E-04	.2603169E-04	.2603169E-04	96	96
50.0	.2681280E+01	.2135805E-04	.2135805E-04	.2135805E-04	99	99
50.0	.2195247E+01	.1751131E-04	.1751131E-04	.1751131E-04	99	99
50.0	.1797316E+01	.1435070E-04	.1435070E-04	.1435070E-04	99	99
50.0	.1471518E+01	.1175686E-04	.1175686E-04	.1175686E-04	99	99
50.0	.1204777E+01	.9629827E-05	.9629827E-05	.9629827E-05	99	99
50.0	.9863879E+00	.7886498E-05	.7886498E-05	.7886498E-05	99	99
50.0	.8075861E+00	.6458161E-05	.6458161E-05	.6458161E-05	99	99
50.0	.6611956E+00	.5288177E-05	.5288177E-05	.5288177E-05	99	99
50.0	.5413411E+00	.4329968E-05	.4329968E-05	.4329968E-05	99	99
50.0	.4432126E+00	.3545283E-05	.3545283E-05	.3545283E-05	99	99

EQUATION 5

	A	B	EXACT	RHANK1	RLAGH1	L1	L2
*	.1	.4000000E+01	.4248354E-15	-.7944747E-05	-.7944742E-05	35	35
	.1	.3274923E+01	.1855621E-09	.1325152E-04	.1325153E-04	34	34
	.1	.2681280E+01	.1048669E-05	-.2151398E-04	-.2151399E-04	33	33
	.1	.2195247E+01	.3214726E-03	.3580420E-03	.3580420E-03	34	34
	.1	.1797316E+01	.1397213E-01	.1391831E-01	.1391831E-01	33	33
	.1	.1471518E+01	.1639422E+00	.1640110E+00	.1640110E+00	32	32
	.1	.1204777E+01	.7996765E+00	.7995999E+00	.7995999E+00	31	31
	.1	.9863879E+00	.2165752E+01	.2165827E+01	.2165827E+01	33	33
	.1	.8075861E+00	.3953819E+01	.3953750E+01	.3953750E+01	33	33
	.1	.6611956E+00	.5541257E+01	.5541319E+01	.5541319E+01	33	33
	.1	.5413411E+00	.6504828E+01	.6504767E+01	.6504767E+01	33	33
	.1	.4432126E+00	.6780675E+01	.6780731E+01	.6780731E+01	33	33
	.5	.4000000E+01	.1341851E-02	.1336826E-02	.1336826E-02	33	33
	.5	.3274923E+01	.1535574E-01	.1536201E-01	.1536201E-01	32	32
	.5	.2681280E+01	.7365691E-01	.7365005E-01	.7365005E-01	31	31
	.5	.2195247E+01	.1972547E+00	.1972613E+00	.1972613E+00	33	33
	.5	.1797316E+01	.3574078E+00	.3574018E+00	.3574018E+00	33	33
	.5	.1471518E+01	.4983823E+00	.4983877E+00	.4983877E+00	33	33
	.5	.1204777E+01	.5830693E+00	.5830641E+00	.5830641E+00	33	33
	.5	.9863879E+00	.6064178E+00	.6064227E+00	.6064227E+00	33	33
	.5	.8075861E+00	.5828629E+00	.5828581E+00	.5828581E+00	35	35
	.5	.6611956E+00	.5313707E+00	.5313752E+00	.5313752E+00	35	35
	.5	.5413411E+00	.4675585E+00	.4675540E+00	.4675540E+00	35	35
	.5	.4432126E+00	.4017504E+00	.4017547E+00	.4017547E+00	41	41
	1.0	.4000000E+01	.1831564E-01	.1831404E-01	.1831404E-01	32	32
	1.0	.3274923E+01	.5606295E-01	.5606491E-01	.5606491E-01	33	33
	1.0	.2681280E+01	.1111010E+00	.1110989E+00	.1110989E+00	33	33
	1.0	.2195247E+01	.1645112E+00	.1645131E+00	.1645131E+00	33	33
	1.0	.1797316E+01	.2003707E+00	.2003688E+00	.2003688E+00	33	33
	1.0	.1471518E+01	.2140938E+00	.2140956E+00	.2140956E+00	33	33
	1.0	.1204777E+01	.2095335E+00	.2095316E+00	.2095316E+00	35	35
	1.0	.9863879E+00	.1933525E+00	.1933543E+00	.1933543E+00	35	35
	1.0	.8075861E+00	.1715211E+00	.1715194E+00	.1715194E+00	35	35

1.0	.6611956E+00	.1481848E+00	.1481865E+00	.1481865E+00	39	39
1.0	.5413411E+00	.1257748E+00	.1257731E+00	.1257731E+00	43	43
1.0	.4432126E+00	.1054931E+00	.1054947E+00	.1054947E+00	47	47
5.0	.4000000E+01	.1797316E-01	.1797298E-01	.1797298E-01	33	33
5.0	.3274923E+01	.1915614E-01	.1915631E-01	.1915631E-01	33	33
5.0	.2681280E+01	.1871670E-01	.1871653E-01	.1871653E-01	35	35
5.0	.2195247E+01	.1725193E-01	.1725209E-01	.1725209E-01	35	35
5.0	.1797316E+01	.1529250E-01	.1529234E-01	.1529234E-01	35	35
5.0	.1471518E+01	.1320521E-01	.1320536E-01	.1320536E-01	39	39
5.0	.1204777E+01	.1120438E-01	.1120423E-01	.1120423E-01	43	43
5.0	.9863879E+00	.9395505E-02	.9395649E-02	.9395649E-02	47	47
5.0	.8075861E+00	.7816756E-02	.7816616E-02	.7816616E-02	53	53
5.0	.6611956E+00	.6468993E-02	.6469129E-02	.6469129E-02	62	62
5.0	.5413411E+00	.5334670E-02	.5334538E-02	.5334538E-02	75	75
5.0	.4432126E+00	.4388808E-02	.4388936E-02	.4388936E-02	99	99
10.0	.4000000E+01	.6703200E-02	.6703173E-02	.6703173E-02	35	35
10.0	.3274923E+01	.6261734E-02	.6261760E-02	.6261760E-02	35	35
10.0	.2681280E+01	.5600487E-02	.5600459E-02	.5600459E-02	35	35
10.0	.2195247E+01	.4865198E-02	.4865224E-02	.4865224E-02	39	39
10.0	.1797316E+01	.4144684E-02	.4144658E-02	.4144658E-02	43	43
10.0	.1471518E+01	.3484941E-02	.3484965E-02	.3484965E-02	45	45
10.0	.1204777E+01	.2904606E-02	.2904582E-02	.2904582E-02	51	51
10.0	.9863879E+00	.2406711E-02	.2406734E-02	.2406734E-02	59	59
10.0	.8075861E+00	.1986313E-02	.1986290E-02	.1986290E-02	71	71
10.0	.6611956E+00	.1635021E-02	.1635043E-02	.1635043E-02	94	94
10.0	.5413411E+00	.1343474E-02	.1343452E-02	.1343452E-02	99	99
10.0	.4432126E+00	.1102603E-02	.1102624E-02	.1102624E-02	99	99
50.0	.4000000E+01	.3692465E-03	.3692439E-03	.3692439E-03	43	43
50.0	.3274923E+01	.3103929E-03	.3103954E-03	.3103954E-03	46	46
50.0	.2681280E+01	.2586610E-03	.2586585E-03	.2586585E-03	51	51
50.0	.2195247E+01	.2142983E-03	.2143007E-03	.2143007E-03	59	59
50.0	.1797316E+01	.1768519E-03	.1768496E-03	.1768496E-03	72	72
50.0	.1471518E+01	.1455672E-03	.1455694E-03	.1455694E-03	95	95
50.0	.1204777E+01	.1196065E-03	.1196043E-03	.1196043E-03	99	99
50.0	.9863879E+00	.9816009E-04	.9816222E-04	.9816222E-04	99	99
50.0	.8075861E+00	.8049568E-04	.8049362E-04	.8049362E-04	99	99
50.0	.6611956E+00	.6597518E-04	.6597718E-04	.6597718E-04	99	99
50.0	.5413411E+00	.5405485E-04	.5405291E-04	.5405291E-04	99	99
50.0	.4432126E+00	.4427775E-04	.4427964E-04	.4427964E-04	99	99