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LINEAR Q MODEL CALCULATIONS

by

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INTRODUCTION

The purpose of this note is to give a brief review of the mathematics of viscoelasticity required to understand calculations of Q, current frequency dependent Q models and to present details of one such model. The following is a short discussion of the context of theories of energy loss.

Three features are observed which can be included in the phenomenological theory of viscoelasticity:

1) Amplitude decay
2) Phase shift between stress and strain
3) Velocity dispersion

The first needs no comment. Number two is associated with the hysteresis loop. Some discussion of the applicability of linear theory of viscoelasticity has turned on whether experiments show an ellipse, as linear theory predicts, or some other shape. The dissapation parameter Q^{-1} is introduced by linear theory in this context as the tangent of the phase difference between the stress and strain when the system is sinusoidally driven. It will be shown below that this number is the same as one calculated from several other definitions.

Dispersion is a more subtle effect and its significance is still discussed. In terms of linear theory arguments can be given that any causal system must have frequency dependent velocity (eg. Futterman (1962)). A fairly simple arguement for the necessity of velocity dispersion is given in Aki and Richards (1980) where it is shown that withoutr dispersion, a propogating delta function pulse will induce a small finite displacement at a finite distance without any time passing. As shown below, Q and velocity are functionally related, so Q also must, in general, be frequency dependent. Kennett (1983) gives a very brief discussion of the need for frequency dependent Q using arguments similar to those for frequency dependent velocity.

The problem is to determine the nature of the frequency dependence. The Minster-Anderson model (Anderson and Minster (1979), Minster and Anderson (1981)) discussed below reflects an attempt to give a weak frequency dependence to Q over a finite frequency band under the constraints of a linear physical theory.

The relevence to three dimensions is that, upon introduction of the angle γ between phase propogation and amplitude decay (Borcherdt (1971,1973), Buchen (1973), Krebes (1980)), one can parametrize all features of a plane wave using the velocity and Q of a homogeneous (i.e. γ=0) wave whose properties are essentially one dimensional.

The discussion which follows occasionally bogs down in the details of calculation, the intention is completeness.

PHENOMENOLOGICAL THEORY

Constitutive Equations

Let σ denote the stress and ϵ the strain. Then the phenomenological stress strain relations for viscoelasticity are:

$$\epsilon = \int_{-\infty}^{t} J(t-u)\,d\sigma(u) \tag{1}$$

$$\sigma = \int_{-\infty}^{t} M(t-u)\,d\epsilon(u)$$

where J and M are known as creep and relaxation functions respectively and are determined from the system response to a step in stress or strain.

To consider steady state vibrations we write

$$\epsilon = \epsilon_0 e^{i\omega t} \qquad \sigma = \sigma_0 e^{i\omega t} \tag{2}$$

where σ_0 and ϵ_0 are complex quantities dependant only on position. After a change of variables of integration in equations (1) one finds

$$\epsilon = J(\omega)\sigma \qquad \sigma = M(\omega)\epsilon \tag{3}$$

where

$$J(\omega) = i\omega \int_0^{\infty} J(t)e^{i\omega t}dt \tag{4}$$

a similar relation holds between $\mathcal{M}$ and M. M and J are known as the complex modulus and the complex compliance, respectively.

In what follows we will follow convention and consider expressions involving complex compliance, the analogous calculations involving the complex modulus are straightforward.

Potential Energy

In addition to measuring the phase shift as described above, one can define Q^{-1} as a measure of the energy loss per cycle of a driven system (MacDonald (1961), O'Connell and Budiansky (1978)). In order for Q to be dimensionless one normalizes to some measure of energy stored during the cycle. It will be shown that normalizing with either peak potential energy or twice the average potential energy gives the same number as the phase shift definition described above. Since there is some discrepency in the literature over this point, in particular O'Connell and Budiansky (1978) state the two normalizations give different results, some space will be used here to detail the calculations of potential energy. The approach taken here follows Gross (1948) and MacDonald (1961) and differs from that of O'Connell and Budiansky (1978) which relies on the development of the theory of viscoelastic theory in Bland (1960).

The arguments presented in the context of elasticity for the rate of change of internal energy of a deformed solid (Hudson (1980), Jeffreys (1931)) hold as well for viscoelastic materials. One has in one dimension the rate of change

$$W' = \sigma_R \epsilon_R' \tag{5}$$

where a prime indicates differentiation with respect to time and subscripts R and I will denote the real and imaginary parts of a complex quantity. Using equation 3 we can write W' in terms of the stress or strain time function. For example, since $\sigma_I' = \omega\sigma_R$, we have

$$W' = \frac{J_R}{2}\left((\sigma_R)^2\right)' - \omega J_I(\sigma_R)^2 \tag{6}$$

If we now make the assumption that the internal energy can be defined completely by the stress and rate of stress, we can intergrate to calculate the internal energy of the solid. One finds that the first term in (6) corresponds to energy that is alternately "stored and returned" (Gross (1948)). The second term integrates to give a constantly increasing function which corresponds to the dissapation process. Thus one identifies the potential energy density of the solid as

$$V(t) = \frac{J_R}{2}\left(\sigma_R(t)\right)^2 \tag{7}$$

As supplementary arguments for this form note that in the elastic limit one sets J_I=0 and finds that the proposed potential energy expression approaches the one used in the theory of elastic vibration. Furthermore, if one starts from the equation of motion

$$\rho u'' = \frac{u_{xx}}{J} \tag{8}$$

one can derive the following energy conservation law

$$\frac{1}{2}\left(\rho(u_R')^2 + J_R(\sigma_R)^2\right)' - \omega J_I(\sigma_R)^2 = (\sigma_R u_R')_x \tag{9}$$

The term on the right hand side is the energy flux (eg.Hudson (1980)) at a point and the kinetic energy density term is obvious. One is lead to the same candidate for potential energy as the one derived above.

Q has several equivalent definitions

One can calculate the maximum or average potential energy density for a one dimensional wave using equation (7). One finds the same average as reportd by O'Connell and Budiansky (1978) and that it is one half the maximum, where

$$Max\left(V\left(t\right)\right) = \frac{J_R}{2}\left|\sigma_0\right|^2 \tag{10}$$

Using equation (6) to calculate the dissapated energy, one gets

$$\Delta W = -\pi J_I \left|\sigma_0\right|^2 \tag{11}$$

If one defines

$$Q^{-1} = \frac{\Delta W}{2\pi(energy \quad stored \quad in \quad a \quad cycle)},$$

then measuring the energy stored by taking the maximum potential energy or twice the average of the energy stored, as suggested by O'Connell and Budiansky (1978), results in the same expression for Q which is also equal to the tangent of the phase difference betweeen stress and strain (as is readily seen from the definition of complex compliance) and can be expressed in terms of the complex wave number as well (O'Connell and Budiansky (1978)). Thus the various physical definitions of Q all yield the same number which can also be calculated in a variety of ways. We have

$$Q^{-1} = \frac{-J_I}{J_R} = \frac{M_I}{M_R} = \frac{-(k^2)_I}{(k^2)_R} \tag{12}$$

where k is the complex wave number, i.e. we write the displacement as

$$u = u_0 e^{i(\omega t - kx)} \tag{12.5}$$

The relationship to three dimensional waves mentioned in the introduction is based on the fact that analogous results to (12.5) can be found for homogeneous plane waves. For inhomogeneous waves, the two definitions of Q in terms of maximum or average potential energy are no longer equal and the analogs to (12.5) no longer hold since the definitions give different dependencies on the degree of inhomogeniety in the general expressions for the Q.

Velocity

Velocity as well as Q is determined by the complex compliance. In order to find the relation one starts with equation (9) and finds, using (12.5),

$$k = \omega\sqrt{\rho J} \tag{13}$$

Since velocity is determined by the real part of k we have

$$v = \frac{\omega}{k_R} = \frac{1}{Re\sqrt{\rho J}} \tag{14}$$

Writing J in terms of its absolute value and phase, , we have

$$J = |J|\, e^{i\beta} \tag{15}$$

where

$$\beta = arctan\left(\frac{J_I}{J_R}\right) = -arctanQ^{-1}, \tag{16}$$

hence

$$Re\sqrt{J_R} = |J|^{\frac{1}{2}} \cos\frac{\beta}{2} \tag{17}$$

Using the trig identity

$$\cos^2\theta = \frac{\cos(2\theta)+1}{2}, \tag{18}$$

one finds the following expression for velocity

$$v = \sqrt{\frac{2}{\rho(J_R + |J|)}} \tag{19}$$

or

$$v = \sqrt{\frac{2}{\rho J_R(1+\sqrt{1+Q^{-2}})}} \tag{20}$$

The negative of the imaginary part of the wave number k is usually denoted as α and is the coefficient of amplitude decay. Calculations similar to those above (sign caution: $\beta < 0$ since $J_R/J_I < 0$) gives

$$\alpha = \omega\sqrt{\frac{\rho J_R}{2}(-1+\sqrt{1+Q^{-2}})} \tag{21}$$

So we have

$$\alpha = \frac{\omega}{v}\sqrt{\frac{-1-\sqrt{1+Q^{-2}}}{1+\sqrt{1+Q^{-2}}}} \tag{22}$$

If one makes the low loss approximation

$$-1+\sqrt{1+\frac{Q^{-2}}{2}} \approx \frac{Q^{-2}}{2} \tag{23}$$

one gets the familiar expression

$$\alpha = \frac{\omega Q^{-1}}{2v} \tag{24}$$

EXAMPLE OF A STANDARD LINEAR SOLID

Finding a creep function

It is clear that the creep or relaxation function tells the whole story. The problem arises; what are plausible forms for these functions? One approach is to construct a physical model and then calculate its creep function. One does this by solving the differential equation of the system using a step function in stress as input (eg. Hudson (1980)).

The differential equation of a standard linear solid

The physical model of a standard linear solid (Hudson (1980), Flugge (1967)) is a spring and viscous element in parallel connected in series with another spring. Mathematically this configuration allows an arbitrary linear combination of stress, strain and their time derivatives. A convenient form for the equation of motion for this system is

$$M_\infty(\epsilon + \tau\epsilon') = \sigma + \tau\frac{M_\infty}{M_0}\sigma' \tag{25}$$

where M_∞ is the spring constant of the two springs as if they were in series and there were no viscous element, M_0 is the spring constant of the spring not in parallel with the viscous element and τ is the relaxation time of the parallel spring and viscous element system if one considered just those two as an isolated system.

Applying a step function in stress of magnitude σ_0 implies solving the following equations

$$M_\infty(\epsilon + \tau\epsilon') = 0 \qquad t < 0$$

$$M_\infty(\epsilon + \tau\epsilon') = \sigma_0 \qquad t > 0 \tag{26}$$

$$M_0\epsilon = \sigma_0 \qquad t = 0$$

The last equation avoids the problem of differentiating a step function and says simply that the instantaneous response of the system is as if it were an elastic system with spring constant M_0.

The creep function, Q and velocity for a standard linear solid

Solving the equations one finds

$$\epsilon = \sigma_0\left(\frac{1}{M_0} + \left(\frac{1}{M_\infty} - \frac{1}{M_0}\right)\left(1 - e^{\frac{-t}{\tau}}\right)\right) \qquad t \geq 0 \tag{27}$$

So we have a creep function

$$J(t) = J_0\left(1 + \Delta(1 - e^{\frac{-t}{\tau}})\right) \tag{28}$$

where J_0 is the instantaneous compliance, $\frac{1}{M_0}$, and Δ is the percentage difference between the relaxed and instantaneous compliance. One can now implement the program outlined above to find

$$J(t) = J_0\left(1 + \frac{\Delta}{1 + i\omega\tau}\right) \tag{29}$$

$$Q^{-1}(\omega) = \frac{\Delta\omega\tau}{\Delta + 1 + \omega^2\tau^2} \tag{30}$$

and

$$v(\omega) = \sqrt{\frac{2}{\rho J_0\left(1 + \frac{\Delta}{1+\omega^2\tau^2}\right)\left(1 + \sqrt{1 + Q^{-2}}\right)}} \tag{31}$$

DISTRIBUTION OF RELAXATION TIMES

Creep functions

The creep function for a standard linear solid can be interpreted as the sum of an instantaneous, elastic-like component and an exponentially decaying creep term. Some authors, eg. Gross (1948,1953), MacDonald (1961), Minster and Anderson (1979,1981), prefer to decompose our creep function into a function of the form

$$J = J_0(1 + \Delta\psi(t)) \tag{32}$$

and refer to ψ as the creep function. In what follows we will be working with ψ to develop what we have called a creep function.

Relaxation distribution function

The construction of the physical model of a standard linear solid is not the end of the story; it is certainly not adequate to describe all physical systems. In order to develop more plausible creep functions and maintain physical sense one introduces a relaxation distribution D(t) which can be thought of as the result of superposing a sum or continuous distribution of standard linear solids in parallel keeping the same elastic parameters, J_0 and Δ, but with different time constants (Gross (1948), Anderson and Minster (1979)) . Thus one represents ψ by the integral

$$\psi(t) = \int_0^\infty D(\tau)(1 - e^{\frac{-t}{\tau}})d\tau \tag{33}$$

This level of complexity is as far as we will go. Clearly one might consider summing not only over relaxation times, but over compliances too. As far as I know, no work has been done along this line.

The integral of D(t) is exactly the infinite time limit of ψ, thus a physical constraint on D(t) is that it have a finite integral, i.e. the system tends to some finite strain under a finite stress. The interpretation of Δ in equation (31) as the difference between the instantaneous and relaxed compliances means the integral should be normalized to one (Gross (1948,1953). Further constraints on D(t) have been determined. See Anderson and Minster (1979) for references.

A note on the literature: The representation of ψ given here is not quite the same as that in Liu, Kanamori and Anderson (1976) due to their having written the creep function for a standard linear solid in a different form.

Compliance and Q from a distribution function

One can now express the complex compliance in terms of the distribution function as follows

$$J = i\omega J_0 \int_0^\infty e^{-i\omega s}\left(1 + \Delta \int_0^\infty D(t)(1 - e^{\frac{-s}{t}})dt\right) ds. \tag{34}$$

Integrating the first term in the sum and interchanging the order of integration for the other we get

$$J = J_0\left(1 + i\omega\Delta \int_0^\infty D(t)\left(\int_0^\infty e^{i\omega s}(1 - e^{\frac{-s}{t}})ds\right) dt\right) \tag{35}$$

or,

$$J = J_0\left(1 + \Delta \int_0^\infty \frac{D(t)dt}{1 + i\omega t}\right) \tag{36}$$

Note the similarity of equation (36) to equation (29).

To calculate Q we write J as the sum of its real and imaginary parts

$$J = J_0\left(1 + \Delta \int_0^\infty \frac{D(t)(1 - i\omega t)}{1 + \omega^2 t^2}dt\right) \tag{37}$$

Thus we have Q in terms of the relaxation distribution

$$Q^{-1} = \frac{\Delta \int_0^\infty \frac{D(t)\omega t}{1+\omega^2 t^2}dt}{1 + \Delta \int_0^\infty \frac{D(t)}{1+\omega^2 t^2}dt} \tag{38}$$

The expression for velocity is easily found using equations (19) and (37) or (20),(37) and (38).

THE MINSTER-ANDERSON MODEL

Introduction

Lomnitz (1957) introduced a logarithmic creep function to fit strain data over a limited time domain. The creep function gave a relatively constant Q over a broad range of frequencies. Jeffreys (1957) generalized Lomnitz' law by introducing one more parameter, α, which allows a somewhat more pronounced frequency dependence for Q, the dependence going as ω^α. He used this modified Lomnitz law to reconcile average geophysical processes over a wide range of frequencies. His estimates for alpha varied around .25.

It was recognized that Lomnitz' law ultimately is unphysical since the infinite time limit gives infinite strain (Kanamori and Anderson (1977)). MacDonald (1961) pointed out, after finding an explicit relaxation distribution for the modified Lomnitz law, that it has the same problem; the integral of the distribution function is not finite. He proposed a further modification of Lomnitz' law to retain the essential features while removing the divergence. Minster and Anderson (1981) have given physical arguments for a simple distribution function which also gives a creep function very similar to Jeffreys' and avoids the problem of the infinite time limit as well. Recently a physically plausible model which approximates Jeffrey's creep function was presented by Strick (1984).

Four parameter model distribution function

The Minster-Anderson model gives an absorption band over which Q varies approximately as ω^{α}. The model is parametrized by α, the two frequencies which determine the boundaries of the band, and the parameter Δ introduced above which can be used to determine the minimum for Q once the other parameters have been chosen. The normalized distribution function has three paramters and has the form

$$D(t) = \left(\frac{\alpha}{t_2^{\alpha} - t_1^{\alpha}}\right) \frac{1}{t^{1-\alpha}} \qquad if \quad t_1 \le t \le t_2 \qquad (39)$$

$$0 \qquad otherwise$$

where t_2 and t_1 are the long and short period boundaries of the absorption band.

Formula for Q

Denoting the normalizing constant by N and explicitly writing out the integrals indicated in the previous section one gets the following expression for the complex compliance

$$J = J_0 \left(1 + N\Delta \int_{t_1}^{t_2} \frac{(1 - i\omega t)t^{\alpha-1}}{1 + \omega^2 t^2} dt\right) \qquad (40)$$

and Q^{-1} is given by

$$Q^{-1} = \frac{\omega N \Delta \int_{t_1}^{t_2} \frac{t^{\alpha} dt}{1+\omega^2 t^2}}{1 + N\Delta \int_{t_1}^{t_2} \frac{t^{\alpha-1} dt}{1+\omega^2 t^2}} \qquad (41)$$

As usual, velocity dispersion can be calculated using equation 19 or 20.

In general, no closed form expression for Q exists for this model. Minster and Anderson (1981) give low and high frequency asymptotics and mid-frequency approximations for α less than, equal to and greater than 0. Closed form expressions for α=0,1 and a simple series expansion which can be used to calculate Q for arbitrary values of alpha are given below.

Closed form expressions (α=1,0)

To calculate the compliance and Q one must evaluate expressions of the form

$$\int_{t_1}^{t_2} \frac{t^{\beta} dt}{1 + \omega^2 t^2} \qquad (42)$$

which can be written in closed form for β=-1,0,1. β=0 gives

$$\int_{t_1}^{t_2} \frac{dt}{1 + \omega^2 t^2} = \frac{1}{\omega} arctan \left(\frac{\omega(t_2 - t_1)}{1 + \omega^2 t_1 t_2}\right) \qquad (43)$$

β=-1 gives

$$\int_{t_1}^{t_2} \frac{dt}{t(1 + \omega^2 t^2)} = ln \frac{t_2}{t_1} - \frac{1}{2} ln \left(\frac{1 + \omega^2 t_2^2}{1 + \omega^2 t_1^2}\right) \qquad (44)$$

β=1 gives

$$\int_{t_1}^{t_2} \frac{tdt}{1+\omega^2 t^2} = \frac{1}{2\omega^2} ln\left(\frac{1+\omega^2 t_2^2}{1+\omega^2 t_1^2}\right) \tag{45}$$

When α=0 we get the normalization constant

$$N = \frac{1}{ln\frac{t_2}{t_1}} \tag{46}$$

and so

$$Q_0^{-1} = \frac{\frac{\Delta}{ln\frac{t_2}{t_1}} arctan\left(\frac{\omega(t_2-t_1)}{1+\omega^2 t_1 t_2}\right)}{1+\Delta\left(1-\frac{1}{2}ln\frac{t_2}{t_1} ln\left(\frac{1+\omega^2 t_1^2}{1+\omega^2 t_2^2}\right)\right)} \tag{47}$$

this is a version of the flat absorption band that is the remnant of Lomnitz' original attempt. Similar expressions are found in Liu et. al. (1976) and in Kanamori and Anderson (1977).
α=1 gives

$$Q_1^{-1} = \frac{\frac{\Delta}{2(t_2-t_1)} ln\left(\frac{1+\omega^2 t_2^2}{1+\omega^2 t_1^2}\right)}{1+\frac{\Delta}{\omega(t_2-t_1)} arctan\left(\frac{\omega(t_2-t_1)}{1+\omega^2 t_1 t_1}\right)} \tag{48}$$

These closed form expressions were used to check the following series expansion.

Series expansion

For arbitrary values of α one must evaluate the integral in equation (42). To get a series expansion for this purpose, one expands the denominator of the integral in a geometric series, multiplies by t^β and integrates the resulting expression term by term. Depending on what frequency one is interested in, one can have different series expansions.

For ωt¿1, one has

$$\frac{1}{1+\omega^2 t^2} = \frac{1}{\omega^2 t^2} \sum_{n=0}^{\infty} (-1)^n (\omega t)^{-2n} \tag{49}$$

and for ωt¡1,

$$\frac{1}{1+\omega^2 t^2} = \sum_{n=0}^{\infty} (-1)^n (\omega t)^{2n} \tag{50}$$

The resulting expansions for equation (42) are

$$\frac{1}{\omega^{\beta+1}} \sum_{n=0}^{\infty} (-1)^n \left(\frac{(\omega t_2)^{\beta+2n+1} - (\omega t_1)^{\beta+2n+1}}{\beta+2n+1}\right) \quad if \quad \omega t_1 \leq \omega t_2 \leq 1$$

$$\frac{1}{\omega^{\beta+1}} \sum_{n=0}^{\infty} (-1)^n \left(\frac{1-(\omega t_1)^{\beta+2n+1}}{\beta+2n+1} + \frac{(\omega t_2)^{\beta-(2n+1)} - 1}{\beta-(2n+1)}\right) \quad if \quad \omega t_1 \leq 1 \leq \omega t_2 \tag{51}$$

$$\frac{1}{\omega^{\beta+1}} \sum_{n=0}^{\infty} (-1)^n \left(\frac{(\omega t_2)^{\beta-(2n+1)} - (\omega t_1)^{\beta-(2n+1)}}{\beta-(2n+1)}\right) \quad if \quad 1 \leq \omega t_1 \leq \omega t_2$$

These series have been tested using 100 terms and work quite well. The attached figure duplicates Minster and Anderson's (1981) figure 2.

Estimation of parameters

Since Δ has some physical meaning it seems worthwhile to retain it as a parameter for the model, but one might like to find a value corresponding to some particular peak value on the Q^{-1} curve. I have used as a rough first approximation the following relation for 0¡α¡1

$$\Delta_\alpha \approx \frac{2}{\pi} Q_\alpha^{-1} (1 - \frac{t_1}{t_2}) \left(\frac{1-\alpha}{\alpha}\right) \left(\frac{1 - \left(\frac{t_1}{t_2}\right)^\alpha}{1 - \left(\frac{t_1}{t_2}\right)^{\alpha-1}}\right) \tag{52}$$

A brief outline of how this formula was derived goes as follows. Starting with the mid-frequency approximation for α=0

$$Q_0^{-1} \approx \frac{\Delta}{ln\frac{t_2}{t_1}} arctan \left(\frac{\omega(t_2 - t_1)}{1 + \omega^2 t_1 t_2}\right) \tag{53}$$

we get the maximum Q^{-1} value for that curve as

$$Q_0^{-1} \approx \frac{\pi \Delta_0}{2ln\frac{t_2}{t_1}} \tag{54}$$

One uses the high frequency asymptotics given by Minster and Anderson (1981) to find, for a given α, t_1 and t_2 what Δ should be so that the α=0 and specified α curves meet at high frequencies. The resulting equation is

$$\frac{\Delta_\alpha}{\omega} \left(\frac{\alpha}{\alpha - 1}\right) \left(\frac{t_1}{t_2}\right)^\alpha \left(\frac{1 - \left(\frac{t_1}{t_2}\right)^{1-\alpha}}{1 - \left(\frac{t_1}{t_2}\right)^\alpha}\right) = \frac{\Delta_0}{\omega t_1} \left(\frac{1 - \frac{t_1}{t_2}}{ln\frac{t_2}{t_1}}\right) \tag{55}$$

If one now assumes that Q goes as ω^α until $\omega=2\pi/t_1$ at which point it meets the α=0 curve, this means that at the meeting point

$$Q_\alpha^{-1} \left(\frac{t_1}{t_2}\right)^{-\alpha} \approx Q_0^{-1} \tag{56}$$

Using the three equations, (54),(55), and (56), one gets the result.

ACKNOWLEDGEMENTS

I would like to thank Chuck Mueller for teaching me how to use the TEX typesetting system.

REFERENCES

The following are references used in the text and others that a person interested in viscoelastic phenomena might find useful. I can't vouch for all of them, some are articles I never got the time to look at.

Aki, K. and Richards, P.G. (1980) *Quantitative Seismology* W.H. Freeman Co.

Anderson, D.L. and Minster, J.B. (1979) *The frequency dependence of Q in the earth and implications for mantle rheology and Chandler wobble.* *GJRAS* **58**, 431-440.

Anderson, D.L. and Given, J.W. (1982) *Absorption band Q model for the earth.* *JGR* **87**, 3893-3904.

Brennan, B.J. (1980) *Pulse propagation in media with frequency dependent Q.GRL* **7**, 211-213.

Brennan, B.J. and Smylie, D.E. (1981) *Linear viscoelasticity and dispersion in seismic wave propagation. Rev Geo. and Sp. Phys.* **19**, 233-246.

Bland, D.R. (1960) *The Theory of Linear Viscoelasticity* Pergamon Press.

Borcherdt, R.D. (1971) *Inhomogeneous Body and Surface Plane Waves in a Generalized Viscoelastic Half-space* Phd thesis University of California at Berkeley.

Borcherdt, R.D. (1973) *Energy and plane waves in viscoelastic media JGR* **78**, 2442-2453.

Buchen, P.W. (1971) *Plane waves in viscoelastic media GJRAS* **23**, 531-542.

Flugge, W. (1967) *Viscoelasticity* Blaisdell Publishing

Futterman, W. I. (1962) *Dispersive body waves* *JGR* **67**, 5279-5291.

Gladwin, M.T and Stacey, F.D. (1974) *Anelastic degradation of acoustic pulses in rock Phys. Earth and Plan. Int.* **8**, 332-336.

Gross, B. (1948) *On Creep and Relaxation II J Appl. Phys.* **19**, 257-264.

Gross, B. (1953) *Mathematical Structure of the Theories of Viscoelasticity* Hermann and Co.

Hudson, J.A. (1980) *The Excitation and Propagation of Elastic Waves* Cambridge University Press

Jeffreys, H. (1931) *Cartesian Tensors* Cambridge University Press

Jeffreys, H. (1957) *Rock creep, tidal friction and the moon's ellipticities* Monthly notices of the Royal *Astronomical Society* **118**, 14-17. (also in his collected papers)

Jeffreys, H. (1971) *Importance of damping in geophysics GJRAS* **40**, 23-27.

Kanamori, H. and Anderson D. L. (1977) *Importance of physical dispersion in surface wave and free oscillation problems: review* Rev. Geo. and Sp. Phys. **15**, 105-112.

Kolsky, H. (1956) *Propagation of stress pulses in viscoelastic solids* Phil. Mag. Royal Soc. **1**, 693-710.

Krebes, E.S. (1980) *Seismic Body Waves in Anelastic Media* PhD thesis University of Alberta, Edmonton, Canada.

Liu, H-S, Anderson, D.L., and Kanamori, H. (1976) *Velocity dispersion due to anelasticity; implications for seismology and mantle compostion GJRAS* **47** 41-58.

Lomnitz, C. (1957) *Linear dissipation in solids J Appl. Phys.* **28**, 201-205.

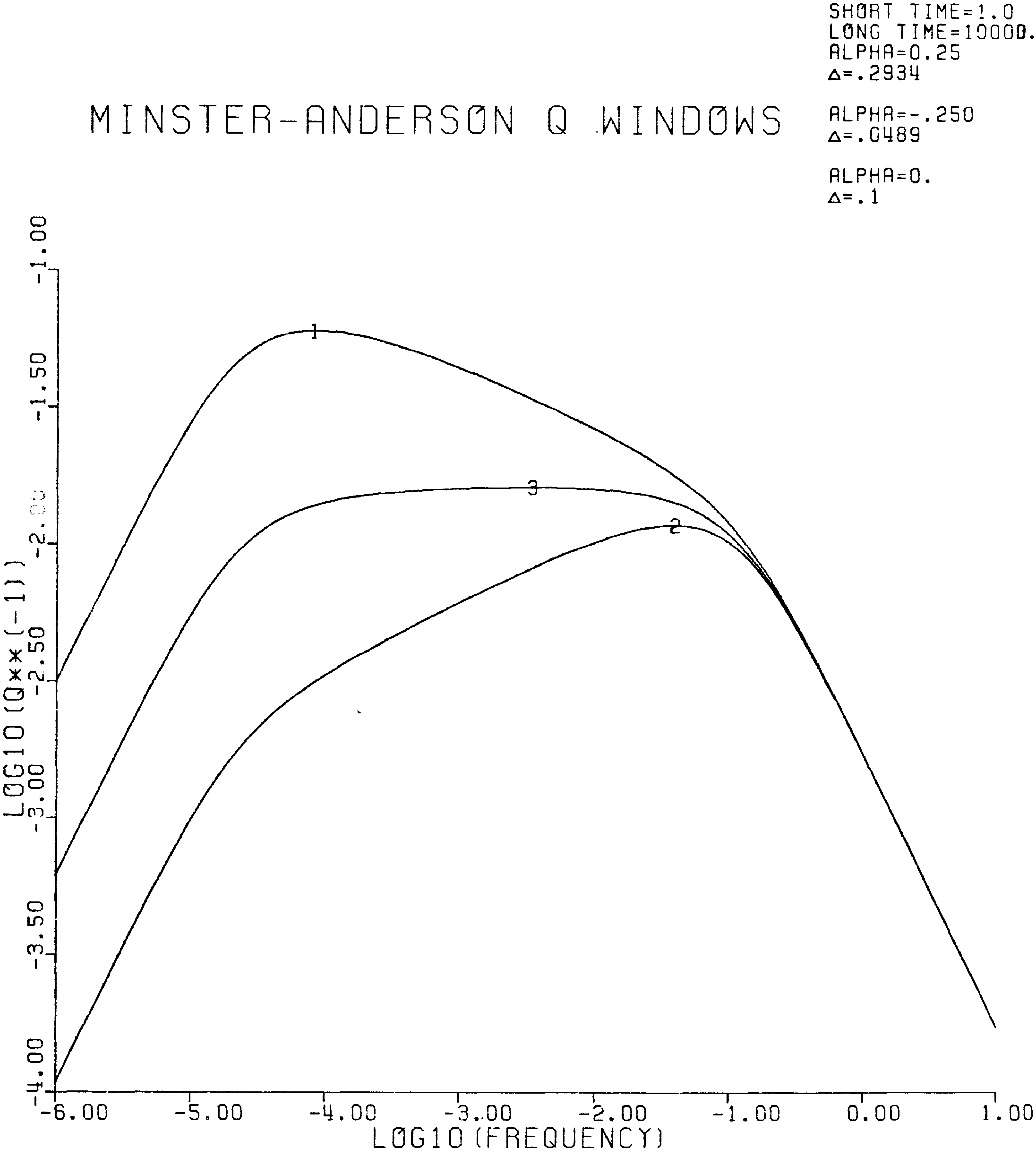
MacDonald, J.R. (1961) *Theory and application of a superposition model of internal friction and creep J Appl. Phys.* **32**, 2385-2398.

Minster, J.B. and Anderson, D.L. (1981) *A model for dislocation controlled rheology for the mantle Phil. Trans. R. Soc. London* **299**, 318-356.

O'Connell, R.J. and Budiansky, B. (1978) *Measures of dissipation in viscoelastic media GRL* **5**, 5-8.

Strick, E. (1984)*Implications of Jeffrey's-Lomnitz Transient Creep JGR* **89**, 437-451.

Toll, J.S. (1956) *Causality and the dispersion relation: logical foundations* Phys. Rev. **104**, 1760-1770.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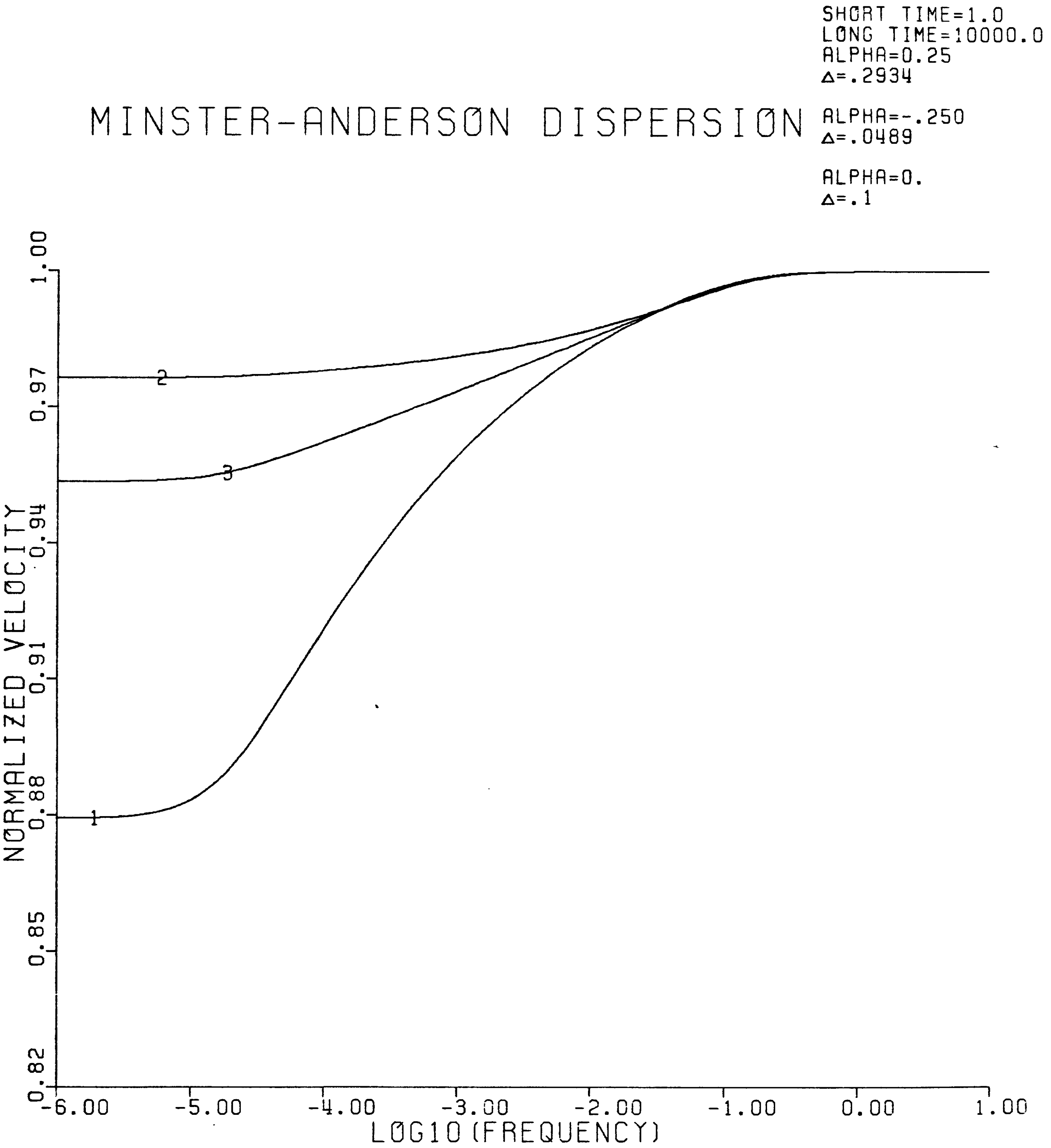
2
3
-1.00
-1.50
-2.00
-2.50
-3.00
-3.50
-4.00
LOG10 (Q**(-1))
-6.00
-5.00
-4.00
-3.00
-2.00
-1.00
0.00
1.00
LOG10 (FREQUENCY)



MINSTER-ANDERSON DISPERSION
SHORT TIME=1.0
LONG TIME=10000.0
ALPHA=0.25
Δ=.2934
ALPHA=-.250
Δ=.0489
ALPHA=0.
Δ=.1
NORMALIZED VELOCITY
1.00
0.97
0.94
0.91
0.88
0.85
0.82
1
2
3
-6.00
-5.00
-4.00
-3.00
-2.00
-1.00
0.00
1.00
LOG10 (FREQUENCY)