UNITED STATES DEPARTMENT OF THE INTERIOR

GEOLOGICAL SURVEY

Two- and Three-Dimensional Low-Frequency Radiation From an Arbitrary Source in a Fluid-Filled Borehole

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Open-File Report 85-550

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1985

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TWO- AND THREE-DIMENSIONAL LOW-FREQUENCY RADIATION FROM AN ARBITRARY SOURCE IN A FLUID-FILLED BOREHOLE

By Myung W. Lee

ABSTRACT

Far-field displacement fields were derived for an impulsive point force acting on a fluid-filled borehole wall under the assumption that the borehole diameter is small compared to the wavelength involved. The displacements due to an arbitrary source can be easily computed by combining the solutions for the impulsive sources.

In general, the borehole source generates not only longitudinal and vertically polarized shear waves but also horizontally polarized shear waves. This study also indicates that only the axis-symmetrical motion around the borehole due to normal stress is affected by the presence of the fluid in the borehole. The tangential stresses acting on a fluid-filled borehole do not affect the radiation into the surrounding medium due to the presence of the fluid in the long-wavelength limit.

INTRODUCTION

The far-field radiation pattern from a seismic source inside a fluid-filed borehole is very important in understanding not only the effect of the borehole fluid on the seismic radiation but also the characteristics of the different borehole sources. Recently, Lee and others (1984) observed an anomalous radiation pattern from a downhole airgun source during a well-to-well vertical seismic profiling experiment and concluded that the borehole fluid had a substantial effect on the measured seismic radiation patterns. Current development of vertical seismic profiles necessitated a better understanding of the borehole sources as well as the borehole and fluid effect on the measured seismic signal.

This investigation focuses on the derivation of the far-field seismic radiation pattern from an arbitrary source acting on the wall of the fluid-filled borehole under the assumption that the borehole diameter is very small compared to the wavelengths of interest.

Heelan (1953) discussed P- and S-wave radiation pattern from axis-symmetric borehole sources acting on the wall of an empty borehole. Lee and Balch (1982) derived the far-field radiation pattern from axis-symmetrical sources in a fluid-filled borehole and discussed in detail the effect of the borehole fluid on the seismic radiation into the surrounding medium. White and Sengbush (1963) also formulated the effect of borehole fluid on the seismic radiation pattern combining Heelan's solution with the tube wave inside the borehole. All of the above-mentioned authors treated only the axis-symmetrical waves propagating around the borehole under the low-frequency assumption.

White (1960) derived the far-field radiation pattern from radial and tangential pairs of forces acting on the wall of an empty borehole using a seismic reciprocity theorem. Greenfield (1978) obtained the seismic displacement fields from a point force applied to the surface of a cylindrical cavity in an elastic medium without any assumption about the size of a borehole relative to the wave length.

In this investigation, the three scalar potentials, which describe the elastic wave propagation in the surrounding medium, were formulated using an infinite sum of Bessel functions and the complex Fourier transform. Retaining the terms proportional to a and a^2 (where "a" is the radius of the borehole), and expanding the azimuthal dependence of the source function into the complex Fourier series, $\exp(ip\theta)$, it is shown that only P = 0, 1, and 2 terms are needed to describe the seismic radiation around the borehole under the low-frequency assumption.

Two-dimensional radiation patterns and some mathematical details are enclosed in the appendix.

DERIVATION OF THE SOLUTION

Consider a cylindrically circular, fluid-filled borehole with a radius "a" in a homogeneous elastic medium of density ρ, compressional speed α, and shear speed β. The fluid medium has a compressional velocity α_f and density ρ_f. The elastic wave field inside the fluid-filled borehole can be described by a scalar potential Φ', and the elastic wave propagating around the borehole can be represented by three scalar potentials (Φ, ψ, X).

The three scalar potentials in the surrounding medium can be written as the following formula in the frequency domain (e. g., Harkride, 1964):

$$
\begin{aligned}
\nabla^2 \bar{\Phi} &= -\frac{\omega^2}{\alpha^2}\bar{\Phi} \\
\nabla^2 \bar{\Psi} &= -\frac{\omega^2}{\beta^2}\Psi \\
\nabla^2 X &= -\frac{\omega^2}{\beta^2} X
\end{aligned}
\tag{1}
$$

and in the fluid the potential is given by:

$$
\nabla^2 \bar{\Phi}' = -\frac{\omega^2}{\alpha_f^2}\bar{\Phi}'
$$

In a cylindrical coordinate system (r, θ, z), shown in figure 1, the displacement field can be derived from the scalar potentials by the following formulae. In the surrounding medium, U_r, U_θ, and U_z, (r, θ, and z direction displacement, respectively) are:

$$
\begin{aligned}
U_r &= \frac{\partial \bar{\Phi}}{\partial r} + \frac{\partial^2 \bar{\Psi}}{\partial r \partial z} + \frac{1}{r}\frac{\partial X}{\partial \theta} \\
U_\theta &= \frac{1}{r}\frac{\partial \bar{\Phi}}{\partial \theta} + \frac{1}{r}\frac{\partial^2 \bar{\Psi}}{\partial z \partial \theta} - \frac{\partial X}{\partial r} \\
U_z &= \frac{\partial^2 \bar{\Phi}}{\partial z} + \frac{\partial^2 \bar{\Psi}}{\partial z^2} + \frac{\omega^2}{\beta^2}\bar{\Psi}
\end{aligned}
\tag{2}
$$

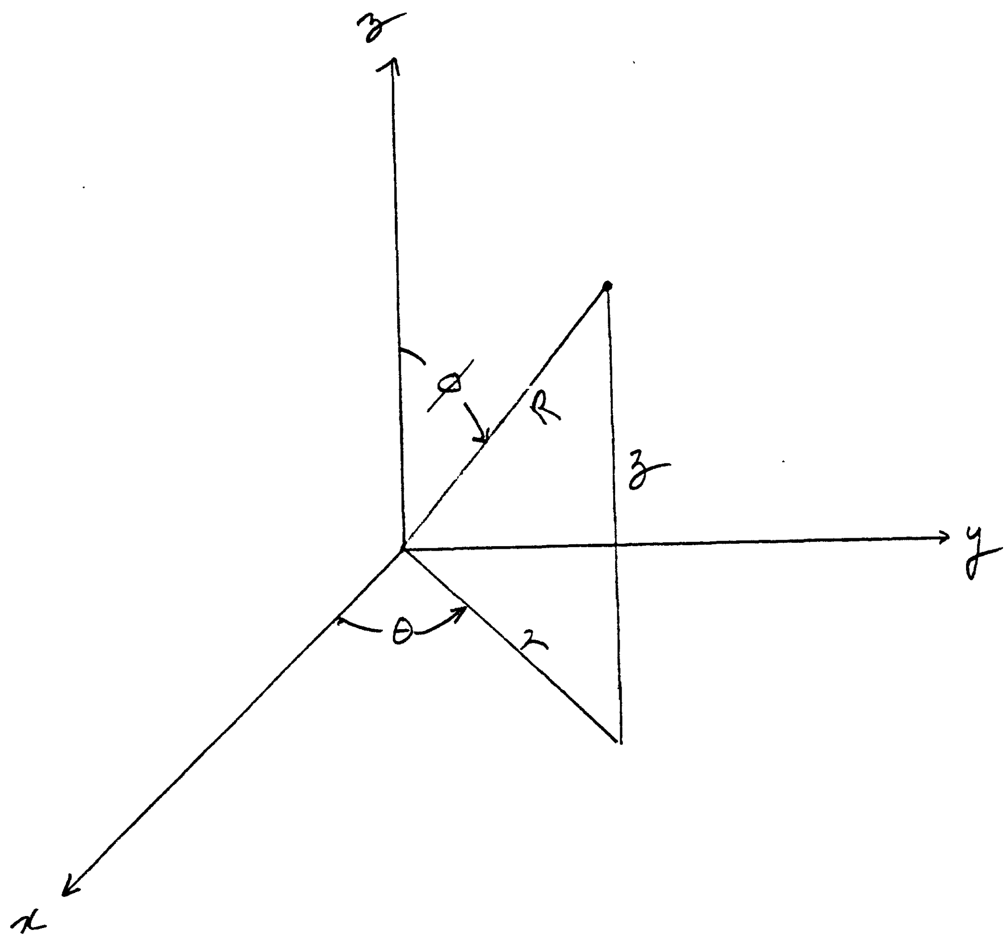

Figure 1.--Coordinate system for solution. An orthogonal cartesian coordinate (x, y, z), a cylindrical coordinate system (r, θ, z), and a spherical coordinate system (R, Ø, θ) are shown

The solutions of the displacement potentials in the surrounding medium may be written, imposing the radiation boundary conditions, as:

$$\Phi = \sum_{p=-\infty}^{\infty} \int_{-\infty}^{\infty} A_p H_p^{(2)}(mr) e^{-ikz} e^{ip\theta} dk, \tag{3}$$

$$\Psi = \sum_{p=-\infty}^{\infty} \int_{-\infty}^{\infty} B_p H_p^{(2)}(nr) e^{-ikz} e^{ip\theta} dk,$$

and

$$X = \sum_{p=-\infty}^{\infty} \int_{-\infty}^{\infty} C_p H_p^{(2)}(nr) e^{-ikz} e^{ip\theta} dk.$$

In equation (3), $H_p^{(2)}$ is the second kind of Hankel function of order P, and the radial wave numbers m and n are given by:

$$m = \left(\frac{\omega^2}{\alpha^2} - k^2 \right)^{1/2} \triangleq \left(k_\alpha^2 - k^2 \right)^{1/2},$$

$$n = \left(\frac{\omega^2}{\beta^2} - k^2 \right)^{1/2} \triangleq \left(k_\beta^2 - k^2 \right)^{1/2}.$$

Inside the fluid-filled borehole, the potential can be written as:

$$\Phi' = \sum_{p=-\infty}^{\infty} \int_{-\infty}^{\infty} D_p J_p(\ell r) e^{-ikz} e^{ip\theta} dk. \tag{4}$$

In equation (4), J_p is the Bessel function of order p, and ℓ is given by:

$$\ell = \left(\frac{\omega^2}{\alpha_f^2} - k^2 \right)^{1/2}.$$

The value of p should be taken only integer values in order that the displacement fields are single valued functions of the azimuthal angle θ. The unknown constants A_p, B_p, and C_p can be evaluated by applying appropriate boundary conditions on the wall of the fluid-filled borehole.

The usual boundary conditions to be satisfied on the borehole wall, at r = a, are:

$$
\begin{aligned}
U_r - U_r' &= 0, \\
P_{rz} - P_{rz}' &= T_z(\theta)\,\delta(z), \\
P_{r\theta} - P_{r\theta}' &= T_\theta(\theta)\,\delta(z), \\
P_{rr} - P_{rr}' &= T_r(\theta)\,\delta(z).
\end{aligned}
\tag{5}
$$

In equation (5), P_{ij} is the stress component in the cylindrical coordinate system and $\delta(z)$ is the Dirac delta function, and $T_i(\theta)$, i = r, z, or θ, is the source stress acting on the borehole wall as a function of θ. The primed quantities in equation (5) represent the quantities in the fluid.

The stresses appropriate in solving boundary conditions are:

$$
\begin{aligned}
P_{rr} &= \lambda\left(\frac{\partial U_r}{\partial r} + \frac{\partial U_z}{\partial z} + \frac{1}{r}\frac{\partial U_\theta}{\partial \theta} + \frac{U_r}{r}\right) + 2\mu\frac{\partial U_r}{\partial r}, \\
P_{rz} &= \mu\left(\frac{\partial U_r}{\partial z} + \frac{\partial U_z}{\partial r}\right), \\
P_{r\theta} &= \mu\left(\frac{1}{r}\frac{\partial U_r}{\partial \theta} + \frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r}\right).
\end{aligned}
\tag{6}
$$

Here λ and μ are Lame's constants and relate to the seismic velocity as:

$$
\alpha = \left(\frac{\lambda + 2\mu}{\rho}\right)^{1/2},
$$

$$
\beta = \left(\frac{\mu}{\rho}\right)^{1/2}.
$$

The first term of equation (5) requires the continuity of the radial stress at the borehole wall, and the remaining three terms require the continuity of the stresses acting on the borehole wall. Because the shear modules μ_f in the fluid are assumed to be zero, the boundary conditions for the tangential stresses (P_{rz} and $P_{r\theta}$) are identical to those for the empty borehole.

Let's define the radial stress on the borehole wall as:

$$T_r(\theta)\,\delta(z) = \frac{1}{2\pi} \sum_{p=-\infty}^{\infty} \int_{-\infty}^{\infty} E_p^r \, e^{-ikz} e^{ip\theta} \, dk. \tag{7}$$

Defining the other tangential stresses like the radial stress, the solution for the unknown constants A_p, B_p, C_p, and D_p can be represented by the following matrix equation.

$$\begin{bmatrix} G_p \end{bmatrix} \begin{bmatrix} A_p \\ B_p \\ C_p \\ D_p \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{E_p^z}{2\pi\mu} \\ \dfrac{E_p^\theta}{2\pi\mu} \\ \dfrac{E_p^r}{2\pi\mu} \end{bmatrix} \tag{8}$$

The matrix elements of g_{ij} of G_p are given by:

$$
\begin{aligned}
g_{11} &= H_p'(ma) \\
g_{12} &= -ik\, H_p'(na) \\
g_{13} &= ip\, H_p(na)/a \\
g_{14} &= J_p'(la) \\
g_{21} &= -2ik\, H_p'(ma) \\
g_{22} &= (k_\beta^2 - 2k^2)\, H_p'(na) \\
g_{23} &= kp\, H_p(na)/a \\
g_{24} &= 0 \\
g_{31} &= 2ip\left[H_p'(ma)/a - H_p(na)/a^2 \right] \\
g_{32} &= 2pk\left[H_p'(na)/a - H_p(na)/a^2 \right] \\
g_{33} &= H_p'(na)/a - p^2 H_p(na)/a^2 - H_p''(na) \\
g_{34} &= 0 \\
g_{41} &= 2\, H_p''(ma) - \lambda\omega^2 H_p(na)/(\alpha^2\mu) \\
g_{42} &= -2ik\, H_p''(na) \\
g_{43} &= 2ip\left[H_p'(na)/a - H_p(na)/a^2 \right] \\
g_{44} &= \rho_f\, \omega^2 J_p(la)/\mu .
\end{aligned}
\tag{9}
$$

The E_p^r in equation (7) or (8) can be derived using the complex Fourier series expansion of the stress function, and given by:

$$E_p^r = \frac{1}{2\pi}\int_0^{2\pi} T_r(\theta)\, e^{-ip\theta}\, d\theta . \tag{10}$$

Equations (2), (3), and (9) provide exact solutions for the borehole sources acting on the fluid-filled borehole wall.

LOW-FREQUENCY APPROXIMATION

The exact solution of the matrix equation shown in equation (8) consists of the infinite series of terms with increasing periodicity around the borehole. The coefficients of the unknown constants A_p, B_p, C_p, and D_p should be solved for each value of p.

When the borehole radius is very small compared with the wavelengths of interest, the coefficients A_p, B_p, C_p, and D_p can be expanded in terms of parameter a. If terms proportional to a and a^2 are kept for the solution, only p = 0, p = ± 1, and p = ± 2 are required. In the following, the approximate solution for equation (8) is presented for each p.

A. p = 0.--When p = 0, the solutions are independent of the azimuthal angle, which are axis-symmetrical solutions. For this case, Lee and Balch (1982) derived the solution for the radial stress source in a fluid-filled borehole, and Heelan (1953) derived the far-field radiation for the radial and tangential stresses acting on an empty borehole.

Retaining only the dominant terms of the matrix equation (8), the zero-order solution can be written as:

$$\begin{pmatrix} A_0 \\ B_0 \\ C_0 \\ D_0 \end{pmatrix} = \begin{pmatrix} \dfrac{-ik}{m k_\beta^2 Q_1} & 0 & \dfrac{a l^2 (n^2 - k^2)}{2m k_\beta^2 T_u Q_1} \\ \dfrac{-1}{n k_\beta^2 S_1} & 0 & \dfrac{i a l^2 k}{n k_\beta^2 T_u S_1} \\ 0 & \dfrac{-a}{2n S_1} & 0 \\ 0 & & \dfrac{1}{T_u} \end{pmatrix} \begin{pmatrix} \dfrac{E_0^z}{2\pi\mu} \\ \dfrac{E_0^\theta}{2\pi\mu} \\ \dfrac{E_0^r}{2\pi\mu} \\ 0 \end{pmatrix} \tag{11}$$

where

$$T_u = -\left(\ell^2 + \frac{\rho_f \omega^2}{\mu}\right),$$

$$Q_p = \frac{i}{\pi} \frac{2^p (p-1)!}{(ma)^p},$$

$$S_p = \frac{i}{\pi} \frac{2^p (p-1)!}{(na)^p}.$$

For an empty borehole, which is $\rho_f = 0$ in equation (11), the solutions are identical to the Heelan's (1953) derivation. In a radial stress source in the fluid-filled borehole, this solution is identical to Lee and Balch (1982).

When $p = 0$, the D_0 term, which is appropriate to the solution in the fluid, is included as a solution, because only in this case does the fluid in the borehole affect the radiation in the surrounding medium.

Substituting D_0 into equation (4), we find that the poles of the integral are determined by $T_u = 0$. This pole corresponds to the tube wave velocity in the low-frequency limit, and this proves the existence of the tube wave in the fluid-filled borehole (Balch and Lee, 1984). Equation (11) also indicates that only normal stress, T_r, can generate the tube wave in the borehole. The above observation implies that "tube wave" in the fluid-filled borehole, which is caused by the cross-sectional area change of the borehole, can affect the seismic radiation pattern in the surrounding medium.

<u>B. p = +1</u>.--The determinant of matrix G_p, Δ_p, can be written as:

$$\Delta_p = -g_{14} \tilde{\Delta}_{14} + g_{44} \tilde{\Delta}_{44}$$

where $\tilde{\Delta}_{ij}$ is the minor of g_{ij}.

The leading term analysis of the determinant indicated that the first term is dominant for the solution when $p \geq 1$. This implies that the solution outside the borehole is independent of the fluid in the borehole. In other words, when $p \geq 1$, the solution for the fluid-filled borehole is identical to the solution for the empty borehole.

If only the leading terms of the matrix element are used, the determinant, when $P \geq 1$, is proved to be identical to zero. One way to derive non-zero determinant in order to solve equation (8) is including the next order terms in the expansion of the matrix element. For example, when $p = 1$, the g_{31} term can be expanded as:

$$g_{31} \cong \frac{-2imQ_2}{a}\left(1 + \frac{m^2a^2}{4}\right),$$

where Q_2 is the leading term of $H_2^{(2)}$ (ma).

After some lengthy algebra, the solution for p = 1 can be written as:

$$\begin{bmatrix} A_1 \\ B_1 \\ C_1 \end{bmatrix} = \begin{bmatrix} \left(\dfrac{-3ki}{m k_\beta^2 Q_2}\right) & \dfrac{i}{a m k_\beta^2 Q_2} & \left(\dfrac{-1}{a m k_\beta^2 Q_2}\right) \\ \left(\dfrac{k^2 - 2n^2}{n^3 k_\beta^2 S_2}\right) & \dfrac{-k}{a n^3 k_\beta^2 S_2} & \left(\dfrac{-ik}{a n^3 k_\beta^2 S_2}\right) \\ \dfrac{-k}{n^3 S_2} & \left(\dfrac{1}{a n^3 S_2}\right) & \dfrac{i}{a n^3 S_2} \end{bmatrix} \begin{bmatrix} \dfrac{E_1^z}{2\pi\mu} \\ \dfrac{E_1^\theta}{2\pi\mu} \\ \dfrac{E_1^r}{2\pi\mu} \end{bmatrix} \qquad (12)$$

In equation (12), the matrix elements with the parentheses indicate the terms which changes the algebraic sign for p = -1.

C. p = +2.--Applying the same method as for p = ± 1, the solution is given by:

$$
\begin{Bmatrix} A_2 \\ \\ B_2 \\ \\ C_2 \end{Bmatrix} = \begin{bmatrix} \dfrac{-2ik}{m(k_\beta^2-k_\alpha^2)Q_3} & \left(\dfrac{4i}{am(k_\beta^2-k_\alpha^2)Q_3}\right) & \dfrac{-4}{am(k_\beta^2-k_\alpha^2)Q_3} \\ \\ \dfrac{2(k_\alpha^2-n^2)}{n^3(k_\beta^2-k_\alpha^2)S_3} & \left(\dfrac{-4k}{an^3(k_\beta^2-k_\alpha^2)S_3}\right) & \dfrac{-4ik}{an^3(k_\beta^2-k_\alpha^2)S_3} \\ \\ \left(\dfrac{-2k\,k_\alpha^2}{n^3(k_\beta^2-k_\alpha^2)S_3}\right) & \dfrac{4k_\beta^2}{an^3(k_\beta^2-k_\alpha^2)S_3} & \left(\dfrac{4i\,k_\beta^2}{an^3(k_\beta^2-k_\alpha^2)S_3}\right) \end{bmatrix} \begin{Bmatrix} \dfrac{E_2^z}{2\pi\mu} \\ \\ \dfrac{E_2^\theta}{2\pi\mu} \\ \\ \dfrac{E_2^r}{2\pi\mu} \end{Bmatrix} \tag{13}
$$

Like the p = ± 1 case, the matrix elements with a parenthesis denotes the terms which changes the algebraic sign when p = -2.

FAR-FIELD APPROXIMATION

Substituting equations (3), (11), (12), and (13) into equation (2), the displacement field can be evaluated under the low-frequency approximation. When an observation point is very far from the source region, the far-field radiation can be formulated simply by applying the following formula to the equation (3).

$$
\int_{-\infty}^{\infty} I(k)\, H_p^{(2)}(mr)\, e^{-ikz}\, dk \xrightarrow[R\to\infty]{} I(k_\alpha \cos\phi)\,(i)^{p+1}\,\frac{2e^{-ik_\beta R}}{R} . \tag{14}
$$

where R, Ø, θ, is the spherical coordinate system.

Retaining the 1/R decay terms and the coordinate transformation, the far-field radiation in the spherical coordinate system can be written as:

$$U_R = \frac{2k_\alpha e^{-ik_\alpha R}}{R} A_0(k_\alpha \cos\phi) \tag{15}$$

$$+ \frac{2k_\alpha e^{-ik_\alpha R}}{R} \sum_{p=1}^{2} A_p(k_\alpha \cos\phi)\left(e^{ip\theta} + S_t e^{-ip\theta}\right)(i)^p ,$$

$$U_\phi = \frac{-2i\, k_\beta^2\, e^{-ik_\beta R} \sin\phi}{R} B_0(k_\beta \cos\phi)$$

$$- \frac{2i\, k_\beta^2\, e^{-ik_\beta R} \sin\phi}{R} \sum_{p=1}^{2} B_p(k_\beta \cos\phi)\left(e^{ip\theta} + S_t e^{-ip\theta}\right)(i)^p ,$$

$$U_\theta = \frac{-2k_\beta\, e^{-ik_\beta R} \sin\phi}{R} C_0(k_\beta \cos\phi)$$

$$- \frac{2k_\beta\, e^{-ik_\beta R} \sin\phi}{R} \sum_{p=1}^{2} C_p(k_\beta \cos\phi)\left(e^{ip\theta} - S_t e^{-ip\theta}\right)(i)^p .$$

Here, $S_t = 1$, when the source stress function is given by E_p^r or E_p^z, and $S_t = -1$, when the source stress function is given by E_p^θ.

Finally, the far-field, low-frequency approximated solutions for the impulse-like borehole sources in the fluid-filled borehole retaining only a and a^2 terms with 1/R decay can be written as:

A) $T_r(\theta)$ is given and defined as $T_r(\theta) = T_r\delta(\theta)$.

$$U_R = - T_{rp} \cos\theta \sin\phi \frac{e^{-ik_\alpha R}}{R}$$

$$- T_{rp} \frac{iwa\alpha^2}{\alpha \beta^2} \left[\frac{T_\alpha^* (1 - 2\beta^2\cos^2\phi/\alpha^2)}{(T_\alpha^* + \rho_f/\rho)} + \frac{2\cos 2\theta \sin^2\phi \beta^2}{(1-\beta^2/\alpha^2) \alpha^2} \right] \frac{e^{-ik_\alpha R}}{R}, \tag{16}$$

$$U_\phi = - T_{rs} \cos\theta \cos\phi \frac{e^{-ik_\beta R}}{R}$$

$$- T_{rs} \frac{iwa}{\beta} \left[\frac{T_\beta^* \sin\phi \cos\phi}{(T_\beta^* + \rho_f/\rho)} + \frac{\cos 2\theta \cos\phi \sin\phi}{(1-\beta^2/\alpha^2)} \right] \frac{e^{-ik_\beta R}}{R},$$

$$U_\theta = T_{rs} \sin\theta \frac{e^{-ik_\beta R}}{R} \frac{\sin 2\theta \sin\phi}{(1-\beta^2/\alpha^2)} \frac{e^{-ik_\beta R}}{R} .$$

$$+ T_{rs} \frac{iwa}{\beta}$$

where

$$T_{rp} = \frac{a T_r}{4\pi\rho\alpha^2},$$

$$T_{rs} = \frac{a T_r}{4\pi\rho\beta^2},$$

$$T_\alpha^* = \frac{\beta^2}{\alpha_f^2}\left(1 - \frac{\alpha_f^2 \cos^2\phi}{\alpha^2}\right),$$

$$T_\beta^* = \frac{\beta^2}{\alpha_f^2}\left(1 - \frac{\alpha_f^2 \cos^2\phi}{\beta^2}\right).$$

B) $T_z(\theta)$ is given and defined as $T_z(\theta) = T_z\,\delta(\theta)$.

$$U_R = -T_{zp}\cos\phi\,\frac{e^{-ik_\alpha R}}{R} - T_{zp}\,\frac{i\omega a\, 3\cos\theta\sin\phi\cos\phi}{\alpha}\,\frac{e^{-ik_\alpha R}}{R},$$

(17)

$$U_\phi = T_{zs}\sin\phi\,\frac{e^{-ik_\beta R}}{R} + T_{zs}\,\frac{i\omega a\cos\theta(2\sin^2\phi - \cos^2\phi)}{\beta}\,\frac{e^{-ik_\beta R}}{R},$$

$$U_\theta = T_{zs}\,\frac{i\omega a \sin\theta\cos\phi}{\beta}\,\frac{e^{-ik_\beta R}}{R},$$

where

$$T_{zp} = \frac{a\,T_z}{4\pi\rho\alpha^2} ,$$

$$T_{zs} = \frac{a\,T_z}{4\pi\rho\beta^2} .$$

C) $T_\theta(\theta)$ is given and defined as $T_\theta(\theta) = T_\theta\delta(\theta)$.

$$U_R = -T_{\theta p} \sin\theta \sin\phi \frac{e^{-ik_\alpha R}}{R}$$

$$- T_{\theta p} \frac{i\omega a \sin 2\theta \sin^2\phi}{\alpha} \frac{e^{-ik_\beta R}}{R} ,$$

$$U_\phi = -T_{\theta s} \sin\theta \cos\phi \frac{e^{-ik_\beta R}}{R}$$

$$- T_{\theta s} \frac{i\omega a \sin 2\theta \sin\phi \cos\phi}{\beta(1-\beta^2/\alpha^2)} \frac{e^{-ik_\beta R}}{R} , \tag{18}$$

$$U_\theta = -T_{\theta s} \cos\theta \frac{e^{-ik_\beta R}}{R}$$

$$- T_{\theta s} \frac{i\omega a}{2\beta} \left[1 + \frac{2\cos 2\theta}{(1-\beta^2/\alpha^2)} \right] \sin\phi \frac{e^{-ik_\beta R}}{R} ,$$

where

$$T_{\theta p} = \frac{a\,T_\theta}{4\pi\rho\alpha^2} ,$$

$$T_{\theta s} = \frac{a\,T_\theta}{4\pi\rho\beta^2} .$$

DISCUSSION

As mentioned previously, the zero-order solution ($p = 0$) from the normal stress acting on the wall of the fluid-filled borehole has a fluid effect on the seismic radiation into the surrounding medium. The detailed description of this case can be found in Lee and Balch (1982). Also when the tangential stresses act on the fluid-filled borehole, the radiation into the surrounding medium is independent of the fluid in the borehole under low-frequency assumption.

As shown in equations (16), (17), and (18), the radiation pattern is a function of frequency, thus the propagating wave changes its waveform depending upon the frequency content of the source function. Table 1 shows magnitude and phase angle of the radial displacement from an impulsive tangential stress, T_θ, with $\phi = 90^\circ$ and $\theta = 45^\circ$. This radiation was computed assuming a Poisson's solid with $a = 10$ cm and $\alpha = 2{,}357$ m/s. At 300 Hz, the amplitude is about 0.6% higher than that at 0 Hz, and its phase angle is about 6.5°. This example indicates that the frequency dependence of the radiation could be ignored for normal seismic exploration purposes.

The solutions presented in equations (16), (17), and (18) offer a simple way to compute radiation patterns for an arbitrary source acting on the wall of the fluid-filled borehole. By performing the complex Fourier series expansion of the source function and substituting only $p = 0$, 1, and 2 terms of the series expansion into equations (16), (17), and (18), the long-wavelength, far-field radiation patterns can be easily obtained.

For example, consider the source distribution shown in figure 2. Three equal normal stresses 120 degrees apart act on the fluid-filled borehole wall. The total displacement fields from this source distribution can be derived simply by summing the individual contribution. That is:

$$\tilde{U}_R = U_R(\theta) + U_R(\theta+120) + U_R(\theta+240),$$

where the quantity with $\sim$ denotes the total displacement field from the three normal stresses. Using the trigonometric relation such as:

$$\sin\theta + \sin(\theta+120) + \sin(\theta+240) = 0,$$

the far-field displacement field can be written as:

$$\tilde{U}_R = -T_{rp}\,\frac{3i\omega a\alpha}{2\beta^2}\,\frac{T_\alpha^*(1-2\beta^2\cos^2\phi/\alpha^2)}{(T_\alpha^* + \rho_f/\rho)}\,\frac{e^{-ik_\alpha R}}{R},$$

$$\tilde{U}_\phi = -T_{rr}\,\frac{3i\omega a}{\beta}\,\frac{T_\beta^*\sin\phi\cos\phi}{(T_\beta^* + \rho_f/\rho)}\,\frac{e^{-ik_\beta R}}{R},$$

$$\tilde{U}_\theta = 0.$$

Table 1.--Magnitude and phase angle of the U_r component with $\emptyset = 90^{\circ}$ and $\theta = 45^{\circ}$ from an impulsive tangential stress at $\theta = 0^{\circ}$

Frequency (Hz)	Amplitude	Phase (degree)
0	1.000	0.00
25	1.000	0.54
50	1.0002	1.18
75	1.0004	1.62
100	1.0007	2.16
125	1.0011	2.70
150	1.0016	3.23
175	1.0022	3.77
200	1.0028	4.31
225	1.0036	4.85
250	1.0044	5.38
275	1.0054	5.92
300	1.0064	6.45

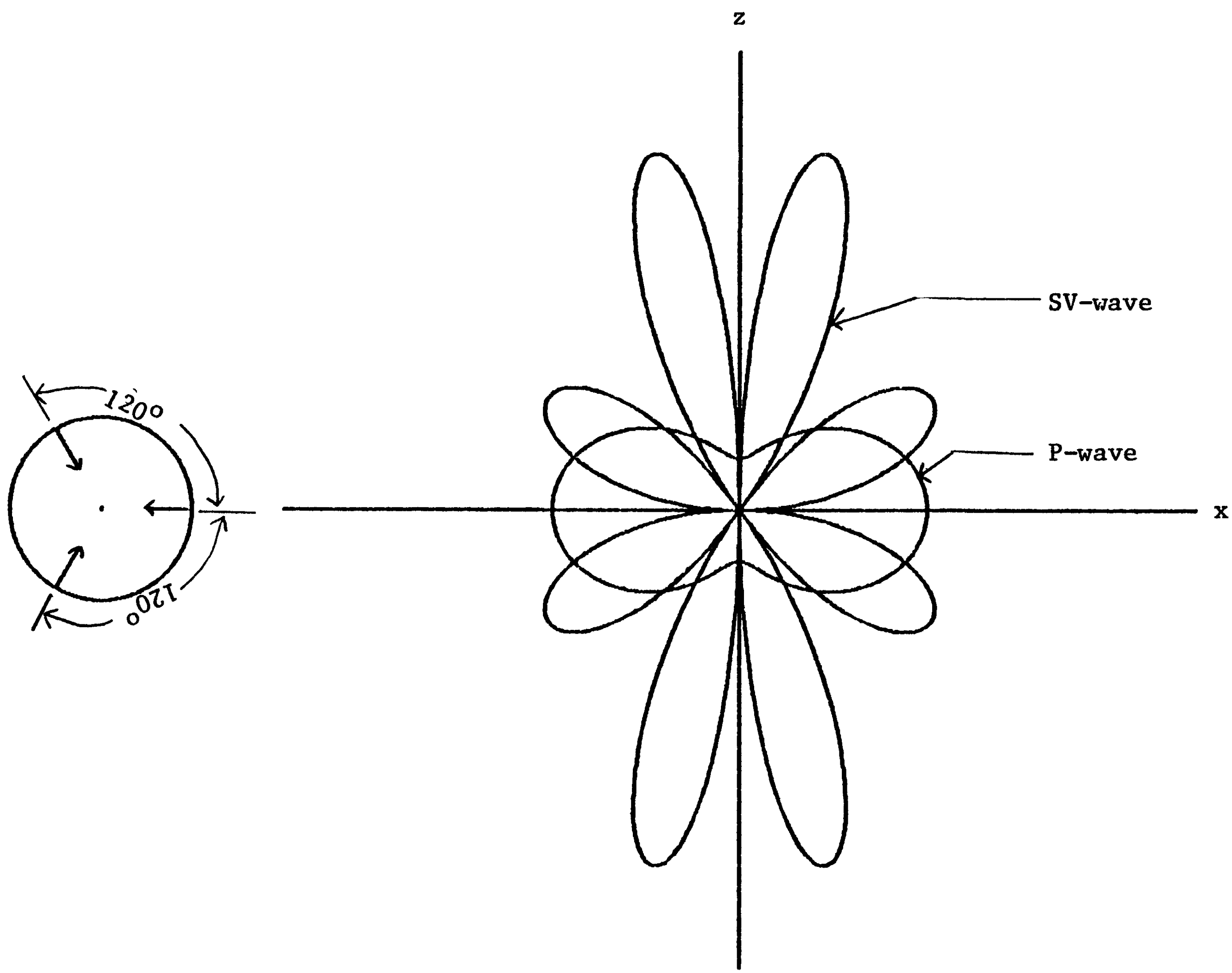

Figure 2.--Radiation patterns for P- and SV-waves from the 3 normal stresses acting on a fluid-filled borehole. Left portion shows the source configuration in x-y plane and the right portion shows the radiation patterns in the x-z plane. Parameters are $\alpha/\alpha_f = 1.4$, $\alpha/\beta = \sqrt{3}$, and $\rho_f/\rho = 0.5$.

The radiation pattern from this source distribution, shown in figure 2, is identical to that from the uniform stress distribution on the borehole wall.

This example illustrates the simplicity in computing radiation patterns for arbitrary borehole sources.

When $T_\theta(\theta) = T_\theta \delta(\theta)$, which is single force, the radiation patterns of P- and SH-waves in the X-Y plane ($\emptyset = 90^o$) are shown in figure 3 with a Poisson solid and $\omega = 0$. The P-wave radiation is proportional to $\cos\theta$ and the SH-wave radiation pattern is proportional to $\sin\theta$. The maximum displacement amplitude of the SH-wave is 3 times greater than that of the P-wave.

When $T_\theta(\theta) = T_\theta \delta(\theta) + T_\theta \delta(\theta + 180^o)$, which is double forces at the opposite side of the borehole, the radiation patterns are shown in figure 3 with a Poisson solid. Because of the property of the trigonometric function, the p = 1 solution is cancelled; thus only p = 0 and p = 2 solutions are retained. As can be seen from figure 4, P-wave radiation is almost negligible compared to the SH-wave motion. The displacement waveform for the single force is the same as the source stress waveform, while the displacement waveform for the couple forces is the derivative of the source waveform.

CONCLUSIONS

In this paper, the far-field radiation patterns from an impulsive source acting on the wall of the fluid-filled borehole were derived under the assumption that the borehole radius is very small compared to the wavelength of interest. The following conclusions can be made from this investigation:

1. The radiation patterns from a point force, or point stress, are almost independent of frequency within the seismic frequency band.
2. The fluid effect on the radiation pattern into the surrounding medium can be detected for the axis-symmetrical waves propagated around the borehole only when the normal forces are acting on the wall.
3. In general, the borehole sources generated not only P- and SV-waves but also SH-waves around the borehole.
4. Combining the solutions from the impulsive sources, the radiation patterns from an arbitrary source can be easily obtained.
4. Applying the seismic reciprocity theory, the solution could be used to evaluate the wall motion from the plane waves incident on the borehole.

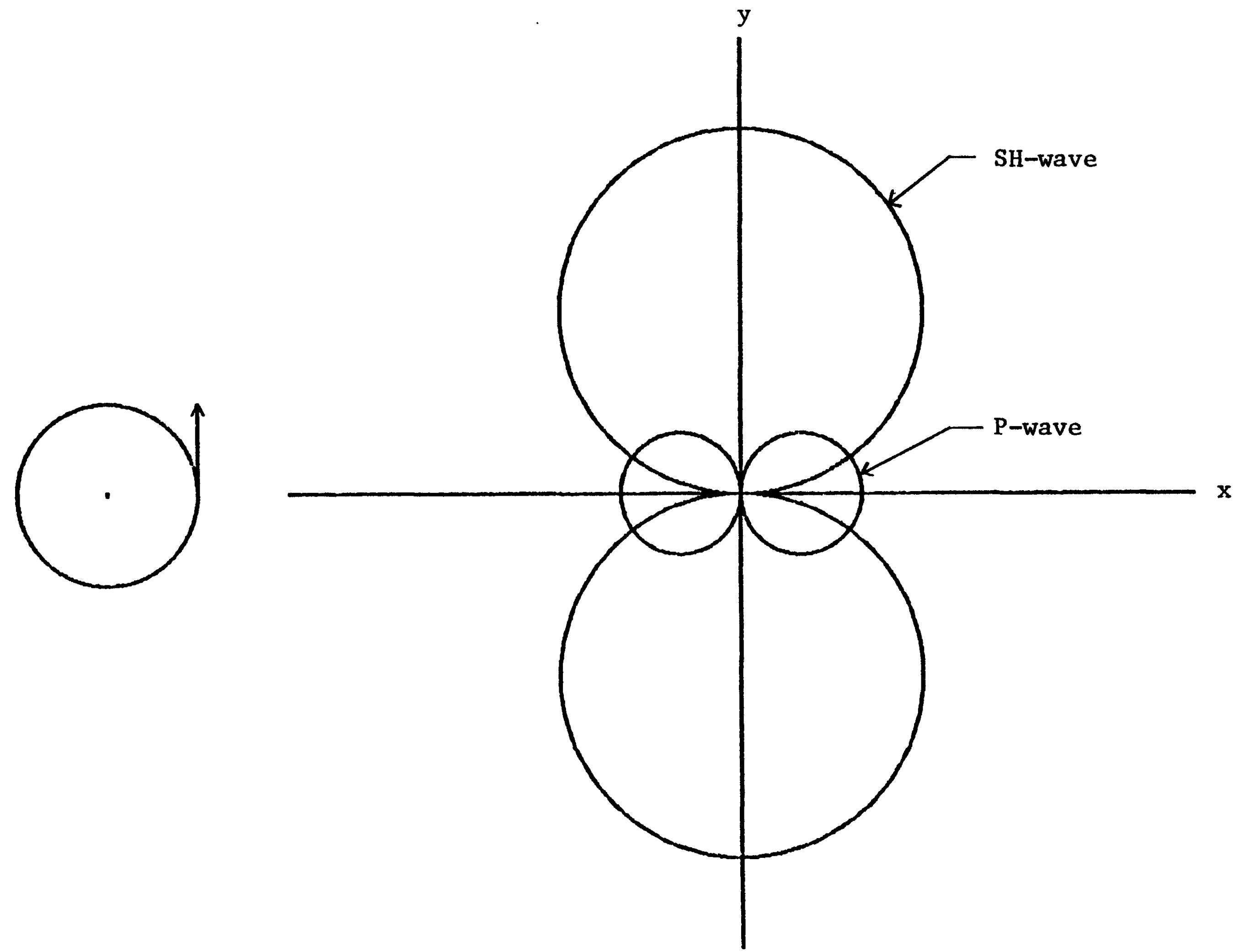

Figure 3.--Radiation patterns for P- and SH-waves from a single force acting on a fluid-filled (or empty) borehole. Left portion shows the source configuration in the x-y plane and the right portion shows the radiation patterns in the x-y plane using a Poisson's solid.

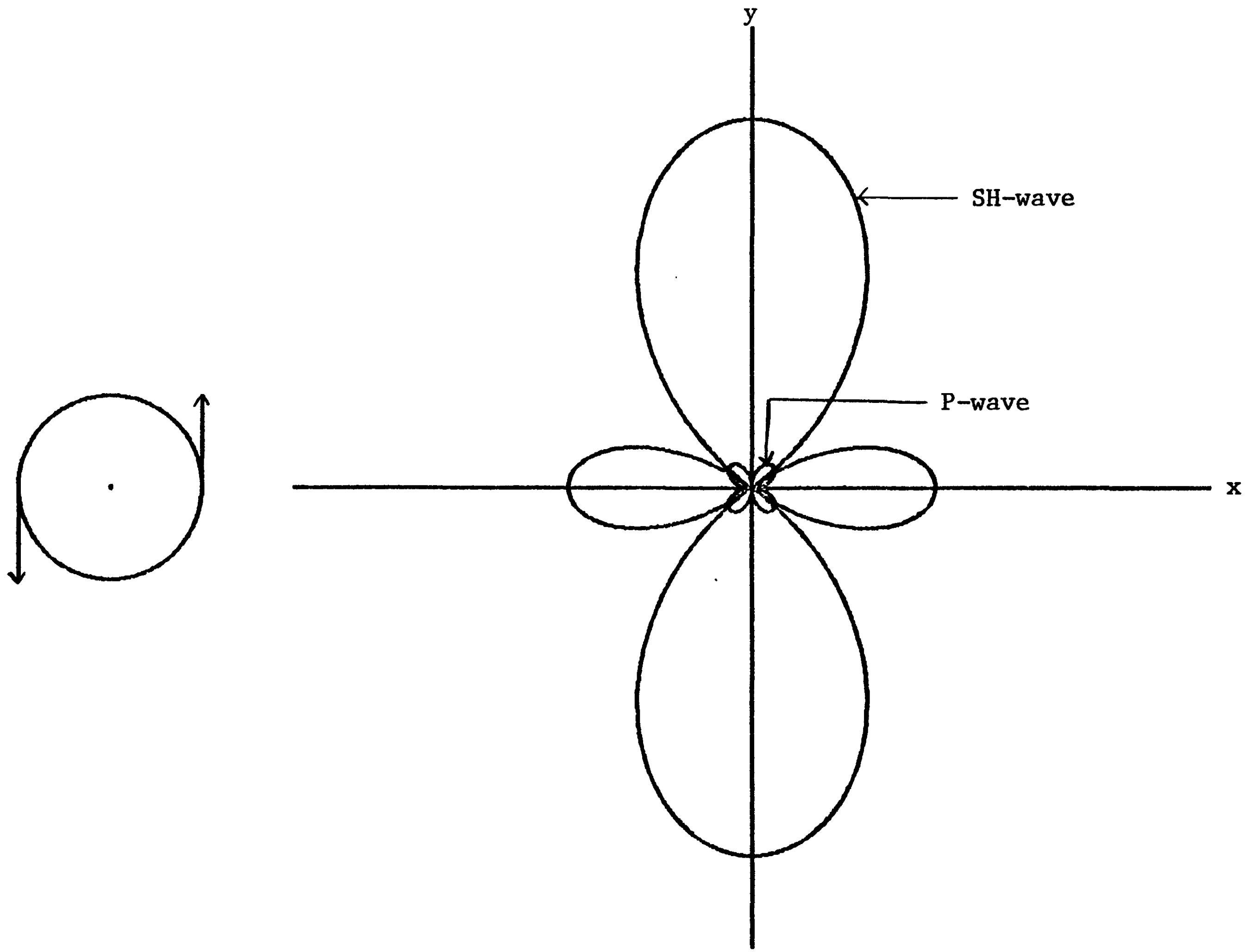

Figure 4.--Radiation patterns for P- and SH-waves from double forces acting on a fluid-filled (or empty) borehole. Left portion shows the source configuration in the x-y plane and the right portion shows the radiation pattern in the x-y plane using a Poisson's solid.

REFERENCES

Balch, A. H., and Lee, M. W., 1984, Borehole seismology, in Balch, A. H., and Lee, M. W., eds., Vertical seismic profiling: International Human Resources Development Company, 488 p.

Greenfield, R. J., 1978, Seismic radiation from a point source on the surface of a cylindrical cavity: Geophysics, v. 43, p. 1071-1082.

Harrington, R. F., 1961, Time harmonic electromagnetic fields: New York, McGraw-Hill, Inc.

Heelan, P. A., 1953, Radiation from a cylindrical source of finite length: Geophysics, v. 18, p. 685-696.

Lee, M. W., and Balch, A. H., 1982, Theoretical seismic wave radiation from a fluid-filled borehole: Geophysics, v. 47, p. 1308-1314.

Lee, M. W., Balch, A. H., and Parrott, K. R., 1984, Radiation from a downhole air gun source: Geophysics, v. 49, p. 27-36.

White, J. E., 1960, Use of reciprocity theorem for computation of low-frequency radiation patterns: Geophysics, v. 25, p. 613-624.

White, J. E., and Sengbush, R. L., 1963, Shear waves from explosive sources: Geophysics, v. 28, p. 1101-1109.

APPENDIX

This appendix provides some of the mathematical details omitted in the main text and derived displacement fields for the two-dimensional case.

A) Proof of equation (9).--The following abbreviation should be understood throughout this derivation:

$$\sum_{p=-\infty}^{\infty} \int_{-\infty}^{\infty} D_p J_p(\ell r)\, e^{-ikz} e^{ip\theta} dk \longrightarrow D_p J_p(\ell r)$$

Using equations (2), (3), and (6), the following displacement and stress components can be derived.

1. Inside the fluid:

$$U_r' = D_p J_p'(\ell r)$$

$$P_{rr}' = -D_p \rho_f \omega^2 J_p(\ell r) \qquad \text{(A-1)}$$

$$P_{rz}' = P_{r\theta}' = 0$$

2. In the surrounding medium:

$$
\begin{aligned}
U_r = {} & A_p H_p''(mr) \\
& - B_p ik H_p''(nr) \\
& + C_p ip H_p(nr) ,
\end{aligned}
$$

$$
\begin{aligned}
P_{rr} = {} & A_p \left[2\mu H_p''(mr) - \frac{\lambda \omega^2}{\alpha^2} H_p(nr) \right] \\
& - B_p 2ik\mu H_p''(nr) \\
& + C_p \left[2\mu ip \frac{H_p'(nr)}{r} - 2\mu ip \frac{H_p(nr)}{r^2} \right] ,
\end{aligned}
\tag{A-2}
$$

$$
\begin{aligned}
P_{rz} = {} & - A_p 2ik\mu H_p''(mr) \\
& + B_p \mu (k_\beta^2 - 2k^2) H_p'(nr) \\
& + C_p p\mu k \frac{H_p(nr)}{r} ,
\end{aligned}
$$

$$
\begin{aligned}
P_{r\theta} = {} & A_p 2i\mu p \left[\frac{H_p'(mr)}{r} - \frac{H_p(mr)}{r^2} \right] \\
& + B_p 2\mu pk \left[\frac{H_p'(nr)}{r} - \frac{H_p(nr)}{r^2} \right] \\
& + C_p \mu \left[\frac{H_p'(nr)}{r} - \frac{p^2 H_p(nr)}{r^2} - H_p''(nr) \right] .
\end{aligned}
$$

Substituting equations (A-1) and (A-2) into equation (6) with r = a and utilizing equation (10), equations (8) and (9) can be obtained.

B. Derivation of equations (11), (12), and (13).--The derivation of equations (11), (12), and (13) requires a good deal of algebra. The derivative formula I used for the coefficients of the determinant is the following.

$$H_p'(mr) = -m\,H_{p+1}(mr) + \frac{p H_p(mr)}{r}$$

$$H_p''(mr) = \frac{p(p-1)}{r^2} H_p(mr) - m^2 H_p(mr) + \frac{m}{r} H_p(mr).$$

The expansion of the Hankel functions can be written, when P > 1, as:

$$H_p^{(2)}(x) \cong \frac{i\,2^p}{\pi x^p}(p-1)!\left[1 + \frac{x^2}{4(p-1)} + \cdots\right]$$

As mentioned in the main text, the expansion of the Hankel function should include the second term in order to get non-zero determinant, when $p \geq 1$. For the Hankel function order 1, which appears in the zero order (p = 0) solution,

$$H_1^{(2)}(x) \cong \frac{2i}{\pi x}$$

was used.

The approximate coefficients, which are valid under the assumption that the radius of the borehole is very small compared to the wavelength of interest, are shown below.

Let's define:

$$Q_p = \frac{i\,2^p}{\pi (ma)^p}(p-1)!\,,$$

$$S_p = \frac{i\,2^p}{\pi (na)^p}(p-1)!\,.$$

When p = 0, the leading term of the matrix elements are given by:

$$
\begin{aligned}
g_{11} &= -mQ_1 \\
g_{12} &= inkS_1 \\
g_{13} &= 0 \\
g_{14} &= \ell^2 a/2 \\
g_{21} &= 2_i kmQ_1 \\
g_{22} &= (k^2 - n^2)nS_1 \\
g_{23} &= 0 \\
g_{24} &= 0 \\
g_{31} &= 0 \\
g_{32} &= 0 \\
g_{33} &= -2nS_1/a \\
g_{34} &= 0 \\
g_{41} &= 2mQ_1/a \\
g_{42} &= -2_i knS_1/a \\
g_{43} &= 0 \\
g_{44} &= \rho_f \omega^2/\mu
\end{aligned}
\qquad \text{(A-3)}
$$

The determinant of the zero-order of G_p, Δ_o, is given by:

$$
\begin{aligned}
\Delta_o &= g_{14} g_{33} (g_{21} g_{42} - g_{41} g_{22}) + g_{44} g_{33} (g_{11} g_{22} - g_{12} g_{21}) \\
&\cong \frac{-2n^2 m k_\beta^2 Q_1 S_1^2}{a} \left(\ell^2 + \frac{\rho_f \omega^2}{\mu} \right).
\end{aligned}
\qquad \text{(A-4)}
$$

Applying Cramer's rule to equation (8) with equations (A-2) and A-3), equation (11) can be obtained.

When $p = 1$, as mentioned in the main text, the leading term of the determinant is:

$$\Delta_1 \cong -g_{14}\,\tilde{\Delta}_{14}$$

where $\tilde{\Delta}_{14}$ is the minor of g_{14}.

Therefore, the following matrix elements are required to solve equations for A, B, and C.

$$g_{21} \cong i k m Q_2$$

$$g_{22} \cong \frac{(k^2-n^2) n S_2}{2} \qquad \text{(A-5)}$$

$$g_{23} \cong \frac{k n S_2}{2}$$

$$g_{31} \cong \frac{-2 i m Q_2}{a}\left(1 + \frac{m^2 a^2}{4}\right)$$

$$g_{32} \cong \frac{-2 k n S_2}{a}\left(1 + \frac{n^2 a^2}{4}\right)$$

$$g_{33} \cong \frac{-2 n S_2}{a}$$

$$g_{41} \cong \frac{2 m Q_2}{a}\left[1 + \frac{(k^2+m^2-n^2)a^2}{4}\right]$$

$$g_{42} \cong \frac{-2 i k n S_2}{a}\left(1 - \frac{n^2 a^2}{4}\right)$$

$$g_{43} \cong \frac{-2 i n S_2}{a}\left(1 + \frac{n^2 a^2}{4}\right).$$

The dominant term of determinant Δ_1 is given by:

$$\Delta_1 \cong g_{14}\, m n^4 k_\beta^2 Q_2 S_2^2 \tag{A-6}$$

Applying Cramer's rule to equation (8) with equations (A-5) and (A-6), equation (12) can be obtained.

When p = 2, the procedures are identical to the case of p = 1. The necessary element of the determinant is given by:

$$\begin{aligned}
g_{21} &\cong i m k Q_3 \\
g_{22} &\cong \frac{(k^2-n^2) n S_3}{2} \\
g_{23} &\cong \frac{k n S_3}{2} \\
g_{31} &\cong \frac{-3 i m Q_3}{a}\left(1 + \frac{m^2 a^2}{12}\right) \\
g_{32} &\cong \frac{-3 k n S_3}{a}\left(1 + \frac{n^2 a^2}{12}\right) \\
g_{33} &\cong \frac{-3 n S_3}{a}\left(1 + \frac{n^2 a^2}{12}\right) \\
g_{41} &\cong \frac{3 m Q_3}{a}\left[1 + \frac{(3m^2+k^2-n^2)a^2}{12}\right] \\
g_{42} &\cong \frac{-3 i k n S_3}{a} \\
g_{43} &\cong \frac{-3 i n S_3}{a}\left(1 + \frac{n^2 a^2}{12}\right)
\end{aligned} \tag{A-7}$$

The dominant term of the determinant Δ_2 is given by:

$$\Delta_2 \cong g_{14} \, \frac{3}{8} \, m n^4 (k_\beta^2 - k_\alpha^2) \, Q_3 \, S_3^2 \tag{A-8}$$

Applying Cramer's rule to equation (8) with equations (A-7) and (A-8), equation (13) can be obtained.

C. Derivation of equation (15).--The leading terms of the displacement fields when the observation point is very far from the source are, from equation (2):

$$\begin{aligned} U_r &\cong \frac{\partial \Phi}{\partial r} + \frac{\partial^2 \Psi}{\partial r \partial z} , \\ U_\theta &\cong -\frac{\partial X}{\partial r} , \\ U_z &\cong \frac{\partial \Phi}{\partial z} + (k_\beta^2 - k^2) \Psi . \end{aligned} \tag{A-9}$$

Using the relation of equation (14), the far-field displacements are:

$$\begin{aligned} U_r &= \sum_{p=-2}^{2} A_p (k_\alpha \cos\phi) \, 2k_\alpha \sin\phi \, (i)^p e^{ip\theta} \, \frac{e^{-ik_\alpha R}}{R} \\ &\quad - \sum_{p=-2}^{2} B_p (k_\beta \cos\phi) \, k_\beta^2 \sin(2\phi) \, (i)^{p+1} e^{ip\theta} \, \frac{e^{-ik_\beta R}}{R} , \\ U_\theta &= -\sum_{p=-2}^{2} C_p (k_\beta \cos\phi) \, 2k_\beta \sin\phi \, (i)^p e^{ip\theta} \, \frac{e^{-ik_\beta R}}{R} \\ U_z &= \sum_{p=-2}^{2} A_p (k_\alpha \cos\phi) \, 2k_\alpha \cos\phi \, (i)^p e^{ip\theta} \, \frac{e^{-ik_\alpha R}}{R} \\ &\quad + \sum_{p=-2}^{2} B_p (k_\beta \cos\phi) \, 2k_\beta^2 \sin^2\phi \, (i)^{p+1} e^{ip\theta} \, \frac{e^{-ik_\beta R}}{R} . \end{aligned} \tag{A-10}$$

By the coordinate transformation, the displacement field in the spherical coordinate system is:

$$
\begin{aligned}
U_R &= U_r \sin\phi + U_z \cos\phi\,, \\
U_\phi &= U_r \cos\phi - U_z \sin\phi\,, \\
U_\theta &= U_\theta\,.
\end{aligned}
\tag{A-11}
$$

Substituting equation (A-10) into (A-11) and utilizing the property of the solution matrix shown in equations (11), (12), and (13), equation 15 can be derived.

D. Two-dimensional case.--In the two-dimensional wave propagation in the X-Y plane, the two scalar potentials are needed to describe the displacement field. It can be written as follows from equation (1):

$$
\begin{aligned}
\nabla^2 \Phi &= -k_\alpha^2 \Phi\,, \\
\nabla^2 X &= -k_\beta^2 X\,.
\end{aligned}
\tag{A-12}
$$

The formal solution of equation (A-12) could be written as:

$$
\Phi = \sum_{p=-\infty}^{\infty} A_p e^{ip\theta} H_p^{(2)}(k_\alpha r)\,, \tag{A-13}
$$

$$
X = \sum_{p=-\infty}^{\infty} C_p e^{ip\theta} H_p^{(2)}(k_\beta r)\,.
$$

In the two-dimensional case, the source function is defined by:

$$
T_r(\theta) = \sum_{p=-\infty}^{\infty} E_p^r e^{ip\theta}
$$

with

$$
E_p^r = \frac{1}{2\pi}\int_0^{2\pi} T_r(\theta)\, e^{-ip\theta}\, d\theta.
$$

The coefficients A_p and C_p are the solution of the following equation.

$$
\begin{bmatrix} g_{31} & g_{33} \\ g_{41} & g_{43} \end{bmatrix}
\begin{Bmatrix} A_p \\ C_p \end{Bmatrix}
=
\begin{Bmatrix} \dfrac{E_p^\theta}{\mu} \\ \dfrac{E_p^r}{\mu} \end{Bmatrix}.
$$

The coefficients of g_{ij} could be obtained by k --> 0 in equation (9).

Following the same procedures in the main text, the coefficient of A_p and C_p can be derived and it is shown as:

$$\begin{bmatrix} A_0 \\ C_0 \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{a}{2mQ_1} \\ \dfrac{-a}{2nS_1} & 0 \end{bmatrix} \begin{bmatrix} \dfrac{E_0^\theta}{\mu} \\ \dfrac{E_0^r}{\mu} \end{bmatrix} \tag{A-14}$$

$$\begin{bmatrix} A_1 \\ C_1 \end{bmatrix} = \begin{bmatrix} \dfrac{i}{amn^2Q_2} & \left(\dfrac{-1}{an^2m\,Q_2}\right) \\ \left(\dfrac{1}{an^3S_2}\right) & \dfrac{i}{an^3S_2} \end{bmatrix} \begin{bmatrix} \dfrac{E_1^\theta}{\mu} \\ \dfrac{E_1^r}{\mu} \end{bmatrix}$$

$$\begin{bmatrix} A_2 \\ C_2 \end{bmatrix} = \begin{bmatrix} \left(\dfrac{-4i}{am\,(m^2-n^2)\,Q_3}\right) & \dfrac{4}{am\,(m^2-n^2)\,Q_3} \\ \dfrac{-4}{an\,(m^2-n^2)\,S_3} & \left(\dfrac{-4i}{an\,(m^2-n^2)\,S_3}\right) \end{bmatrix} \begin{bmatrix} \dfrac{E_2^\theta}{\mu} \\ \dfrac{E_2^r}{\mu} \end{bmatrix}$$

In equation (A-14), the matrix elements with a parenthesis denote the terms which change the algebraic sign when p is negative.

The far-field displacement can be derived utilizing the following formulae:

$$U_r \cong \frac{\partial \Phi}{\partial r},$$

$$U_\theta \cong -\frac{\partial X}{\partial r},$$

with

$$\frac{\partial H_0^{(2)}(kr)}{\partial r} \cong -k\, H_1^{(2)}(kr), \tag{A-15}$$

$$\frac{\partial H_1^{(2)}(kr)}{\partial r} \cong -k\, H_2^{(2)}(kr),$$

$$\frac{\partial H_2^{(2)}(kr)}{\partial r} \cong k\, H_1^{(2)}(kr).$$

The far-field displacement fields were obtained by substituting equation (A-14) into equation (A-15). The desired displacements are given below.

1) $T_r(\theta)$ is given and defined as $T_r(\theta) = T_r\,\delta(\theta)$.

$$U_r = \frac{T_r\, i a^2 k_\alpha}{8\mu}\left[1 + \frac{2\cos 2\theta\, \beta^2}{(1-\beta^2/\alpha^2)\,\alpha^2}\right] H_1^{(2)}(k_\alpha r) - \frac{T_r\, i a \beta^2 \cos\theta}{4\mu\alpha^2} H_2^{(2)}(k_\alpha r), \tag{A-16}$$

$$U_\theta = \frac{T_r\, i a \sin\theta}{4\mu} H_2^{(2)}(k_\beta r) - \frac{T_r\, i a^2 \sin 2\theta}{4\mu}\,\frac{1}{(1-\beta^2/\alpha^2)} H_1^{(2)}(k_\beta r).$$

2) $T_\theta(\theta)$ is given and defined as $T_\theta(\theta) = T\theta^\delta(\theta)$.

$$U_r = \frac{-T_\theta\, i a \beta^2}{4\mu\alpha^2} \sin\theta\, H_2^{(2)}(k_\alpha r)$$

$$+ \frac{i a^2 k_\alpha \quad \beta^2}{4\mu(1-\beta^2/\alpha^2)\alpha^2} \sin 2\theta\, H_1^{(2)}(k_\alpha r)$$

$$U_\theta = \frac{T_\theta\, i a^2 k_\beta}{8\mu}\left[1 + \frac{2\cos 2\theta}{(1-\beta^2/\alpha^2)}\right] H_1^{(2)}(k_\beta r) \tag{A-17}$$

$$- \frac{T_\theta\, i a}{4\mu} \cos\theta\, H_2^{(2)}(k_\beta r)$$

When the normal stresses act on at $\theta = 0$ and $\theta = 180^\circ$ with equal magnitude, then the radial and tangential displacement can be written as:

$$\tilde{U}_r = U_r(\theta) + U_r(\theta + 180)$$

$$\doteq \frac{T_r\, i\, a^2 k_\alpha}{4\mu}\left(\frac{2}{\pi k_\alpha r}\right)^{1/2} e^{\frac{3}{4}\pi i}\, e^{-i k_\alpha r}\left[1 + \frac{2\cos 2\theta\ \beta^2}{(1-\beta^2/\alpha^2)\alpha^2}\right], \tag{A-18}$$

$$\tilde{U}_\theta = \frac{-T_r\, i a^2}{2\mu}\left(\frac{2}{\pi k_\beta r}\right)^{1/2} e^{\frac{3}{4}\pi i}\, e^{-i k_\beta r} \cdot \frac{1}{(1-\beta^2/\alpha^2)}.$$

Here the following relation was used.

$$H_1^{(2)}(k_\alpha r) \xrightarrow[k_\alpha r \to \infty]{} e^{-i(k_\alpha r - \frac{3}{4}\pi)}\left(\frac{2}{\pi k_\alpha r}\right)^{1/2}.$$

The displacements shown in equation (A-18) are identical to those of White (1960), who derived the same equation using the seismic reciprocity theorem.

E. Source description.--Let's define spatial Fourier transform as:

$$\tilde{S}(k) \triangleq \int_{-\infty}^{\infty} S(z)\, e^{ikz}\, dz$$

and its inverse Fourier transform as:

$$S(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{S}(k)\, e^{-ikz}\, dk.$$

The source located at r = a and z = 0 can be written as:

$$T(\theta, z) = T(\theta)\,\delta(z). \tag{A-19}$$

Therefore, using complex Fourier analysis and Fourier transfer of $\delta(z)$, we can write equation (A-19) as:

$$T(\theta, z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikz} \sum_{p=-\infty}^{\infty} E_p\, e^{ip\theta}\, dk,$$

with

$$E_p = \frac{1}{2\pi}\int_0^{2\pi} T(\theta)\, e^{-ip\theta}\, d\theta.$$

Let's define other source distribution such as

$$T^{*}(\theta, z) = T(\theta)\, L_d(z)$$

where

$$L_d(z) = 1 \quad \text{when} \quad 0 < |z| < \frac{d}{2}$$

= 0 otherwise.

Using the following relation,

$$\tilde{L}_d(k) = \int_{-\infty}^{\infty} L_d(z)\, e^{ikz}\, dz = \frac{2\sin\left(\frac{kd}{2}\right)}{k},$$

we can show that

$$T^*(\theta,z) = \sum_{p=-\infty}^{\infty} E_p e^{ip\theta} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\sin\left(\frac{kd}{2}\right)}{k} e^{-ikz} dk.$$

As far as the far-field displacement field is concerned, the contribution of the source T* can be evaluated by substituting $k = k_\alpha \cos\phi$ or $k_\beta \sin\phi$ depending on the wave type considered.

Let's use $k_\beta \sin\phi$ for k.

Then,

$$T^*(\theta,z) \xrightarrow[R\to\infty]{} \sum_{p=-\infty}^{\infty} E_p d \frac{\sin\left(\frac{k_\beta d}{2}\right)}{\frac{k_\beta d}{2}} e^{ip\theta}.$$

Using the following formula:

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1, \text{ we can show that}$$

$$\lim_{k_\beta d\to 0} T^*(\theta,z) \longrightarrow \sum_{p=-\infty}^{\infty} E_p d e^{ip\theta}.$$

This implies that as long as $\omega << 2\beta/d$, we can treat the distributed source in the z-direction as a point source at z = 0.