ADJUSTED MAXIMUM LIKELIHOOD ESTIMATION OF THE MOMENTS OF LOGNORMAL POPULATIONS FROM TYPE I CENSORED SAMPLES

by Timothy A. Cohn

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Dallas L. Peck, Director

For additional information
write to:

Branch of Systems Analysis
U.S. Geological Survey
410 National Center
Reston, Virginia 22092

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ADJUSTED MAXIMUM LIKELIHOOD ESTIMATION OF THE MOMENTS OF LOGNORMAL POPULATIONS FROM TYPE I CENSORED SAMPLES

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ABSTRACT

An adjusted maximum likelihood estimator is presented for the moments of lognormal populations given type I censored samples. The estimator is shown to be efficient, and nearly-unbiased in moderate-size samples.

1. INTRODUCTION: THE PROBLEM

This paper considers estimation of lognormal population moments from type I censored samples. Such samples arise, for example, in estimating loads of trace pollutants, since sample concentrations often fall below the analytical detection limits of the laboratory [Kushner, 1976; Owen and DeRouen, 1980; Gilliom and Helsel, 1986; and Helsel and Gilliom, 1986]. In the general case, one has a set of N independent observations, $\{X_1 . . . X_N\}$, K of which are censored because they did not exceed the censoring threshold. The censoring thresholds are denoted $\{T_i\}$. It is assumed that the X_i come from an uncensored lognormal population with parameters $\{\mu,\sigma\}$, and probability density function (PDF) given by:

$$f(x) = \begin{cases} 0 & x \le 0 \\ \dfrac{1}{x\cdot\sqrt{2\cdot\pi\cdot\sigma^2}}\exp\left\{\dfrac{-[\ln(x)-\mu]^2}{2\cdot\sigma^2}\right\} & x > 0 \end{cases} \qquad (1.1)$$

One wants to estimate the population mean or other moments. The expression for the r^{th} non-central population moment, M_r, of a lognormal variate is given by Aitchison and Brown [1981]:

$$M_r = \exp(r\cdot\mu + r^2\cdot\sigma^2/2) \qquad (1.2)$$

Maximum likelihood estimators, first developed by Hald [1949] and Cohen [1950], are usually recommended for use with type I censored data [David, 1981]. They are efficient and nearly-unbiased in many applications [David, 1981 p. 138]. The log-likelihood function, given type I censored samples and assuming a lognormal distribution, is:

$$L = \ln\tilde{L} = \sum_{X_i > T_i}\left\{-\ln(\sigma) - \frac{[\ln(X_i)-\mu]^2}{2\cdot\sigma^2}\right\} + \sum_{X_k < T_k}\left\{\ln\left(\Phi\left[\frac{\ln(T_k)-\mu}{\sigma}\right]\right)\right\} \qquad (1.3)$$

where $\Phi[\cdot]$ is the standard normal CDF. The first derivatives of equation (1.3) with respect to the parameters can be set to zero and solved for the MLE parameter estimates $\hat{\mu}$ and $\hat{\sigma}$:

$$\frac{\partial L}{\partial \mu} = \sum_{X_i > T_i}\{[\ln(X_i)-\mu]/\sigma^2\} + \sum_{X_k < T_k}\{\varphi[\xi_k]/\Phi[\xi_k]\}(-1/\sigma) = 0 \qquad (1.4)$$

$$\frac{\partial L}{\partial \sigma} = \sum_{X_i > T_i} \{-1/\sigma + [\ln(X_i)-\mu]^2/\sigma^3\} + \sum_{X_k < T_k} \{\varphi[\xi_k]/\Phi[\xi_k]\}(-\xi_k/\sigma) = 0 \tag{1.5}$$

where $\xi_k \equiv [\frac{(\ln(T_k)-\mu)}{\sigma}]$ is a standardized censoring threshold, and $\varphi[\cdot]$ is the standard normal PDF. $\hat{\mu}$ and $\hat{\sigma}$, the solutions of equations (1.4) and (1.5), can be expressed as functions of sufficient statistics [van Zwet, 1966], and are often negligibly-biased estimates of μ and σ even in small samples [Stedinger and Cohn, 1986].

Several authors [see Cohen, 1976; Rukhin, 1986] discuss the corresponding MLE estimator for the r^{th} population moment, $\hat{M}_r$:

$$\hat{M}_r = \exp(r \cdot \hat{\mu} + r^2 \cdot \hat{\sigma}^2/2) \tag{1.6}$$

which is consistent, asymptotically normal and asymptotically efficient [Kendall and Stuart, 1979, p. 41-43]. However, $\hat{M}_r$ has poor properties in small samples even when the complication of censoring is not present. Kendall and Stuart [1979, p. 74] note that:

$$\begin{aligned} E\{(\hat{M}_r)\} &= E\{[\exp(r \cdot \hat{\mu} + r^2 \cdot \hat{\sigma}^2/2)]\} \\ &= \exp(r \cdot \mu + r^2 \cdot \sigma^2/(2 \cdot N)) \cdot (1 - r^2 \cdot \sigma^2/N)^{-(N-1)/2} \\ &> M_r \end{aligned} \tag{1.7}$$

Thus $\hat{M}_r$ is always upwardly biased, and lacks a finite mean if $N \le r^2 \cdot \sigma^2$.

Finney [1941] derives an adjusted moment estimator (in fact, a Rao-Blackwell estimator) which solves the moment-estimation problem for uncensored samples. He defines a function,

$$f(t) \equiv 1 + t + \frac{(n-1)}{(n+1)} \cdot \frac{t^2}{2!} + \frac{(n-1)^2}{(n+1)(n+3)} \cdot \frac{t^3}{3!} + \sum_{p=4}^{\infty} \frac{(n-1)^{p-1}}{(n+1)\cdot \ldots \cdot (n+2\cdot p-3)} \left(\frac{t^p}{p!} \right) \qquad (1.8)$$

It can then be shown [Finney, 1941; Kendall and Stuart, 1979, p. 74] that

$$E[\exp(\bar{y}) \cdot f(s_Y^2/2)] = \exp(\mu + \sigma^2/2) = M_1 \qquad (1.9)$$

where $\bar{y}$ and s_Y^2 are the unbiased sample mean and variance estimators of the logarithms of the X's. Since any function of jointly sufficient statistics, given suitable regularity conditions, is a minimum variance estimator of its expectation [Aitchison and Brown, 1981, p. 45], the Finney estimator is optimal (UMVUE) in the class of unbiased estimators. Figure 1 (which corresponds to Finney's [1941] Figure 1) displays the relative efficiency (the ratio of the variances) of the unbiased sample moment estimators ($\bar{X} \equiv \sum_{i=1}^{N} \frac{X_i}{N}$) and ($S_X^2 \equiv \sum_{i=1}^{N} \frac{(X_i - \bar{X})^2}{N-1}$) compared with the corresponding UMVUEs. The sample-moments estimators are substantially less efficient than the UMVUE for large values of σ^2.

Rukhin [1986] derives downward-biased Bayesian estimators that, in small samples, have substantially lower mean square errors (MSE) than the UMVUE.

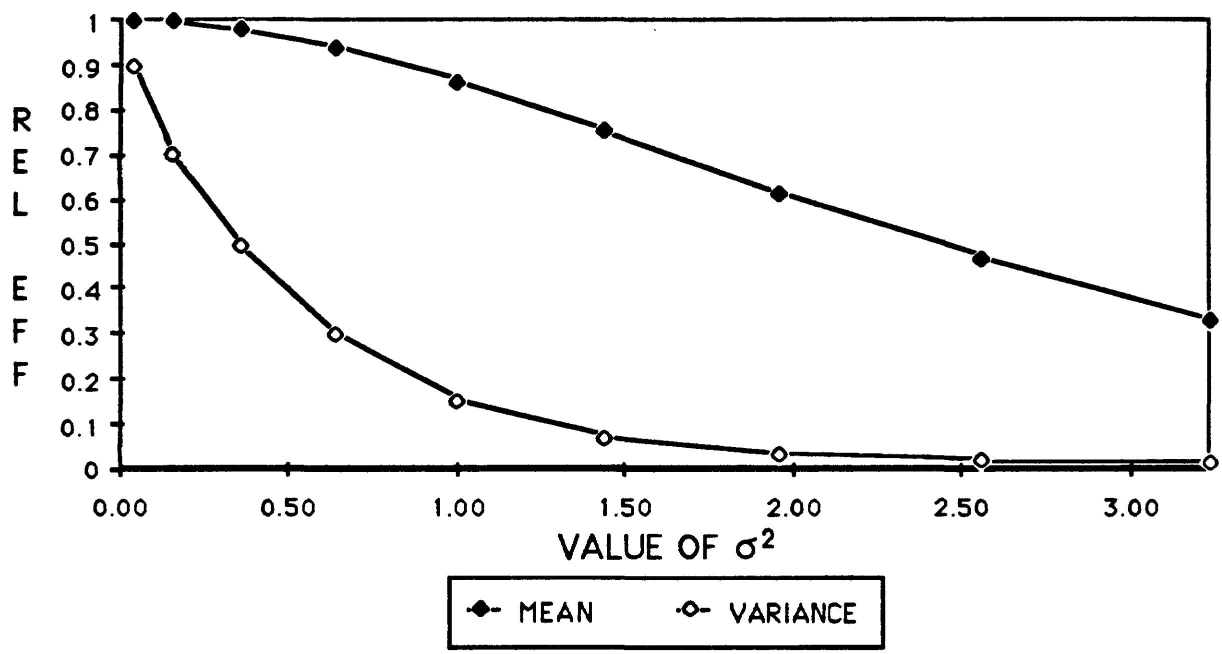

FIGURE 1: Relative efficiency of the sample mean and sample variance estimators to the UMVUE (Finney) estimator, as a function of σ^2.

No comparable Rao-Blackwell estimator exists in the case of type I censored data. In fact, van Zwet [1966] demonstrates that virtually all "unbiased estimation is impossible" with type I censored normal samples, since there is always a finite probability of zero threshold exceedances. However, it may still be possible to derive an estimator whose bias is negligibly small for cases of interest. It is the purpose of this paper to present such an estimator.

This paper proceeds along the following lines. Results from Shenton and Bowman [1977] are used to obtain first-order estimates of the bivariate distribution of $\hat{\mu}$ and $\hat{\sigma}^2$. Asymptotically independent functions of $\hat{\mu}$ and $\hat{\sigma}^2$ are derived. The distribution of one function is asymptotically normal. The distribution of the second function is asymptotically gamma, with shape parameter (α) given as a function only of the standardized censoring level, $\{\xi_i\} \equiv \left\{\frac{\ln(T_i)-\mu}{\sigma}\right\}$, and sample size. An Adjusted Maximum Likelihood Estimator (AMLE), similar to Finney's [1941], is then derived for estimating the lognormal moments. The AMLE is asymptotically equivalent to the MLE, but is found in Monte Carlo experiments to be substantially less biased than the MLE if $\frac{\sigma^2}{N}$ is large.

2. THE COVARIANCE AND BIAS OF THE ESTIMATORS

The derivation of the first-order biases and variances requires a few results which are provided in the Appendix. However, some notational conventions will facilitate discussion. Let:

$$L_{\theta_i} \equiv E[\frac{\partial L}{\partial \theta_i}]$$

$$L_{\theta_i\theta_j} \equiv E[\frac{\partial^2 L}{\partial \theta_i \partial \theta_j}]$$

$$L_{\theta_i\theta_j\theta_k} \equiv E[\frac{\partial^3 L}{\partial \theta_i \partial \theta_j \partial \theta_k}]$$

$$L_{\theta_i\theta_j,\theta_k} \equiv \left[\frac{\partial L_{\theta_i\theta_j}}{\partial \theta_k}\right]$$

where the θ's are arbitrary parameters.

The first-order covariance matrix, denoted $(\sigma^2 \cdot V)$, of $\hat{\mu}$ and $\hat{\sigma}$ is given by:

$$\sigma^2 \cdot V \equiv \sigma^2 \cdot \begin{bmatrix} V_{\mu\mu} & V_{\mu\sigma} \\ V_{\mu\sigma} & V_{\sigma\sigma} \end{bmatrix} \equiv \begin{bmatrix} L^{\mu\mu} & L^{\mu\sigma} \\ L^{\mu\sigma} & L^{\sigma\sigma} \end{bmatrix} \equiv \begin{bmatrix} L_{\mu\mu} & L_{\mu\sigma} \\ L_{\mu\sigma} & L_{\sigma\sigma} \end{bmatrix}^{-1} \quad (2.1)$$

The matrix elements, $L_{\theta_i\theta_j}$, are given in the Appendix.

A first-order estimate of the bias of the MLE estimator $\hat{\theta}_i$ of θ_i is given by Shenton and Bowman [1977, equation 3.12b; see Hosking, 1985]:

$$E[\hat{\theta}_i - \theta] = \sum_{j=1}^{2} \sum_{k=1}^{2} \sum_{l=1}^{2} L^{\theta_i\theta_j} L^{\theta_k\theta_l} \left(L_{\theta_j\theta_k,\theta_l} + \frac{L_{\theta_j\theta_k\theta_l}}{2} \right) \; ; \; i=1,2 \quad (2.2)$$

Adapting the procedure described in Kendall and Stuart [1979, p. 66-8], first-order biases, which depend on μ and σ^2 only through $\{\xi_i\}$, can be computed for "standardized" estimators:

$$B\left(\frac{\hat{\mu}}{\sigma}\right) \equiv E\left[\frac{\hat{\mu}-\mu}{\sigma}\right] \qquad B\left(\frac{\hat{\sigma}}{\sigma}\right) \equiv E\left[\frac{\hat{\sigma}-\sigma}{\sigma}\right]$$

$$B\left(\frac{\hat{\sigma}^2}{\sigma^2}\right) \equiv E\left[\frac{\hat{\sigma}^2-\sigma^2}{\sigma^2}\right] \qquad B\left(\frac{\hat{\sigma}^4}{\sigma^4}\right) \equiv E\left[\frac{\hat{\sigma}^4-\sigma^4}{\sigma^4}\right]$$

The set $\{\xi_1 \ldots \xi_N\}$ are estimated by $\{\hat{\xi}_i \equiv \frac{\ln(T_i)-\hat{\mu}}{\hat{\sigma}}\ ;\ i=1,\ldots N\}$. First-order estimates of the variance of $\hat{\xi}_i$ can be obtained from the parameter variances [Kendall and Stuart, 1977, p. 247]:

$$\mathrm{Var}[\hat{\xi}_i] = \mathrm{Var}\left[\frac{(\ln(T_i)-\hat{\mu})}{\hat{\sigma}}\right]$$

$$\approx V_{\mu\mu} - 2\cdot\xi_i\cdot V_{\mu\sigma} + \xi_i^{2}\cdot V_{\sigma\sigma} \qquad (2.3)$$

For moderate sample sizes (N>50), and a single censoring threshold between the 20th and 60th percentiles, the standard error of estimates of ξ_i is approximately $1.5/\sqrt{N}$. Figures 2 and 3 show that the first-order standardized biases and variances remain relatively constant over short ranges of ξ_i, even for high levels of censoring. Thus $B[(\frac{\theta}{\sigma})]$ and V_θ, respectively the first-order bias and variance of $(\frac{\theta}{\sigma})$, are known quite accurately. Although the cases shown correspond to a single censoring threshold ($\xi_i=\xi$ for all i), the same results apply for multiple thresholds.

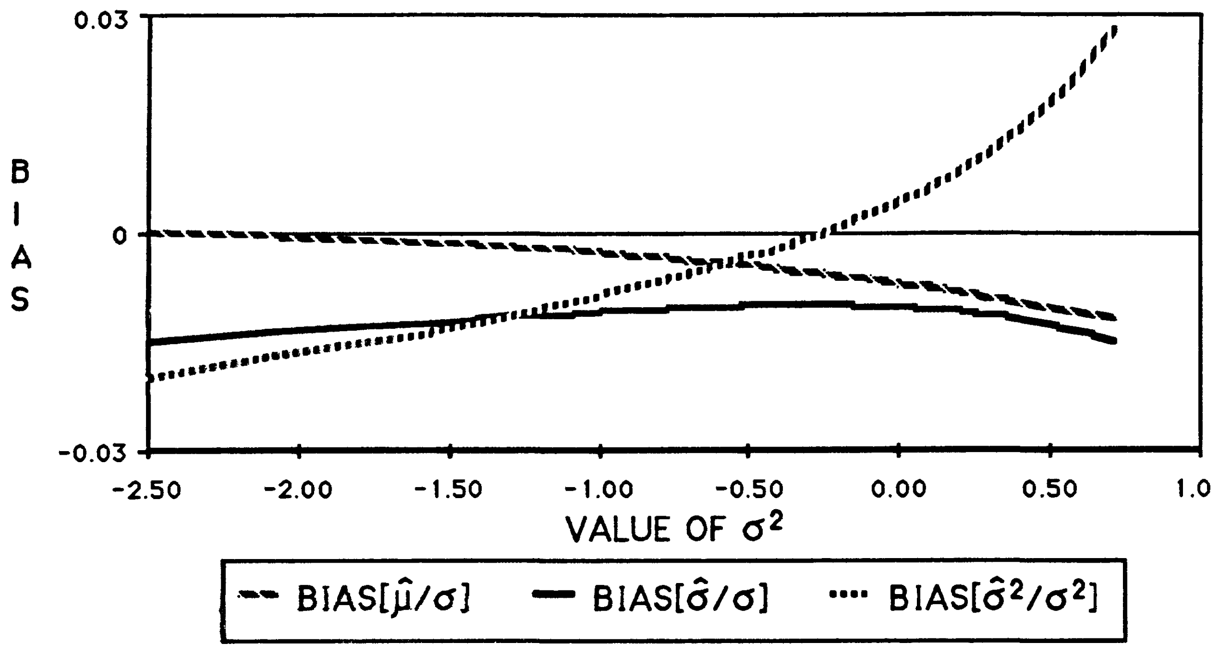

FIGURE 2: First-order bias of the standardized MLE parameter estimators, given a singly-censored sample of size 50, as a function of the standardized censoring threshold.

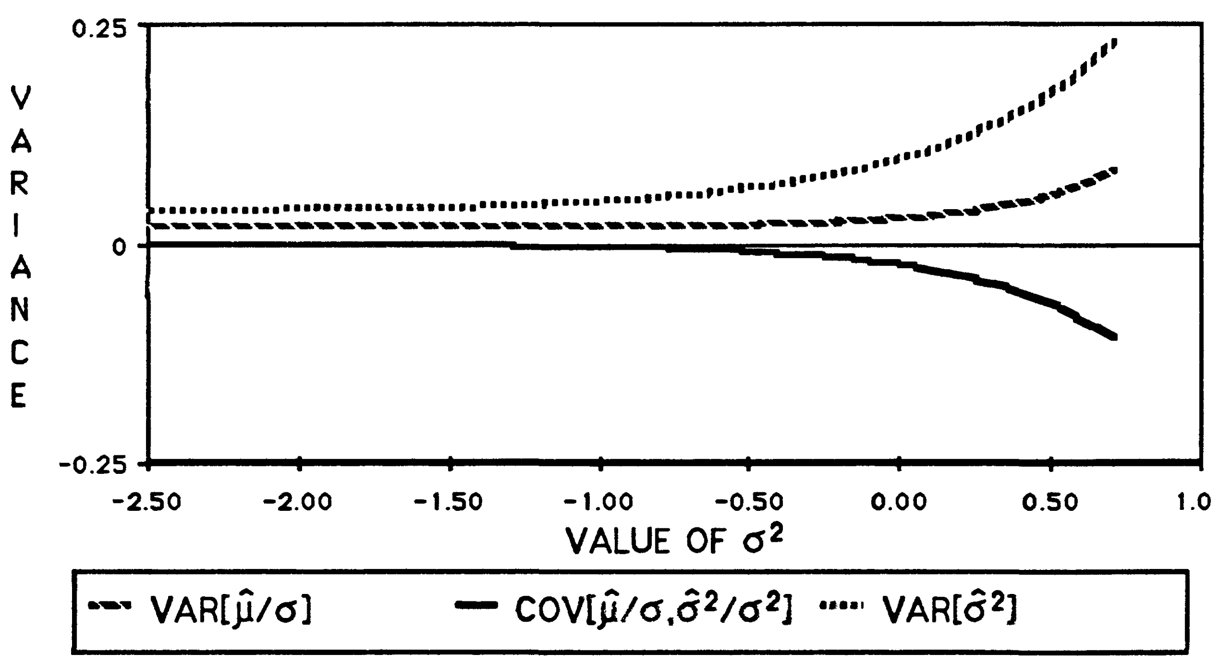

FIGURE 3: First-order covariances of the standardized MLE parameter estimators, given a singly-censored sample of size 50, as a function of the standardized censoring threshold.

3. THE FIRST-ORDER SAMPLING DISTRIBUTIONS OF $\hat{\mu}$ AND $\hat{\sigma}^2$

With uncensored data, $(\frac{\hat{\sigma}^2}{\sigma^2})$ has a gamma distribution with PDF given by:

$$f(x) = \begin{cases} 0 & x \leq 0 \\ \dfrac{x^{\alpha-1} \cdot e^{-x/\beta}}{\beta^{\alpha} \cdot \Gamma(\alpha)} & x > 0 \end{cases} \tag{3.1}$$

where the parameters are given by $\alpha = \frac{N-1}{2}$, and $\beta = \frac{2}{N}$. With censored data, the exact distribution of $(\frac{\hat{\sigma}^2}{\sigma^2})$ is not known. However, the asymptotic distribution of $(\frac{\hat{\sigma}^2}{\sigma^2})$ is gamma, with parameters $\{\alpha, \beta\}$ available from the asymptotic biases:

$$\hat{\alpha} = \text{Mean}^2/\text{Variance}$$

$$= \frac{E^2\left[(\frac{\hat{\sigma}^2}{\sigma^2})\right]}{\text{Var}[(\frac{\hat{\sigma}^2}{\sigma^2})]}$$

$$= \frac{(1+\text{Bias}[\frac{\hat{\sigma}^2}{\sigma^2}])^2}{\{1+\text{Bias}[\frac{\hat{\sigma}^4}{\sigma^4}]-(1+\text{Bias}[\frac{\hat{\sigma}^2}{\sigma^2}])^2\}}$$

$$\approx \quad \frac{(1+B[\frac{\hat{\sigma}^2}{\sigma^2}])^2}{\{1+B[\frac{\hat{\sigma}^4}{\sigma^4}]-(1+B[\frac{\hat{\sigma}^2}{\sigma^2}])^2\}} \qquad (3.2)$$

$$\hat{\beta} \quad = \quad \text{Mean}/\alpha$$

$$= \quad E[\frac{\hat{\sigma}^2}{\sigma^2}]/\alpha$$

$$\approx \quad (1+B[\frac{\hat{\sigma}^2}{\sigma^2}])/\hat{\alpha} \qquad (3.3)$$

As N increases, the biases go to zero and $\hat{\beta} \approx (\frac{1}{\alpha})$. With complete (uncensored) data, the (second-order) bias of $(\frac{\hat{\sigma}^4}{\sigma^4})$ is $(\frac{1}{N^2})$ and the estimators are a function of sample size alone:

$$\hat{\alpha} \quad = \quad \frac{(N-1)}{2} \qquad (3.4)$$

$$\hat{\beta} \quad = \quad 2/N \qquad (3.5)$$

Equations (3.1), (3.2) and (3.3) then give the exact distribution of $(\frac{\hat{\sigma}^2}{\sigma^2})$.

With complete samples $\hat{\mu}$ is normally distributed and independent of $(\frac{\hat{\sigma}^2}{\sigma^2})$. Its distribution is described exactly by first-order results.

With censored samples, $\hat{\mu}$ is neither normally-distributed nor uncorrelated with $(\frac{\hat{\sigma}^2}{\sigma^2})$. However, one can define a new variate,

$$\hat{\omega} \equiv \hat{\mu} - \{ \frac{V_{\mu\sigma^2}}{V_{\sigma^2\sigma^2}} \}\cdot\hat{\sigma}$$

$$\approx \hat{\mu} - \{ \frac{Cov[\hat{\mu},\hat{\sigma}^2]}{Var[\hat{\sigma}^2]} \}\cdot\sigma^2 \qquad (3.6)$$

which is asymptotically normal and uncorrelated with $\hat{\sigma}^2$, and is thus asymptotically independent of $\hat{\sigma}^2$.

To first order, the expected value of $\hat{\omega}$ is:

$$E[\hat{\omega}] = E[\hat{\mu}] + E[-\{ \frac{V_{\mu\sigma^2}}{V_{\sigma^2\sigma^2}} \}\cdot\hat{\sigma}^2]$$

$$\approx \mu + \varepsilon\cdot\sigma^2 \qquad (3.7)$$

where

$$\varepsilon \equiv \frac{\left[Bias[\frac{\hat{\mu}}{\sigma}] - \frac{Cov[\frac{\hat{\mu}}{\sigma},\frac{\hat{\sigma}^2}{\sigma^2}]}{Var[\frac{\hat{\sigma}^2}{\sigma^2}]} \right]}{\sigma} \qquad (3.8)$$

ε has the same units as σ^{-1}. However, ε is small in magnitude (zero for complete samples), and can be estimated by

$$\hat{\varepsilon} = \frac{\left[B[\frac{\hat{\mu}}{\sigma}] - [\frac{V_{\mu\sigma^2}}{V_{\sigma^2\sigma^2}}] \right]}{\hat{\sigma}} \qquad (3.9)$$

Neglecting the bias correction, the first-order variance of $\hat{\omega}$ is then:

$$\mathrm{Var}[\hat{\omega}] \approx \mathrm{Var}[\hat{\mu}] - 2\cdot\{\frac{\mathrm{Cov}[\hat{\mu},\hat{\sigma}^2]}{\mathrm{Var}[\hat{\sigma}^2]}\}\cdot\mathrm{Cov}[\hat{\mu},\hat{\sigma}^2] + \{\frac{\mathrm{Cov}[\hat{\mu},\hat{\sigma}^2]}{\mathrm{Var}[\hat{\sigma}^2]}\}^2\cdot\mathrm{Var}[\hat{\sigma}^2]$$

$$\approx A\cdot\sigma^2 \qquad (3.10)$$

where

$$A \equiv \{V_{\mu\mu} - \frac{V^2_{\mu\sigma^2}}{V_{\sigma^2\sigma^2}}\}$$

$$\leq V_{\mu\mu} \qquad (3.11)$$

The final inequality emphasizes that $\hat{\omega}$ is less variable than $\hat{\mu}$.

4. ESTIMATING THE POPULATION MOMENTS

Given that $(\frac{\hat{\sigma}^2}{\sigma^2})$ is a gamma (α,β) variate, it can be shown that for integer p>0 [Rohatgi, 1976, p. 207]:

$$E\{\hat{\sigma}^{2p}\} = \sigma^{2p}\cdot\beta^{p}\cdot\frac{\Gamma(\alpha+p)}{\Gamma(\alpha)}$$

$$= \sigma^{2p}\cdot\beta^{p}\cdot\{\alpha\cdot(\alpha+1)\cdot(\alpha+2)\cdots(\alpha+p-1)\} \qquad (4.1)$$

Following the reasoning of Finney [1941] and Bradu and Mundlak [1970], let

$$H(t,\alpha,\beta) = \sum_{p=0}^{\infty}[\frac{t^p}{p!}][\frac{\Gamma(\alpha)}{\beta^p\cdot\Gamma(\alpha+p)}]$$

$$= \sum_{p=0}^{\infty}[\frac{t^p}{p!}][\frac{1}{\beta^p\cdot\{\alpha\cdot(\alpha+1)\cdot(\alpha+2)\cdots(\alpha+p-1)\}}] \qquad (4.2)$$

Combining equations (4.1) and (4.2), one obtains:

$$E\{H(\delta\cdot\hat{\sigma}^2,\alpha,\beta)\} = \sum_{p=0}^{\infty}[\frac{(\delta\cdot\sigma^2)^p}{p!}]$$

$$= \exp(\delta\cdot\sigma^2) \qquad (4.3)$$

where δ is an arbitrary constant. Now consider the estimator:

$$\tilde{M}_r \equiv \exp(r\cdot\hat{\omega})\cdot H((r^2/2 - r\cdot\varepsilon - r^2\cdot A/2)\cdot\hat{\sigma}^2, \alpha, \beta) \qquad (4.4)$$

Because $\hat{\omega}$ and $\hat{\sigma}^2$ are assumed to be independent,

$$E[\tilde{M}_r] = E\{\exp(r\cdot\hat{\omega})\} \cdot E\{H((r^2/2 - r\cdot\varepsilon - r^2\cdot A/2)\cdot\hat{\sigma}^2, \alpha, \beta)\}$$

$$\approx \exp[r\cdot\mu + r\cdot\varepsilon\cdot\sigma^2 + r^2\cdot A\cdot\sigma^2/2]\cdot\exp[r^2\cdot\sigma^2/2 - r\cdot\varepsilon\cdot\sigma^2 - r^2\cdot A/2]$$

$$= \exp(r\cdot\mu + r^2\cdot\sigma^2/2)$$

$$= M_r \qquad (4.5)$$

$\tilde{M}_r$ is called an Adjusted Maximum Likelihood Estimator (AMLE). With large N: **A** goes to zero; α becomes large; β converges to $(\frac{1}{\alpha})$; $H(t,\alpha,\beta)$ converges to $\exp(t)$; and therefore $\tilde{M}_r$, the AMLE, converges to $\hat{M}_r$, the MLE. While the method has been derived for the r^{th} non-central moment, it can be generalized to any function of the form:

$$\exp(c_1\cdot\mu + c_2\cdot\sigma^2) \qquad (4.6)$$

where c_1 and c_2 are arbitrary constants, including estimation of the variance of a lognormal population [see Likeš, 1980].

5. VARIANCE OF THE AMLE ESTIMATOR

The variance of the AMLE estimator can be obtained in terms of the true values of the parameters [see Bradu and Mundlak, 1970]. Note that:

$$\mathrm{Var}(\tilde{M}_r) = E[\tilde{M}_r^2] - (E[\tilde{M}_r])^2$$

$$\approx E[\tilde{M}_r^2] - \{\exp(r\cdot\mu + r^2\cdot\sigma^2/2)\}^2 \qquad (5.1)$$

For large N,

$$E[\tilde{M}_r^2] = E[\{\exp(r\cdot\hat{\omega})\cdot H((r^2/2 - r\cdot\varepsilon - r^2\cdot A/2)\cdot\hat{\sigma}^2,\alpha,\beta)\}^2]$$

$$= E[\exp(2\cdot r\cdot\hat{\omega})]\cdot E[H^2(\delta\cdot\hat{\sigma}^2,\alpha,\beta)]$$

$$\approx \exp\left(2\cdot r\cdot E[\hat{\omega}] + \frac{4\cdot r^2\cdot \mathrm{Var}[\hat{\omega}]}{2}\right)\cdot E[H^2(\delta\cdot\hat{\sigma}^2,\alpha,\beta)] \qquad (5.2)$$

where $E[\hat{\omega}]$ and $\mathrm{Var}[\hat{\omega}]$ are known, and

$$\delta \equiv \left(r^2/2 - r\cdot\varepsilon - \frac{r^2\cdot A}{2}\right)$$

From equation (4.2), $H^2(t,\alpha,\beta)$ can be expressed as an infinite series:

$$H^2(t,\alpha,\beta) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} [\frac{t^{p+q}}{p!q!}][\frac{\Gamma^2(\alpha)}{\beta^{p+q} \cdot \Gamma(\alpha+p) \cdot \Gamma(\alpha+q)}]$$

$$= \sum_{h=0}^{\infty} W_h \cdot [t/\beta]^h \qquad (5.3)$$

where

$$W_h \equiv \sum_{p+q=h} \frac{\Gamma^2(\alpha)}{[\Gamma(\alpha+p) \cdot \Gamma(\alpha+q) \cdot p! \cdot q!]}$$

$$= \{\frac{\Gamma^2(\alpha)}{\Gamma^2(\alpha+h)}\} \cdot \binom{2(\alpha+h-1)}{h} \qquad (5.4)$$

Expanding equation (5.3) yields:

$$H^2(t,\alpha,\beta) = \sum_{h=0}^{\infty} \{\frac{\Gamma^2(\alpha)}{\Gamma^2(\alpha+h)}\} \cdot \binom{2(\alpha+h-1)}{h} \cdot (\frac{t}{\beta})^h \qquad (5.5)$$

Recalling equation (4.1), one now obtains:

$$E\{H^2(\delta \cdot \hat{\sigma}^2,\alpha,\beta)\} = \sum_{h=0}^{\infty} \{\frac{\Gamma(\alpha)}{\Gamma(\alpha+h)}\} \cdot \binom{2(\alpha+h-1)}{h} \cdot (\delta \cdot \sigma^2)^h$$

$$= \exp(2 \cdot \delta \cdot \sigma^2) \cdot H(\delta^2 \cdot \sigma^4,\alpha,1) \qquad (5.6)$$

This can be substituted into equation (5.2), and then into equation (5.1), to yield an estimate of the variance of $\tilde{M}_r$ in terms of μ and σ^2.

6. MONTE CARLO RESULTS

A Monte Carlo experiment was conducted to determine the finite-sample properties of the AMLE estimator when estimating the mean and variance of a lognormal population. Separate experiments were run for samples of size N=50 independent (pseudo-random) observations drawn from lognormal populations with shape parameter, σ, between 0.2 and 2.0. These values of σ correspond to coefficients of variation ranging from 0.2 to 7.3, and coefficients of skew between 0.6 and 414. The scale parameter, μ, was set to zero, which entails no loss of generality. Each experiment was conducted for censoring levels of 0% (uncensored), 50% and 70%. Experiments were also conducted with several distinct censoring thresholds in each sample, but the results were not qualitatively different than what one sees given a single threshold. In each generated sample, any observation failing to exceed the specified censoring threshold was censored.

Two criteria were used to assess the estimators' performance: bias and variance. Given 10000 sample estimates for each case, the estimators' standardized bias and variance were estimated by:

$$\text{Standard bias} \equiv \sum_{j=1}^{10000} \frac{\left\{\frac{(\hat{M}_{rj} - M_r)}{M_r}\right\}}{10000}$$

$$\text{Standard variance} \equiv \sum_{j=1}^{10000} \frac{\left\{\frac{(\hat{M}_{rj} - \overline{M}_r)}{M_r}\right\}^2}{10000}$$

Two alternative estimators were tested for comparison: the linear regression LR method [Gilliom and Helsel, 1986], which involves using standard sample moment estimates after "filling in" the censored values from a regression of the above-threshold values on their plotting positions; and MLE estimators. The LR method is equivalent to the standard moment estimates in the case of no censoring. The closed-form variance of the AMLE estimator given in equations (5.2) and (5.6) was also computed for the estimator of the mean.

Tables 1-4 report the results of the experiments. Table 1 displays the biases of the three estimators of the mean. In no case does the standardized bias of the AMLE exceed 1%, although the bias is statistically significant (alpha=5% level) in four of the fifteen cases tabulated. The LR method also shows negligible bias. The MLE, as in the uncensored case, becomes highly biased for large values of σ^2. Table 2 contains the variances of estimators of the mean. The AMLE has the lowest variance for all cases tested, although it does not differ greatly from the LR method for small values of σ. Table 2 also contains the closed-form 'EXACT AMLE' results worked out in section 6. In all but two of the fifteen cases the Monte Carlo results do not differ significantly (alpha=5% level) from the closed-form results. Tables 3 and 4 correspond to the same cases that appear in Tables 1 and 2, but give results for the performance of estimators of the population variance rather than of the population mean.

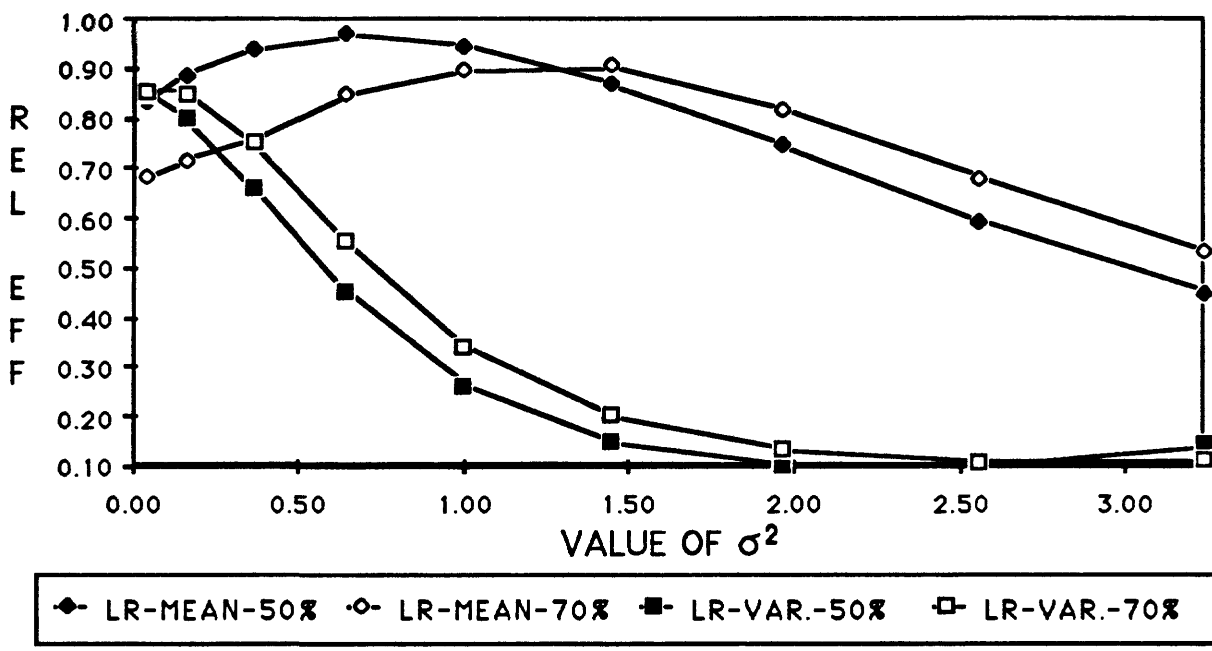

FIGURE 4: Relative efficiency of the linear regression (LR) mean and variance estimators to the corresponding AMLE estimators as a function of σ^2. From Monte Carlo experiments, with censoring thresholds at the 50th and 70th percentiles.

Figure 4 compares the efficiency of the LR estimator to the AMLE with data censored at the 50th and 70th percentiles, as a function of σ^2. This figure can be compared with Figure 1. For large values of σ^2, the results are nearly identical to those found by Finney. For smaller values of σ^2, the LR estimator is substantially less efficient than the AMLE estimator. As in Figure 1, the differences are more pronounced for the variance estimator than they are for the mean estimator.

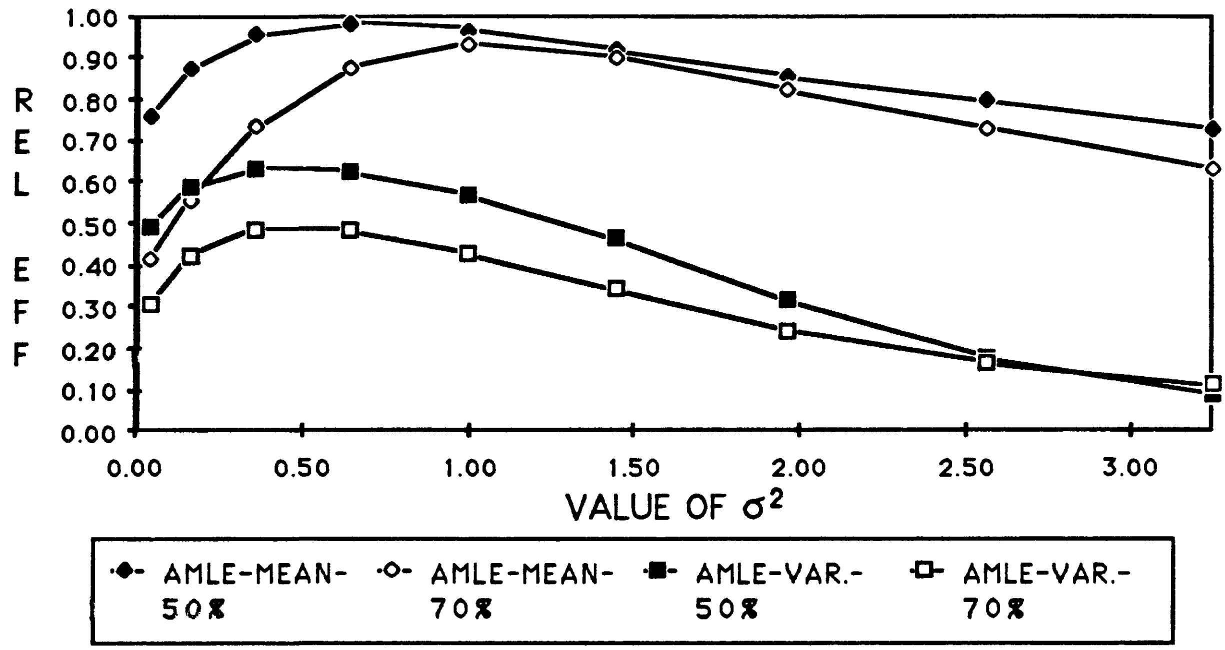

FIGURE 5: Relative efficiency of the censored-data AMLE estimators to the complete-data UMVUE, as a function of σ^2. From Monte Carlo experiments, with censoring thresholds for the AMLE at the 50^{th} and 70^{th} percentiles.

Figure 5 compares the efficiency of the AMLE censored-data estimators to the AMLE with uncensored data (Finney's estimator). This indicates the loss of information that goes with failing to observe the exact values of the below-threshold observations. Even for considerable amounts of censoring, the censored-data AMLE estimator is nearly as efficient as the complete-data estimator. For example, for σ^2=1.0 and 50%-censoring, one expects 25 above-threshold observations. Yet the relative efficiency of the AMLE estimator, compared to the complete-data estimator, is above 95% when estimating the population mean, and close to 70% when estimating the population variance.

7. AN EXAMPLE

Approximately 100 analytical laboratories around the world participate in a quality assurance program run by the U.S. Geological Survey. In one study [Janzer, 1985], natural river water from the South Platte (Colorado) was sent to the laboratories for analysis. Thirteen of the laboratories analyzed their samples by direct atomic absorption, and reported the following concentrations for dissolved iron [µg/l]:

11	<1	4	100	5	<50	<10
42	220	<20	30	<10		
70						

The '<T' values indicates that the laboratory was unable to detect iron at the level of T [µg/l]. Estimates of the mean and standard deviation of the reported concentrations were required, assuming that the reported values are independent, identically-distributed lognormal variates.

Log-space parameter estimates were computed using both the Linear Regression method and Maximum Likelihood:

Method Employed	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\sigma}^2$
Linear Regression	2.432	1.827	--
Maximum Likelihood	2.336	1.890	3.573^2

The corresponding estimates of the population mean and variance, and standard deviation were computed:

Method Employed	Mean	Variance		(Std. Dev.)^2
Linear Regression	38.6	3906.9	=	$(62.5)^2$
Adjusted Max. Likelihood	48.1	13331.1	=	$(115.5)^2$
Maximum Likelihood	61.7	131915.4	=	$(363.2)^2$

Without knowing the true distribution, it is difficult to say which estimator performed best. The point here is that the estimates can differ greatly.

8. SUMMARY

This paper considers estimation of the moments of lognormal populations from moderate-sized type I censored samples. An adjusted maximum likelihood method is derived which seems to have negligible bias. It is shown to be more efficient than either the MLE or a linear regression-based estimator. Also, it is observed that moderate levels of type I censoring on the left may not substantially reduce the information content of a sample if one is interested in estimating population moments.

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APPENDIX

This appendix provides the derivations for some of the terms employed in the main body of the paper. The results are based entirely on the derivatives of the log-likelihood function.

After some manipulation of equation (1.3), one obtains:

$$L_{\mu} = 0 \qquad (A.1)$$

$$L_{\sigma} = 0 \qquad (A.2)$$

$$L_{\mu\mu} = \frac{1}{\sigma^2}\sum_{i=1}^{N} \{-(1-\Phi) + \Phi\cdot B\} \qquad (A.3)$$

$$L_{\mu\sigma} = \frac{1}{\sigma^2}\sum_{i=1}^{N} \{-2\cdot\varphi + \Phi\cdot(\xi\cdot B + Q)\} \qquad (A.4)$$

$$L_{\sigma\sigma} = \frac{1}{\sigma^2}\sum_{i=1}^{N} \{(1-\Phi)\cdot[1-3\cdot D] + \Phi\cdot[\xi^2\cdot B + 2\cdot\xi\cdot Q]\} \qquad (A.5)$$

$$L_{\mu\mu\mu} = \frac{1}{\sigma^3}\sum_{i=1}^{N} \{-\Phi\cdot C\} \qquad (A.6)$$

$$L_{\mu\mu\sigma} = \frac{1}{\sigma^3}\sum_{i=1}^{N} \{2\cdot(1-\Phi) - \Phi\cdot[\xi\cdot C + 2\cdot B]\} \qquad (A.7)$$

$$L_{\mu\sigma\sigma} = \frac{1}{\sigma^3}\sum_{i=1}^{N} \{6\cdot\varphi - \Phi\cdot[\xi^2\cdot C + 4\cdot\xi\cdot B + 2\cdot Q]\} \qquad (A.8)$$

$$L_{\sigma\sigma\sigma} = \frac{1}{\sigma^3}\sum_{i=1}^{N} \{(1-\Phi)\cdot[12\cdot D - 2] - \Phi\cdot[\xi^3\cdot C + 6\cdot\xi^2\cdot B + 6\cdot\xi\cdot Q]\} \quad (A.9)$$

$$L_{\mu\mu,\mu} = \frac{1}{\sigma^3}\sum_{i=1}^{N} \{\varphi + \Phi\cdot[Q\cdot B + C]\} \quad (A.10)$$

$$L_{\mu\mu,\sigma} = \xi\cdot L_{\mu\mu,\mu} - (\frac{2}{\sigma})\cdot L_{\mu\mu} \quad (A.11)$$

$$L_{\mu\sigma,\mu} = \frac{1}{\sigma^3}\sum_{i=1}^{N} \{2\cdot\xi\cdot\varphi + \varphi\cdot[\xi\cdot B + Q] + \Phi\cdot[2\cdot B + \xi\cdot C]\} \quad (A.12)$$

$$L_{\mu\sigma,\sigma} = \xi\cdot L_{\mu\sigma,\mu} - (\frac{2}{\sigma})\cdot L_{\mu\sigma} \quad (A.13)$$

$$L_{\sigma\sigma,\mu} = \frac{1}{\sigma^3}\sum_{i=1}^{N} \{-\varphi + 3\cdot\xi^2\cdot\varphi + 2\cdot\Phi\cdot\xi\cdot B + [\varphi\cdot B + \Phi\cdot C]\cdot\xi^2$$

$$+ 2\cdot[\varphi + \xi\cdot(\varphi\cdot Q + \Phi\cdot B)]\} \quad (A.14)$$

$$L_{\sigma\sigma,\sigma} = \xi\cdot L_{\sigma\sigma,\mu} - (\frac{2}{\sigma})\cdot L_{\sigma\sigma} \quad (A.15)$$

where

$$\xi \equiv \xi_i \equiv \frac{\ln(T_i)-\mu}{\sigma}$$

$$\varphi \equiv \varphi_i \equiv \varphi(\xi_i)$$

$$\Phi \equiv \Phi_i \equiv \Phi(\xi_i)$$

$$Q \equiv Q_i \equiv \frac{\partial \Phi_i}{\partial \xi_i} \equiv \frac{\varphi(\xi_i)}{\Phi(\xi_i)}$$

$$B \equiv B_i \equiv \frac{\partial Q_i}{\partial \xi_i} \equiv -Q_i \cdot (\xi_i - Q_i)$$

$$C \equiv C_i \equiv \frac{\partial B_i}{\partial \xi_i} \equiv -\{B_i \cdot (\xi_i - Q_i) + Q_i \cdot (1 - B_i)\}$$

$$D \equiv D_i \equiv 1 + \varphi \cdot [\frac{\xi_i}{(1-\Phi_i)}]$$

The terms required for estimating the bias and variance of $\hat{\sigma}^2$ and $\hat{\sigma}^4$ can be obtained by repeated application of the chain rule to the above results.

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TABLE 1

Standardized Bias of LR, MLE and AMLE Estimators of the Mean of a Lognormal Population Based on 10000 Monte Carlo Experiments
(N=50; Censoring Threshold at 0%, 50%, and 70%)

Results for σ = 0.2

	LR	(ST. ERR.)	MLE	(ST. ERR.)	AMLE	(ST. ERR.)
UNCENSORED	0.0002	(0.0003)	0.0002	(0.0003)	0.0002	(0.0003)
50% CENS.	0.0045	(0.0004)	-0.0005	(0.0003)	0.0005	(0.0003)
70% CENS.	0.0070	(0.0005)	-0.0005	(0.0005)	0.0020	(0.0004)

Results for σ = 0.6

	LR	(ST. ERR.)	MLE	(ST. ERR.)	AMLE	(ST. ERR.)
UNCENSORED	0.0006	(0.0009)	0.0012	(0.0009)	0.0006	(0.0009)
50% CENS.	0.0093	(0.0010)	0.0017	(0.0009)	0.0013	(0.0009)
70% CENS.	0.0190	(0.0012)	0.0041	(0.0011)	0.0056	(0.0011)

Results for σ = 1.0

	LR	(ST. ERR.)	MLE	(ST. ERR.)	AMLE	(ST. ERR.)
UNCENSORED	0.0010	(0.0019)	0.0061	(0.0018)	0.0010	(0.0017)
50% CENS.	0.0098	(0.0018)	0.0121	(0.0018)	0.0018	(0.0018)
70% CENS.	0.0240	(0.0019)	0.0176	(0.0019)	0.0081	(0.0018)

Results for σ = 1.4

	LR	(ST. ERR.)	MLE	(ST. ERR.)	AMLE	(ST. ERR.)
UNCENSORED	0.0021	(0.0036)	0.0212	(0.0029)	0.0014	(0.0028)
50% CENS.	0.0086	(0.0035)	0.0440	(0.0034)	0.0022	(0.0031)
70% CENS.	0.0227	(0.0034)	0.0603	(0.0037)	0.0091	(0.0031)

Results for σ = 1.8

	LR	(ST. ERR.)	MLE	(ST. ERR.)	AMLE	(ST. ERR.)
UNCENSORED	0.0055	(0.0075)	0.0577	(0.0047)	0.0017	(0.0043)
50% CENS.	0.0094	(0.0074)	0.1290	(0.0069)	0.0027	(0.0050)
70% CENS.	0.0201	(0.0073)	0.1970	(0.0101)	0.0085	(0.0054)

TABLE 2

Standardized Standard Error of LR, MLE and AMLE Estimators of the Mean of a Lognormal Population Based on 10000 Monte Carlo Experiments
(N=50; Censoring Threshold at 0%, 50%, and 70%)

Results for σ = 0.2

	LR (ST. ERR.)	MLE (ST. ERR.)	AMLE (ST. ERR.)	EXACT AMLE
UNCENSORED	0.0283(0.0002)	0.0283(0.0002)	0.0283(0.0002)	0.0286
50% CENS.	0.0356(0.0003)	0.0328(0.0003)	0.0325(0.0003)	0.0326
70% CENS.	0.0534(0.0006)	0.0450(0.0004)	0.0441(0.0004)	0.0435

Results for σ = 0.6

	LR (ST. ERR.)	MLE (ST. ERR.)	AMLE (ST. ERR.)	EXACT AMLE
UNCENSORED	0.0926(0.0007)	0.0918(0.0007)	0.0916(0.0007)	0.0923
50% CENS.	0.0967(0.0008)	0.0942(0.0007)	0.0937(0.0007)	0.0945
70% CENS.	0.1225(0.0010)	0.1079(0.0009)	0.1074(0.0009)	0.1091

Results for σ = 1.0

	LR (ST. ERR.)	MLE (ST. ERR.)	AMLE (ST. ERR.)	EXACT AMLE
UNCENSORED	0.1863(0.0022)	0.1754(0.0015)	0.1733(0.0015)	0.1742
50% CENS.	0.1812(0.0021)	0.1818(0.0017)	0.1762(0.0016)	0.1754
70% CENS.	0.1886(0.0018)	0.1852(0.0017)	0.1794(0.0016)	0.1791

Results for σ = 1.4

	LR (ST. ERR.)	MLE (ST. ERR.)	AMLE (ST. ERR.)	EXACT AMLE
UNCENSORED	0.3601(0.0119)	0.2942(0.0031)	0.2823(0.0029)	0.2833
50% CENS.	0.3531(0.0120)	0.3412(0.0056)	0.3045(0.0039)	0.2979
70% CENS.	0.3444(0.0118)	0.3741(0.0080)	0.3113(0.0045)	0.2973

Results for σ = 1.8

	LR (ST. ERR.)	MLE (ST. ERR.)	AMLE (ST. ERR.)	EXACT AMLE
UNCENSORED	0.7484(0.0681)	0.4734(0.0070)	0.4271(0.0059)	0.4284
50% CENS.	0.7444(0.0683)	0.6874(0.0359)	0.5000(0.0126)	0.4793
70% CENS.	0.7352(0.0687)	1.0080(0.0688)	0.5375(0.0148)	0.4947

TABLE 3

Standardized Bias of LR, MLE and AMLE Estimators of the Variance of a Lognormal Population Based on 10000 Monte Carlo Experiments (N=50; Censoring Threshold at 0%, 50%, and 70%)

Results for σ = 0.2

	LR	(ST. ERR.)	MLE	(ST. ERR.)	AMLE	(ST. ERR.)
UNCENSORED	0.0005	(0.0024)	-0.0164	(0.0022)	0.0007	(0.0022)
50% CENS.	-0.0402	(0.0035)	-0.0003	(0.0032)	-0.0002	(0.0032)
70% CENS.	-0.0430	(0.0044)	0.0006	(0.0040)	0.0008	(0.0040)

Results for σ = 0.6

	LR	(ST. ERR.)	MLE	(ST. ERR.)	AMLE	(ST. ERR.)
UNCENSORED	0.0022	(0.0051)	0.0136	(0.0037)	0.0015	(0.0036)
50% CENS.	-0.0148	(0.0056)	0.0508	(0.0051)	0.0013	(0.0046)
70% CENS.	-0.0251	(0.0060)	0.0630	(0.0061)	0.0044	(0.0052)

Results for σ = 1.0

	LR	(ST. ERR.)	MLE	(ST. ERR.)	AMLE	(ST. ERR.)
UNCENSORED	0.0132	(0.0162)	0.1165	(0.0075)	0.0020	(0.0063)
50% CENS.	0.0083	(0.0163)	0.3018	(0.0152)	0.0032	(0.0084)
70% CENS.	0.0035	(0.0164)	0.5297	(0.0317)	0.0040	(0.0097)

Results for σ = 1.4

	LR	(ST. ERR.)	MLE	(ST. ERR.)	AMLE	(ST. ERR.)
UNCENSORED	0.0538	(0.0629)	0.4545	(0.0205)	0.0021	(0.0114)
50% CENS.	0.0530	(0.0630)	2.0140	(0.3265)	0.0083	(0.0203)
70% CENS.	0.0520	(0.0630)	16.2400	(5.4950)	-0.0034	(0.0231)

Results for σ = 1.8

	LR	(ST. ERR.)	MLE	(ST. ERR.)	AMLE	(ST. ERR.)
UNCENSORED	0.1254	(0.1993)	1.7940	(0.1094)	0.0009	(0.0225)
50% CENS.	0.1253	(0.1993)	88.0100	(60.5700)	0.0335	(0.0754)
70% CENS.	0.1253	(0.1994)	4.55E+04	(3.01E+04)	-0.0169	(0.0672)

TABLE 4

Standardized Standard Error of LR, MLE and AMLE Estimators of the Variance of a Lognormal Population Based on 10000 Monte Carlo Experiments
(N=50; Censoring Threshold at 0%, 50%, and 70%)

Results for σ = 0.2

	LR (ST. ERR.)	MLE (ST. ERR.)	AMLE (ST. ERR.)
UNCENSORED	0.2354 (0.0022)	0.2195 (0.0018)	0.2227 (0.0018)
50% CENS.	0.3455 (0.0034)	0.3194 (0.0029)	0.3187 (0.0029)
70% CENS.	0.4368 (0.0046)	0.4020 (0.0039)	0.4038 (0.0039)

Results for σ = 0.6

	LR (ST. ERR.)	MLE (ST. ERR.)	AMLE (ST. ERR.)
UNCENSORED	0.5145 (0.0171)	0.3736 (0.0041)	0.3615 (0.0039)
50% CENS.	0.5591 (0.0170)	0.5059 (0.0080)	0.4545 (0.0062)
70% CENS.	0.5986 (0.0173)	0.6084 (0.0113)	0.5188 (0.0077)

Results for σ = 1.0

	LR (ST. ERR.)	MLE (ST. ERR.)	AMLE (ST. ERR.)
UNCENSORED	1.6180 (0.2376)	0.7545 (0.0154)	0.6314 (0.0115)
50% CENS.	1.6300 (0.2370)	1.5220 (0.1797)	0.8376 (0.0392)
70% CENS.	1.6470 (0.2367)	3.1800 (0.3977)	0.9666 (0.0430)

Results for σ = 1.4

	LR (ST. ERR.)	MLE (ST. ERR.)	AMLE (ST. ERR.)
UNCENSORED	6.2980 (1.9330)	2.0450 (0.1110)	1.1410 (0.0442)
50% CENS.	6.3010 (1.9330)	3.27E+01(1.45E+01)	2.0320 (0.4307)
70% CENS.	6.3070 (1.9330)	5.50E+02(2.02E+02)	2.3060 (0.3505)

Results for σ = 1.8

	LR (ST. ERR.)	MLE (ST. ERR.)	AMLE (ST. ERR.)
UNCENSORED	19.9400 (8.0940)	1.09E+01(1.55E+00)	2.2390 (0.1906)
50% CENS.	19.9400 (8.0940)	6.06E+03(3.00E+03)	7.5460 (3.1390)
70% CENS.	19.9400 (8.0950)	3.01E+06(1.10E+06)	6.7290 (2.0130)