AN EFFICIENT DETERMINISTIC-PROBABILISTIC APPROACH TO MODELING REGIONAL GROUND-WATER FLOW: THEORY

U.S. GEOLOGICAL SURVEY

Open-File Report 88-90

Prepared in cooperation with
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LOS ANGELES DEPARTMENT OF WATER AND POWER

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By *Chung-Cheng Yen* and *Gary L. Guymon*

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CONVERSION FACTORS

Metric (SI) units are used in this report. For readers who prefer inch-pound units, the conversion factors for the terms used in this report are listed below.

Multiply	By	To obtain
cm (centimeter)	0.0328	feet

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ABSTRACT

An efficient probabilistic model is developed and cascaded with a deterministic model for predicting dynamic water-table elevations in regional aquifers. The objective is to quantify model uncertainty where precise estimates of water-table elevations may be required. The probabilistic model is based on the two-point probability method which only requires prior knowledge of uncertain variables mean and coefficient of variation. The two-point estimate method is theoretically developed and compared with the Monte Carlo simulation method. The results of comparisons using hypothetical deterministic problems indicate that the two-point estimate method is only generally valid for linear problems where the coefficients of variation of uncertain parameters (for example, storage coefficient and hydraulic conductivity) are small. The two-point estimate method may be applied to slightly nonlinear problems with good results, provided coefficients of variation are small. In such cases, the two-point estimate

method is much more efficient than the Monte Carlo method provided the number of uncertain variables is less than eight.

INTRODUCTION

Increasingly, workers in ground-water hydrology are recognizing the need to systematically consider uncertainty when modeling regional ground-water flow. A variety of deterministic ground-water flow models based on finite-element or finite-difference approaches now are routinely and widely used. During the last decade, ground-water modeling research has focused on methods for systematically dealing with ground-water modeling uncertainty.

This study was undertaken as part of a larger study to develop models as management tools for the ground-water basin in Owens Valley, California which provides a major part of the water supply for the city of Los Angeles, California. The overall study involved the development of several mathematical models to provide tools to study management strategies for minimizing effects on desert floor vegetation while providing required water for the city of Los Angeles. Because of the assumed sensitivity of desert vegetation to water-table elevations, mathematical models may have to be able to predict future water-table elevations to a high level of precision. The objective of the research reported in this study was to develop suitable models that would assess efficient models of uncertainty and would define areas of the ground-water basin, where more study is needed in order to obtain required precision in estimating water-table elevations.

The overall study of Owens Valley ground-water was a cooperative study involving the city of Los Angeles, Department of Water and Power, Inyo County, and the U.S. Geological Survey.

Modeling uncertainty may be grouped into four categories of error, as

follows [Guymon et al., 1981]:

1. Errors due to the choice of a mathematical representation of an aquifer and the choice of a numerical analog to solve the partial differential equation that describe ground-water motion,
2. Errors due to spatial and temporal discretization and averaging that arise when applying regional numerical ground-water flow models,
3. Errors due to boundary and initial conditions that are required to solve the deterministic equation of ground-water motion,
4. Errors due to the choice of parameters that necessarily arise in any physics-based model of ground-water motion.

Total model uncertainty or error is some function of the above errors. Errors may be interrelated and total modeling uncertainty may be nonstationary in a dynamic system.

Research relating to ground-water modeling uncertainty can be classified as: (1) direct methods, where the statistics of the response function, that is, water levels, can be directly obtained by reformulating the governing partial differential equation, such as: first- and second-order error analysis [Dettinger, 1979; Dettinger and Wilson, 1981], spectral analysis [Gelhar, 1974 and 1977; Mizell et al., 1982], perturbation and linearization [Tang and Pinder, 1977 and 1979], and sensitivity analysis [McElwee and Yukler, 1978], and (2) cascade methods, where an assumed probability model is cascaded into a deterministic numerical model, such as: Monte Carlo simulation [Warren and Price, 1961; Wu et al., 1973; Freeze, 1975], nearest neighbor model [Smith and Freeze, 1979 a and b], geostatistic approach [Delhomme, 1978 and 1979], method of embedding matrix [Dagan, 1979, 1981, 1982 a, b, and c], and inverse problems and

conditional simulation [Neuman and Yakowitz, 1979; Neuman et al., 1980].

The advantages of using the direct methods are that simple assumptions can be used concerning uncertain data or parameters and tedious computation is not needed. The drawbacks of these methods are that the reformulation process is complicated and only linear or quasi-linear systems have been studied. The advantages of using the cascade methods are that they are simple to formulate and applicable to any system. The major drawbacks to current cascade methods are complete information is required for the probability model and tremendous computation efforts usually are required. The objective of this paper is to develop a probability model that systematically evaluates parameter uncertainty and avoids the drawbacks commonly encountered in the cascade methods. To do this, a new probability approach based on the point estimate of the probability moments [Rosenblueth, 1975] is proposed. The new approach, which uses the advantages of both the direct and cascade methods, is presented.

REVIEW OF DETERMINISTIC GROUND-WATER FLOW MODEL

For purposes of clarity and completeness, the example deterministic model is reviewed. In general, flow through a porous medium domain is three dimensional. The governing partial differential equation, derived from the linear Darcy's law and continuity equation, has the form

$$\frac{\partial}{\partial x}\left(K_{xx}\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_{yy}\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_{zz}\frac{\partial \phi}{\partial z}\right) + Q = S\,\frac{\partial \phi}{\partial t}\ , \qquad (1)$$

where K_{xx}, K_{yy}, K_{zz} are the principal components of the hydraulic conductivity tensor; $\phi = \phi(x,y,z,t)$ is the total potential hydraulic head;

$$Q = \sum_{i=1}^{n} Q_i\ \delta(x-x_i)\ \delta(y-y_i)\ \delta(z-z_i)\ \delta(t-t_i)$$

are the sources or sinks and δ is the Dirac delta function; and S is defined either as the specific yield (S_y) of an unconfined aquifer or as the storage coefficient (S_s) of a confined aquifer.

However, because the saturated thicknesses of most aquifers are thin relative to their horizontal dimensions, a simple approach, that is, concept of horizontal flow, usually can be used to represent the complicated phenomenon of flow through porous media. This approach assumes that the flow in an aquifer is everywhere approximately horizontal and vertical flow components are negligible. The error introduced by this assumption is small in most cases of practical interest [Bear, 1979].

Integrating (1) in the z-direction, one obtains

$$\frac{\partial}{\partial x}\left(K_{xx}\bar{B}\,\frac{\partial\bar{\phi}}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_{yy}\bar{B}\,\frac{\partial\bar{\phi}}{\partial y}\right) + Q(x,y,t) + q_z\Big|_{b_1(x,y)}^{b_2(x,y)} = S\,\frac{\partial\bar{\phi}}{\partial t}\ , \tag{2}$$

where $\bar{B}$ is the averaged thickness of an aquifer and $\bar{\phi}$ is defined as:

$$\bar{\phi} = \bar{\phi}\,(x,y,t) = \frac{1}{\bar{B}}\int_{b_1(x,y)}^{b_2(x,y)} \phi(x,y,z,t)\,dz\ , \tag{3}$$

where $b_1(x,y)$ and $b_2(x,y)$ are the lower and upper boundaries of an aquifer, respectively.

By using the nodal domain integration method [Hromadka et al., 1981], the final compact matrix representation of (2) is

$$[S]\left\{\phi\right\} + [P]\left\{\dot{\phi}\right\} = \left\{F\right\}\ , \tag{4}$$

where [S] is a symmetrical stiffness matrix which depends only on the geometry

and hydraulic conductivity, $\{\phi\}$ is a column matrix, which stores the averaged potential hydraulic head of the aquifer; [P] is a symmetrical capacitance matrix which depends on the geometry as well as on the specific yield for an unconfined aquifer or the storage coefficient for a confined aquifer; $\{\dot{\phi}\}$ is a column matrix, which consists of the time derivatives of the averaged potential hydraulic head; and $\{F\}$ is a column matrix, which stores the sinks or sources and specified boundary conditions.

For a triangular-finite element (fig. 1), the element matrices for [S] and [P] are

$$[S]^e = \frac{K^e B^e}{4A^e}\begin{bmatrix} (x_{23}^2 + y_{23}^2) & -(x_{13}x_{23} + y_{13}y_{23}) & (x_{12}x_{23} + y_{12}y_{23}) \\ & (x_{13}^2 + y_{13}^2) & -(x_{12}x_{13} + y_{12}y_{13}) \\ \text{(symmetric)} & & (x_{12}^2 + y_{12}^2) \end{bmatrix}, \qquad (5)$$

and

$$[P]^e = \frac{S^* A^e}{3(\eta+2)}\begin{bmatrix} \eta & 1 & 1 \\ 1 & \eta & 1 \\ 1 & 1 & \eta \end{bmatrix}, \qquad (6)$$

where K^e is the averaged saturated hydraulic conductivity of an element; B^e is the averaged saturated aquifer thickness of an element; A^e is the area of an element, $x_{ij} = x_j - x_i$; i,j = 1,2,3 (where x is the x-coordinate for node i); S^* is the averaged specific yield or storage coefficient for unconfined or confined aquifer, and η is the mass lumping factor for a nodal domain integration model (η equal to 2, 22/7, + ∞ for a Galerkin, subdomain, or integrated finite difference scheme, respectively).

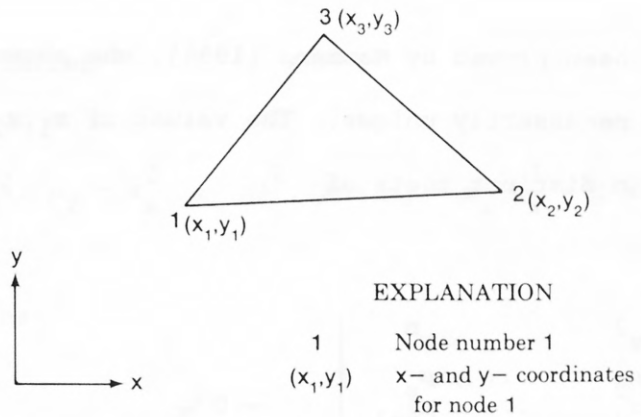

FIGURE 1.—Triangular finite element representation.

A fully explicit time domain advancement scheme [Pinder and Gray, 1977], is used in this study to solve the deterministic part of the problem, and thus (4) is written as

$$\left([S] + \frac{1}{\Delta t} [P] \right)_{t+\Delta t} \{\phi\}_{t+\Delta t} = \frac{1}{\Delta t} [P]_t \{\phi\}_t + \{F\}_{t+\Delta t} , \qquad (7)$$

where Δt is the time step increment, and the subscripts represent the reference time at which the matrix is approximated.

ALGEBRAIC MOMENT PROBLEM

The basic concept of the two-point estimate method is similar to the algebraic moment problem [Mammana, 1954 and Tucker, 1963], which states that given 2n real numbers $\mu_o, \mu_1, \ldots, \mu_{2n-1}$, one can find n distinct real numbers $x_1, x_2, \ldots, x_n$ and n positive real numbers $P_1, P_2, \ldots, P_n$ such that

$$\sum_{i=1}^{n} P_i x_i^k = \mu_k; \quad k = 0, 1, \ldots, 2n-1 . \tag{8}$$

This statement has been proved by Mammana [1954], who showed that the solutions of x's and P's are necessarily unique. The values of $x_1, x_2, \ldots, x_n$ are obtained by solving for the n distinct roots of

$$\begin{vmatrix} 1 & x & x^2 & \cdots & x^n \\ \mu_0 & \mu_1 & \mu_2 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_{n+1} \\ \vdots & & & & \vdots \\ \mu_{n-1} & \mu_n & \mu_{n+1} & \cdots & \mu_{2n-1} \end{vmatrix} = 0 . \tag{9}$$

The n positive numbers $P_1, P_2, \ldots, P_n$ are then obtained by solving

$$\sum_{i=1}^{n} P_i x_i^k = \mu_k; \quad k = 0, 1, \ldots, n-1 . \tag{10}$$

If we let X be a random variable taking values $x_1, x_2, \ldots, x_n$ with corresponding probabilities $P_1, P_2, \ldots, P_n$, then (10) represents the moments of random variable X. On the other hand, if one can estimate the first 2n-1 moments of random variable X, then one can use (8) and (9) to find a discrete probability density function of n mass densities for the random variable X. In practice, only the first few moments can be estimated from successive experiments or observations for any random variable. Thus, limited n values of x's and P's can be obtained from (8) and (9), respectively.

For example, let random variable X have a finite mean $\bar{x}$, variance s_x^2,

and skewness coefficient $\nu_x = E[(X - \bar{x})^3]/s_x^3)$. If we further assume that random variable Y is a function of X, that is, Y = f(X), then the approximate expressions for the moments of random variable Y can be obtained by solving (8) and (9).

First, substituting

$$\mu_0 = 1,\ \mu_1 = \bar{x},\ \mu_2 = s_x^2 + (\bar{x})^2,\ \text{and}\ \mu_3 = \nu_x s_x^3 + 3\bar{x} s_x^2 + (\bar{x})^3,$$

into (9) one obtains

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & \bar{x} & s_x^2 + (\bar{x})^2 \\ \bar{x} & s_x^2 + (\bar{x})^2 & \nu_x s_x^3 + 3\bar{x} s_x^2 + (\bar{x})^3 \end{vmatrix} = 0 . \tag{11}$$

By solving (11), the two distinct roots are

$$\begin{aligned} x_1 &= \bar{x} + \frac{1}{2}\left(\nu_x s_x - \sqrt{\nu_x^2 s_x^2 + 4s_x^2}\right), \\ x_2 &= \bar{x} + \frac{1}{2}\left(\nu_x s_x + \sqrt{\nu_x^2 s_x^2 + 4s_x^2}\right), \end{aligned} \tag{12}$$

then, the two positive numbers P_1 and P_2 can be obtained by solving the following simultaneous equations

$$\begin{aligned} &P_1 + P_2 = 1, \\ &x_1 P_1 + x_2 P_2 = \bar{x} . \end{aligned} \tag{13}$$

Thus, substituting (12) into (13) one obtains

$$P_1 = \frac{1}{2}\left(1 - \frac{\nu_x}{\sqrt{\nu_x^2 + 4}}\right), \qquad P_2 = 1 - P_1, \tag{14}$$

and substituting (14) into (12), one obtains

$$x_1 = \bar{x} + s_x\left(\sqrt{P_2/P_1}\right), \qquad x_2 = \bar{x} - s_x\left(\sqrt{P_1/P_2}\right) . \tag{15}$$

Because Y = f(X), the approximate expression for the n^{th} moment of the random variable Y is given as

$$E[Y^n] = P_1 f^n(x_1) + P_2 f^n(x_2) .$$

This is a two-point estimate representation of moments of the random variable Y and the first two central moments of Y can be approximated as

$$\bar{y} = P_1 f(x_1) + P_2 f(x_2) , \tag{16}$$

and

$$s_y^2 = P_1 P_2 [f(x_1) - f(x_2)]^2 .$$

Obviously, the algebraic moment problem represents the M-point estimate method of a single random variable.

To further illustrate the derivation of the point estimate method, consider the following example for two random variables. Let X_1, X_2, and Y be real random variables and $Y = f(X_1, X_2)$ be a well behaved function. Given the means $\bar{x}_1$ and $\bar{x}_2$, standard deviations s_1 and s_2, skewness coefficients ν_1 and ν_2, and correlation coefficient ρ_{12} ($= \left\{E[X_1 X_2] - \bar{x}_1\bar{x}_2\right\}/s_1 s_2$), we want to find an approximate expression for the moments of the random variable Y.

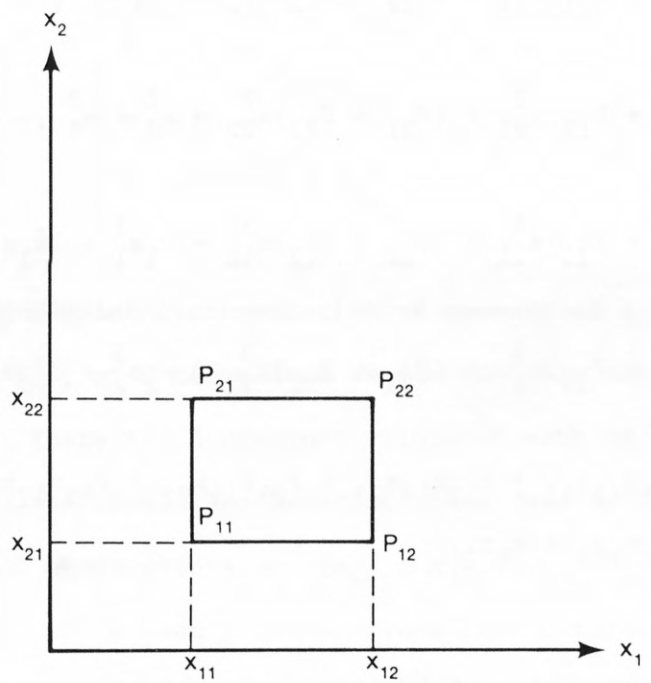

FIGURE 2.—Two-point representation for a function of two random variables.

Using the marginal density function (with the help of fig. 2), the problem can be formulated as

$$P_{11} + P_{12} + P_{21} + P_{22} = 1,$$

$$E[x_1] = (P_{11} + P_{21})x_{11} + (P_{22} + P_{12})x_{12} = \bar{x}_1 ,$$

$$E[x_2] = (P_{11} + P_{12})x_{21} + (P_{21} + P_{22})x_{22} = \bar{x}_2 ,$$

$$E[x_1^2] = (P_{11} + P_{21})x_{11}^2 + (P_{22} + P_{12})x_{12}^2 = \bar{x}_1^2 + s_1^2 ,$$

$$E[x_2^2] = (P_{11} + P_{12})x_{21}^2 + (P_{21} + P_{22})x_{22}^2 = \bar{x}_2^2 + s_2^2 , \qquad (17)$$

$$E[x_1^3] = (P_{11} + P_{21})x_{11}^3 + (P_{22} + P_{12})x_{12}^3 = \nu_1 s_1^3 + 3\bar{x}_1 s_1^2 + \bar{x}_1^3 ,$$

$$E[x_2^3] = (P_{11} + P_{12})x_{21}^3 + (P_{21} + P_{22})x_{22}^3 = \nu_2 s_2^3 + 3\bar{x}_2 s_2^2 + \bar{x}_2^3 ,$$

$$E[x_1 x_2] = P_{11}x_{11}x_{21} + P_{12}x_{12}x_{21} + P_{21}x_{11}x_{22} + P_{22}x_{12}x_{22},$$
$$= \rho_{12} s_1 s_2 + \bar{x}_1 \bar{x}_2 .$$

By solving the above system of equations, we obtain

$$\begin{aligned} x_{11} &= \bar{x}_1 - s_1\sqrt{(P_{12}/P_{11})}, \\ x_{12} &= \bar{x}_1 + s_1\sqrt{(P_{11}/P_{12})}, \\ x_{21} &= \bar{x}_2 - s_2\sqrt{(P_{22}/P_{21})}, \\ x_{22} &= \bar{x}_2 + s_2\sqrt{(P_{21}/P_{22})}, \end{aligned} \qquad (18)$$

where

$$P_{11} = \frac{1}{4}(1 + \rho_{12})[1 - (\frac{1-\rho_{12}}{1+\rho_{12}})A_1],$$

$$P_{12} = \frac{1}{4}(1-\rho_{12})\ (1+A_1), \tag{19}$$

$$P_{21} = \frac{1}{4}(1+\rho_{12})(1+A_2),$$

$$P_{22} = \frac{1}{4}(1+\rho_{12})[1- (\frac{1-\rho_{12}}{1+\rho_{12}})A_2],$$

in which

$$A_i = \frac{\nu_i}{\sqrt{4 + \nu_i^2}} \quad ; \quad i = 1,2.$$

In general, an M-point representation of moments of a random variable, $Y = f(X_1, X_2, \ldots, X_N)$, of N correlated random variable can be obtained as illustrated above. There are M distinct values on each of the N coordinate axes, giving MN points on the axes. The N-variate mass density involves M^N points. We may represent any of these points by $(x_{1i_o}, x_{2i_1}, \ldots, x_{Ni_{N-1}})$ for all N-tuples $(i_o, i_1, \ldots, i_{N-1})$, where each i takes values from 0 through M - 1, and where the projections of all the points on the k^{th} axis are $x_{ko}, x_{k1}, \ldots, x_{kN-1}$, and the mass density at each such point is denoted by $P(i_o, i_1, \ldots, i_{N-1})$. Thus, the approximate expression of an M-point estimate method for a function of N random variables can be expressed as

$$E[Y^n] = \sum_{i_o=0}^{M-1} \sum_{i_1=0}^{M-1} \cdots \sum_{i_{N-1}=0}^{M-1} P(i_o, i_1, \ldots, i_{M-1}) \cdot f^n(x_{1i_o}, x_{2i_1}, \ldots, x_{Ni_{N-1}}) . \tag{20}$$

This means we need to solve $MN + M^N$ unknowns in order to have a unique expression of (20). The solution of the $MN + M^N$ unknowns primarily depends on the prior information of each random variable. If the first K moments of each random variable can be estimated, then it provides $1 + (N - 1)N/2 + KN$ simultaneous equations, such as

(1) The sum of M concentrations (mass densities) must equal unity, that is,

$$\sum_{i_o=o}^{M-1} \sum_{i_1=o}^{M-1} \cdots \sum_{i_{N-1}=o}^{M-1} = 1 . \tag{21}$$

(2) The $(N - 1)N/2$ correlation equations between N variables (assuming cross-moments only up to order 2 for simplicity), that is,

$$\rho_{i \neq j} = \frac{E[(x_i - \bar{x}_i)(x_j - \bar{x}_j)]}{s_i s_j} , \tag{22}$$

for $i,j = 1,2, \ldots, N$.

(3) The KN moment equations for N variables, that is,

$$E[\ (x_i - \bar{x}_i)^n\ ] = \text{Estimated Values} \tag{23}$$

for $n = 1,2, \ldots, K$ and $i = 1,2, \ldots, N$.

In order to have a unique representation of (20), the number of the system equations must equal the number of unknowns, that is,

$$MN + M^N = 1 + \frac{(N-1)N}{2} + KN \ . \tag{24}$$

If the number of unknowns is greater than the number of the system equations, some unknowns can be arbitrarily assigned before solving the system equations. On the other hand, if the number of unknowns is less than the number of the system equations, relation between the random variables have to be assumed before solving the system equations. Table 1 shows some combinations of N random variables and M points of concentrations for minimum first K moments needed in order to have a unique representation for (20). From table 1, it is clear that it is impractical to find a unique representation of (20) such that $0 \leq P_{ij} < 1$ when the number of random variables is greater than 2. It is also tedious to estimate the 4^{th} or higher order moments for each random variable. To avoid these problems, a simplified two-point estimate method is proposed and will be discussed in the following section.

Table 1. Minimum first K moments required for N random variables by M-point estimate method

N	M	$MN + M^N$	Minimum first K moments required	$1 + \frac{N(N-1)}{2} + KN$	Redundant
2	2	8	3	8	0
2	3	15	7	16	1
2	4	24	11	24	0
3	2	14	4	16	2
3	3	36	11	37	1
3	4	76	24	76	0
4	2	24	5	27	3

TWO-POINT ESTIMATE METHOD

The two-point estimate method for a function $Y = f(X_1, X_2, \ldots, X_N)$ can be formulated as was demonstrated in the last section. The approximate expression for the moments of the random variable Y is

$$E[Y^n] = \Sigma \, (P_{\underline{+++}\ldots\underline{+}}) \, f^n(x_{1\underline{+}}, x_{2\underline{+}}, \ldots, x_{N\underline{+}}), \tag{25}$$

where $P_{+\ldots+}$ denotes the value of mass density corresponding to the point $(x_{1+}, x_{2+}, \ldots, x_{N+})$ and $x_{i\underline{+}}$ denotes the two mass density concentrated points for the random variable i.

In (25), there are 2^N values of P and 2N real values of $x_{i\underline{+}}$. The total unknowns of (25) are $2N + 2^N$. Table 2 shows the combinations of the number of

random variables and the minimum first K moments required to construct (25). Table 2 shows that the method presented in the last section is tedious and impractical to apply when the number of random variables exceeds three. Guymon et al. (1981) proposed a simplified two-point estimate method which assumed the skewness coefficients are small and the two concentrated points take the positions as

$$x_{i\pm} = \bar{x}_i \pm s_i \quad ; \quad i = 1,2, \ldots, N \ . \tag{26}$$

Table 2. Minimum first K moments for N random variables by a two-point estimate method

N	Minimum K moments required	Redundant
2	3	0
3	4	2
4	5	3
5	7	4
6	10	0
7	18	6
8	31	6
9	55	2
10	100	2

Then the simplified two-point estimate method for (25) has the following two forms:

(1) When all random variables are uncorrelated

$$E[Y^n] = \frac{1}{2^N}\left[(Y_{++\ldots+})^n + (Y_{-+\ldots+})^n + \ldots + (Y_{--\ldots-})^n\right], \tag{27}$$

where $\{Y_{\pm\ldots\pm}\}$ indicates all possibilities of

$$Y = f(x_{1j}, x_{2j}, \ldots, x_{Nj}) \text{ for } j = + \text{ or } -.$$

(2) When all random variables are correlated

$$E[Y^n] = \frac{1}{2^N}\left[(P_{++\ldots+})(Y_{++\ldots+})^n + (P_{-+\ldots+})(Y_{-+\ldots+})^n + \right.$$
$$\left. + \ldots + (P_{--\ldots-})(Y_{--\ldots-})^n\right], \tag{28}$$

where

$$P_{++\ldots+} = 1 + \sum_{i,j=1}^{N} \alpha_i \alpha_j \delta_{ij} \rho_{ij},$$

with

$$\delta_{ij} = \begin{cases} 0, & i \geq j \\ 1, & i < j \end{cases}; \quad i,j = 1,2, \ldots, N,$$

and

$$\alpha_i = \begin{cases} 1, & \text{if } i^{th} \text{ subscript of } P_{++\ldots+} \\ & \text{is a positive sign.} \\ -1, & \text{if } i^{th} \text{ subscript of } P_{++\ldots+} \\ & \text{is a negative sign.} \end{cases}$$

COMPARISON OF SIMPLIFIED TWO-POINT ESTIMATE METHOD AND FIRST-ORDER ERROR ANALYSIS

In this section, we prove that the simplified two-point estimate method for the expected value and the variance of a system response function Y, becomes equivalent to the first-order error analysis [Dettinger, 1979] when the variances of the random variables become small, such as

$$E[Y] \simeq f(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n),$$

$$\text{Var}[Y] \simeq \sum_{i=1}^{N} \frac{\partial f(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)^2}{\partial x_i} \text{Var}[X_i]. \tag{29}$$

First, consider a single random variable case. For a function of a single component variable, the approximate expression for the first two moments of the response function of the two-point estimate method can be expressed as

$$E[Y] = \frac{1}{2}\left\{f(x_1) + f(x_2)\right\}, \tag{30}$$

$$\text{Var}[Y] = \frac{1}{4}\left\{f(x_1) - f(x_2)\right\}^2, \tag{31}$$

where $x_1 = \bar{x} - s$ and $x_2 = \bar{x} + s$, for $Y = f(X)$.

If $Y = f(X)$ is expanded in a Taylor series about x_1 and x_2, we obtain

$$f(x_1) = f(\bar{x} - s)$$
$$= f(\bar{x}) - f'(\bar{x})s + \frac{1}{2} f''(\bar{x})s^2 - \frac{1}{3!} f'''(\bar{x})s^3 + \ldots \quad , \tag{32}$$

and

$$f(x_2) = f(\bar{x} + s)$$
$$= f(\bar{x}) + f'(\bar{x})s + \frac{1}{2} f''(\bar{x})s^2 + \frac{1}{3!} f'''(\bar{x})s^3 + \ldots \quad . \tag{33}$$

Adding (32) and (33)

$$f(x_1) + f(x_2) = 2\, f(\bar{x}) + f''(\bar{x})s^2 + \frac{1}{12} f''''(\bar{x})s^4 + \ldots \quad . \tag{34}$$

If we ignore second and higher order terms, the above equation can be simplified to

$$\frac{1}{2}\left\{f(x_1) + f(x_2)\right\} \simeq f(\bar{x}). \tag{35}$$

The right hand side of (35) is equivalent to the first-order analysis (29) for the mean response and the left hand side is equivalent to the simplified two-point estimate method for the mean value (30).

By subtracting (32) from (33)

$$f(x_2) - f(x_1) = 2\, f'(\bar{x})s + \frac{1}{3} f'''(\bar{x})s^3 + \ldots \quad . \tag{36}$$

Again, by ignoring third order and higher terms, the above equation can be simplified to

$$f(x_2) - f(x_1) \simeq 2\, f'(\bar{x})s \ . \tag{37}$$

By squaring (37) and dividing through by 4

$$\frac{1}{4}\left\{f(x_2) - f(x_1)\right\}^2 \simeq \left\{f'(\bar{x})s\right\}^2 \quad . \tag{38}$$

From (38), it is clear that the variance from the two-point estimate method is equivalent to the first-order analysis.

Second, the two-point estimate method for a function of two component variables can be expressed as

$$\begin{aligned} E[Y^n] = {} & P_{11}f^n(x_{11},x_{21}) + P_{21}f^n(x_{12},x_{21}) + P_{12}f^n(x_{11},x_{22}) \\ & + P_{22}f^n(x_{12},x_{22}). \end{aligned} \tag{39}$$

If it is assumed that the skewness coefficients of X_1 and X_2 are zero and the coefficient of correlation ρ_{12} is zero, then

$$P_{11} = P_{12} = P_{21} = P_{22} = \frac{1}{4} \ ,$$

and

$$\begin{aligned} x_{11} &= \bar{x}_1 - s_1 \ , \\ x_{21} &= \bar{x}_2 - s_2 \ , \\ x_{12} &= \bar{x}_1 + s_1 \ , \\ x_{22} &= \bar{x}_2 + s_2 \ . \end{aligned} \tag{40}$$

Finally, (39) can be written as

$$E[Y^n] = \frac{1}{4}\left\{f^n(\bar{x}_1-s_1,\bar{x}_2-s_2) + f^n(\bar{x}_1-s_1,\bar{x}_2+s_2) + f^n(\bar{x}_1+s_1,\bar{x}_2-s_2) + f^n(\bar{x}_x+s_1,\bar{x}_2+s_2)\right\} . \tag{41}$$

Expanding each individual term on the right hand side of (41) by Taylor series about $(\bar{x}_1 \pm s_1, \bar{x}_2 \pm s_2)$ for the first moment yields

$$f(\bar{x}_1 \pm s_1, \bar{x}_2 \pm s_2) = f(\bar{x}_1,\bar{x}_2) + \left\{(\pm)s_1\left(\frac{\partial f}{\partial x_1}\right) + (\pm)s_2\left(\frac{\partial f}{\partial x_2}\right)\right\} + \frac{1}{2}\left\{s_1\left(\frac{\partial^2 f}{\partial x_1^2}\right) + 2(\pm)s_1\ (\pm)s_2\left(\frac{\partial^2 f}{\partial x_1 \partial x_2}\right) + s_2\left(\frac{\partial^2 f}{\partial x_2^2}\right)\right\} + \ldots \tag{42}$$

By neglecting the second and higher order derivative terms, (42) can be simplified as

$$f(\bar{x}_1 \pm s_1, \bar{x}_2 \pm s_2) = f(\bar{x}_1,\bar{x}_2) + \left\{(\pm)s_1\left(\frac{\partial f}{\partial x_1}\right) + (\pm)s_2\left(\frac{\partial f}{\partial x_2}\right)\right\} . \tag{43}$$

Finally, the first moment of Y is

$$E[Y] \simeq \frac{1}{4}\left\{f(\bar{x}_1,\bar{x}_2) - s_1\left(\frac{\partial f}{\partial x_1}\right) - s_2\left(\frac{\partial f}{\partial x_2}\right) + f(\bar{x}_1,\bar{x}_2) - s_1\left(\frac{\partial f}{\partial x_1}\right) + s_2\left(\frac{\partial f}{\partial x_2}\right) + f(\bar{x}_1,\bar{x}_2) + s_1\left(\frac{\partial f}{\partial x_1}\right) + s_2\left(\frac{\partial f}{\partial x_2}\right) + f(\bar{x}_1,\bar{x}_2) + s_1\left(\frac{\partial f}{\partial x_1}\right) - s_2\left(\frac{\partial f}{\partial x_2}\right)\right\} ,$$

or

$$E[Y] \simeq f(\bar{x}_1, \bar{x}_2) \ , \tag{44}$$

which is equivalent to the first-order analysis for the mean response function.

The second moment of response function Y can be represented as

$$\begin{aligned} E[Y^2] = \frac{1}{4} \Big\{ & f^2(\bar{x}_1 - s_1, \bar{x}_2 - s_2) + f^2(\bar{x}_1 - s_1, \bar{x}_2 + s_2) \\ & + f^2(\bar{x}_1 + s_1, \bar{x}_2 - s_2) + f^2(\bar{x}_1 + s_1, \bar{x}_2 + s_2) \Big\} \ . \end{aligned} \tag{45}$$

Also, we know that

$$Var[Y] = E[Y^2] - (E[Y])^2 . \tag{46}$$

By substituting (44) and (45) into (46)

$$\begin{aligned} Var[Y] = \frac{1}{4} \Big\{ & f^2(\bar{x}_1 - s_1, \bar{x}_2 - s_2) + f^2(\bar{x}_1 - s_1, \bar{x}_2 + s_2) \\ & + f^2(\bar{x}_1 + s_1, \bar{x}_2 - s_2) + f^2(\bar{x}_1 + s_1, \bar{x}_2 + s_2) \Big\} \\ & - \Big\{ f(\bar{x}_1, \bar{x}_2) \Big\}^2 . \end{aligned} \tag{47}$$

After some algebraic manipulations

$$Var[Y] \simeq s_1 \left(\frac{\partial f}{\partial x_1}\right)^2 + s_2 \left(\frac{\partial f}{\partial x_2}\right)^2 , \tag{48}$$

which is equivalent to the first-order analysis for the variance of the system response function Y. By induction, one can show that the first-order analysis for the mean and variance for the system response function Y is equivalent to the simplified two-point estimate method.

Insensitivity of the skewness coefficient which leads to the simplified two-point estimate method is examined in Example 1.

Example 1

Let random variables X and Y have a functional relation as $Y = X^2$. Assume that $\bar{x} = 0$, $s = 0.1, 0.2, 0.5$, and $\upsilon = 0, 0.2, 0.4, 0.8$. Results from the two-point estimate method are given in table 3. It is interesting to note that the moments of Y are less sensitive to the skewness coefficient than to the standard deviation of X.

The accuracy of the simplified two-point estimate method is demonstrated by considering two examples.

Example 2

Consider the random variable $Y = \mathrm{Exp}(X)$, and let X have the normal distribution $N(\mu,\sigma^2)$. Then Y has a log normal distribution. The expected value and variance of the log normal variate Y are

$$\begin{aligned} E[Y] &= \mathrm{Exp}\left(\mu + \frac{\sigma^2}{2}\right) \\ \mathrm{Var}[Y] &= \left\{E[Y]\right\}^2\left\{\mathrm{Exp}(\sigma^2) - 1\right\} . \end{aligned} \tag{49}$$

If we let $\mu = 1.0$ and $\sigma = 0.1, 0.2$, and 0.5, table 4 shows the results of the exact solution and the simplified two-point estimate method for the mean and variance of Y. It is clear that the estimated variance is sensitive to the effect of the coefficient of variation of the random variable X. This indicates that the two-point estimate method is accurate for estimating the mean value but less accurate for variance estimation using a high coefficient of variation.

Table 3. Results of two-point estimate method for $Y = X^2$

υ_x	$E[Y^n]$	$s_x = 0.1$	$s_x = 0.2$	$s_x = 0.5$
0	$\bar{Y}$	1.01	1.04	1.25
	s_y	0.20	0.40	1.00
	υ_y	0	0	0
0.2	$\bar{Y}$	1.01001	1.03999	1.24994
	s_y	0.19800	0.39200	0.94999
	υ_y	0.20020	0.20020	0.20020
0.4	$\bar{Y}$	1.01003	1.03992	1.25004
	s_y	0.19600	0.38400	0.90000
	υ_y	0.40020	0.40010	0.40020
0.8	$\bar{Y}$	1.00999	1.03999	1.24998
	s_y	0.19200	0.36800	0.80020
	υ_y	0.79999	0.79999	0.80000

Example 3

Consider the example: $Y = X_1^2 X_2^2$, $\bar{x}_1 = \bar{x}_2 = 1$, , $s_1 = 0.2$, $s_2 = 0.4$, $\nu_1 = 0.1$, $\nu_2 = 0.2$ and $\rho_{12} = 0.5$. Using (18) and (19), we obtain $P_{11} = 0.3688$, $P_{12} = 0.1312$, $P_{21} = 0.1374$, $P_{22} = 0.3626$ and $x_{11} = 0.8807$, $x_{12} = 1.3353$, $x_{21} = 0.3502$, $x_{22} = 1.2462$. Substituting all the P's and x's into (39), we obtain $\bar{y} = 1.2333$ and $s_y = 1.2118$. By using the simplified two-point estimate method, $\bar{y} = 1.3664$ and $s_y = 1.1706$, which represents about a 10-percent difference from the standard two-point estimate method. If we use the Taylor

series expansion, the results are $\bar{y} = 1.36$ and $s_y = 1.097$, which is close to th simplified two-point estimate method. For small coefficients of variation of X_1 and X_2, the simplified two-point estimate method is equivalent to the Taylor series expansion method. However, this does not indicate that the simplified two-point estimate method is better than the standard two-point estimate method for this particular problem because the relation between Y and X_1 and X_2 is nonlinear.

Table 4. Comparison of exact solution and two-point estimate method for Y = Exp(X)

Item	$\sigma_x = 0.1$		$\sigma_x = 0.2$		$\sigma_x = 0.5$	
	$\bar{Y}$	σ_y	$\bar{Y}$	σ_y	$\bar{Y}$	σ_y
Exact solution	2.7319	0.2739	2.7732	0.5602	3.0802	1.6416
Two-point estimate method	2.7318	0.2723	2.7728	0.5473	3.0652	1.4165
Relative error[1](%)	8.5×10^{-4}	5.8×10^{-1}	1.32×10^{-2}	2.31×10^{0}	4.87×10^{-1}	1.37×10^{1}
Coefficient of variation	0.1		0.2		0.5	

1. Relative Error (%) = $\frac{|\text{Exact Value} - \text{Estimated Value}|}{\text{Exact Value}} \times 100\%$

A random variable with a discontinuity in the first derivative must be treated with care as is illustrated in the following example.

Example 4

For example, let

$$Y = \begin{cases} 0, & 0 < x \leq 1 \\ x-1, & 1 < x < 2 \end{cases}$$

with probability density function $P(x) = 0.5$ for $0 < X < 2$. By direct integration, we obtain $\bar{x} = 1$, $s_x = \sqrt{1/3}$, $\nu_x = 0$ and $\bar{y} = 0.25$, $s_y = 0.3227$. Using the two-point estimate method, we have $P_1 = P_2 = 0.5$, $x_1 = 0.4226$, $x_2 = 1.5774$ and $\bar{y} = 0.2887$, $s_y = 0.5$. If we recalculate the mean and variance of Y from individual segments $0 < X \leq 1$ and $1 < X < 2$, separately, we get for segment $1 < X < 2$ $\bar{x} = 1.5$ and $s_x = 0.2887$ by direct integration. From the two-point estimate method, we obtain $P_1 = P_2 = 0.25$, $x_1 = 1.2113$, $x_2 = 1.7887$. Notice that $P_1 + P_2 = 0.5$ is equal to the probability of X on $1 < X < 2$. In segment $0 < X \leq 1$, it is clear from the probability density function of Y that $\bar{y} = 0$ and $s_y = 0$. Finally, the two-point estimate method gives $\bar{y} = 0.25$ and $s_y = 0.3227$, which coincides with the exact solution. This is because in each segment, Y(X) and P(X) are linear functions of X.

DETERMINISTIC-PROBABILISTIC MODEL

Information on assumed uncertain variables, parameters, and boundary conditions are applied to a deterministic model in a sampling sense. In the Monte Carlo method, a probability density function must be known or assumed for each selected uncertain variable from which random variables are generated to

apply to the deterministic model. Usually a large number of samples (perhaps as many as 500) are required to develop reliable statistics on the dependent variable (for example, water table-elevations). In the simplified two-point estimate method, each uncertain variable selected is represented by the assumed or known mean and coefficient of variation. No a priori knowledge of the probability density function is required. Statistics of the dependent variable are generated from the deterministic model by using only two sample points for each random variable.

By use of the output moments, statistical inference is used to determine the reliability of simulated aquifer response. The statistics from the deterministic-probabilistic model can be estimated by the unbiased estimators as

$$\bar{y} = \frac{1}{N} \sum_{i=1}^{N} y_i \ ,$$

and

$$s_y^2 = \frac{1}{N-1} \sum_{i=1}^{N} (y_i - \bar{y})^2 \ , \tag{50}$$

for the mean and variance of the response function. The most interesting part of the statistic is the confidence interval for the mean of the response function. A discussion on the confidence interval is included for completeness.

If we know nothing about the probability density function for the solution, denoted as Y, Chebyshev's inequality [Hogg and Tanis, 1977] can be used to obtain the confidence interval as

$$\text{Prob} \left[\bar{y} - hs_y \leq y \leq \bar{y} + hs_y\right] \geq 1 - \frac{1}{h^2} \ , \tag{51}$$

where $\bar{y}$ and s_y are the mean and standard deviation for Y, respectively. For example, if two standard deviations are assumed (h = 2), the probability that y is bounded by $[\bar{y} - 2s_y, \bar{y} + 2s_y]$ is greater or equal to 75 percent. Now if Y is assumed to be symmetrically distributed, Gauss' inequality may be applied

$$\text{Prob}\,[\bar{y} - hs_y \le y \le \bar{y} + hs_y] \ge 1 - \frac{4}{9h^2}\,, \tag{52}$$

If h = 2, there is at least an 89 percent of probability that y is bounded by $[\bar{y} - 2s_y, \bar{y} + 2s_y]$. Or if Y is assumed normally distributed, a 95-percent confidence interval of y is given as $[\bar{y} - 1.96s_y, \bar{y} + 1.96s_y]$.

The versatility of the Beta (Pearson's type I) distribution [Harr, 1977] permits representation of a wide diversity of distributional shapes whose measures must be positive quantities and whose ranges are of rather limited extent. The Beta distribution is given by the expression

$$f(y) = \frac{1}{(b - a)B(\alpha + 1, \beta + 1)} \left(\frac{y - a}{b - a}\right)^{\alpha} \left(\frac{b - y}{b - a}\right)^{\beta}\,, \tag{53}$$

where a and b are the lower and upper bounds of the distribution, respectively, and α and β are the Beta distribution parameters; $a \le y \le b$, $\alpha > -1$, $\beta > -1$, and

$$B(\alpha+1, \beta+1) = \frac{\Gamma(\alpha + 1)\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 2)}\,.$$

The expected value and variance are

$$E[Y] = a + \frac{\alpha + 1}{\alpha + \beta + 2}(b - a)\,, \tag{54}$$

and

$$\mathrm{Var}[Y] = \frac{(b - a)^2(\alpha + 1)(\beta + 1)}{(\alpha + \beta + 2)^2(\alpha + \beta + 3)} . \tag{55}$$

On the basis of available data or intuition, one can usually estimate the mean $\bar{y}$, the variance, s_y, the upper limit b and the lower limit a for the Beta distribution. Using the above information, one can obtain the parameters α and β from

$$\alpha = \frac{\tilde{x}^2}{\tilde{\nu}} (1 - \tilde{x}) - (1 + \tilde{x}), \tag{56}$$

$$\beta = \frac{\alpha + 1}{\tilde{x}} - (\alpha + 2), \tag{57}$$

where

$$\tilde{x} = \frac{\bar{x} - a}{b - a} ,$$

$$\tilde{\upsilon} = \left(\frac{s_x}{b - a}\right)^2 . \tag{58}$$

Once a beta distribtution is determined, confidence intervals and other desired statistical properties can be established [Harr, 1977].

A simple steady-state, two-dimensional vertical slice ground-water flow model is used to compare the results from the two-point probability estimate scheme with results from the "traditional" Monte Carlo method. Figure 3 shows the porous media geometry, divided into 18 triangular finite elements. Additionally, a 72 triangular finite element model (in the same pattern as shown in fig. 3) was studied.

It is assumed that there are three horizontal layers of equal thickness in the model, and it is also assumed that within each layer the saturated hydraulic conductivity is log normal, $\Lambda(-2, 0.43^2)$. It is assumed that hydraulic conductivity is the only uncertain parameter. Figure 4 shows the results from the Monte Carlo simulation method with 500 sample solutions. The mean and variance of the saturated hydraulic conductivity, which are generated from the Monte Carlo simulation method, were used as input information for the simplified two-point estimated method. Results of this simulation are shown in figure 4.

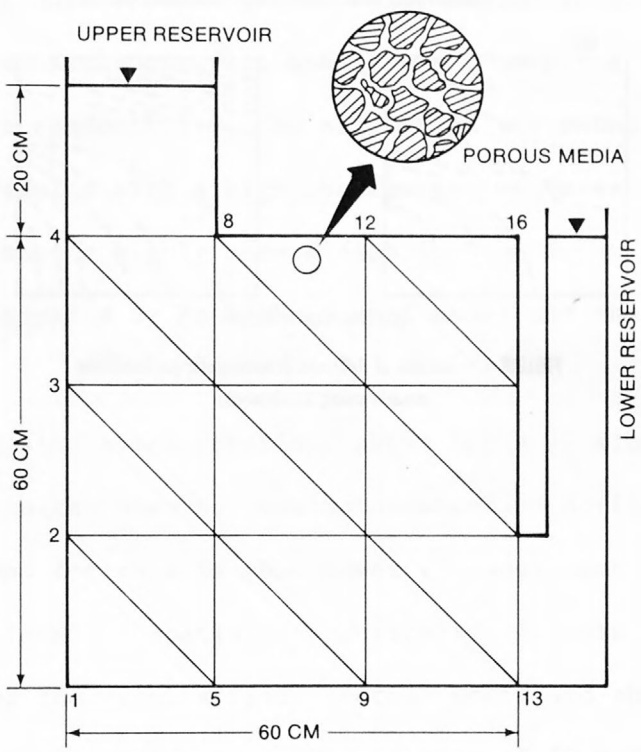

FIGURE 3.—Two-dimensional, vertical slice ground-water flow model (18 elements).

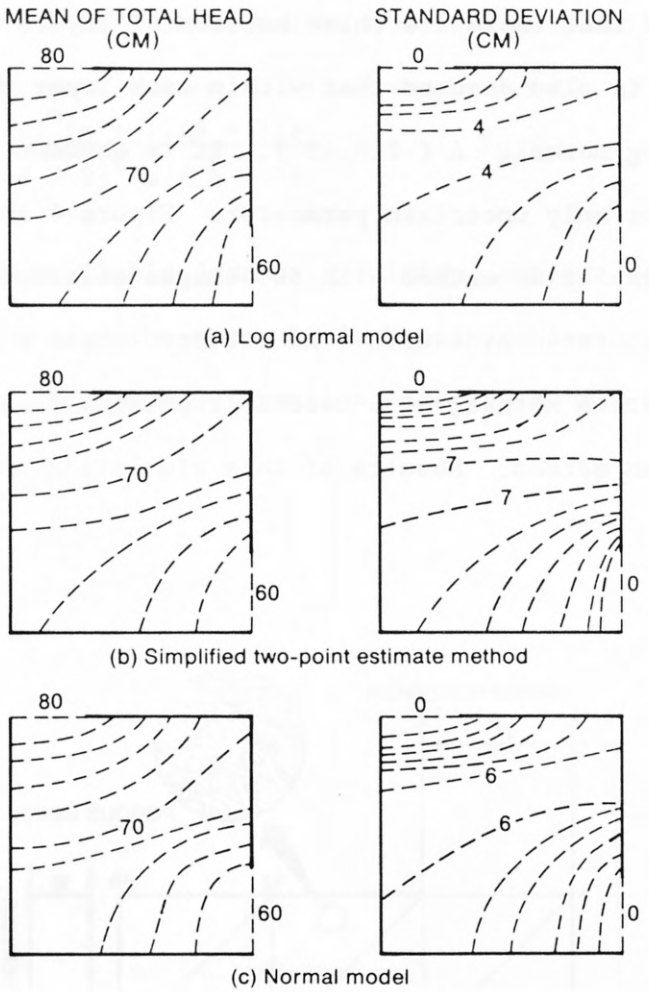

FIGURE 4.—Results of different deterministic-probabilistic models using 18 elements.

Also, a 500-sample solution run of the Monte Carlo simulation method was done on a normal distribution by using the same statistics as the simplified two-point estimate method. These results also are shown in figure 4. Comparing the results of these three models, the log normal model distribution had smaller standard deviations for all regions. The simplified two-point estimate method had higher standard deviations for all regions. As was mentioned previously, the success of the simplified two-point estimate method requires a small coefficient of variation of the random variable; that is, the saturated hydraulic conductivity in this case. In this case, the coefficient of variation of the saturated hydraulic conductivity was about 123 percent. Using a normal model to approximate the log normal model gave results that also were poor. Since the solution (total potential hydraulic head) is not merely a linear function of the saturated hydraulic conductivity, the simplified two-point estimate method will not provide good results with a high coefficient of variation. Also, this example indicates that a highly skewed distribution function cannot be successfully approximated by both the normal model and the simplified two-point estimate method.

Repeating the simulation described above for a 72-element representation yielded similar results; however, estimated standard deviations were somewhat higher. The reasons for this is that fewer elements tend to smooth variability. The level of spatial discretization affects statistical results.

Figure 5 shows the results from a normal model and the simplified two-point estimate method using a coefficient of variation of 50 percent for hydraulic conductivity. In this case, the mean hydraulic conductivity was 1. It is clear that both models are comparable and provide good results when compared to the deterministic model. If we further assume a positive coefficient of correlation between each layer, the solution (fig. 6) tends to converge to the deterministic

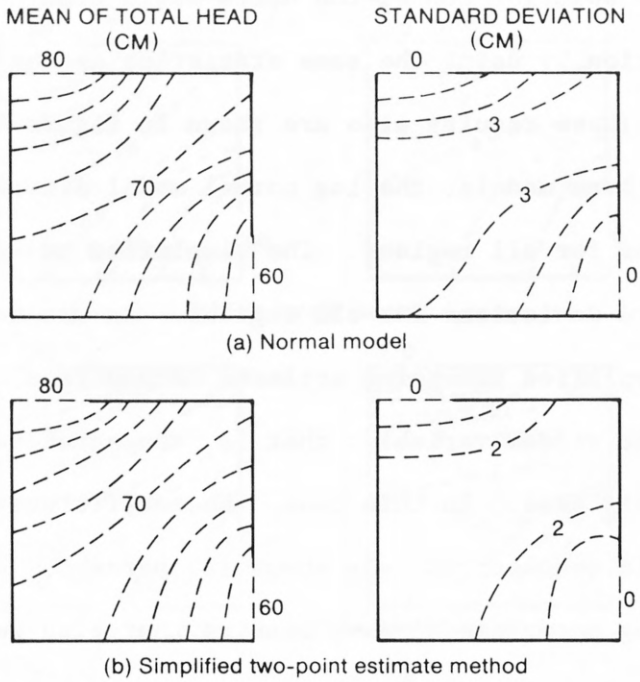

FIGURE 5.—Results of two-dimensional, vertical slice ground-water flow model for small coefficient of variation.

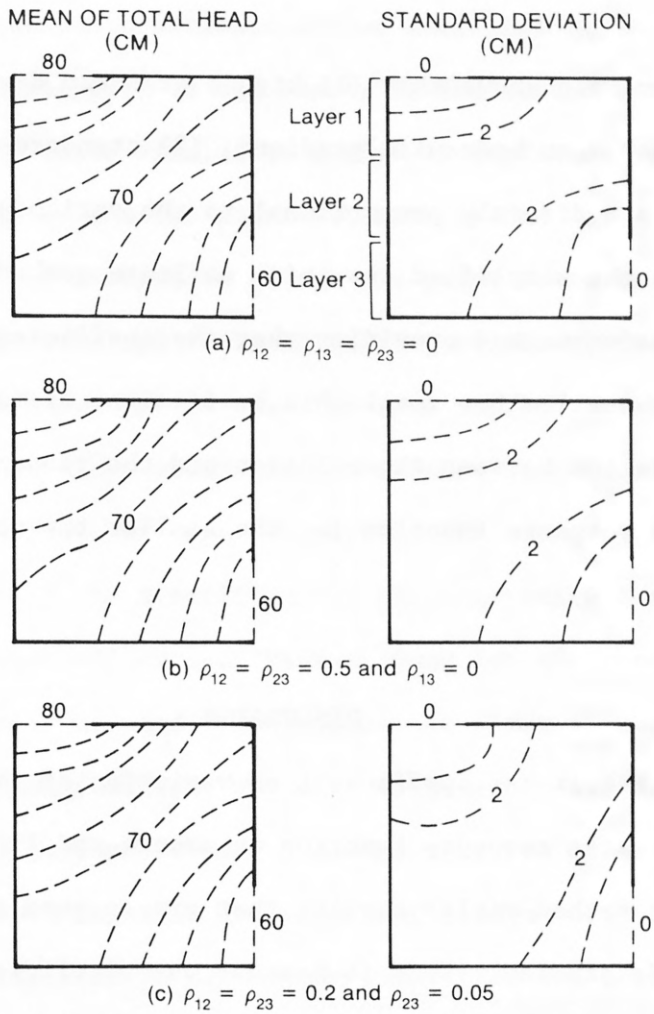

FIGURE 6.—Results of two-dimensional, vertical slice ground-water flow model by simplified two-point estimate method.

model.

This simple example shows that (1) higher standard deviations occur at nodal points with a higher mean hydraulic gradient, (2) standard deviations of the response function are directly proportional to the variances of the component variables, and (3) the simplified two-point estimate method is comparable to the Monte Carlo simulation method providing that the coefficient of variation of the random variable is small. How small this coefficient of variation should be depends on the relation between the solution and the random variable. The more nonlinear a system response function is, the smaller the coefficient of variation must be.

DISCUSSION

It is clear that if the coefficient of variation of the random variable is small and if the system response function is smooth and linear the simplified two-point estimate method yields results that are as good as the computationally more costly methods studied. This is because the simplified two-point estimate method is equivalent to the first-order analysis. Under the assumption that the response function is a linear function of the component variables, the first-order analysis yields the exact solutions for the mean and variance for the response function. If the response function is not linear, then the coefficients of variation of the component variables must be small in order to achieve a satisfactory result from the first-order analysis [Cornell, 1972].

In order to apply the simplified two-point estimate method, all we need to know are the mean and variance of each variable, and the coefficient of correlation between each pair of variables. Here we only need to perform 2^N iterations (N is the number of random variables) instead of usually assumed many sample solutions for the Monte Carlo simulation method. But the iteration number

of the simplified two-point estimate method increases rapidly when the number of random variables increases. If the number of the random variables is greater than nine, the Monte Carlo simulation method is preferred because the total number of iterations by the two-point estimate method will be greater than 512. In order to reduce the iterations of the two-point estimate method, a sensitivity analysis needs to be done and rational lumping of uncertain parameters needs to be made. This will allow one to treat several random variables as a group of random variables. In other words, we assume that these random variables have either a perfect positive or negative correlation.

The advantages of the simplified two-point estimate method are (1) no prior assumption of the probability density functions for the random variables are required, (2) there is no need to calculate the first or second order derivatives of the system response function, which is usually not explicitly known, and (3) there is no need to perform as many iterations as the Monte Carlo simulation method requires, providing that the number of random variables is small.

Disadvantages related to use of the simplified two-point estimate method relate to an a priori assumption of near linearity of the response function or small coefficient of variation of the random variable. If this assumption cannot be made, or is suspect, a Monte Carlo method may provide better results in critical areas of concern in a ground-water basin or in the areas of the basin where the response function is definitely nonlinear.

REFERENCES

Bear, J., Hydraulics of Groundwater. McGraw-Hill Inc., 1979.

Cornell, C.A., First-order analysis of model and parameter uncertainty. International Symposium on Uncertainties in Hydrologic and Water Resources Systems, Univ. of Arizona, Tucson, 1972.

Dagan, G., Models of groundwater flow in statistically homogeneous porous formations. Water Resources Research, 15(1), 47-63, 1979.

Dagan, G., Analysis of flow through heterogeneous random aquifers by the method of embedding matrix: 1. Steady flow. Water Resources Research, 17(1), 107-121, 1981.

Dagan, G., Stochastic modeling of groundwater flow by unconditional and conditional probabilities: 1. Conditional simulation and the direct problem. Water Resources Research, 18(4), 813-833, 1982(a).

Dagan, G., Stochastic modeling of groundwater flow by unconditional and conditional probabilities: 2. The solute transport. Water Resources Research, 18(4), 835-848, 1982(b).

Dagan, G., Analysis of flow through heterogeneous random aquifers: 2. Unsteady flow in confined formations. Water Resources Research, 18(5), 1571-1585, 1982(c).

Delhomme, J.P., Kriging in the hydrosciences. Advances in Water Resources, 1(5), 251-266, 1978.

Delhomme, J.P., Spatial variability and uncertainty in groundwater flow parameters: A geostatistical approach. Water Resources Research, 15(2), 269-280, 1979.

Dettinger, M.D., Numerical modeling of aquifer system under uncertainty: A second moment analysis. M.S. Thesis, Dept. of Civil Eng., MIT, Cambridge, 1979.

Dettinger, M.D. and J.L. Wilson, First-order analysis of uncertainty in numerical models of ground-water flow: 1. Mathematical development. Water Resources Research, 17(1), 149-161, 1981.

Freeze, R.A., A stochastic-conceptual analysis of one-dimensional groundwater flow in nonuniform homogeneous media. Water Resources Research, 11(5), 725-741, 1975.

Gelhar, L.W., Stochastic analysis of phreatic aquifers. Water Resources Research, 10(3), 539-535, 1974.

Gelhar, L.W., Effects of hydraulic conductivity variation on groundwater flow, in Hydraulic Problem Solved by Stochastic Hydraulics Methods, Proceedings, Second International, IAHR Symposium on Stochastic Hydraulics, Water Resources Publications, Fort Collins, Colorado, 409-431, 1977.

Guymon, G.L., M.E. Harr, R.L. Berg, and T.V. Hromadka, II, A probabilistic-deterministic analysis of one-dimensional ice segregation in a freezing soil column. Cold Regions Science and Technology, Vol. 5, 127-140, 1981.

Harr, M.E., Mechanics of Particulate Media, McGraw-Hill, Inc., 1977.

Hogg, R.V. and E.A. Tanis, Probability and Statistical Inference, Macmillan Publishing Co., Inc., 1977.

Hromadka, II, T.V., G.L. Guymon, and G.C. Pardoen, Nodal domain integraiton model of two-dimensional unsaturated soil-water flow: Development. Water Resources Research, 17(5), 1425-1430, 1981.

Mammana, C., Sul problema algebraicv dei moment, Ann. Scuola, Norm. Sup., Pisa, Vol. 8, 133-140, 1954.

McElwee, C.D. and M.A. Yukler, Sensitivity of groundwater models with respect to variations in transmissivity and storage. Water Resources Research, 14(3), 451-459, 1978.

Mizell, S.A., A.L. Gutjahr, and L.W. Gelhar, Stochastic analysis of spatial variability in two-dimensional steady groundwater flow assuming stationary and nonstationary head. Water Resources Research, 18(4), 1053-1067, 1982.

Neuman, S.P. and S.Y. Yakowitz, A statistical approach to the inverse problem of aquifer hydrology: 1. Theory. Water Resources Research, 15(4), 845-860, 1979.

Neuman, S.P., G.E. Fogg, and E.A. Jacobson, A statistical approach to aquifer hydrology: 2. Case study. Water Resources Research, 16(1), 33-58, 1980.

Pinder, G.F. and W.G. Gray, Finite Element Simulation in Surface and Subsurface Hydrology, Academic Press, 1977.

Rosenblueth, E., Point estimates for probability moments. Proc. Natl. Acad. Sci., USA, 72(10), 3812-3814, 1975.

Smith, L. and R.A. Freeze, Stochastic analysis of steady state groundwater flow in a bounded domain: 1. One-dimensional simulations. Water Resources Research, 15(3), 521-528, 1979(a).

Smith, L. and R.A. Freeze, Stochastic analysis of steady state groundwater flow in a bounded domain: 2. Two-dimensional simulation. Water Resources Research, 15(6), 1543-1559, 1979(b).

Tang, D.H. and G.F. Pinder, Simulation of groundwater flow and mass transport under uncertainty. Advances in Water Resources, 1(1), 25-29, 1977.

Tang, D.H. and G.F. Pinder, Analysis of mass transport with uncertain physical parameters. Water Resources Research, 15(5), 1147-1155, 1979.

Tucker, H.G., An estimate of the compounding distribution of a compound Poisson distribution. Teoriia Veroyatnostyei i ee Primeneniia, Vol. III, 211-216, 1963.

Warren, J.E. and H.C. Price, Flow in heterogeneous porous media. Soc. of Petrol Eng. J., Vol. 1, 153-169, 1961.

Wu, T.H., S.K. Vyas, and N.Y. Chang, Probabilistic analysis of seepage. J. Soil Mech. Fund. Div., A.S.C.E., 99(SM4), 324-340, 1973.