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A MODEL FOR HYDROSTATIC CONSOLIDATION OF PIERRE SHALE

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INTRODUCTION

Braddock and Machette (1976) have done hydrostatic consolidation tests on Pierre Shale. What follows is an attempt at modeling their experimental results using a special case of Biot's (1941) theory of three-dimensional consolidation. The objective is to provide a theoretical basis for predicting the variation with time of pore pressure, volume of expelled pore fluid, and axial strains, resulting from hydrostatic consolidation of Pierre Shale.

BIOT'S THEORY OF THREE-DIMENSIONAL CONSOLIDATION

Biot (1941) considered the stress-strain response of a porous material having the following properties: (1) isotropy, (2) reversibility of stress-strain relations under final equilibrium conditions, (3) linearity of stress-strain relations, (4) an incompressible pore fluid, (5) small strains, and (6) water flow through the porous skeleton according to Darcy's Law. A brief review of Biot's isotropic theory will be given and applied to the particular case of hydrostatic consolidation. Biot's theory will then be applied to Braddock and Machette's (1976) experimental results.

The first of Biot's constitutive (stress-strain) relations are in standard tensor notation:

$$e_{ij} = \frac{1}{2\mu}\left(\sigma_{ij} - \frac{\lambda}{3\lambda+2\mu}\delta_{ij}\sigma_{kk}\right) + \frac{P}{3H}\delta_{ij} \qquad (1)$$

where e_{ij} represents strains, σ_{ij} represents stresses, δ_{ij} is the Kronecker delta, μ and λ are respectively the shear modulus and Lamé's constant for the elastic skeleton, P, is the pore pressure, and the coefficient 1/H is a measure of compressibility of the porous material for a given change in water pressure. As can be seen, equations 1 reduce to the usual elastic relations when the pore pressure, P, vanishes.

Because consolidation involves removal of pore water, an additional variable specifying the change in the amount of pore fluid per unit volume of porous medium must be defined. For a saturated material containing an incompressible fluid, this variable, θ, is equal to $\eta - \eta_0$, where η and η_0 are porosities in the strained and unstrained states. Biot assumed the relation between change in water content (porosity), pore pressure, and mean stress $(\frac{\sigma_{ii}}{3H})$ to be

$$\theta = P/R + \frac{\sigma_{ii}}{3H} \tag{2}$$

where the coefficient 1/R measures the change in water content (porosity) for a given change in water pressure. This second constitutive relation (equation 2) predicts that an increase in pore-water pressure causes an increase in porosity, and an increase in compression causes a decrease in porosity in a water-saturated body.

The constitutive equations (1) and (2) may be inverted for stress in terms of strain to give

$$\sigma_{ij} = \lambda\delta_{ij}e_{kK} + 2\mu e_{ij} + \alpha\delta_{ij}P \tag{3}$$

where $\alpha = \frac{3\lambda + 2\mu}{3H} = \frac{K}{H}$ where K is the bulk modulus and

$$\theta = \alpha e_{ii} + P/Q \tag{4}$$

where $1/Q = 1/R - \alpha/H$.

The stresses given by equation (3) must satisfy equilibrium, or

$$\frac{\partial\sigma_{ij}}{\partial x_j} + \rho F_i = 0 \tag{5}$$

where F_i represents the components of body force per unit mass at the point x_i. Substitution of equation (3) into (5) and use of the strain displacement relation $e_{ij} = {}^1/{}_2[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}]$ leads to the equations of equilibrium in terms of displacements

$$(\lambda + \mu)\frac{\partial}{\partial x_j}(\frac{\partial u_i}{\partial x_i}) + \mu\frac{\partial}{\partial x_i}(\frac{\partial u_j}{\partial x_i}) - \alpha\frac{\partial P}{\partial x_j} + \rho F_j = 0 \qquad (6).$$

We see that the pressure gradients $\frac{\partial P}{\partial x_j}$ affect the displacements like a body force.

An additional relation is needed to describe the flow of pore fluid in response to changes in pore pressure. According to Darcy's Law, the rate of flow of fluid, V_i, at a point, defined by the volume of fluid crossing a unit area per unit time, is proportional to the gradient of pore pressure at that point, or

$$V_i = -k \frac{\partial P}{\partial x_i} \qquad (7)$$

where k is a constant called the permeability.

Assuming the pore fluid to be incompressible, continuity requires the increase of fluid content per unit time in a volume of porous solid, or $\int_v \frac{\partial \theta}{\partial t} dv$, to equal the volume of fluid entering per unit time through the surface of the volume, or $-\int_s V_i \eta_i ds$, where η_i is an outward normal to s. We have, then,

$$\int_v \frac{\partial \theta}{\partial t} dv = -\int_s V_i \eta_i ds$$

which, by the divergence theorem of Gauss, is

$$\int_v \frac{\partial \theta}{\partial t} dv = -\int_v \frac{\partial V_i}{\partial x_i} dv$$

or, finally, the statement of continuity:

$$\frac{\partial \theta}{\partial t} + \frac{\partial V_i}{\partial x_i} = 0 \qquad (8).$$

substituting equations (4) and (7) in (8) leads to,

$$k \frac{\partial^2 P}{\partial x_i \partial x_i} = \alpha \frac{\partial \varepsilon_{kk}}{\partial t} + 1/Q \frac{\partial P}{\partial t} \qquad (9).$$

The four equations (6) and (9) in the four unknowns u_i and P constitute the basic equations in Biot's (1941) theory of consolidation.

HYDROSTATIC CONSOLIDATION OF ISOTROPIC POROUS SOLID

Consider the cylindrical specimen of porous rock shown in figure 1. The specimen is jacketed so that fluid flow is parallel to the x_1 axis. Under hydrostatic conditions $\sigma_{11} = \sigma_{22} = \sigma_{33} = -\sigma$ and because the porous rock is assumed to be isotropic $e_{11} = e_{22} = e_{33}$ or $e_{kk} = 3e_{11}$ and from equations (3),

$$e_{kk} = 3e_{11} = \frac{\alpha P - \sigma}{K} \qquad (10).$$

Because fluid flow occurs only in the x_1 direction equation (9), the statement of continuity, reduces to,

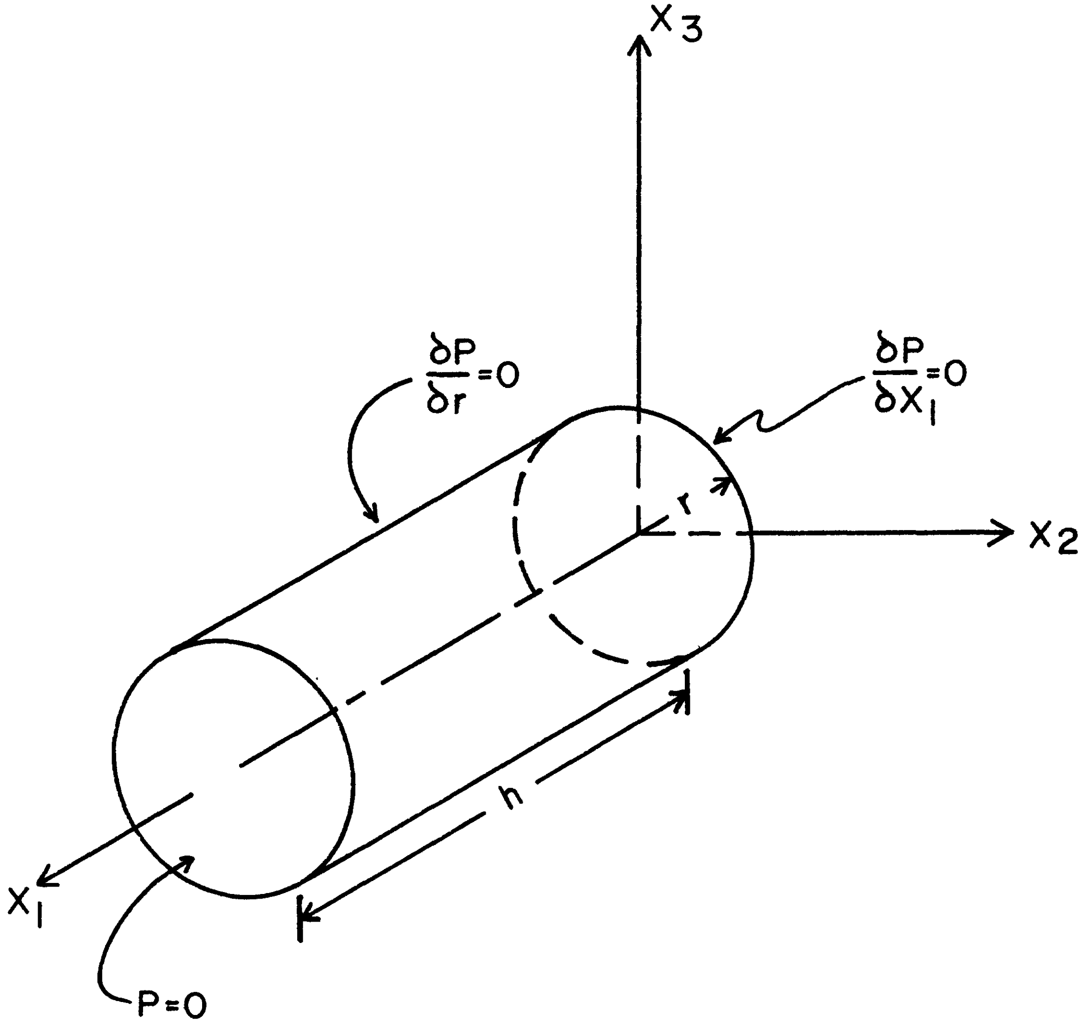

Figure 1.--Cylindrical specimen of length h. Boundary conditions on pore pressure are as shown.

$$k \frac{\partial^2 P}{\partial x_1^2} = \frac{\alpha}{K} \frac{\partial}{\partial t} [\alpha P - \sigma] + 1/Q \frac{\partial P}{\partial t} \qquad (11),$$

where we have used equation (10) for e_{kk}. Also, because the confining pressure, σ, is independent of time equation (11) reduces to,

$$k \frac{\partial^2 P}{\partial x_1^2} = [\frac{\alpha}{K} + \frac{1}{Q}] \frac{\partial P}{\partial t} \qquad (12).$$

In equations (9) and (12), the coefficient 1/Q is a measure of the amount of water than can be forced into a porous material when the volume of material is kept constant. For most rocks the compressibility of the mineral constituents is small; thus $1/Q \approx 0$. Further, because of this small compressibility and because of the small compressibility of water, changes in water content and volumetric strains are approximately equal. Such a condition is satisfied when α=1 and 1/Q=0 in equation (4). Then, since $1/Q = 1/R - \alpha/H$ and $\alpha = K/H$, we have for rocks with α=1 and 1/Q=0, R=H=K where K is the bulk modulus of the porous matrix and in particular equation (12) becomes,

$$\frac{\partial^2 P}{\partial x_1^2} = \frac{1}{C} \frac{\partial P}{\partial t} \qquad (13)$$

where C, the coefficient of consolidation for this case, equals kK.

In Braddock and Machette's (1976) experimental apparatus, a number of different drainage configurations are used. In the present model, we will assume that the end of the specimen x_3=h (fig. 1) is drained so that here P=0. Also, no flow occurs at the end x_1=0 (where the pore pressure is measured) and thus by Darcy's Law (equation 7) $\frac{\partial P}{\partial x} = 0$ at $x_1 = 0$. This is equivalent to their drainage configuration 1a (Braddock and Machette, 1976, p. 24).

Thus, the distribution of pore pressure during hydrostatic consolidation is given by a diffusion-type equation for pore pressure (equation 13) subject to the initial condition,

$$P = -\sigma \qquad t = 0,\ 0 < x_1 < h,$$

and boundary conditions,

$$P = 0 \qquad x_1 = h,\ t > 0$$

$$\frac{\partial P}{\partial x_1} = 0 \qquad x_1 = 0,\ t > 0.$$

The solution to equation (13) subject to these conditions is found in Carslaw and Jaeger (1959, p. 97) and is,

$$P = \frac{4\sigma}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} e^{-C(2n+1)^2 \pi^2 t/4h^2} \cos \frac{(2n+1)\pi x_1}{2h}$$

$$= \sigma \left[1 - \sum_{n=0}^{\infty} (-1)^n \left\{\operatorname{erfc} \frac{(2n+1)h-x_1}{2(Ct)^{1/2}} + \operatorname{erfc} \frac{(2n+1)h+x_1}{2(Ct)^{1/2}}\right\}\right] \qquad (14),$$

where erfc is the complementary error function. The second form of the solution (14) is most suitable for small times.

The pore pressure measured at $x_1=0$ is,

$$P = \frac{4\sigma}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} e^{-C(2n+1)^2 \pi^2 t/4h^2}$$

$$= \sigma \left[1-2 \sum_{n=0}^{\infty} (-1)^n \operatorname{erfc} \frac{(2n+1)h}{2(Ct)^{1/2}} \right] \qquad (15).$$

Defining the dimensionless variable $\tau = \frac{\pi C t}{4h^2}$ we write the dimensionless expressions for the pore pressure at $x_1=0$,

$$P/\sigma = 1 - 2 \sum_{n=0}^{\infty} (-1)^n \operatorname{erfc} (2n+1) \pi/4\tau^{1/2}$$

$$= \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} e^{-(2n+1)^2\tau} \qquad (16).$$

The first form of equation (16) converges rapidly for small τ and the second form converges rapidly for large τ. The variation of P/σ with τ is shown in figure 2. From equations (16) and figure 2 we see that $P = \sigma$ when $\tau = 0$ and $P/\sigma \to 0$ as $\tau \to \infty$.

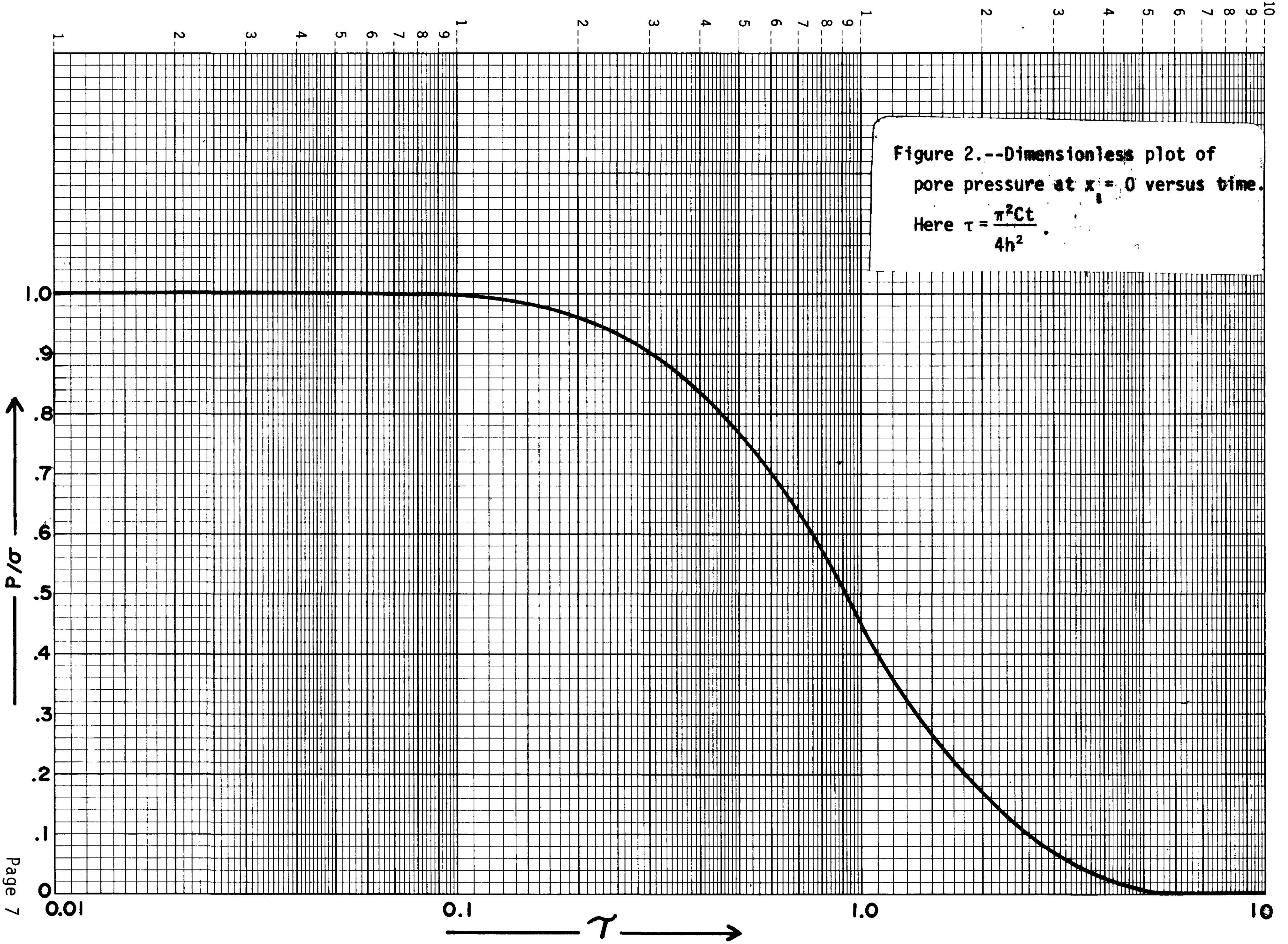

Figure 2.--Dimensionless plot of pore pressure at $x_1 = 0$ versus time. Here $\tau = \frac{\pi^2 Ct}{4h^2}$.

The axial strain field is given by substituting equation 14 in equation (10) or,

$$e_{11} = \frac{P - \sigma}{3K}$$

$$= \frac{\sigma}{3K}\left[\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)} e^{-C(2n+1)^2\pi^2 t/4h^2} \cos\frac{(2n+1)x_1}{2h} - 1\right]$$

$$= \frac{-\sigma}{3K}\sum_{n=0}^{\infty}(-1)^n\left[\operatorname{erfc}\frac{(2n+1)h-x_1}{2(Ct)^{1/2}} + \operatorname{erfc}\frac{(2n+1)h+x_1}{2(Ct)^{1/2}}\right] \qquad (17).$$

The axial displacement field is obtained by integrating equations (17) with respect to x_1. At $x_1=h$ the axial displacement is $u_1=h-h_0$, where h_0 is the original length of the specimen. The total axial strain measured in the consolidation experiment is then given by $\varepsilon_a = \frac{h-h_0}{h_0}$ (change in length of the specimen over the original length) or,

$$\varepsilon_a = \frac{h-h_0}{h_0} = \frac{\sigma}{3K}\left[\frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} e^{-C(2n+1)^2\pi^2 t/4h^2} - 1\right]$$

$$= \frac{2\sigma}{3K}\left(\frac{Ct}{h^2}\right)^{1/2}\left[\pi^{-1/2} + 2\sum_{n=0}^{\infty}(-1)^n \operatorname{ierfc}\frac{nh}{(Ct)^{1/2}}\right] \qquad (19),$$

where ierfc represents the integral of the complementary error function (Carslaw and Jaeger, 1959, p. 483).

Again, with $\tau = \frac{\pi^2 Ct}{4h^2}$, we have the dimensionless expressions for the total axial strain,

$$\frac{3K\varepsilon_a}{\sigma} = \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} e^{-(2n+1)^2\tau} - 1$$

$$= \frac{-4}{\pi}\left[\tau^{1/2}\right]\left[\pi^{-1/2} + 2\sum_{n=0}^{\infty}(-1)^n \operatorname{ierfc}\frac{n\pi}{2\tau^{1/2}}\right] \qquad (20),$$

where the second form is most suitable for small τ. The variation of $\frac{3K\varepsilon_a}{\sigma}$ is shown in figure 3 (left-hand side). Note from equations (20) and figure 3

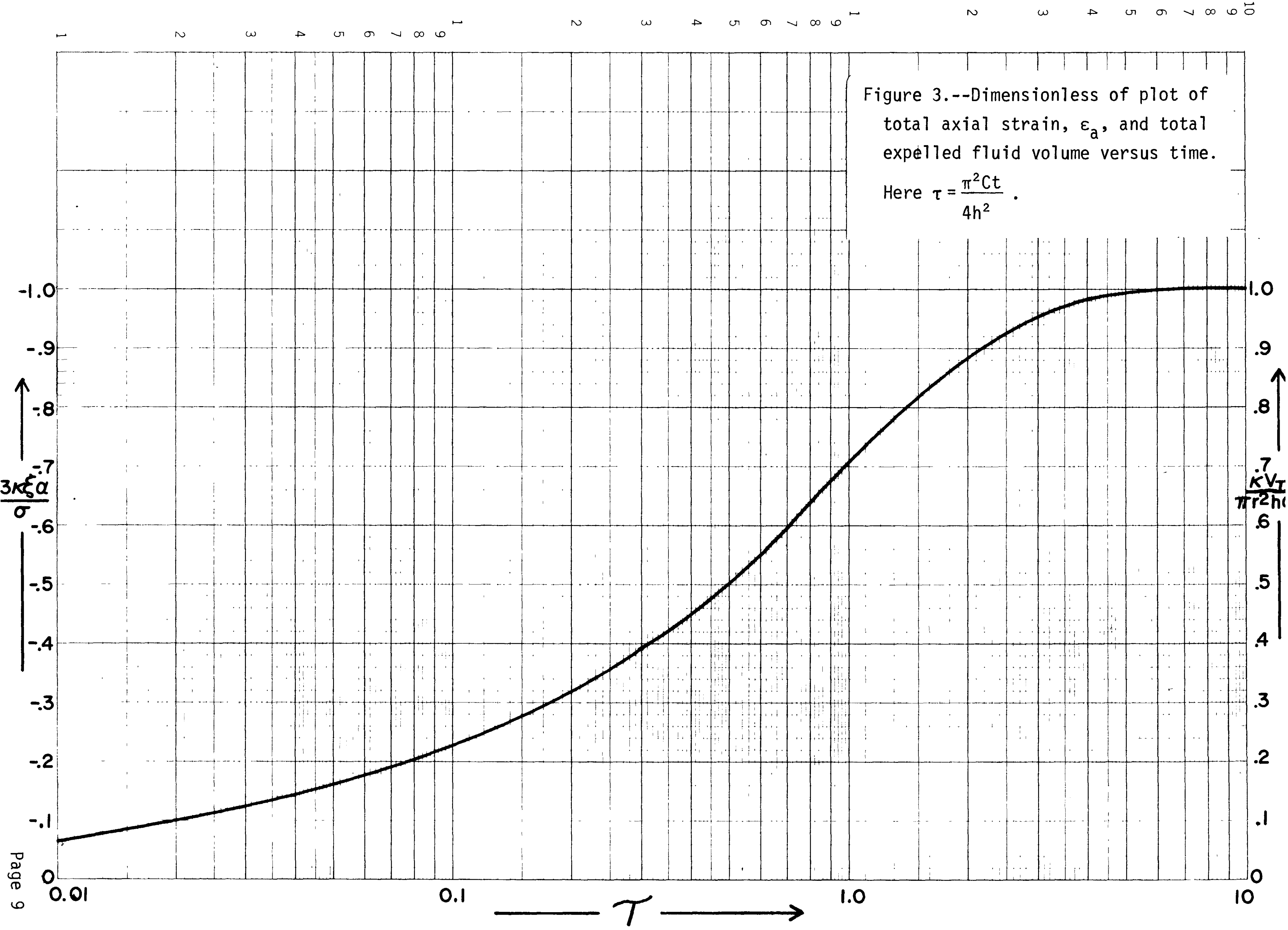

Figure 3.--Dimensionless of plot of total axial strain, ε_a, and total expelled fluid volume versus time. Here $\tau = \frac{\pi^2 Ct}{4h^2}$.

that $\varepsilon_a = 0$ for $\tau = 0$ and $\varepsilon_a = \frac{-\sigma}{3K}$ as $\tau \to \infty$. Thus for large times, when the pore pressure (fig. 2) vanishes, the specimen reaches a final total axial elastic compressive strain given by $\varepsilon_a = -\sigma/3K$.

Braddock and Machette (1976) measured the volume of water expelled from samples during consolidation. This was done at the end x_1=h (fig. 1) in their experimental drainage configuration 1a. The volume of fluid crossing unit area per unit time, V, during consolidation is given by Darcy's Law (equation 7) where $\frac{\partial P}{\partial x_1}$ is given by the gradient with respect to x_1 of equation (14). At x_1=h then,

$$V = \frac{-2k\sigma}{h} \sum_{n=0}^{\infty} e^{-C\ (2n+1)^2\pi^2 t/4h^2}$$

$$= \frac{-k\sigma}{(\pi C t)^{1/2}} [\ 1 + 2 \sum_{n=1}^{\infty} (-1)^n e^{\ -n^2 h^2/Ct}] \qquad (21).$$

The total volume of fluid, V_T, crossing the end x_3=h of the sample up to a time t (the volume measured by Braddock and Machette) is then given by integration of equations (21) or,

$$V_T = \pi r^2 \int_0^t V\ dt$$

$$= \frac{\pi r^2 h\sigma}{K} [\ 1 - \frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{e^{-C(2n\ +\ 1)^2\pi^2 t/4h^2}}{(2n+1)^2}\ ]$$

$$= \frac{2\pi r^2 h}{K} [\ \frac{Ct}{h^2}\ ]^{1/2} [\pi^{-1/2} + \sum_{n=0}^{\infty} (-1)^n \text{ ierfc } \frac{nh}{(Ct)^{1/2}} \qquad (22).$$

Equations (22) are written in dimensionless form as

$$\frac{KV_T}{\pi r^2 h\sigma} = 1 - \frac{8}{\pi^2} \sum_{n=0}^{\infty} e^{-(2n+1)^2\tau}$$

$$= \frac{4\tau^{1/2}}{\pi} [\ \pi^{-1/2} + 2 \sum_{n=0}^{\infty} (-1)^n \text{ ierfc } \frac{n\pi}{2\tau^{1/2}}\ ] \qquad (23).$$

We see that equations (23) are identical in form but opposite in sign to equations (20) for the dimensionless total axial strains. This is a consequence of the assumption that changes in water content and volumetric strains are approximately equal. The variation of $\frac{KV_T}{\pi r^2 h\sigma}$ with τ is shown on the right-hand side of figure 3. We see that the amount of fluid expelled is zero at $\tau=0$ and approaches a fixed value at large τ. This fixed value is equal to the negative of the total change in fluid content of the sample as can be seen from the definition of change in fluid content θ, the change in volume of fluid per unit volume, and equations (2) and (10). As the pore pressure goes to zero in equations (2) and (10) the final change in fluid content and porosity in a saturated sample is then $\theta = \eta - \eta_0 = -\sigma/K$ where K is the bulk modulus of the porous matrix. The total change in fluid content is then $-\pi r^2 h\ \sigma/K$ which is the negative of the total volume of fluid expelled at $x_1=h$. Hence, the volume of fluid that can be removed depends on the compressibility of the porous matrix, 1/K, and the confining pressure σ. Increasing σ or decreasing K leads to an increase in the volume of fluid removed and a further decrease in porosity.

HYDROSTATIC CONSOLIDATION OF A TRANSVERSELY ISOTROPIC SOLID

Up to this point in the theoretical development for hydrostatic consolidation the porous solid has been assemed to be isotropic. However, Pierre Shale, like most sedimentary rocks, is transversely isotropic. Such a material has one set of properties in the plane of the bedding and a different set of properties normal to the plane of the bedding. To be consistent with the experimental procedures of Braddock and Machette (1976) we will take planes parallel to the x_1x_2 plane on figure 1 as the plane of the bedding and directions parallel to the x_3 axis as normal to the bedding. For this case, where x_3 is the axis of elastic symmetry, equations 1 are written,

$$e_{11} = \frac{\sigma_{11}}{E_1} - \frac{\nu_1\sigma_{22}}{E_1} - \frac{\nu_2\sigma_{33}}{E_2} + P/3H_1$$

$$e_{22} = \frac{-\nu_1}{E_1}\,{}_{11} + \frac{\sigma_{22}}{E_1} - \frac{\nu_2}{E_2}\sigma_{33} + P/3H_1$$

$$e_{33} = \frac{-\nu_2}{E_2}\sigma_{11} - \frac{\nu_2}{E_2}\sigma_{22} + \frac{\sigma_{33}}{E_2} + P/3H_2$$

$$e_{12} = \left[\frac{2(1+\nu_1)}{E_1}\right]\sigma_{12}$$

$$e_{13} = \frac{1}{\mu_2}\sigma_{13}$$

$$e_{23} = \frac{1}{\mu 2}\sigma_{23} \qquad (24),$$

where E_1 is Young's modulus and ν_1 is Poisson's ratio in the plane of the bedding, E_2 is Young's modulus, ν_2 is Poisson's ratio, and μ_2 is the shear modulus normal to the bedding plane. The coefficient $1/H_1$ is a measure of the compressibility of the porous material in the bedding plane for a given change in pore water pressure and $1/H_2$ has a similar meaning for porous material normal to the plane of the bedding.[1]

The relation between water content, pore pressure and stress given by equation (2) for an isotropic porous solid becomes in this case,

$$\theta = \frac{\sigma_{11} + \sigma_{22}}{3H_1} + \frac{\sigma_{33}}{3H_2} + P/R \qquad (25),$$

where $1/R$ measures the change in water content for a given change in pore water pressure.

Consider a hydrostatic jacketed test in which the pore pressure is equal to the negative of the hydrostatic pressure; $\sigma_{11} = \sigma_{22} = \sigma_{33} = -\sigma$, $P = \sigma$. For a saturated porous solid $\theta = \eta - \eta_0$. If the pore fluid and the mineral constituents are assumed to be incompressible there will be no change in porosity or $\theta = 0$ in this test and from equation (25) we find,

$$1/R = \frac{2H_2 + H_1}{3H_1 H_2},$$

and in general,

$$\theta = \frac{\sigma_{11} + \sigma_{22}}{3H_1} + \frac{\sigma_{33}}{3H_2} + \left[\frac{2}{3H_1} + \frac{1}{3H_2} \right] P \qquad (26).$$

If the pore fluid is now allowed to drain out under hydrostatic stress the change in volume, e_{kk}, will be approximately equal to the volume of water squeezed out, θ. After a sufficient time the pore pressure will vanish and from equations (25) and (26),

$$e_{kk} = e_{11} + e_{22} + e_{33} = \left[\frac{1}{E_1} - \frac{\nu_1}{E_1} - \frac{\nu_2}{E_2} \right] (\sigma_{11} + \sigma_{22})$$

$$+ \left[\frac{1 - 2\nu_2}{E_2} \right] \sigma_{33} = \theta = \frac{\sigma_{11} + \sigma_{22}}{3H_1} + \frac{\sigma_{33}}{3H_2},$$

and we see that,

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[1] To interpret $1/H_1$ and $1/H_2$ consider a jacketed sample under no external loads; $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{12} = \sigma_{23} = \sigma_{13} = 0$. A tube is attached to the jacketed sample and water under a known pressure is pumped in. The resulting strains are measured and $1/H_1$ and $1/H_2$ are found from equations (24) or,

$$1/3H_1 = \frac{e_{11}}{P} = \frac{e_{22}}{P} \text{ and } 1/3H_2 = \frac{e_{33}}{P}.$$

$$\frac{1}{3H_1} = \frac{1-\nu_1}{E_1} - \frac{\nu_2}{E_2}$$

$$\frac{1}{3H_2} = \frac{1-2\nu_2}{E_2}.$$

Then, for the special case of a saturated porous solid with incompressible mineral constituents and an incompressible pore fluid, equations (25) and (26) become,

$$e_{11} = \frac{\sigma_{11}}{E_1} - \frac{\nu_1}{E_1}\sigma_{22} - \frac{\nu_2}{E_2}\sigma_{33} + [\frac{1-\nu_1}{E_1} - \frac{\nu_2}{E_2}]\, P$$

$$e_{22} = \frac{-\nu_1}{E_1}\sigma_{11} + \frac{\sigma_{22}}{E_1} - \frac{\nu_2}{E_2}\sigma_{33} + [\frac{1-\nu_1}{E_1} - \frac{\nu_2}{E_2}]\, P$$

$$e_{33} = \frac{-\nu_2}{E_2}\sigma_{11} - \frac{\nu_2}{E_2}\sigma_{22} + \frac{\sigma_{33}}{E_2} + [\frac{1-2\nu_2}{E_2}]\, P$$

$$e_{12} = [\frac{2(1+\nu_1)}{E_1}]\, \sigma_{12}$$

$$e_{13} = \frac{1}{\mu_2}\sigma_{13}$$

$$e_{23} = \frac{1}{\mu}\sigma_{23}$$

$$\theta = [\frac{1-\nu_1}{E_1} - \frac{\nu_2}{E_2}]\,(\sigma_{11} + \sigma_{22}) + [\frac{1-2\nu_2}{E_2}]\, \sigma_{33}$$

$$+ [\frac{2(1-\nu_1)}{E_1} + \frac{(1-4\nu_2)}{E_2}]\, P \qquad (27).$$

Darcy's Law governing the flow of fluids in porous solids becomes for the case of transverse isotropy,

$$V_1 = -k_1 \frac{\partial P}{\partial x_1}$$

$$V_2 = -k_1 \frac{\partial P}{\partial x_2}$$

$$V_3 = -k_2 \frac{\partial P}{\partial x_3} \qquad (28).$$

where k_1 is the permeability in the plane of the bedding and k_2 is the permeability in directions normal to the plane of the bedding.

The equation of continuity (equation 8) for the case of transverse isotropy is,

$$\frac{\partial \theta}{\partial t} = k_1 \left[\frac{\partial^2 P}{\partial x_1^2} + \frac{\partial^2 P}{\partial x_2^2} \right] + k_2 \frac{\partial^2 P}{\partial x_3^2} \qquad (29).$$

Substituting equation (27) for θ in equation (29) we arrive at,

$$k_1 \left[\frac{\partial^2 P}{\partial x_1^2} + \frac{\partial^2 P}{\partial x_2^2} \right] + k_2 \frac{\partial^2 P}{\partial x_3^2} = \left[\frac{1-\nu_1}{E_1} - \frac{\nu_2}{E_2} \right] \frac{\partial}{\partial t} (\sigma_{11} + \sigma_{22})$$

$$+ \left[\frac{1-2\nu_2}{E_2} \right] \frac{\partial \sigma_{33}}{\partial t} + \left[\frac{2(1-\nu_1)}{E_1} + \frac{(1-4\nu_2)}{E_2} \right] \frac{\partial P}{\partial t} \qquad (30).$$

Since the cylindrical sample is jacketed so that fluid flow occurs only in the x_1 direction, and the confining pressure ($\sigma_{11} = \sigma_{22} = \sigma_{33} = -\sigma$) is independent of time, equation (30) reduces to the form of equation (13) except that the coefficient of consolidation, C, is replaced by C_1, which is defined by,

$$1/C_1 = \frac{1}{k_1} \left[\frac{2(1-\nu_1)}{E_1} + \frac{(1-4\nu_2)}{E_2} \right] = \frac{1}{k_1 \bar{K}} \qquad (31).$$

Again we have the initial and boundary conditions,

$P = -\sigma \quad t = 0,\ 0 \leq x_1 \leq h$

$P = 0 \quad t > 0,\ x_1 = h$

$\frac{\partial P}{\partial x_1} = 0 \quad t > 0,\ x_1 = 0,$

and pore pressures are given by equations (14) and (15) with C replaced by C_1. The pore pressure at x_1=0 for the transversely isotropic case is the, in dimensionless form,

$$P/\sigma = 1-2 \sum_{n=0}^{\infty} (-1)^n \mathrm{erfc}\ (2n + 1)\ \pi/4\tau^{1/2}$$

$$= \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)} e^{-(2n + 1)^2 \tau} \qquad (32).$$

Equations (32) are identical to equations (16), except that $\tau = \dfrac{\pi^2 C_1 t}{4h^2}$. Hence, figure 2 can be used to obtain the pore pressure in the transversely isotropic case.

The strains resulting from hydrostatic consolidation are obtained from equations (27) and can be written,

$$e_{11} = e_{22} = \frac{1}{3K_1}[\,P - \sigma\,]$$

$$e_{33} = \frac{1}{3K_2}[\,P - \sigma\,] \qquad (33),$$

where $1/3K_1 = \dfrac{1}{3H_1} = \dfrac{1 - \nu_1}{E_1} - \dfrac{\nu_2}{E_2}$ and $1/3K_2 = \dfrac{1 - 2\nu_2}{E_2}$. Following the same arguments that led to equations (19) for the total axial strain we arrive at the dimensionless form,

$$\frac{3K_1 \varepsilon_a}{\sigma} = \frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n + 1)^2} e^{-(2n + 1)^2 \tau} - 1$$

$$= \frac{-4}{\pi} [\,\tau^{1/2}\,] [\,\pi^{-1/2} + 2 \sum_{n=0}^{\infty} (-1)^n \operatorname{ierfc} \frac{n\pi}{2\tau^{1/2}}\,] \qquad (34).$$

Equation 34 is of the same form as equation (20) except that K is replaced by K_1 and $\tau = \dfrac{\pi^2 C_1 t}{4h^2}$ and hence the time variation of ε_a in the transversely isotropic case can be obtained by using figure 3.

Arguments similar to those given in the isotropic case lead directly to the dimensionless expression for the total volume of fluid expelled at $x_1 = h$,

$$\frac{\bar{K} V_T}{\pi r^2 h \sigma} = 1 - \frac{8}{\pi^2} \sum_{n=0}^{\infty} e^{-(2n + 1)^2 \tau}$$

$$= \frac{4\tau^{1/2}}{\pi} [\,\pi^{-1/2} + 2 \sum_{n=0}^{\infty} (-1)^n \operatorname{ierfc} \frac{n\pi}{2\tau^{1/2}}\,] \qquad (35),$$

where $\bar{K}$ is given by equation (31) and $\tau = \dfrac{\pi^2 C_1 t}{4h^2}$. Again this is seen to be of the same form as the expression (equation 23) for the isotropic case and thus the total volume of expelled fluid in the transversely isotropic case can be obtained from figure 3.

In summary, the time variations of pore pressures at $x_1=0$, total axial strains, and total expelled fluid volumes for hydrostatic consolidation of a transversely isotropic porous solid can all be obtained from figures 2 and 3 provided appropriate bulk moduli (K_1 and $\bar{K}$) and an appropriate coefficient of consolidation, C_1, can be found. We will now consider how these constants can be found from a particular hydrostatic consolidation test.

APPLICATION TO A PARTICULAR HYDROSTATIC CONSOLIDATION TEST

Figures 4, 5, and 6 show respectively the variation with time of the pore pressure, the axial strain and the total expelled fluid volume for Braddock and Machette's (1976, fig. 13, p. 36) hydrostatic-consolidation test 152. The test was carried out under a confining pressure of 400 bars (40 MPa) with the bedding parallel to the x_1x_2 plane on figure 1, and with a drainage and pore pressure measuring configuration consistent with that shown on figure 1.

Following the usual procedures for matching theoretical and experimental curves the data in figures 4, 5, and 6 was plotted on translucent semilogarithmic paper on comparable scales to figures 2 and 3 and horizontally shifted until "best fit" was obtained. The resulting ratio of τ to t yields the consolidation coefficient,

$$C_1 = \frac{4h^2}{\pi^2}\tau/t,$$

where h is the original sample length (5 cm). Comparing the experimental points of figures 4, 5, and 6 with the theoretical curves of figures 2 and 3 we find that in all cases the experimental and theoretical points are within 15 percent of each other when $C_1 = 2.81 \times 10^{-6}$ cm²/sec.

To find the constant K_1 we match points between the dimensionless theoretical curve (fig. 3) for ε_a and the "best fit" curve of figure 5. For example, at 700 hours (fig. 5) $\varepsilon_a = -0.0101$ and the comparable point on figure 3 at $\tau = 0.7$ is given by $\frac{3K_1\varepsilon_a}{\sigma} = 0.6$. Substituting the known values $\sigma = -400 \times 10^6 \frac{\text{dyne}}{\text{cm}^2}$ (-40 MPa) and $\varepsilon_a = 0.0101$ and solving for K_1 yields

$$K_1 = 7.92 \times 10^9 \frac{\text{dyne}}{\text{cm}^2} \text{ (0.792 GPa).}$$

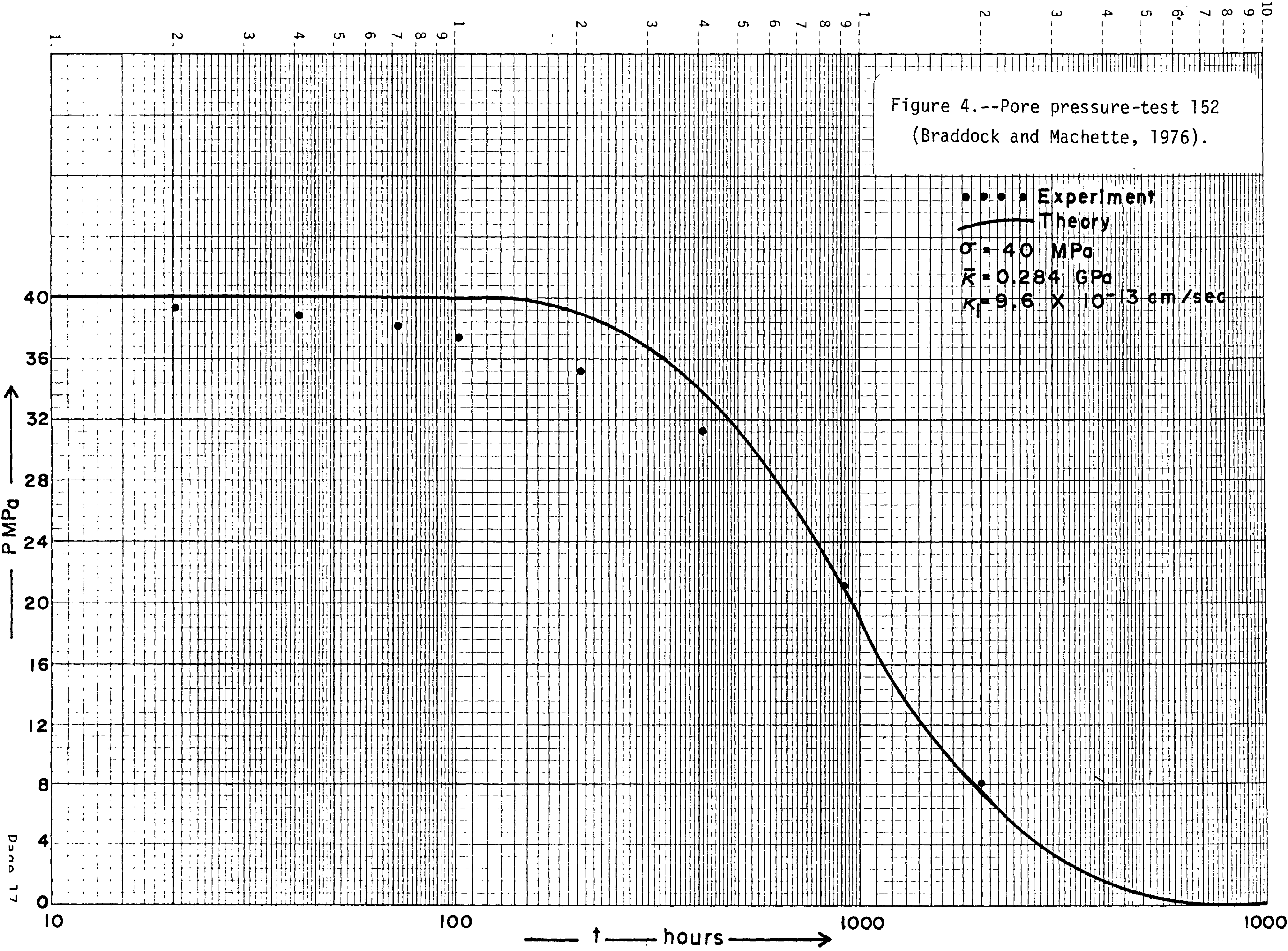

Figure 4.--Pore pressure-test 152 (Braddock and Machette, 1976).

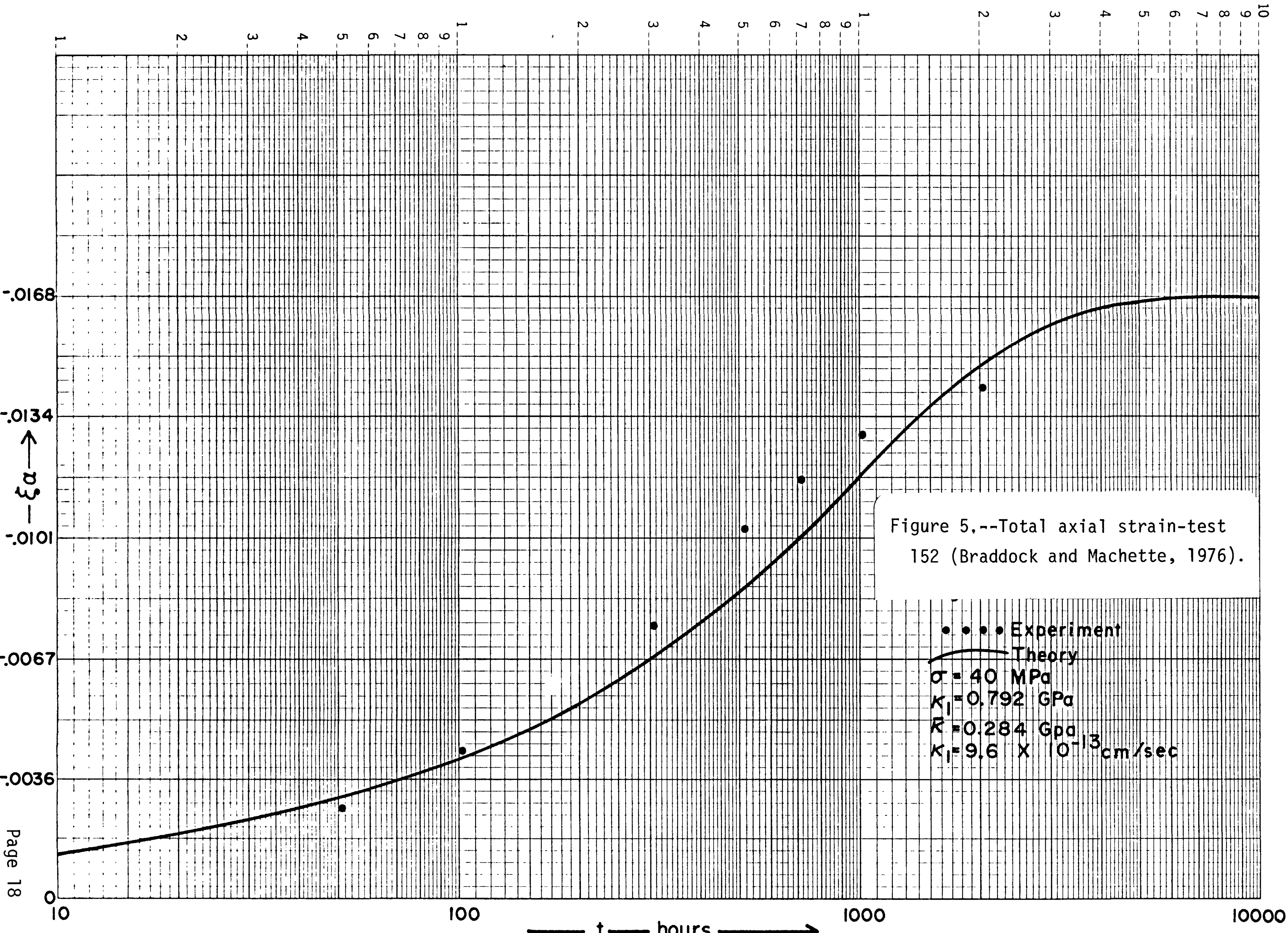

Figure 5.--Total axial strain-test 152 (Braddock and Machette, 1976).

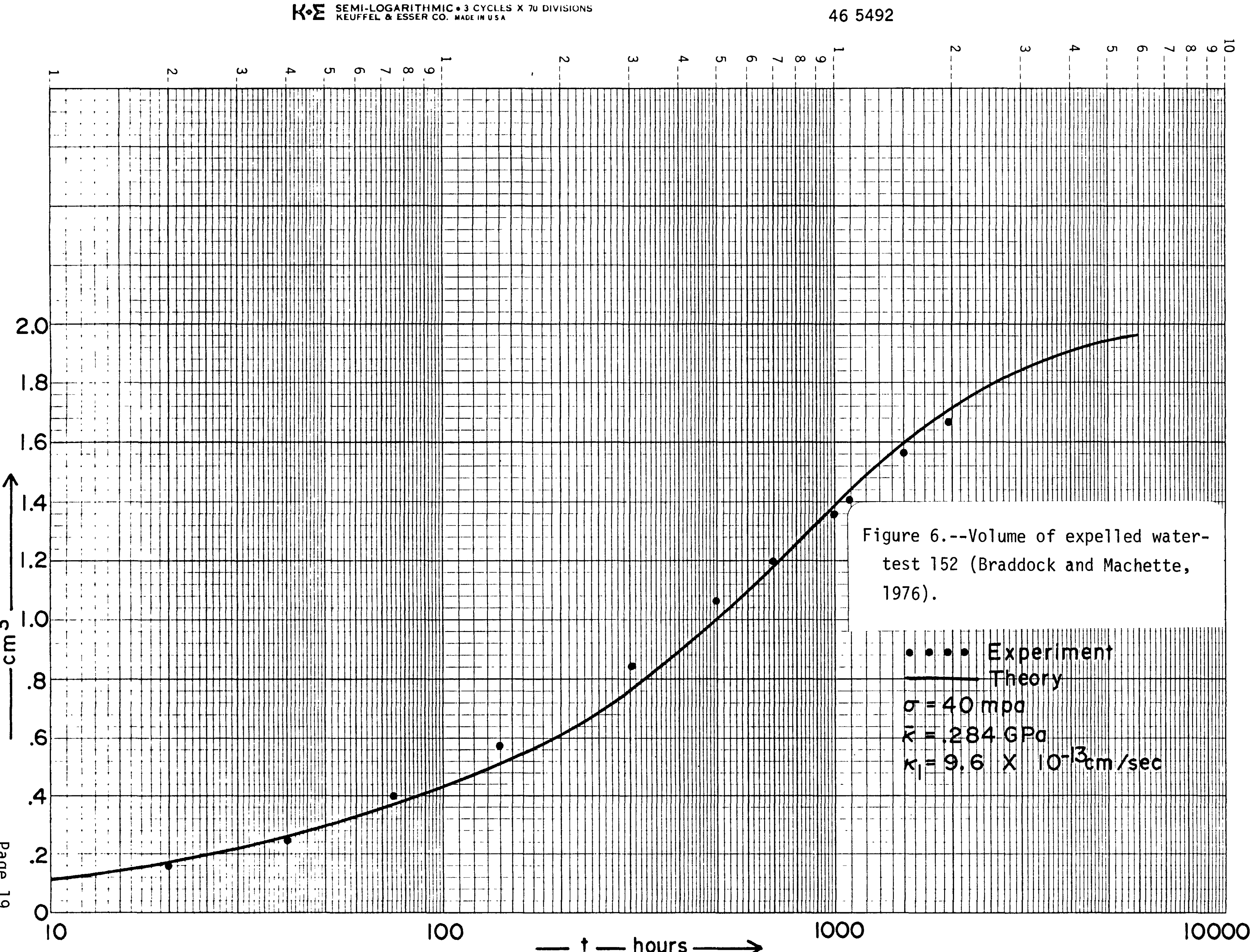

Figure 6.--Volume of expelled water-test 152 (Braddock and Machette, 1976).

In the same manner the constant $\bar{K}$ can be found by comparing the dimensionless theoretical curve for V_T (fig. 3) and the "best fit" curve of figure 6. We find that $K = 2.84 \text{ X } 10^9 \frac{\text{dyne}}{\text{cm}^2}$ (0.284 GPa).

Having determined C_1 and $\bar{K}$ we can next determine the permeability in the axial direction, k_1, from the relation $C_1 = k_1\bar{K}$ (equation 31). Solving for k_1 we find it to be $9.86 \text{ X } 10^{-16}\text{cm}^3/\text{sec/dyne/cm}$ which is $9.99 \text{ X } 10^{-10}$ darcys or $9.60 \text{ X } 10^{-13}\text{cm/sec}$. This permeability is similar to but somewhat larger than the value ($5.6 \text{ X } 10^{-14}\text{cm/sec}$) obtained by Braddock and Machette (1976) in test 152.

Also, as $\bar{K}$ and K_1 are known, $K_2 = \frac{3(1 - \nu_2)}{E_2}$ can be found from equation (31), that is, from,

$$1/\bar{K} = [\, \frac{2(1 - \nu_1)}{E_1} + \frac{(1 - 4\nu_2)}{E_2} \,].$$

$$= [\, \frac{2(1 - \nu_1)}{E_1} - \frac{2\nu_2}{E_2} + \frac{1 - 2\nu_2}{E_2} \,]$$

$$= 1/3\ [\, \frac{2}{K_1} + \frac{1}{K_2} \,]\ .$$

Solving for K_2 we see that $K_2 = 1.24 \text{ X } 10 \frac{9 \text{ dyne}}{\text{cm}^2}$ (0.124 GPa). Hence, at the end of consolidation when $P = 0$, e_{33}, the strain perpendicular to bedding, is predicted from the last of equations (33) to be -0.107. This is roughly 6.4 times greater than the final axial strain predicted by equation (34) and shown on figure 4. Braddock and Machette (1976) found the strain perpendicular to bedding to be 1.1 to 2.6 times as large as strains parallel to bedding in hydrostatic consolidation at 600-bars confining pressure, so the predicted value may be somewhat large.

CONCLUSIONS

Although the theory presented is only a first attempt at modeling laboratory hydrostatic consolidation of Pierre Shale, it appears that it can be used to estimate the time variation of pore pressure, volume of expelled water, and axial strains and especially the long-time behavior of these quantities. Also, because the theory has been applied with moderate success to only one hydrostatic consolidation test, comparisons with other test data should be made which, no doubt, will force further theoretical refinements.

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