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Radiative transfer from a homogeneous half-space: A fast algorithm solution

by

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Radiative transfer from a homogeneous half-space:

A fast algorithm solution.

A computer algorithm is provided for the solution of a periodically heated half-space with radiative transfer. It takes advantage of matrix hardware commands to perform the Laplace transform solution and can be adapted to other systems and languages which provides fast matrix arithmetic.

Introduction

Analysis of thermal infrared data from aircraft, satellites or ground-based platforms commonly requires the use of a model. Because there is a non linear radiative transfer term in the boundary condition three methods are generally applied: linearization, finite differences and Laplace transform. The purpose of this paper is to describe a numerical solution based on the Laplace transform method developed by Jaegar (1953). This work represents an updating of an earlier treatment (Watson, 1971) and takes advantage of current computer technology in matrix arithmetic, either using machine hardware or array-processor technology.

The Problem:

Given a periodically heated homogeneous half-space, the surface temperature variation is related to the surface heat flux by the formula

$$F_n = \frac{P}{\sqrt{\pi T}} \sum_{s=1}^{m} p_{n-s+1} \cdot V_s \qquad (1)$$

where $F_n = G_n - \sigma V_n^4 \quad , \quad p_{m+n} = p_n \qquad (2)$

G_n is a discrete representation of the periodic heating function at times $t=n\Delta t; n = 1,2,...m$ and $m\Delta t = T$, the heating period.

V_n is the discrete surface temperature in the nth interval.

p_n is a set of numerical coefficients determined solely by the number of intervals in the period, m (Jaegar, 1953).

P is the thermal inertia of the homogeneous half-space.

σ is the Stefan-Boltzmann constant.

F_n is the surface flux in the nth interval.

In its most general form the flux G_n contains the radiative heating from the sun and sky and the sensible heat exchange between the ground and the atmosphere, the latter being a function of the surface temperature V_n. For purposes of this example G_n will be treated as independent of V_n. Extensions of other cases using appropriate empirical formulas (Watson, 1980) can be developed from this example.

An iterative solution to equation 1, subject to the boundary condition (equation 2) can be expressed using the following matrix notation.

$$F = \begin{bmatrix} F_1 \\ F_2 \\ \cdot \\ \cdot \end{bmatrix} \qquad G = \begin{bmatrix} G_1 \\ G_2 \\ \cdot \\ \cdot \end{bmatrix} \qquad V = \begin{bmatrix} V_1 \\ V_2 \\ \cdot \\ \cdot \end{bmatrix}$$

$$Q = \frac{P}{\sqrt{\pi T}} \begin{bmatrix} p_1 & p_m & p_{m-1} & \cdots \\ p_2 & p_1 & p_m & \cdots \\ p_3 & p_2 & p_1 & \cdots \\ \cdot & & \cdot & \cdot \end{bmatrix}$$

Using this notation, equation 1 becomes

$$F = Q.V \qquad (3)$$

For the ith iteration let us introduce

$$\Delta_n^{(i)} = V_n^{(i)} - V_n^{(i-1)} \qquad (4)$$

Then the linear expansion for the quartic term in equation 2 can be written

$$[\, V_n^{(i)} \,]^4 \cong [\, V_n^{(i-1)} \,]^4 + 4[\, V_n^{(i-1)} \,]^3 . \Delta_n^{(i)} \qquad (5)$$

Equation 1 for the ith iteration can then be re-written as

$$\begin{aligned} F_n^{(i)} &= \sum_S Q_{ns} . V_s^{(i)} - Q_{11} V_n^{(i)} + Q_{11} V_n^{(i)} \\ &\cong \sum_S Q_{ns} . V_s^{(i-1)} - Q_{11} . V_n^{(i-1)} + Q_{11} . V_n^{(i)} \\ &\cong \sum_S Q_{ns} . V_s^{(i-1)} + Q_{11} \Delta_n^{(i)} \end{aligned} \qquad (6)$$

Substituting equations 5 and 6 in equation 2 yields

$$\Delta_n^{(i)} = \frac{G_n - F_n^{(i)} - \sigma [V_n^{(i-1)}]^4}{Q_{11} + 4\sigma [V_n^{(i-1)}]^3} \qquad (7)$$

A summary of the iteration equations follows

$$F^{(i)} = Q . V^{(i-1)} \tag{8}$$

$$\Delta_n^{(i)} = \frac{G_n - F_n^{(i)} - \sigma [V_n^{(i-1)}]^4}{Q_{11} + 4\sigma [V_n^{(i-1)}]^3} \qquad n = 1, 2 \ldots m \tag{9}$$

$$V_n^{(i)} = \Delta_n^{(i)} + V_n^{(i-1)} \tag{10}$$

The iterations (equations 8,9, and 10) are repetitively solved until some convergence criterion has been satisfied. The following method provides a fairly straightforward approach.

Let us introduce a convergence term, $E^{(i)}$ and an error term $\epsilon^{(i)}$ for the ith iteration where

$$E^{(i)} \equiv \mathrm{Max} | \Delta_n^{(i)} | \qquad n = 1, 2 .. m \tag{11}$$

$$\epsilon^{(i)} \equiv \mathrm{Max} | U_n - \Delta_n^{(i)} | \tag{12}$$

$$\text{where } U_n^{(i)} \equiv \lim_{i \to \infty} V_n^{(i)} \tag{13}$$

For $E^{(i)} < .1$ we have found that $\epsilon < E$ and hence the accuracy desired in the solution can be used to determine the iteration limit.

The final step to the solution of the problem is selection of initial iteration values. One method is to linearize the boundary condition and solve using a Fourier series (Carslaw & Jaegar, 1959, p. 74; Watson, 1975)

Thus

$$V_s^{(i)} = U_o + \sum_{p=1} A'_p \cos(\omega pt + \theta_p) \tag{14}$$

where $U_o = [(1/m\sigma) \sum_{r=1}^{m} G_r]^{1/4}$

and

$$A'_p = A_p/\sqrt{(1+X_p)^2+X_p^2}$$

$$\theta_p = \epsilon_p - \delta_p$$

$$X_p = P\sqrt{\omega p/2}/4\sigma U_o^3$$

$$\delta_p = \tan^{-1}\{X_p/(1+X_p)\}$$

$$G_s + 3\sigma U_o^4 = 4\sigma U_o^3 \sum_{p=1} A_p \cos(\omega p t_s + \epsilon_p) .$$

Often a satisfactory solution requires only the first harmonic term (p=1).

Example

A sample numerical solution is provided for the case where G is a half wave plus a constant. This corresponds approximately to the conventional diurnal heating of the earth's surface.

$$\text{Let } G(t) = A\cos\omega t + B \qquad -\pi/2 \le \omega t \le \pi/2$$

$$= B \qquad \pi/2 < |\omega t|$$

Choose $A = 400\ W.m^{-2}$
$B = 200\ W.m^{-2}$

A set of temperature values is computed for several values of P (1000, 1500, 2000, 2500 TIU) and the curves plotted (figure 1) using m=48. Temperatures are listed at noon (n=1) and midnight (n=25) for reference in Table 1.

Table 1.--Selected temperature values for various thermal inertias.

Thermal inertias	Temperatures	
P	T(n=1)	T(n=25)
1000	292.9	265.0
1500	288.2	268.0
2000	285.4	269.8
2500	283.6	270.9

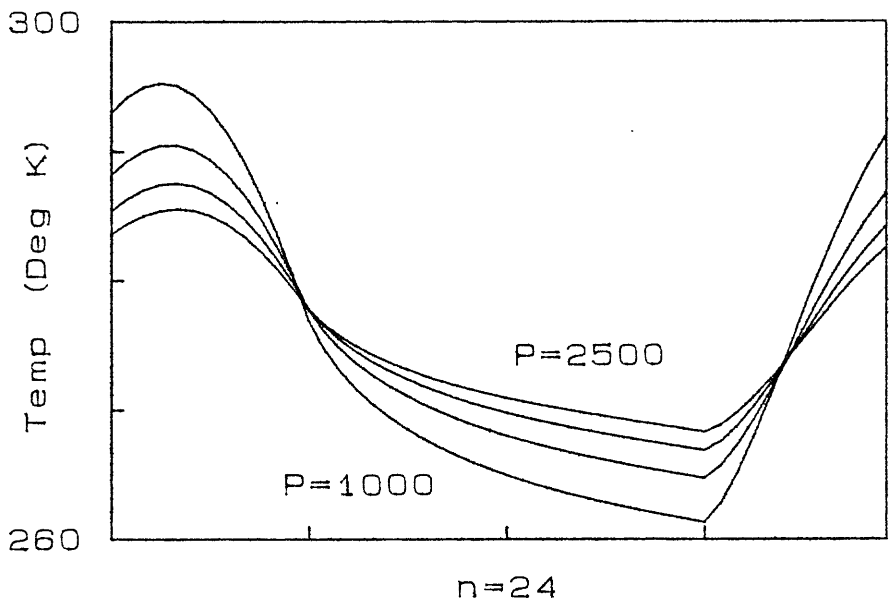

Figure 1.--Diurnal temperature variations for a range of thermal inertias (P). P=1000, 1500, 2000, 2500 TIU. The time increment is 1/48th of the heating cycle. n=1 corresponds to zero time.

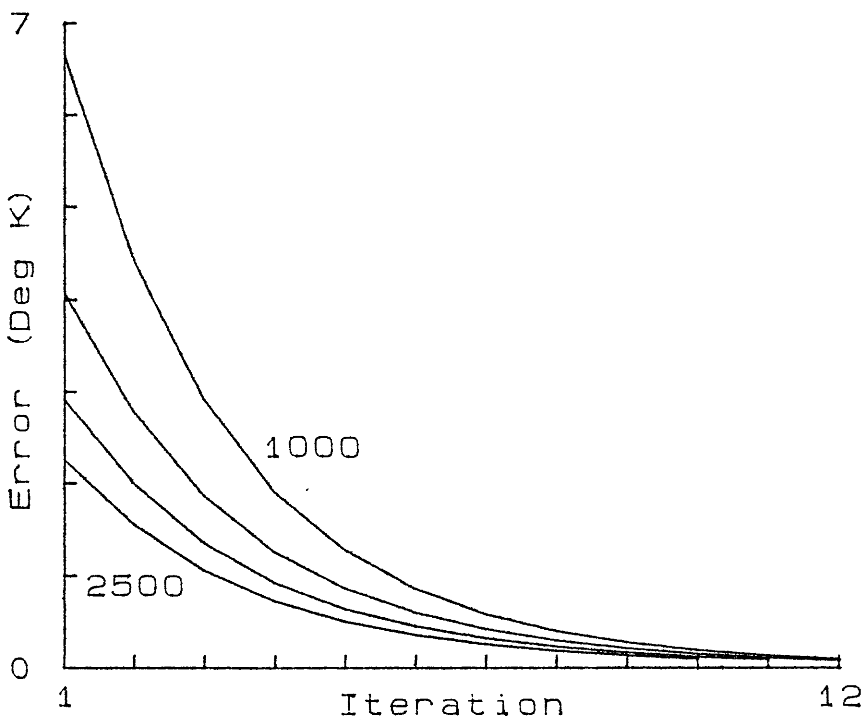

Figure 2.--The error term $E^{(i)}$ (see equation 11 versus iteration numbers i for varying thermal inertias (as in Figure 1).

A convergence factor of 0.1 degrees as used to terminate the iterations. A plot (figure 2) of the convergence term $E^{(i)}$ versus the error $\epsilon^{(i)}$ is provided to illustrate the convergence of the solution.

Program Listing

The listing provided is in extended BASIC from an HP 9845 minicomputer and illustrates the use of matrix commands to solve the problem. It can be readily adapted to other systems with similar firmware features.

```
SUB Transf (Cons, Q (*) , F0 (*) , V1 (*) , V2 (*) )
!  INPUT:    Arrays: Q (*) , F0 (*) , V1 (*)    Const: Cons
!  OUTPUT:   V2 (*)
!  Cons=P/SQR (PI*T)
OPTION BASE 1
DIM F (48, 1) , V3 (48, 1) , V4 (48, 1) , Del (48, 1)
Sigma=5.67E-8
!  ITERATIONS
Iter=1
Next_iter: MAT F=Q*V1              !      Equation 8
MAT F=F* (Cons)
MAT F=F0-F
MAT V4=V1.V1
MAT V3=V4.V1
MAT V4=V3.V1
MAT V4=V4* (Sigma)
MAT F=F-V4
MAT V3=V3* (4*Sigma)
MAT V3=V3+ (Q (1, 1) *Cons)
MAT Del=F/V3                !           Equation 9
MAT SEARCH Del (*, 1) , MAX; Dmax
MAT SEARCH Del (*, 1) , MIN; Dmin
MAT V2=V1+Del        !                  Equation 10
Dum=MAX (Dmax, ABS (Dmin) )        !    Equation 11
IF Dum<.1 THEN SUBEXIT
Iter=Iter+1
MAT V1=V2
GOTO Next_iter
SUBEND
```

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