Reflection and Refraction of P and Type-I S Waves at Plane Interfaces in Elastic and Anelastic Media

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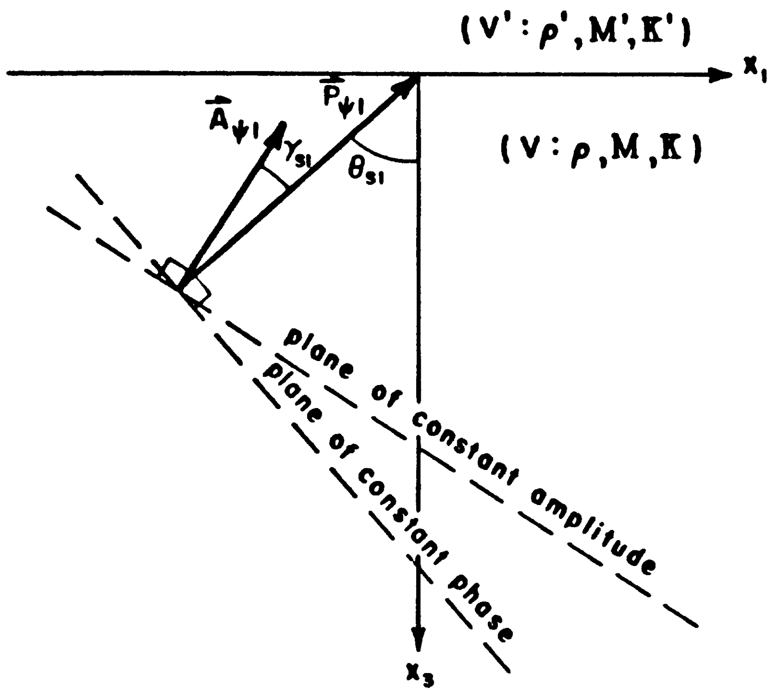
Roger D. Borcherdt



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SUMMARY

The classic problems concerning the reflection and refraction of plane P and SV body waves at plane boundaries are considered using the general theory of linear viscoelasticity, which accounts for the behavior of both elastic and linear anelastic media. Laws of reflection and refraction are derived for general (homogeneous or inhomogeneous) P and type-I S waves (SV waves) incident on a free surface and a plane welded boundary.

For anelastic media, the general theory predicts that plane P and SV waves reflected and refracted at plane boundaries, in general,

a) are inhomogeneous,

b) have elliptical particle motions,

c) have velocities and maximum attenuations that depend on the angle of incidence and frequency,

d) transport energy in a different direction and at a different velocity than that of phase propagation,

d) propagate parallel to the boundary for at most one angle of incidence.

The general theory predicts each of these characteristics for the waves whenever a plane P or SV wave interacts with a plane boundary between materials with different intrinsic attenuations; such as a bedrock-soil, mantle-crust, or core-mantle interface. None of the above physical characteristics of the waves are predicted by elasticity theory.

INTRODUCTION

The theory of general linear viscoelasticity accounts for elastic as well as linear anelastic behavior of materials. The phenomenological theory, based on Boltzmann's superposition principle, accounts for energy absorption by the material due to anelasticity and includes as special cases any model with a linear constitutive relation, such as the one proposed by Lomnitz (1957), or the infinite number of models derivable from various configurations of springs and dashpots, such as elastic, Voigt, Maxwell, standard linear, and generalized Voigt.

As recently as 1960 Hunter concluded that application of the general theory to other than one-dimensional wave propagation problems was incomplete. Lockett (1962) applied the general theory to the two-dimensional problem of a homogeneous plane wave incident on a welded boundary. He concluded that a new type of wave was required to satisfy the boundary conditions and that the results exhibited several features not obtained in the elastic solution. The type of wave that Lockett found necessary to consider is an inhomogeneous plane wave (that is, a plane wave whose planes of constant amplitude are not parallel to planes of constant phase; Brekhovskikh (1960)). Cooper and Reiss (1966), Cooper (1967), Shaw and Bugl (1969), and Schoenberg (1971) also have considered the problem treated by Lockett. Their results confirm those of Lockett's and show that in general, a homogeneous plane P or SV wave upon interacting with a plane boundary produces inhomogeneous plane waves. Their results show that to consider wave propagation in anelastic media with more than one boundary one must consider the more general problem of an inhomogeneous wave incident on a plane boundary. This problem is treated in detail in this paper. Characteristics are derived for the reflected and refracted waves resulting from a general SV wave (either homogeneous or inhomogeneous) incident on a welded boundary and on a free surface. Solutions for the problem of a P wave incident on a welded boundary are also presented. Results of the previous investigators are derived as a special case.

Detailed treatments of the physical characteristics and energy associated with general plane waves in linear-viscoelastic media have been developed by Borcherdt (1971, 1973) and Buchen (1971). The treatment presented by Borcherdt (1973) provides the mathematical framework for this study and equations given in this earlier paper, for example, 21, are referred to here as I-21. These equations show that the physical characteristics of plane body waves in anelastic media are distinctly different from those of plane body waves in elastic media.

Previously, anelasticity has been incorporated into studies of the internal structure of the Earth by introducing an attenuation factor of the form e^{-ax} where x is distance measured in the direction of propagation and a is the absorption coefficient. Introduction of such a factor assumes that the waves are homogeneous and that except for attenuation the physical characteristics of the wave are analogous to those predicted by elasticity. Theoretical work (Lockett 1962; Borcherdt 1971, 1972) shows that the boundary conditions at an interface between anelastic materials, such as those in the Earth, cannot be satisfied by considering only homogeneous waves. Theoretical implications of this result are presented in this paper; numerical implications for the anelastic interal structure of the Earth remain to be evaluated.

GENERAL PLANE-WAVE SOLUTIONS

Let V and V' represent two infinite homogeneous isotropic linear viscoelastic (HILV) media with a common plane boundary in welded contact. For reference let (x_1, x_2, x_3) denote a set of orthogonal coordinate axes

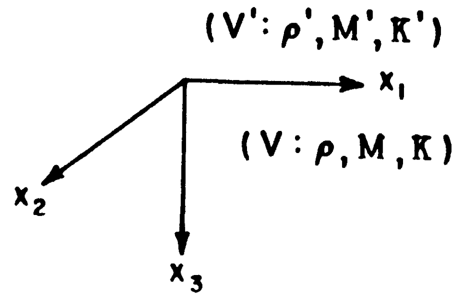

Fig. 1. Cartesian coordinate system and plane interface ($x_1 x_2$ plane) between two general viscoelastic media.

chosen such that the space occupied by medium V is described by $x_3 \geq 0$ (see Figure 1). Parameters used to characterize medium V are $\rho \equiv$ mass density, $\mu \equiv$ complex shear modulus (I-10), and $\kappa \equiv$ complex bulk modulus (I-11). Primes are used to denote the parameters of medium V'. Algebraically, parameters more convenient for characterizing the material are the complex wave numbers corresponding to homogeneous P and S waves (I-25), given by

$$k_P \equiv \omega/\alpha \equiv \omega/p.v.[(\kappa + \frac{4}{3}\mu)/\rho]^{1/2} \tag{1a}$$

and

$$k_S \equiv \omega/\beta \equiv \omega/p.v.(\mu/\rho)^{1/2}, \tag{1b}$$

where $p.v.(z)^{1/2}$ denotes the principal value of the square root of the complex number z (Kreysig 1967, p. 535 and 550). Additional relations useful for later reference between the complex wave number and the corresponding velocity and Q^{-1} for homogeneous P and S waves are

$$k_P{}^2 = \frac{\omega^2}{v_{HP}{}^2}\ 2(1 - iQ_P{}^{-1})(1 + \sqrt{1 + Q_P{}^{-2}})^{-1} \tag{2a}$$

and

$$k_S{}^2 = \frac{\omega^2}{v_{HS}{}^2}\ 2(1 - iQ_S{}^{-1})(1 + \sqrt{1 + Q_S{}^{-2}})^{-1} \tag{2b}$$

where v_{HP}, v_{HS} and $2\pi Q_P{}^{-1}$, $2\pi Q_S{}^{-1}$ are the phase velocities (I-30a, I-32a) and fractional losses in energy density (I-72), respectively, for homogeneous P and S waves.

Harmonic motions of the media are governed by the equation of motion (I-14). Solutions of the Helmholtz equations (I-24), which satisfy the divergenceless gage (I-23b), provide solutions for the equation of motion (I-14).

To consider only two-dimensional reflection-refraction problems, we shall restrict the plane-wave solutions for the displacement potentials to those that have propagation and attenuation vectors in a plane perpendicular to the plane of contact. In addition, only those solutions representing plane waves propagating in the positive x_1 direction are considered with the incident waves assumed to originate in V.

For medium V, a set of such plane-wave solutions for the displacement potentials is given by the following:

$$\phi = \phi_1 + \phi_2 = \sum_{j=1}^{2} B_j \exp[-\vec{A}_{\phi j}\cdot\vec{r}] \exp[i(\omega t - \vec{P}_{\phi j}\cdot\vec{r})] \tag{3a}$$

and

$$\vec{\psi} = \vec{\psi}_1 + \vec{\psi}_2 = \sum_{j=1}^{2} \vec{C}_j \exp[-\vec{A}_{\psi j}\cdot\vec{r}] \exp[i(\omega t - \vec{P}_{\psi j}\cdot\vec{r})], \tag{3b}$$

where the propagation vectors are defined by

$$\vec{P}_{\phi j} \equiv k_R \hat{x}_1 + (-1)^j d_{\alpha R} \hat{x}_3 \tag{3c}$$

$$\vec{P}_{\psi j} \equiv k_R \hat{x}_1 + (-1)^j d_{\beta R} \hat{x}_3 \quad (j = 1,2) \tag{3d}$$

with definitions

$$d_\alpha \equiv p.v.(k_P^2 - k^2)^{1/2} \tag{3e}$$

$$d_\beta \equiv p.v.(k_S^2 - k^2)^{1/2} \tag{3f}$$

and the attenuation vectors are defined by

$$\vec{A}_{\phi j} \equiv - k_I \hat{x}_1 + (-1)^{j+1} d_{\alpha I} \hat{x}_3 \tag{3g}$$

$$\vec{A}_{\psi j} \equiv - k_I \hat{x} + (-1)^{j+1} d_{\beta I} \hat{x}_3 \quad (j = 1,2), \tag{3h}$$

where $\vec{C}_1$ and $\vec{C}_2$ are arbitrary complex vectors chosen such that $\nabla \cdot \vec{\psi} = 0$ and k is an arbitrary complex number chosen such that $k_R \geq 0$ to ensure propagation in the plus x_1 direction. (The subscripts R and I denote the real and imaginary parts of the corresponding complex quantities; the signs of d_α and d_β are determined by requiring that the mean energy fluxes associated with the incident and transmitted waves are in the negative x_3 direction.)

A set of solutions in medium V', corresponding to plane waves propagating away from the boundary with circular frequency ω, is specified by attaching primes to each of the wave parameters in 3a through 3h and setting $B_2' = \vec{C}_2' = 0$.

The most general solutions of the Helmholtz equations, I-24a and I-24b, are ones for which separate complex wave numbers k are taken for each of the scalar components of $\vec{\psi}_j (j = 1,2)$. However, the condition $\nabla \cdot \vec{\psi} = 0$ immediately implies that those for each of the components of ψ_j must be equal. In addition, the boundary conditions for the problems under consideration will imply that the resulting k for ψ_j and that for ϕ_j must be equal (Borcherdt 1971, p. 52). Hence, they are assumed equal in 3a and 3b without loss of generality. This important assumption leads directly to an extended version of Snell's law for the reflected and refracted waves to be discussed later.

The type of inhomogeneous S wave described by the general solution for $\vec{\psi}_j$ in equation 3b depends on the nature of the complex vector $\vec{C}_j$. If $\vec{C}_j$ is of the form $Z\hat{n}$ where Z is a complex number and $\hat{n}$ is a real vector, then $\nabla \cdot \vec{\psi} = 0$ implies $\hat{n}$ is perpendicular to a plane containing the propagation and attenuation vectors, and that the particle motion is elliptically polarized in this plane. The characteristics of this type of S wave are derived in detail (Borcherdt, 1973a). This type will be referred to as type I and upon introduction of a boundary perpendicular to the plane containing the propagation and attenuation vectors it can be called an SV wave, as an extension of the definition given for elastic media.

If $\vec{C}_j$ is a general complex vector of the form $c_{j_1}\hat{x}_1 + c_{j_2}\hat{x}_2 + c_{j_3}\hat{x}_3$, where c_{j_1}, c_{j_2}, and c_{j_3} are arbitrary complex numbers chosen such that $\nabla\cdot\vec{\psi} = 0$, then the particle motion is not necessarily in the plane defined by the propagation and attenuation vectors. A special case of interest is that of $c_{j_2} = 0$ with the propagation and attenuation vectors in the x_1x_3 plane. The displacement field corresponding to such a wave is given by

$$\vec{u}_{Rj} = |D_j| \exp[-\vec{A}_{\psi j}\cdot\vec{r}] \cos(\omega t - \vec{P}_{\psi j}\cdot\vec{r} + \arg[D])\hat{x}_2 \qquad (4a)$$

where

$$D_j = (c_{j_3}\hat{x}_1 - c_{j_1}\hat{x}_3)\cdot(\vec{A}_{\psi j} + i\vec{P}_{\psi j}) \qquad (4b)$$

and $\nabla\cdot\vec{\psi} = 0$ implies

$$c_{j_1}(\vec{A}_{\psi j} + i\vec{P}_{\psi j})\cdot\hat{x}_1 = -c_{j_3}(\vec{A}_{\psi j} + i\vec{P}_{\psi j})\cdot\hat{x}_3 \quad . \qquad (5)$$

Hence, the particle motion for this type of inhomogeneous plane S wave is not elliptical as in the case of type I, but linear and perpendicular to the plane containing the propagation and attenuation vectors. This type of inhomogeneous S wave will be referred to as type II and upon introduction of an appropriate boundary it can be called an SH wave. The physical properties and the reflection and refraction of SH waves are considered in a concurrent paper (Borcherdt, in press).

The solutions 3a and 3b represent <u>general</u> (that is, either homogeneous or inhomogeneous) P and S waves, respectively. The physical characteristics of these waves specified by 3 for anelastic media are in strong contrast to those considered in the classical problems with only elastic media (Borcherdt 1973a). Each of the waves has a velocity and maximum attenuation dependent on the angle between the propagation and attenuation vectors which, for angles near 90°, approach zero and infinity, respectively (I-30, I-31). The velocity and maximum attenuation of an inhomogeneous wave are less than and greater than those, respectively, for homogeneous waves (I-33, I-34), and upon introduction of a particular viscoelastic model they assume a corresponding dependence on frequency. For inhomogeneous waves in anelastic media, the particle motions for P and type-I S waves are elliptical (I-37, I-42), the directions and velocities of maximum energy flow are not the same as those for phase propagation (I-51, I-53), the mean kinetic density is not equal to the mean potential energy density (I-58, I-59), and the mean rate of energy dissipation depends on the component of the mean intensity in the direction of the attenuation vector.

BOUNDARY CONDITIONS

A welded contact between media V and V' is ensured by requiring that the stress and displacement across the boundary are continuous. These restrictions imply that the following relations between the parameters of the solutions for V and V' must be satisfied at $x_3 = 0$;

$$u_1 = u_1' \quad , \quad k(B_1 + B_2) + d_\beta(c_{12} - c_{22}) = kB_1' + d_\beta' c_{12}' \quad , \tag{6a}$$

$$u_2 = u_2' \quad , \quad d_\beta(c_{11} - c_{21}) + k(c_{13} - c_{23}) = d_\beta' c_{11}' + kc_{13}' \quad , \tag{6b}$$

$$u_3 = u_3' \quad , \quad d_\alpha(B_1 - B_2) - k(c_{12} + c_{22}) = d_\alpha' B_1' - kc_{12}' \quad , \tag{6c}$$

$$p_{31} = p_{31}', \ \mu[2kd_\alpha(B_1 - B_2) + (d_\beta^2 - k^2)(c_{12} + c_{22})] =$$

$$\mu'[2kd_\alpha' B_1' + (d_\beta'^2 - k^2)c_{12}'] \quad , \tag{7a}$$

$$p_{32} = p'_{32} \ , \ \mu[d_\beta(d_\beta c_{11} + kc_{13}) + d_\beta(d_\beta c_{21} - kc_{23})] =$$

$$\mu' d'_\beta \, [d'_\beta c'_{11} + kc'_{13}] \ , \tag{7b}$$

$$p_{33} = p'_{33} \ , \ \mu[-(d_\beta{}^2 - k^2)(B_1 + B_2) + 2d_\beta k(c_{12} - c_{22}) =$$

$$\mu'[-(d'^2_\beta - k^2)B'_1 + 2d'_\beta kc'_{12}] \quad . \tag{7c}$$

where c_{ji} denotes the i component of the complex vector $\vec{C}_j$. Consideration of a single incident wave (P, SV, or SH) allows one to solve these equations for the complex amplitudes of the transmitted and reflected waves in terms of those for the assumed incident wave.

GENERAL SV WAVE INCIDENT ON A WELDED BOUNDARY

The physical problem to be considered in this section is that of a general type-I S wave incident on the welded boundary at an arbitrary angle and polarized in a plane perpendicular to the boundary. The problem is specified by setting

$$B_1 = B'_2 = \vec{C}'_2 = c_{ij} = c'_{ij} = 0 \ \text{ for } i = 1,2 \text{ and } j = 1,3 \tag{8}$$

in the general solutions 3a and 3b and in the boundary conditions 6 and 7. The resulting incident plane wave is given by

$$\vec{\psi}_1 = c_{12} \, \hat{x}_2 \ \exp\,[-\vec{A}_{\psi_1}\cdot\vec{r}] \ \exp[i(\omega t - \vec{P}_{\psi_1}\cdot\vec{r})] \ . \tag{9}$$

The particle motion of the incident wave is in general elliptical with no x_2 component (I-43). As an extension of the definition given for elastic media, the incident wave is referred to as an SV wave.

For ease of physical interpretation, the incident general SV wave also is described in terms of its angle of incidence θ_{S_1} and the angle between its attenuation and propagation vectors γ_{S_1} (see Figure 2). In terms of these angles k may be written as

$$k = |\vec{P}_{\psi_1}| \sin \theta_{S_1} - i|\vec{A}_{\psi_1}| \sin(\theta_{S_1} - \gamma_{S_1}) \qquad (10a)$$

which for the special case of an incident homogeneous SV wave simplifies to

$$k = (|\vec{P}_{\psi_1}| - i|\vec{A}_{\psi_1}|) \sin \theta_{S_1} = k_S \sin \theta_{S_1} . \qquad (10b)$$

Expression 9, together with 3c-3h and 10a, provides a complete specification of the incident general SV wave in terms of a given angle of incidence, θ_{S_1}, a given angle between its attenuation and propagation vectors, γ_{S_1}, and a given complex amplitude c_{12}. The range of values for γ_{S_1} is

$0 \leq |\gamma_{S_1}| < \pi/2$.

For later reference the propagation and attenuation vectors for the incident wave are written here in terms of the given angle γ_{S_1} and the parameters of the material (I-31),

$$|\vec{P}_{\psi_1}| = \sqrt{\frac{1}{2}\left(\mathrm{Re}[k_S{}^2] + \left(\mathrm{Re}[k_S{}^2]^2 + \frac{\mathrm{Im}[k_S{}^2]^2}{\cos^2\gamma_{S_1}}\right)^{1/2}\right)} \qquad (11a)$$

$$|\vec{A}_{\psi_1}| = \sqrt{\frac{1}{2}\left(-\mathrm{Re}[k_S{}^2] + \left(\mathrm{Re}[k_S{}^2]^2 + \frac{\mathrm{Im}[k_S{}^2]^2}{\cos^2\gamma_{S_1}}\right)^{1/2}\right)} \qquad (11b)$$

For the case of a general SV wave incident on a welded boundary, substitution of 8 into the boundary conditions 6 and 7 gives four equations involving the four unknown complex amplitudes of the reflected and transmitted waves. The solutions for these complex amplitudes in terms of those specified for the incident wave are easily derived using the correspondence principle (Bland 1960, p. 67). Using these solutions and relations 10 and 11, the parameters of the reflected and transmitted waves may be expressed in terms of those

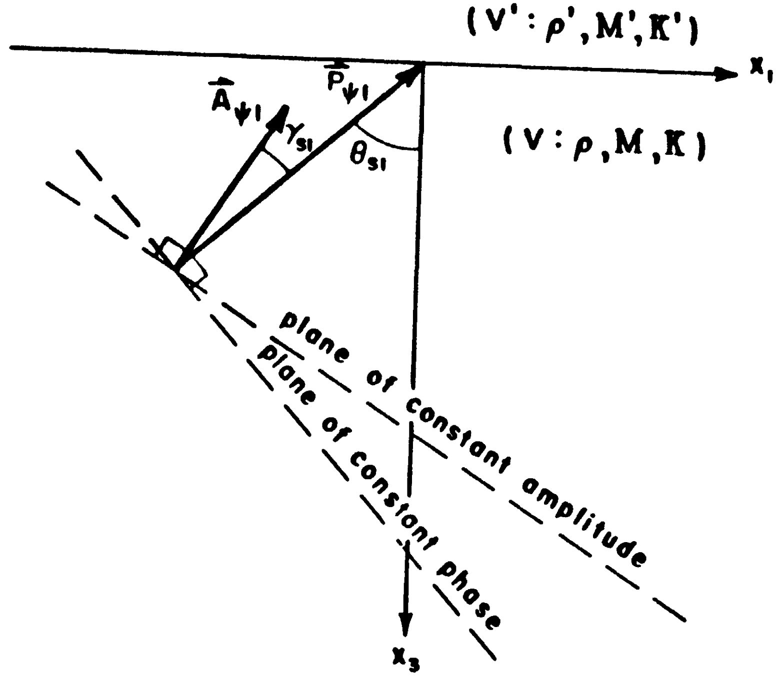

Fig. 2. Propagation and attenuation vectors for a general SV wave incident on a welded boundary between viscoelastic media. For $\gamma_{S_1} = 0$, the incident wave is homogeneous and planes of constant phase are parallel to planes of constant amplitude.

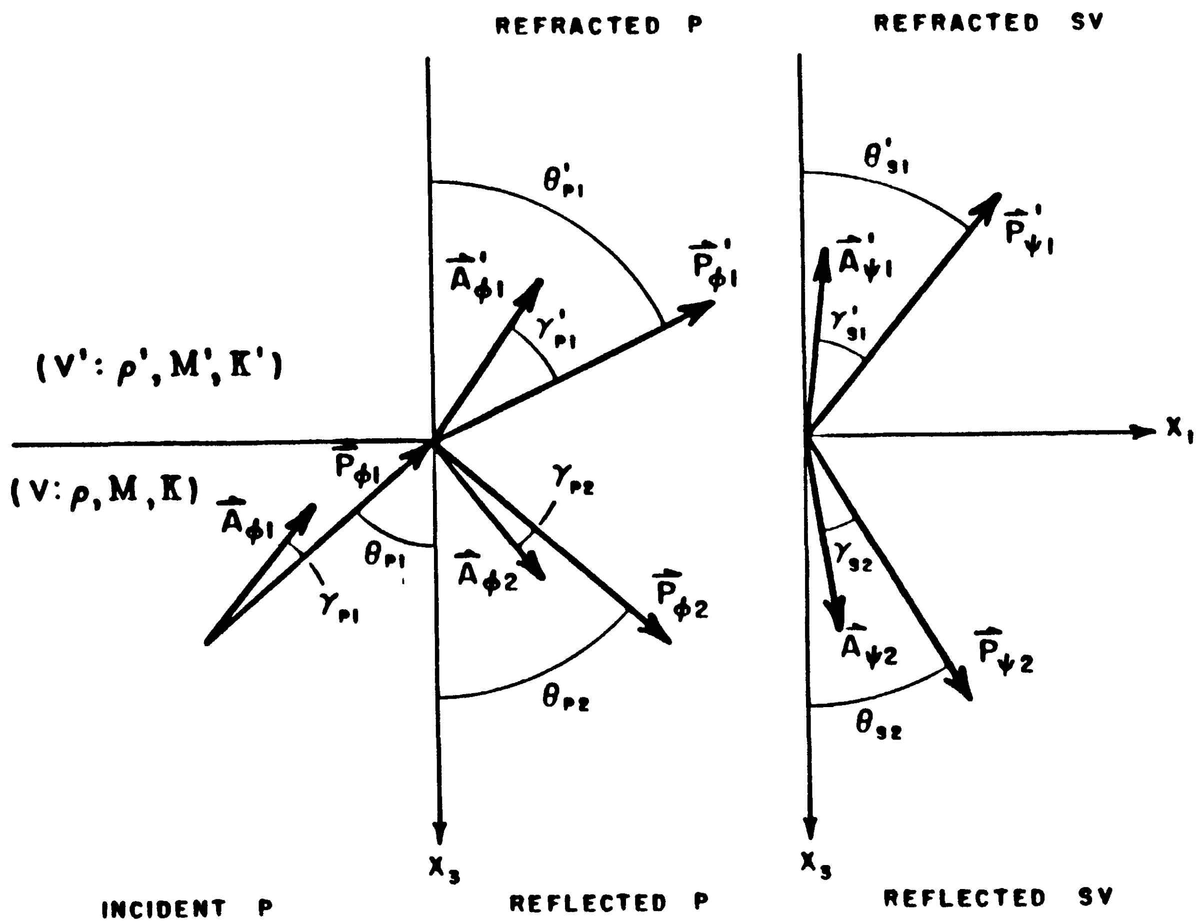

Fig. 3. Definitions of angles which define the directions of the propagation and attenuation vectors for the incident, reflected, and refracted waves for a general type-I S wave (SV wave) incident on a welded boundary between general viscoelastic media.

for the assumed incident wave. Hence, the boundary conditions can be satisfied with the general solutions 3a and 3b for the problem specified by 8. Therefore, from a mathematical point of view, the general solution of the problem is determined and the physical characteristics of the reflected and transmitted waves can be expressed in terms of those for the incident waves using the expressions derived by Borcherdt (1973a).

Notation useful for deriving the laws of reflection and refraction in anelastic media is given in Figure 3. For example, with this notation the propagation and attenuation vectors for the transmitted P wave are given by

$$\vec{P}'_{\phi_1} = k_R \hat{x}_1 - d'_{\alpha R}\, \hat{x}_3 = |\vec{P}'_{\phi_1}|(\sin\theta'_{P_1} \hat{x}_1 + \cos\theta'_{P_1} \hat{x}_3) \qquad (12a)$$

and

$$\vec{A}'_{\phi_1} = -k_I \hat{x}_1 + d'_{\alpha I} \hat{x}_3 = |\vec{A}'_{\phi_1}|[\sin(\theta'_{P_1} - \gamma'_{P_1})\, \hat{x}_1 + \cos(\phi'_{P_1} - \gamma'_{P_1}) \hat{x}_3]. \qquad (12b)$$

The form of the general solutions given by 3a-3h yields an extension of Snell's law, namely

$$k_R = |\vec{P}_{\psi_1}| \sin\theta_{S_1} = |\vec{P}_{\psi_2}| \sin\theta_{S_2} = |\vec{P}_{\phi_2}| \sin\theta_{P_2} = |\vec{P}'_{\psi_1}| \sin\theta'_{S_1} = |\vec{P}'_{\phi_1}| \sin\theta'_{P_1} \qquad (13a)$$

and

$$-k_I = |\vec{A}_{\psi_1}| \sin(\theta_{S_1} - \gamma_{S_1}) = |\vec{A}_{\psi_2}| \sin(\theta_{S_2} - \gamma_{S_2}) = |\vec{A}_{\phi_2}| \sin(\theta_{P_2} - \gamma_{P_2}) =$$

$$|\vec{A}'_{\psi_1}| \sin(\theta'_{S_1} - \gamma'_{S_1}) = |\vec{A}'_{\phi_1}| \sin(\theta'_{P_1} - \gamma'_{P_1}) \quad . \qquad (13b)$$

Introducing the phase velocities of the various waves given by I-30a, 13a may be rewritten as

$$\frac{|\vec{v}_{S_1}|}{\sin\theta_{S_1}} = \frac{\omega}{|\vec{P}_{\psi_1}|\sin\theta_{S_1}} = \frac{|\vec{v}_{S_2}|}{\sin\theta_{S_2}} = \frac{|\vec{v}_{P_2}|}{\sin\theta_{P_2}} = \frac{|\vec{v}'_{S_1}|}{\sin\theta'_{S_1}} = \frac{|\vec{v}'_{P_1}|}{\sin\theta'_{P_1}} \qquad (14)$$

Relations 14 and 13b show, respectively, that the apparent velocities and apparent attenuations along the boundary of the reflected and transmitted waves are equal to that of the incident wave. Expression 14 for the velocities is similar in appearance to expressions that have been derived for elastic media (equation 3-9, Ewing, et al. 1957). However, in the case of anelastic media, the physical meaning of 14 is different. It will be shown that the phase velocities of the transmitted waves and the reflected P wave expressed in 14 are in general phase velocities of inhomogeneous waves and they depend on the angle of incidence. In addition to phase velocity other physical characteristics of inhomogeneous waves are also significantly different from those of homogeneous waves; hence, it is of considerable interest to examine under what conditions the reflected and transmitted waves are inhomogeneous.

For the reflected SV wave equations 3d and 3h show that $|\vec{P}_{\psi_1}| = |\vec{P}_{\psi_2}|$ and $|\vec{A}_{\psi_1}| = |\vec{A}_{\psi_2}|$, hence 13a shows that the angle θ_{S_2} at which the general SV is reflected, equals the angle of incidence θ_{S_1} and 13b shows $\gamma_{S_1} = \gamma_{S_2}$. Hence, the reflected SV wave is homogeneous if and only if the incident SV wave is homogeneous. The phase velocity of the reflected SV wave is independent of the angle of incidence. It equals that of the incident SV wave and varies between 0 and ω/k_{SR} depending on the given angle γ_{S_1}.

For anelastic media it is clear that if the incident SV wave is inhomogeneous then in general the reflected and transmitted waves are also inhomogeneous. A basic result for the case in which the incident SV wave is homogeneous is

Theorem 1. If the incident SV wave is homogeneous and not normally incident, then

1) the reflected P wave is homogeneous if and only if

$$Q_S^{-1} = Q_P^{-1} \text{ and } \sin^2\theta_{S_1} \leq k_P^2/k_S^2 = \mu_R/(\kappa_R + \tfrac{4}{3}\mu_R) = v_{HS}^2/v_{HP}^2 \quad , \qquad (15a)$$

2) the transmitted SV wave is homogeneous if and only if

$$Q_S^{-1} = Q_S'^{-1} \text{ and } \sin^2\theta_{S_1} \leq k_S'^2/k_S^2 = \rho'\mu_R/\rho\mu_R' = v_{HS}^2/v_{HS}'^2 \quad , \qquad (15b)$$

3) The transmitted P wave is homogeneous if and only if

$$Q_S^{-1} = Q_P'^{-1} \text{ and } \sin^2\theta_{S_1} \leq k_P'^2/k_S^2 = \rho'\mu_R/[\rho(\kappa_R' + \tfrac{4}{3}\mu_R')] = v_{HS}^2/v_{HP}'^2 \quad . \qquad (15c)$$

Theorem 1 is proved in appendix 1. (The restriction involving the angle of incidence θ_{S_1} in 15b is automatically satisfied if media V and V' are such that $Q_S^{-1} = Q_S'^{-1}$ and $v_{HS}' \leq v_{HS}$. Similarly, the restriction on θ_{S_1} , in 15c is satisfied if $Q_S^{-1} = Q_P^{-1}$ and $v_{HP}' \leq v_{HS}$.) If the incident SV wave is both normally incident and homogeneous, then 13 shows that the reflected and transmitted waves are also homogeneous with propagation and attenuation vectors perpendicular to the boundary.

For materials in the Earth $Q_S^{-1} \neq Q_P^{-1}$ and for most seismic boundaries $Q_S^{-1} \neq Q_S'^{-1}$ and $Q_S^{-1} \neq Q_P'^{-1}$. Hence, the contrapositive of theorem 1 shows that in general the reflected P wave and both transmitted waves generated at boundaries within the Earth are inhomogeneous for all non-normal angles of incidence. This important result, first derived in part by Lockett (1962) shows that the physical characteristics of reflected and refracted waves in anelastic media are significantly different from those of corresponding waves in elastic media. For example, the reflected P wave and both transmitted waves each have in general a velocity less than and maximum attenuation greater than that of a corresponding homogeneous wave, an elliptical particle motion, and a direction of maximum energy flow different from that of phase propagation (see the second section and Borcherdt (1973a) for other distinctive properties). In addition, the theorem shows that to consider plane wave propagation problems in multilayered anelastic media, one must consider the problem of inhomogeneous waves incident on a plane boundary. Therefore, the theorem justifies the generality assumed in 3 for the incident waves.

The velocities of the reflected P wave and the transmitted waves are given by

$$|\vec{v}_{P_2}| = \omega/|\vec{P}_{\phi_2}| = \omega/(k_R{}^2 + d_{\alpha R}{}^2)^{1/2} \quad , \tag{16a}$$

$$|\vec{v}'_{P_1}| = \omega/|\vec{P}'_{\phi_1}| = \omega/(k_R{}^2 + d'_{\alpha R}{}^2)^{1/2} \quad , \tag{16b}$$

$$|\vec{v}'_{S_1}| = \omega/|\vec{P}'_{\psi_1}| = \omega/(k_R{}^2 + d'_{\beta R}{}^2)^{1/2} \quad . \tag{16c}$$

Substituting 10a into these expressions shows that for anelastic media the velocities of the reflected P and the transmitted waves depend on the angle of incidence, in contrast to the situation for elastic media. As a result, 14 shows that the ratios $\sin\theta_{P_2}/\sin\theta_{S_1}$, $\sin\theta'_{P_1}/\sin\theta_{S_1}$ and $\sin\theta'_{S_1}/\sin\theta_{S_1}$, which are often referred to as reflection and refraction indices, in general, are not constant as in the elastic case but also depend on the angle of incidence.

The refraction angle for the transmitted SV wave is given by

$$\tan\theta'_{S_1} = k_R/d'_{\beta R} = (\sqrt{2}k_R)/\sqrt{|k'^2_S - k^2| + \mathrm{Re}[k'^2_S - k^2]} \quad , \tag{17}$$

from which it is easily inferred that an SV wave composed of several different frequencies incident at a fixed angle will be refracted as a fan of SV waves if either medium V or V' is anelastic. If both media are elastic, then all of the transmitted SV waves will be refracted at the same angle. Similarly, an incident SV wave composed of several different frequencies will generate fans of reflected and transmitted P waves if either medium is anelastic.

Critical angles have played an important role in seismology based on elasticity theory. However, the conditions under which they exist for anelastic media have been somewhat obscure. Different results have been obtained by Lockett (1962), Cooper (1967), and Schoeberg (1971) for the case of incident homogeneous waves. The remainder of this section is devoted to establishing conditions in some detail under which critical angles exist for both homogeneous and inhomogeneous waves incident on the interface.

For explicitness, a critical angle for the reflected P wave is defined here as an angle of incidence for which the reflected P wave propagates parallel to the interface. Analogous definitions are used for the transmitted waves. To investigate the existence of critical angles it is useful to recall that d_β , d_α , d'_β and d'_α were defined as the principal values of the square roots of complex numbers (Kreysig, 1967, p. 550), for example, d_α may be written as

$$d_\alpha = \sqrt{(|d_\alpha|^2 + \mathrm{Re}[d_\alpha{}^2])/2} + i\ \underline{\mathrm{sign}}\ [\mathrm{Im}(d_\alpha{}^2)]\sqrt{(|d_\alpha|^2 - \mathrm{Re}[d_\alpha{}^2])/2} \quad (18a)$$

where

$$d_\alpha^2 = k_S^2[(k_P^2/k_S^2) - \sin^2\theta_{S_1}] \qquad (18b)$$

for an incident homogeneous SV wave.

The classic results of elasticity theory are given by theorem 1 with $Q_S^{-1} = Q_P^{-1} = Q_S'^{-1} = Q_P'^{-1} = 0$ and the definitions of d_α, d_β, d_β' and d_α'. Namely, for elastic media;

1) each angle of incidence satisfying

$$\sin\theta_{S_1} \geq v_{HS}/v_{HP} \qquad (19a)$$

is a critical angle for the reflected P wave,

2) if $v_{HP}' < v_{HS}$ then no critical angles exist for the transmitted waves,

3) if $v_{HS}' < v_{HS} < v_{HP}'$ then the angles satisfying

$$\sin\theta_{S_1} \geq v_{HS}/v_{HP}' \qquad (19b)$$

are critical angles for the transmitted P wave,

4) if $v_{HS} < v'_{HS}$ then the angles satisfying

$$\sin\theta_{S_1} \geq v_{HS}/v_{HP}' \qquad (19c)$$

are critical angles for the transmitted P wave and those satisfying

$$\sin\theta_{S_1} \geq v_{HS}/v_{HS}' \qquad (19d)$$

are critical angles for the transmitted SV wave.

Theorem 1 shows that for elastic media those waves corresponding to equality in 19a-d are homogeneous and those corresponding to strict inequality are inhomogeneous with propagation and attenuation vectors parallel and perpendicular, respectively, to the interface. The inhomogeneous waves propagate with a phase velocity equal to the apparent phase velocity along the interface of the incident wave. (Note, this situation shows that the phase velocity of an inhomogeneous wave in elastic media is not unique for a given value of γ_S as it is for anelastic media (see I-30 and I-31).)

The preceding results show that if elasticity is used as a model for the earth, then critical angles exist in a wide variety of situations. However, we shall see that for an anelastic earth critical angles, as defined above, in general do not exist.

If the incident medium is elastic and the refraction-medium is anelastic, then the situation is summarized by

Theorem 2. V elastic and V' anelastic implies

1) each angle of incidence satisfying

$$\sin \theta_{S_1} \geq v_{HS}/v_{HP}$$

is a critical angle for the reflected P wave,

2) no critical angles exist for either transmitted wave.

The first part of this theorem follows immediately from the definition of d_α and theorem 1. The second part follows from 13a and the fact that the propagation and attenuation vectors for the transmitted waves in an anelastic medium cannot be perpendicular (Borcherdt 1973a, p. 2445).

If the incident medium is anelastic, then the situation is described by theorem 3 and its corollaries.

Theorem 3. If V is anelastic and if $\theta_{S_1} (\theta_{S_1} \neq \pi/2)$ is a critical angle for:

1) the reflected P wave, then

$$\tan \gamma_{S_1} = (\sin^2\theta_{S_1} - \frac{k_{PR} k_{PI}}{k_{SR} k_{SI}})/(\sin \theta_{S_1} \cos \theta_{S_1}) \quad , \qquad (20a)$$

2) the transmitted SV wave, then

$$\tan \gamma_{S_1} = (\sin^2\theta_{S_1} - \frac{k'_{SR}k'_{SI}}{k_{SR}k_{SI}})/(\sin \theta_{S_1}\cos \theta_{S_1}), \qquad (20b)$$

3) the transmitted P wave, then

$$\tan \gamma_{S_1} = (\sin^2\theta_{S_1} - \frac{k'_{PR}k'_{PI}}{k_{SR}k_{SI}})/(\sin \theta_{S_1}\cos \theta_{S_1}) , \qquad (20c)$$

The proof of theorem 3 is given in appendix 2.

If we do not restrict the nature of the incident wave (that is, it may be either homogeneous or inhomogeneous), then we may deduce immediately the following results from theorem 3.

Corollary 3-1. If V' is elastic and V is anelastic, then for a given value of γ_{S_1} there exists at most one angle of incidence, namely $\theta_{S_1} = \gamma_{S_1}$ such that the transmitted waves are interface waves (that is, if $\theta_{S_1} \neq \gamma_{S_1}$, then θ_{S_1} is not a critical angle for either transmitted wave).

Corollary 3-2. If both media are anelastic then:

1) for a given value of γ_{S_1} there exists at most two critical angles for the reflected P wave and each of the transmitted waves,

2) if $\theta_{S_1} - \gamma_{S_1} < 0$, then no critical angles exist,

3) no critical angles θ_{S_1} ($\theta_{S_1} \neq \pi/2$) exist

a) for the reflected P wave if

$$\cos \gamma_{S_1} k_{PR}k_{PI}/(k_{SR}k_{SI}) > 1 ,$$

b) for the transmitted SV wave if

$$\cos \gamma_{S_1} k'_{SR}k'_{SI}/(k_{SR}k_{SI}) > 1 ,$$

c) for the transmitted P wave if

$$\cos \gamma_{S_1} k'_{PR}k'_{PI}/(k_{SR}k_{SI}) > 1 ,$$

4) at most, one critical angle given by

$$\tan \theta_{S_1} = -1/\tan \gamma_{S_1}$$

a) exists for the transmitted SV wave if the parameters of V and V' satisfy

$$\mathrm{Im}[k_S'^2] = \mathrm{Im}[k_S^2] \neq 0 \quad ,$$

b) exists for the transmitted P wave if the parameters of V and V' satisfy $\mathrm{Im}[k_P'^2] = \mathrm{Im}[k_S^2] \neq 0$.

If we consider only incident SV waves that are homogeneous (that is, $\gamma_{S_1} = 0$), then we have the following additional results from theorem 3.

Corollary 3-3. If V' is elastic, V is anelastic, and the incident wave is homogeneous, then no nonzero critical angles exist for either transmitted wave. A result of special interest for materials in the Earth is

Corollary 3-4. If both media are anelastic, the incident SV wave is homogeneous, and $\theta_{S_1} (\theta_{S_1} \neq \pi/2)$ is a critical angle for;

1) the reflected P wave, then $Q_S^{-1} \leq Q_P^{-1}$.
2) the transmitted SV wave, then $Q_S^{-1} \leq Q_S'^{-1}$, $v_{HS} \leq v'_{HS}$, and the critical angle is given by

$$\sin^2\theta_{S_1} = \mathrm{Im}[k_S'^2]/\mathrm{Im}[k_S^2] = v_{HS}^2 F(Q_S'^{-1})/[v_{HS}'^2 F(Q_S^{-1})] \quad , \tag{21a}$$

where F is defined by

$$F(Q_S^{-1}) \equiv Q_S^{-1}/[1 + (1 + Q_S^{-2})^{1/2}] \quad , \tag{21b}$$

3) the transmitted P wave, then $Q_S^{-1} \leq Q_P^{-1}$, $v_{HS} \leq v'_{HP}$, and the critical angle is given by

$$\sin^2\theta_{S_1} = \mathrm{Im}[k_P'^2]/\mathrm{Im}[k_S^2] = v_{HS}^2 F(Q_P'^{-1})/[v_{HP}'^2 F(Q_S^{-1})] \quad , \tag{21c}$$

where the function F is defined by 21b.

Corollary 3-4 is proved in appendix 3. A portion of this result was first derived by Lockett (1962).

For a SV wave incident at oblique angles on a boundary such as a soil-bedrock, crust-mantle, or mantle-core interface, theorem 1 shows the reflected P wave and both transmitted waves will be inhomogeneous and the contrapositives of corollary 3-4 show that these inhomogeneous waves don't propagate parallel to the boundary. These results are in contrast to those derived assuming the Earth is elastic.

In the special case, that the intrinsic attenuations on the two sides of the boundary are equal, we have the following result.

Corollary 3-5. If both media are anelastic and the incident SV wave is homogeneous, then:

1) if $Q_P^{-1} = Q_S^{-1}$ there is one and only one critical angle for the reflected P wave given by

$$\sin^2\theta_{S_1} = k_P^2/k_S^2 = \mu_R/(\kappa_R + \frac{4}{3}\mu_R) = v_{HS}^2/v_{HP}^2 \quad , \tag{22a}$$

2) if $Q_S'^{-1} = Q_S^{-1}$ and $v_{HS} \le v'_{HS}$, there is one and only one critical angle for the transmitted SV wave given by

$$\sin^2\theta_{S_1} = k_S'^2/k_S^2 = (\rho'\mu_R)/(\rho\mu'_R) = v_{HS}^2/v_{HS}'^2 \tag{22b}$$

3) if $Q_P'^{-1} = Q_S^{-1}$ and $v_{HS} \le v'_{HP}$, there is one and only one critical angle for the transmitted P wave given by

$$\sin^2\theta_{S_1} = k_P'^2/k_S^2 = (\rho'\mu_R)/[\rho(\kappa'_R + \frac{4}{3}\mu'_R)] = v_{HS}^2/v_{HP}'^2 \quad . \tag{22c}$$

The proof of corollary 3-5 is given in appendix 4. Corollary 3-5 and 10 show that elastic solids are the only viscoelastic solids for which equality of the intrinsic attenuation factors implies that the reflected P wave and the transmitted waves will propagate parallel to the boundary for more than one angle of incidence.

For purposes of mathematical completeness a partial converse of theorem 3 for the reflected P wave is

Theorem 4. If medium V is anelastic and (20a) is satisfied, then either θ_{S_1} is a critical angle or the direction of maximum attenuation for the reflected P wave is parallel to the boundary. The proof of theorem 4 is given by reversing the steps in the proof of the first part of theorem 3. An immediate corollary is

Corollary 4-1. For every angle of incidence $\theta_{S_1} (\theta_{S_1} \neq \pi/2)$ and for every anelastic solid there exists one and only one γ_{S_1} associated with an incident SV wave such that either θ_{S_1} is a critical angle for the reflected P wave or the direction of maximum attenuation for the reflected P wave is parallel to the boundary.

Similar results may be easily derived for the transmitted waves.

GENERAL SV WAVE INCIDENT ON A FREE SURFACE

If V' is taken to be a vacuum, the boundary of V becomes a free surface on which the stress vanishes. The problem of a general SV wave incident on the free surface is specified by 8 and by setting the amplitudes corresponding to waves in V' equal to zero. As before, the form of the general solutions 3a-3h implies a modified form of Snell's law, namely;

$$|\vec{v}_{S_1}/ \sin \theta_{S_1} = |\vec{v}_{S_2}|/ \sin \theta_{S_2} = |\vec{v}_{P_2}|/ \sin \theta_{P_2} \qquad (23a)$$

and

$$|\vec{A}_{\psi_1}| \sin (\theta_{S_1} - \gamma_{S_1}) = |\vec{A}_{\psi_2}| \sin (\theta_{S_2} - \gamma_{S_2}) = |\vec{A}_{\phi_2}| \sin (\theta_{P_2} - \gamma_{P_2}) \qquad (23b)$$

from which it follows that the velocity of the reflected P wave depends on the angle of incidence. Each of the properties of the waves reflected from a welded boundary derived in the preceding section for the reflected waves is also valid as stated for the waves reflected in this case from a free surface. These properties are not restated here (see Borcherdt (1971) for details).

The boundary conditions specified by 6 and 7 simplify significantly for the free surface problem and permit some additional conclusions. Equations 6 and 7 simplify to

$$\mu[-2k\,d_\alpha B_2 + (d_\beta^2 - k^2)(c_{12} + c_{22})] = 0 \tag{24a}$$

and

$$\mu[-(d_\beta^2 - k^2)B_2 + 2d_\beta k(c_{12} - c_{22})] = 0 \tag{24b}$$

and they readily admit solutions

$$B_2/c_{12} = 4d_\beta k(d_\beta^2 - k^2)/g(k) \tag{25a}$$

and

$$c_{22}/c_{12} = (4d_\alpha d_\beta k^2 - (d_\beta^2 - k^2)^2)/g(k) \tag{25b}$$

where

$$g(k) \equiv 4d_\alpha d_\beta k^2 + (d_\beta^2 - k^2)^2 \quad . \tag{25c}$$

For a non-trivial incident wave $c_{12} \neq 0$ and equations (24a) and (24b) show that $g(k) \neq 0$. Since the root of the equation $g(k) = 0$ is the complex wave number for a Rayleigh-type surface wave on a viscoelastic half-space (Borcherdt 1973b), this result shows that a general plane SV wave upon interacting with a free surface does not generate a Rayleigh-type surface wave.

The amplitude and phase of each reflected wave may be written down immediately in terms of those for the incident wave from expressions 25a and 25b.

In the case of an elastic medium, a normally incident SV wave does not generate a dilatational disturbance upon interacting with the free surface. However, in the case of a vertically incident inhomogeneous wave in anelastic media, a dilatational disturbance is reflected from the free surface. To see this result suppose no dilatational disturbance is generated, then equation 25a implies either $k = 0$, $d_\beta = 0$ or $d_\beta{}^2 - k^2 = 0$. The first alternative is not possible since the assumed form of the incident wave implies $k = i\ |\vec{A}_{\psi_1}| \sin \gamma_{S_1} \neq 0$, and the latter two alternatives are not possible since the medium is anelastic. Hence, the amplitude of the reflected P wave is not zero and a dilatational disturbance is reflected from the free surface. If the normally incident SV wave is homogeneous, then $k = k_S \sin \theta_{S_1}$, and equation 31a shows that there is no dilatational disturbance reflected from the free surface.

For an elastic half-sapce angles of incidence exist such that the incident SV wave is entirely reflected as a dilatational wave. The following theorem shows that such angles exist for only a restricted class of visco-elastic solids of which elastic is a special case.

Theorem 5. If the incident SV wave is homogeneous and there is a nonzero angle of incidence for which the amplitude of the reflected SV wave is zero, then the solid is such that $Q_S{}^{-1} = Q_P{}^{-1}$.

Theorem 5 is proved

in appendix 5. The contrapositive of theorem 5 shows that for anelastic materials in the earth for which $Q_S^{-1} \neq Q_P^{-1}$, the amplitude of the reflected SV wave is nonzero for every nonzero angle of incidence of the incident homogeneous SV wave.

GENERAL P WAVE INCIDENT ON A WELDED BOUNDARY

The problem of a general P wave incident on a welded boundary is specified by setting

$$\vec{C}_1 = \vec{C}_2' = B_2' = c_{1j}' = c_{2j} = 0 \; (j = 1,3) \tag{26a}$$

in 3, 6, and 7, which results in the incident general P wave being given by

$$\phi_1 = B_1 \exp[-\vec{A}_{\phi_1} \cdot \vec{r}] \exp[i(\omega t - \vec{P}_{\phi_1} \cdot \vec{r})] \quad . \tag{26b}$$

Notation for the problem is summarized in Figure 4. k is given by

$$k = |\vec{P}_{\phi_1}| \sin \theta_{P_1} - i|\vec{A}_{\phi_1}| \sin(\theta_{P_1} - \gamma_{P_1}) \tag{27}$$

where the given parameters of the incident wave are B_1,

$$\theta_{P_1} (0 \leq \theta_{P_1} \leq \pi/2) \text{ , and } \gamma_{P_1} (0 \leq |\gamma_{P_1}| < \pi/2) \quad .$$

Proofs of the following results are similar to those given for the incident SV problem.

The extension of Snell's law is

$$|\vec{v}_{P_1}|/\sin \theta_{P_1} = \omega/k_R = \omega/(|P_{\phi_1}|\sin \theta_{P_1}) = |\vec{v}_{P_2}|/ \sin \theta_{P_2} = |\vec{v}_{S_2}|/ \sin \theta_{S_2} =$$

$$|\vec{v}_{P_1}'|/ \sin \theta_{P_1}' = |\vec{v}_{S_1}'|/ \sin \theta_{S_1}' \tag{28a}$$

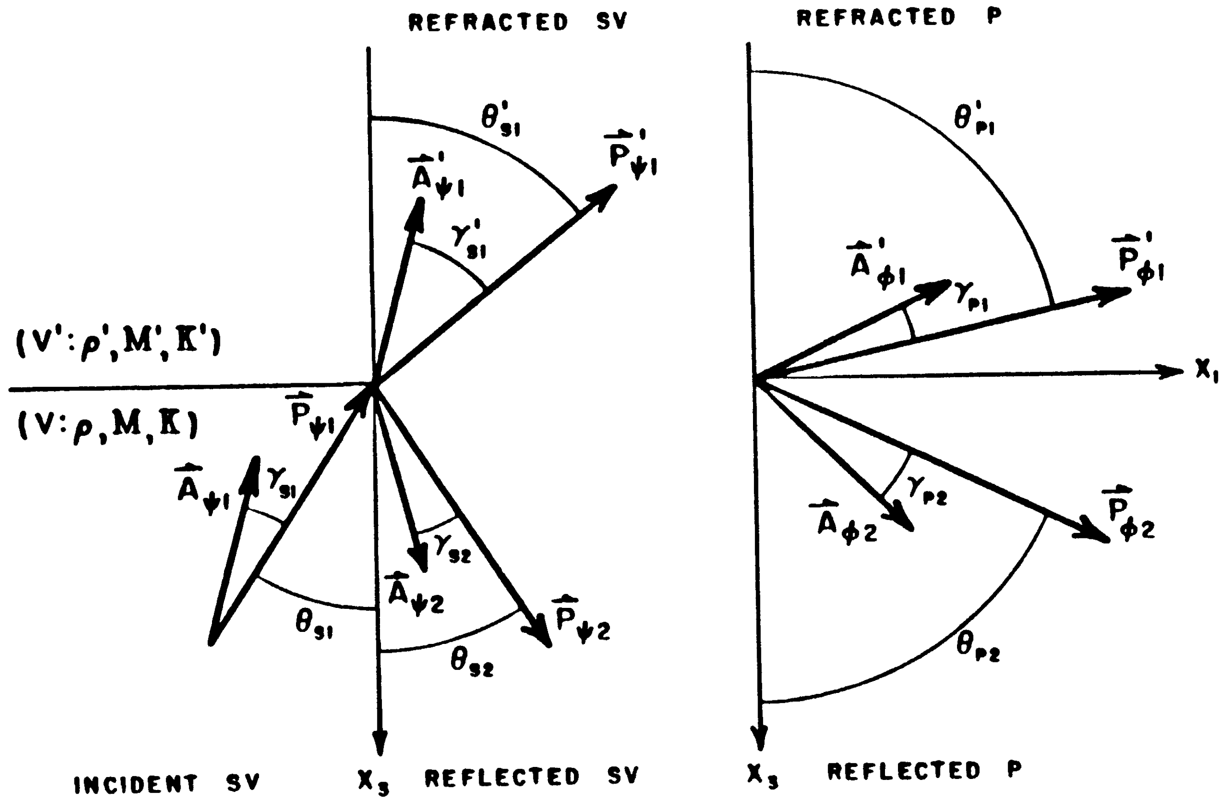

Fig. 4. Definitions of angles that define the directions of the propagation and attenuation vectors for the incident, reflected, and refracted waves for a general wave incident on a welded boundary between general viscoelastic media.

and

$$-k_I = |\vec{A}_{\phi_1}| \sin(\theta_{P_1} - \gamma_{P_1}) = |\vec{A}_{\phi_2}| \sin(\theta_{P_2} - \gamma_{P_2}) = |\vec{A}_{\psi_2}| \sin(\theta_{S_2} - \gamma_{S_2})$$

$$|\vec{A}'_{\phi_1}| \sin(\theta'_{P_1} - \gamma'_{P_1}) = |\vec{A}'_{\psi_1}| \sin(\theta'_{S_1} - \gamma'_{S_1}) \quad . \tag{28b}$$

The reflected P wave is homogeneous if and only if the incident P wave is homogeneous, and its phase velocity is the same as that of the incident P wave. If the incident P wave is inhomogeneous, then it is clear that the reflected SV wave and both transmitted waves in general will be inhomogeneous.

Theorem 6. If the incident P wave is homogeneous and not normally incident, then:

1) the reflected SV wave is homogeneous if and only if

$$Q_P^{-1} = Q_S^{-1} \quad , \tag{29a}$$

2) the transmitted P wave is homogeneous if and only if

$$Q_P^{-1} = Q_S^{-1} \quad \text{and} \quad \sin^2\theta_{P_1} \le k_P'^2/k_P^2 = \rho'(\kappa_R + \tfrac{4}{3}\mu_R)/[\rho(\kappa_R' + \tfrac{4}{3}\mu_R')]$$

$$= v_{HP}^2/v_{HP}'^2 \quad , \tag{29b}$$

3) and the transmitted SV wave is homogeneous if and only if

$$Q_P^{-1} = Q_S'^{-1} \quad \text{and} \quad \sin^2\theta_{P_1} \le k_S'^2/k_P^2 = (\rho'/\rho)(\kappa_R + \tfrac{4}{3}\mu_R)/\mu_R' = v_{HP}^2/v_{HS}'^2 \quad . \tag{29c}$$

If the incident P wave is homogeneous and normally incident, then the reflected and transmitted waves are also homogeneous with propagation and attenuation vectors perpendicular to the boundary.

Theorem 6 shows that both transmitted waves and the reflected SV wave generated by a P wave incident on a boundary between anelastic materials in the Earth are in general inhomogeneous for each oblique angle of incidence. Thus, the physical characteristics of reflected and refracted waves in anelastic media are different from those in elastic media (see the second section and Borcherdt 1973a).

Equations I-30, 3c-h, and 27 show that for anelastic media the velocities of the reflected and transmitted SV waves and the transmitted P wave depend on the angle of incidence. As before, if the incident wave is composed of several different frequencies, then fans of reflected SV and transmitted P and SV waves will be generated in anelastic media.

If both media are elastic, then the classic results for critical angles are easily derived from theorem 6 and the definitions of d_α, d_β, d_α' and d_β'. If medium V is elastic and medium V' anelastic, then no critical angles exist for the transmitted waves.

Theorem 7. If V is anelastic and if θ_P ($\theta_{P_1} \neq \pi/2$) is a critical angle for:

1) the reflected SV wave, then

$$\tan \gamma_{P_1} = (\sin^2\theta_{P_1} - \frac{k_{SR}k_{SI}}{k_{PR}k_{PI}})/(\sin \theta_{P_1} \cos \theta_{P_1}) , \qquad (30a)$$

2) the transmitted P wave, then

$$\tan \gamma_{P_1} = (\sin^2\theta_{P_1} - \frac{k'_{PR}k'_{PI}}{k_{PR}k_{PI}})/(\sin \theta_{P_1} \cos \theta_{P_1}) , \qquad (30b)$$

3) the transmitted SV wave, then

$$\tan \gamma_{P_1} = (\sin^2\theta_{P_1} - \frac{k'_{SR}k'_{SI}}{k_{PR}k_{PI}})/(\sin \theta_{P_1} \cos \theta_{P_1}) . \qquad (30c)$$

Corollaries analogous to corollaries 3-1, 3-2, and 3-3 are implied immediately by theorem 7. A corollary of special interest for anelastic materials in the Earth is

Corollary 7-1. If the incident P wave is homogeneous and if both media are anelastic, then:

1) if medium V is such that $v_{HS} < v_{HP}$, the angle of reflection θ_{S_2} for the reflected SV wave is less than the angle of reflection

θ_{P_2} for the reflected P wave and there are no critical angles for the reflected SV wave,

2) if θ_{P_1} $(\theta_{P_1} \neq \pi/2)$ is a critical angle for the transmitted P wave, it follows that $Q_P^{-1} \leq Q_P^{'-1}$, $v_{HP} \leq v'_{HP}$ and the critical angle is given by

$$\sin^2\theta_{P_1} = \mathrm{Im}[k_P'^2]/\mathrm{Im}[k_P^2] = v_{HP}^2 F(Q_P'^{-1})/(v_{HP}'^2 F(Q_P^{-1})) \ , \qquad (31a)$$

3) if θ_{P_1} $(\theta_{P_1} \neq \pi/2)$ is a critical angle for the transmitted SV wave, it follows that $Q_P^{-1} \leq Q_S^{'-1}$, $v_{HP} \leq v'_{HS}$ and the critical angle is given by

$$\sin^2\theta_{P_1} = \mathrm{Im}[k_S'^2]/\mathrm{Im}[k_P^2] = v_{HP}^2 F(Q_S'^{-1})/(v_{HS}'^2 F(Q_P^{-1})) \ , \qquad (31b)$$

where the function F in 31 a and 31 b is defined by 21b.

An additional corollary analogous to 3-5 is

Corollary 7-2. If both media are anelastic, and the incident P wave is homogeneous, then:

1) if $Q_P^{-1} = Q_S^{-1}$ no critical angles exist for the reflected waves,

2) if $Q_P'^{-1} = Q_P^{-1}$ and $v_{HP}^2 \leq v_{HP}'^2$ there is one and only one critical angle for the transmitted P wave given by

$$\sin^2\theta_{P_1} = k_P'^2/k_P^2 = (\rho'/\rho)(\kappa_R + \frac{4}{3}\mu_R)/(\kappa_R' + \frac{4}{3}\mu_R') = v_{HP}^2/v_{HP}'^2 \ , \qquad (32a)$$

3) if $Q_S'^{-1} = Q_P^{-1}$ and $v_{HP} \leq v'_{HS}$ there is one and only one critical angle for the transmitted SV wave given by

$$\sin^2\theta_{P_1} = k_S'^2/k_P^2 = (\rho'/\rho)(\kappa_R + \frac{4}{3}\mu_R)/\mu_R' = v_{HP}^2/v_{HS}'^2 \ . \qquad (32b)$$

Results analogous to theorem 4 and corollary 4-1 may also be derived for the problem of an incident P wave.

DISCUSSION

The general theory of linear viscoelasticity, based on Boltzman's principal of superposition, accounts for the behavior of both elastic and linear anelastic materials and is independent of any particular viscoelastic model. As a result the general theory provides a general mathematical framework for considering wave propagation in the earth (Savage and Hasegawa, 1967), and in particular the classic P-SV problems solved herein.

The physical characteristics predicted for body waves in layered anelastic media are significantly different from those for corresponding body waves in elastic media. For anelastic media, the reflectd and refracted P and SV waves are, in general, predicted to be inhomogeneous for all non-normal angles of incidence and as a result exhibit: elliptical particle motions, velocities and maximum attenuations which depend on frequency and the angle of incidence, and velocities and directions of maximum energy flow different from those of phase propagation. These physical characteristics predicted for P and SV waves refracted at boundaries between materials with different intrinsic attenuations such as a bedrock-soil, mantle-crust, or core-mantle interface provide insight into the nature of anelastic waves and may be useful for inferring anelastic properties of earth materials.

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APPENDIX 1

The purpose of this section is to prove theorem 1. A useful lemma is

Lemma 1. $k_P{}^2/k_S{}^2$ is a real number if and only if $Q_P{}^{-1} = Q_S{}^{-1}$, in which case $k_P{}^2/k_S{}^2 = \mu_R/(\kappa_R + \frac{4}{3}\mu_R) = v_{HS}{}^2/v_{HP}{}^2$.

Lemma 1 follows immediately from the following relations which are easily derived from 1 and 2;

$$k_P{}^2/k_S{}^2 = [\mu_R/(\kappa_R + \frac{4}{3}\mu_R)]\,[(1 + Q_S{}^{-1})/(1 + Q_P{}^{-1})]$$

$$k_P{}^2/k_S{}^2 = \frac{v_{HS}{}^2}{v_{HP}{}^2}\,[(1 + \sqrt{1 + Q_S{}^{-2}})/(1 + \sqrt{1 + Q_P{}^{-2}})][(1 - iQ_P{}^{-1})/(1 - iQ_S{}^{-1})] \quad .$$

To prove the "only if" part of theorem 1 for the reflected P wave assume the reflected P wave is homogeneous $(\gamma_{P_2} = 0)$, then 13 and I-29 imply

$$k^2 = k_P{}^2 \sin^2\theta_{P_2} \tag{1-1}$$

and

$$d_\alpha{}^2 = k_P{}^2 - k^2 = k_P{}^2 \cos^2\theta_{P_2} \quad . \tag{1-2}$$

These relations, together with 10b, imply $k_P{}^2/k_S{}^2$ is a real number for $\theta_{S_1} \neq 0$ and

$$\frac{d_\alpha{}^2}{k^2} = \frac{k_P{}^2}{k_S{}^2 \sin^2\theta_{S_1}} - 1 \geq 0 \quad . \tag{1-3}$$

Hence, lemma 1 and 1-3 give the desired conclusion that

$$Q_S{}^{-1} = Q_P{}^{-1} \quad \text{and} \quad \sin^2\theta_{S_1} \leq k_P{}^2/k_S{}^2 = \mu_R/(\kappa_R + \frac{4}{3}\mu_R) = v_{HS}{}^2/v_{HP}{}^2 \ . \tag{1-4}$$

Converseley, if 1-4 is valid, then $k_P{}^2/k_S{}^2$ is a real number from which it follows that $d_\alpha{}^2/k^2$ is a non-negative real number. Therefore d_α/k is a real number, say c, which implies

$$d_{\alpha R} = ck_R \quad \text{and} \quad d_{\alpha I} = ck_I \quad . \tag{1-5}$$

Substitution of 1-5 into 3c and 3g with $j = 2$ shows that the propagation and attenuation vectors for the reflected P wave are parallel, that is to say the reflected P wave is homogeneous. The results stated in theorem 1 for the transmitted waves may be proved in a similar fashion.

APPENDIX 2

The purpose of this section is to prove theorem 2 for the reflected P wave. The proofs for the transmitted P and SV waves are similar.

If θ_{S_1} is a critical angle for the reflected P wave, then $\vec{P}_{\theta_2}$ is parallel to $\hat{x}_1$ and 3c implies $d_{\alpha R} = 0$ which implies

$$\mathrm{Im}[d_\alpha{}^2] = 2\, d_{\alpha R} d_{\alpha I} = 2\, k_{PR} k_{PI} - 2\, k_R k_I = 0 \quad . \tag{2-1}$$

Equations 10a and I-29 show that for V anelastic $k_R k_I = -|\vec{P}_{\psi_1}||\vec{A}_{\psi_1}| \cdot \sin\theta_{S_1} \sin(\theta_{S_1} - \gamma_{S_1})$, may be written as

$$k_R k_I = k_{SR} k_{SI} \quad \sin\ \theta_{S_1} \quad \sin(\theta_{S_1} - \gamma_{S_1}) / \cos\gamma_{S_1} \quad . \tag{2-2}$$

Substitution of 2-2 into 2-1 yields

$$\sin\ \theta_{S_1} \quad \sin(\theta_{S_1} - \gamma_{S_1}) = (\cos\gamma_{S_1})(k_{PR} k_{PI} / k_{SR} k_{SI}) \quad , \tag{2-3}$$

which simplifies with trigonometric identities for $\theta_{S_1} \neq \pi/2$ to the desired relation 20a.

APPENDIX 3

Part (2) or corollary 3-4 for the transmitted SV wave is proved in this section. Analogous proofs are deduced easily for the reflected and transmitted P waves.

Suppose θ_{S_1} is a critical angle for the transmitted SV wave and that the incident wave is homogeneous, that is $\gamma_{S_1} = 0$. Theorem 3 implies for $\theta_{S_1} \neq \pi/2$ that the critical angle is given by

$$\sin^2 \theta_{S_1} = \mathrm{Im}[k_S'^2]/\mathrm{Im}[k_S^2] \quad , \tag{3-1}$$

and simplification with 2b gives part of the desired result:

$$\sin^2 \theta_{S_1} = \mathrm{Im}[k_S'^2]/\mathrm{Im}[k_S^2] = v_{HS}^2 F(Q_S'^{-1})/(v_{HS}'^2 F(Q_S^{-1})\ ) \tag{3-2}$$

where

$$F(Q_S^{-1}) \equiv Q_S^{-1}/(1 + \sqrt{1 + (Q_S^{-1})^2}) \quad . \tag{3-3}$$

In addition, 3c implies $d_{\beta R}' = 0$, hence the definition of d_β' (3f) implies

$$d_\beta'^2 = -d_{\beta I}'^2 = \mathrm{Re}[k_S'^2] - \mathrm{Re}[k^2] \tag{3-4}$$

which shows

$$\mathrm{Re}[k_S'^2] - \mathrm{Re}[k^2] \leq 0 \quad . \tag{3-5}$$

By assumption, the incident wave is homogeneous, hence 10b implies that 3-5 simplifies to

$$\mathrm{Re}[k_S'^2] - \sin^2 \theta_{S_1} \mathrm{Re}[k_S^2] \leq 0 \ . \tag{3-6}$$

Substituting (3-1) into (3-6) yields

$$\mathrm{Re}[k_S'^2] \leq \mathrm{Im}[k_S'^2]\mathrm{Re}[k_S^2]/\mathrm{Im}[k_S^2] \quad . \tag{3-7}$$

Since $\mathrm{Re}[k_S'^2] \geq 0$ (Borcherdt 1971, p. 25), I-72 shows that 3-7 implies an additional part of the desired result

$$Q_S^{-1} \leq Q_S'^{-1} \quad . \tag{3-8}$$

Equation 3-2 shows that

$$(v_{HS}^2/v_{HS}'^2)(F(Q_S'^{-1})/F(Q_S^{-1})) \leq 1 \tag{3-9}$$

and (3-8) implies

$$F(Q_S^{-1}) \leq F(Q_S'^{-1}) \quad . \tag{3-10}$$

Equations 3-9 and 3-10 yield the desired final conclusion

$$v_{HS} \leq v_{HS}' \quad . \tag{3-11}$$

APPENDIX 4

The purpose of this section is to prove corollary 3-5 for the reflected P wave. Proofs for the transmitted waves are similar.

From 3c, the component of the propagation vector for the reflected P wave is $d_{\alpha R}$ given by (see 18)

$$d_{\alpha R} = \sqrt{(|k_P{}^2 - k^2| + \mathrm{Re}[k_P{}^2 - k^2])/2} \quad . \tag{4-1}$$

Since the incident SV wave is homogeneous, 10b shows $k^2 = k_S{}^2 \sin^2\theta_{S_1}$ hence,

$$k_P{}^2 - k^2 = k_S{}^2\left(\frac{k_P{}^2}{k_S{}^2} - \sin^2 \theta_{S_1} \right) \quad . \tag{4-2}$$

By lemma 1 (appendix 1), $Q_P^{-1} = Q_S^{-1}$ implies $k_P{}^2/k_S{}^2$ is a non-negative real number and

$$k_P{}^2/k_S{}^2 = \mu_R/(\kappa_R + \frac{4}{3}\mu_R) = v_{HS}{}^2/v_{HP}{}^2 \quad . \tag{4-3}$$

$Q_S^{-1} = Q_P^{-1}$ implies $v_{HS}{}^2/v_{HP}{}^2 \leq 1$, and substitution of 4-2 into 4-1 together with 4-3 shows there is one and only one angle of incidence, namely $\sin^2 \theta_{S_1} = \mu_R/[\kappa_R + (4\mu_R/3)] = v_{HS}{}^2/v_{HP}{}^2$, such that $d_{\alpha R} = 0$, that is, such that the reflected P wave is an interface wave.

APPENDIX 5

The purpose of this section is to prove theorem 5.

Suppose θ_{S_1} is a non-zero angle of incidence such that the amplitude, c_{22}, of the reflected SV wave is zero, then 25b implies

$$16(d_\alpha^2/k^2)(d_\beta^2/k^2) = (d_\beta^2/k^2 - 1)^4 \quad . \tag{5-1}$$

By assumption the incident SV wave is homogeneous, hence,

$$k = k_S \sin \theta_{S_1} \tag{5-2}$$

and the definition of d_β (3f) implies

$$d_\beta^2/k^2 = \sin^{-2}\theta_{S_1} - 1 = \tan^{-2}\theta_{S_1} \quad . \tag{5-3}$$

Equation 5-3 shows d_β^2/k^2 is a real number, hence 5-1 shows d_α^2/k^2 is a real number, which, using 3e and 5-2, imply

$$\mathrm{Im}[d_\alpha^2/k^2] = \mathrm{Im}[(k_P^2/(k_S^2 \sin^2\theta_{S_1})) - 1] = \mathrm{Im}[k_P^2/k_S^2]/\sin^2\theta_{S_1} = 0 \quad . \tag{5-4}$$

Equation 5-4 shows k_P^2/k_S^2 is a real number, hence lemma 1 (appendix 1) implies the desired result

$$Q_P^{-1} = Q_S^{-1} \quad . \tag{5-5}$$

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