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Fast evaluation of squared-Hankel transforms of order-1 by linear digital filtering (Subprogram SQJ1)

by

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DISCLAIMER

This program was written in FORTRAN-77 for a VAX-11/780 system*. Although program tests have been made, no guarantee (expressed or implied) is made by the author regarding program correctness, accuracy, or proper execution on all computer systems.

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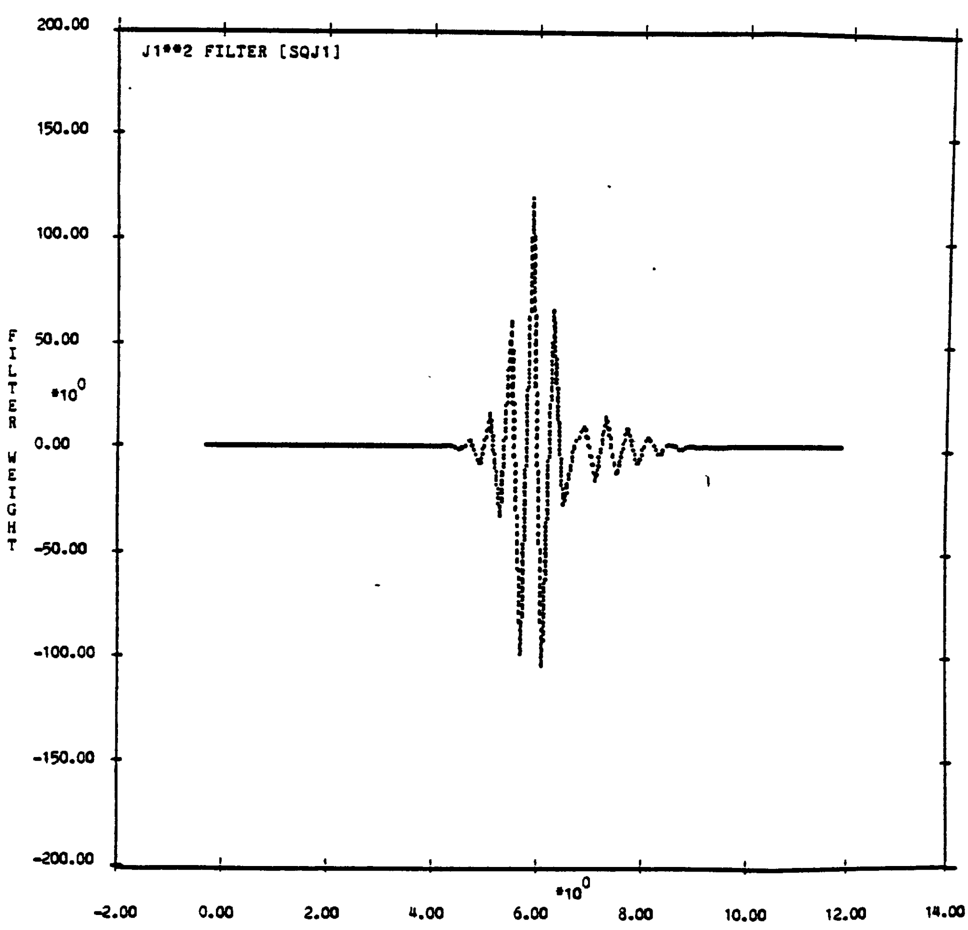
The squared-Hankel transform of order-1 of the function h(g) has the general form,

$$\int_0^\infty h(g)\ [J_1^2(bg)]\ dg = H(b),$$

where b>0 is the (real) transform argument and J_1 is the Bessel function of the first kind of order-1. Numerical integration of squared-Hankel transforms for various continuous bounded kernel functions h(g) are sometimes required in electromagnetic (EM) applications. For example, see Morrison et al (1969), Ryu et al (1970), and Raiche and Spies (1981) for some typical J_1^2-integral forms occurring in various EM calculations. Direct numerical evaluation is generally time-consuming due to the oscillatory behavior of the Bessel function $J_1(bg)$. In constrast, digital filter techniques (e.g., see Anderson, 1979) used to compute linear convolution-type integrals are usually orders-of-magnitude faster than direct quadrature methods. The purpose of this report is to provide a new computer subprogram (SQJ1) for the rapid and accurate evaluation of squared-Hankel transforms of order-1.

Subprogram SQJ1 (listed in Appendix 2--complete with the parameter-calling requirements) uses an adaptive linear digital filter algorithm to rapidly evaluate H(b) without the need to compute any Bessel J-functions. The details of the filter design process (using eq.(1) below) and algorithm follows the method published in Anderson (1979), and will not be repeated here. Best expected relative errors by SQJ1 on a VAX computer are approximately 10^{-6} using single-precision arithmetic and 32-bit floating-point computer words. A plot of the filter weights (taken from Appendix 2) is given in Figure 1, which shows the filter response characteristics for the J_1^2-filter.

Figure 1.--The J_1^2 - filter response from Appendix 2.



Some squared-Hankel transforms of order-1 were tested using SQJ1 on a VAX-11/780 computer for the following known transform pairs (see Gradshteyn and Ryzhik, 1965, p. 718, p. 672, and p. 715),

$$\int_0^\infty x \exp(-a^2 x^2) \, [J_1^2(bx)] \, dx = \exp(-\{b^2/2a^2\}) \, I_1(b^2/2a^2) \, /(2a^2) \qquad (1)$$

$$\int_0^\infty x \, K_0(ax) \, [J_1^2(bx)] \, dx = (R-a)/[aR(R+a)], \; R=(a^2+4b^2)^{1/2} \qquad (2)$$

$$\int_0^\infty x^2 \exp(-2ax) \, [J_1^2(bx)] \, dx = \frac{3b^2}{4\pi}\int_0^{\pi/2} (\cos^2 p)/(a^2+ b^2\cos^2 p)^{5/2} \, dp \qquad (3)$$

where a>0, b>0 in equations (1)-(3). Appendix 1 lists several runs made for various values of a and b, where an accuracy tolerance TOL=.1E-9 was chosen (see the discussion of parameter TOL in Appendix 2).

CONCLUSION

A squared-Hankel transform filter of order-1 has been designed and successfully tested to yield approximately single-precision accuracy on a VAX-11 system. Applications for this filter may be found, for example, in computing coincident loop soundings (Raiche and Spies, 1981), and other similar calculations involving a J_1^2 Hankel transform.

REFERENCES

Anderson, W.L., 1979, Numerical integration of related Hankel transforms of orders 0 and 1 by adaptive digital filtering: Geophysics, v. 44, no. 7, p. 1287-1305.

Gradshteyn, I.S., and Ryzhik, I.M., 1965, Tables of integrals, series, and products: Academic Press, New York, 1086 p.

Morrison, H.F., Phillips, R.J., and O'Brien, D.P., 1969, Quantitative interpretation of transient electromagnetic fields over a layered half-space: Geophys. Prosp., v. 17, p. 82-101.

Raiche, A.P., and Spies, B.R., 1981, Coincident loop transient electromagnetic master curves for interpretation of two-layer earths: Geophysics, v. 46, no. 1, p. 53-64.

Ryu, J., Morrison, H.F., and Ward, S.H., 1970, Electromagnetic fields about a loop source of current: Geophysics, v. 35, no. 5, p. 862-896.

Appendix 1.--Test results using subprogram SQJ1.

The following definitions are used in the table below:

ABS.ERROR = |EXACT - FILTERED|

REL.ERROR = ABS.ERROR / EXACT

EXACT = right-side of respective eqs.(1)-(3)

FILTERED = result from SQJ1

NF = resulting filter weights used for TOL=.1E-9.

TEST RESULTS FOR SQJ1 FILTER:

INTEGRAL *	b	a	EXACT	FILTERED	ABS.ERROR	REL.ERROR	NF
1	0.10000E-01	0.10000E-01	0.78210402E+03	0.7S210474E+03	0.72040377E-03	0.92110992E-06	80
2	0.10000E-01	0.10000E+01	0.99960015E-04	0.99960271E-04	0.25575081E-09	0.25585311E-05	251
3	0.10000E-01	0.10000E-01	0.5216417SE+05	0.52164148E+05	0.29167130E-01	0.55914098E-06	68
1	0.30000E-01	0.31623E-01	0.73564298E+02	0.73564323E+02	0.25424963E-04	0.34561552E-06	80
2	0.30000E-01	0.10000E+01	0.89677090E-03	0.89677307E-03	0.21698523E-08	0.24196284E-05	213
3	0.30000E-01	0.31623E-01	0.16296381E+04	0.16296382E+04	0.12857411E-03	0.78897339E-07	68
1	0.10000E+00	0.10488E+00	0.67280076E+01	0.67280087E+01	0.11212203E-05	0.16664968E-06	80
2	0.10000E+00	0.10000E+01	0.96144602E-02	0.96144713E-02	0.11109212E-07	0.11554691E-05	121
3	0.10000E+00	0.10488E+00	0.44729674E+02	0.44729671E+02	0.21218013E-05	0.47436101E-07	68
1	0.30000E+00	0.31780E+00	0.72400990E+00	0.72400987E+00	0.25658550E-07	0.35439502E-07	80
2	0.30000E+00	0.10000E+01	0.65786957E-01	0.65786920E-01	0.36961020E-07	0.56182899E-06	93
3	0.30000E+00	0.31780E+00	0.16032554E+01	0.16032552E+01	0.25202840E-06	0.15719791E-06	68
1	0.10000E+01	0.10493E+01	0.67182523E-01	0.67182519E-01	0.49114448E-08	0.73105989E-07	80
2	0.10000E+01	0.10000E+01	0.17082039E+00	0.17082042E+00	0.21770052E-07	0.12744410E-06	83
3	0.10000E+01	0.10493E+01	0.44663308E-01	0.44663306E-01	0.20284552E-08	0.45416592E-07	68
1	0.30000E+01	0.31782E+01	0.72389495E-02	0.72389483E-02	0.12726365E-08	0.17580404E-06	80
2	0.30000E+01	0.10000E+01	0.11797671E+00	0.11797670E+00	0.60725924E-08	0.51472808E-07	78
3	0.30000E+01	0.31782E+01	0.16029947E-02	0.16029944E-02	0.31758611E-09	0.19812050E-06	68
1	0.10000E+02	0.10493E+02	0.67181549E-03	0.67181542E-03	0.71709783E-10	0.10674029E-06	80
2	0.10000E+02	0.10000E+01	0.45187305E-01	0.45187309E-01	0.43557343E-08	0.96392877E-07	76
3	0.10000E+02	0.10493E+02	0.44662645E-04	0.44662633E-04	0.12705370E-10	0.28447419E-06	68
1	0.20000E+02	0.22585E+02	0.13234226E-03	0.13234212E-03	0.14682494E-09	0.11094335E-05	81
2	0.20000E+02	0.10000E+01	0.23773431E-01	0.23773450E-01	0.19005252E-07	0.79943244E-06	74
3	0.20000E+02	0.22585E+02	0.43707442E-05	0.43707464E-05	0.21699968E-11	0.49648222E-06	68
1	0.25000E+02	0.33691E+02	0.46479911E-04	0.46479785E-04	0.12602009E-09	0.27112809E-05	85
2	0.25000E+02	0.10000E+01	0.19211998E-01	0.19211851E-01	0.14694017E-06	0.76483544E-05	72
3	0.25000E+02	0.33691E+02	0.11954654E-05	0.11954675E-05	0.20967921E-11	0.17539547E-05	69

* TEST INTEGRALS USED:

(1) $$\int_0^\infty x e^{-(ax)^2}[J_1^2(bx)]\,dx = \frac{1}{2a^2} e^{-\left(\frac{b^2}{2a^2}\right)} I_1\left(\frac{b^2}{2a^2}\right) \qquad (a>0, b>0);$$

(2) $$\int_0^\infty x K_0(ax)[J_1^2(bx)]\,dx = \frac{R-a}{aR(R+a)}, \quad R=\sqrt{a^2+4b^2} \qquad (a>0, b>0);$$

(3) $$\int_0^\infty x^2 e^{-2ax}[J_1^2(bx)]\,dx = \frac{3b^2}{4\pi}\int_0^{\pi/2} \frac{\cos^2\phi\,d\phi}{(a^2+b^2\cos^2\phi)^{5/2}} \qquad (a>0, b>0).$$

Appendix 2.--Source listing of subprogram SQJ1.

```
      REAL*4 FUNCTION SQJ1(B,FUN,TOL,NF)
C=======================================================================
C** THIS IS A REAL*4 VERSION WRITTEN FOR THE VAX-11/780 BY
C   W.L.ANDERSON, U.S.GEOLOGICAL SURVEY, DENVER, COLORADO, USA.
C=======================================================================
C  SUBPROGRAM SQJ1 WILL COMPUTE THE FOLLOWING INFINITE INTEGRAL:
C  THE REAL*4 HANKEL TRANSFORM-SQUARE OF ORDER-1 FOR BOUNDED CONTINUOUS
C  KERNEL FUNCTIONS AND A FIXED TRANSFORM ARGUMENT B.GT.0.  THE
C  METHOD IS SIMILAR TO THE NEW=1 CASE FOR SINGLE-POWER J0,J1-FILTERS
C  DESIGNED AND PUBLISHED IN THE FOLLOWING REFERENCE:
C
C--REF: ANDERSON, W.L., 1979, GEOPHYSICS, VOL. 44, NO. 7, P. 1287-1305.
C
C--SPECIFICALLY,  SQJ1 EVALUATES THE INTEGRAL FROM 0 TO INFINITY OF
C  FUN(G)*[J1(G*B)]**2 *DG, DEFINED AS THE J1**2 HANKEL TRANSFORM OF
C  ORDER N=1 AND TRANSFORM ARGUMENT B.GT.0.  THE METHOD IS BY
C  ADAPTIVE DIGITAL FILTERING OF THE REAL*4 KERNEL FUNCTION FUN (SEE
C  THE ABOVE REFERENCE FOR ADDITIONAL INFORMATION).
C
C--PARAMETERS (ALL INPUT, EXCEPT NF)
C
C     B      = REAL*4 TRANSFORM ARGUMENT B>0.0 OF THE HANKEL TRANSFORM.
C     FUN(G)= EXTERNAL DECLARED REAL*4 FUNCTION NAME (USER SUPPLIED)
C             OF A REAL*4 ARGUMENT G>0. THIS REFERENCE MUST BE SUPPLIED.
C             IF PARAMETERS OTHER THAN G ARE REQUIRED IN FUN, USE COMMON
C             IN THE CALLING PROGRAM AND IN SUBPROGRAM FUN.  FUN(G)
C             MUST BE A CONTINUOUS BOUNDED FUNCTION FOR G.GT.0.
C             THE VALUE OF G IN FUN(G) MUST NOT BE CHANGED BY THE USER.
C             (G>0.0 WILL BE ASSIGNED AN ABSCISSA VALUE BY SQJ1.)
C     TOL    = REQUESTED REAL*4 TRUNCATION TOLERANCE USED AT THE FILTER
C             TAILS FOR ADAPTIVE FILTERING.  A TRUNCATION CRITERION IS
C             DEFINED DURING CONVOLUTION IN A FIXED ABSCISSA RANGE AS
C             THE MAX. ABSOLUTE CONVOLVED PRODUCT TIMES TOL.  TYPICALLY,
C             TOL.LE.0.00001E0 WOULD GIVE ABOUT .01 PER CENT ACCURACY
C             FOR WELL-BEHAVED KERNELS AND MODERATE VALUES OF B.  FOR
C             VERY LARGE OR SMALL B, A VERY SMALL TOL SHOULD BE USED.
C             IN GENERAL, DECREASING THE TOLERANCE WOULD PRODUCE HIGHER
C             ACCURACY IN THE CONVOLUTION SINCE MORE FILTER WEIGHTS ARE
C             USED (UNLESS EXPONENT UNDERFLOWS OCCUR IN THE KERNEL
C             EVALUATION -- SEE NOTE (1) BELOW).
C             FOR MAXIMUM ACCURACY POSSIBLE, TOL=0.0E0 MAY BE USED.
C     NF     = TOTAL NUMBER OF FUNCTION CALLS USED DURING CONVOLUTION.
C             NF IS IN THE RANGE 39.LE.NF.LE.441.  USUALLY,
C             NF IS MUCH LESS THAN 441 FOR TOL>0.
C
C=======================================================================
C--SUBPROGRAM USAGE--
```

```
C  FUNCTION SQJ1 IS CALLED AS FOLLOWS (ASSUMES B>0.0, TOL>=0.0):
C     ...
C     EXTERNAL FUN
C     ...
C     ANS=SQJ1(B,FUN,TOL,NF1)
C     ...
C     END
C     REAL*4 FUNCTION FUN(G)
C     ...USER SUPPLIED CODE FOR EVALUATION OF FUN(G), G.GT.0.
C     END
C======================================================================
C--NOTES
C        (1).  EXP-UNDERFLOW MAY OCCUR IN EXECUTING THIS SUBPROGRAM.
C              THIS IS OK PROVIDED THE MACHINE SYSTEM CONDITIONALLY SETS
C              EXP-UNDERFLOW TO 0.0D0.
C        (2).  ANSI FORTRAN (AMERICAN STANDARD X3.9-1978) IS USED, EXCEPT
C              DATA STATEMENTS MAY NEED TO BE CHANGED FOR SOME COMPILERS.
C        (3).  THE FILTER ABSCISSA CORRESPONDING TO EACH FILTER WEIGHT
C              IS GENERATED IN DOUBLE-PRECISION (TO REDUCE ROUND-OFF),
C              BUT IS USED IN SINGLE-PRECISION IN FUNCTION FUN.
C        (4).  NO CHECKS ARE MADE ON CALLING PARAMETERS (TO SAVE TIME),
C              HENCE UNPREDICTABLE RESULTS COULD OCCUR IF SQJ1
C              IS CALLED INCORRECTLY (OR IF FUNCTION FUN IS IN ERROR).
C======================================================================
C
      DOUBLE PRECISION E,ER,Y1,Y
      DIMENSION WT(441)
      EQUIVALENCE (C,T),(CMAX,TMAX)
C-----E=DEXP(.2D0), ER=1.0D0/E
      DATA E/1.221402758160169834 D0/,ER/.818730753077981859 D0/
C--J1**2 TRANSFORM FILTER WEIGHT ARRAY WT:
      DATA
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     *WT(441)/-4.937813102320317E-28/
C
C FOLLOWING CODE FOR STARTING WEIGHT=214 FROM TOTAL WTS=441.
C
      NONE=0
C-----INITIALIZE KERNEL ABSCISSA GENERATION FOR GIVEN B
      Y1=0.1314258239822337910D1/DBLE(B)
  100 SQJ1=0.0E0
      CMAX=0.0E0
      NF=0
      Y=Y1
C-----BEGIN RIGHT-SIDE CONVOLUTION AT WEIGHT 214
      ASSIGN 110 TO M
      I=214
      Y=Y*E
      GO TO 200
  110 TMAX=AMAX1(ABS(T),TMAX)
      I=I+1
      Y=Y*E
      IF(I.LE.250) GO TO 200
      IF(TMAX.EQ.0.0E0) NONE=1
C-----ESTABLISH TRUNCATION CRITERION (CMAX=TMAX)
```

```
      CMAX=TOL*CMAX
      ASSIGN 120 TO M
      GO TO 200
C-----CHECK FOR FILTER TRUNCATION AT RIGHT END
  120 IF(ABS(T).LE.TMAX) GO TO 130
      I=I+1
      Y=Y*E
      IF(I.LE.441) GO TO 200
  130 Y=Y1
C-----CONTINUE WITH LEFT-SIDE CONVOLUTION AT WEIGHT 213
      ASSIGN 140 TO M
      I=213
      GO TO 200
C-----CHECK FOR FILTER TRUNCATION AT LEFT END
  140 IF(ABS(T).LE.TMAX.AND.
     * NONE.EQ.0) GO TO 190
      I=I-1
      Y=Y*ER
      IF(I.GT.0) GO TO 200
C-----NORMALIZE BY B TO ACCOUNT FOR INTEGRATION RANGE CHANGE
  190 SQJ1=SQJ1/B
      RETURN
  200 C=FUN(SNGL(Y))*WT(I)
      NF=NF+1
      SQJ1=SQJ1+C
      GO TO M,(110,120,140)
      END
```