CONCEPTS AND MODELING IN GROUND-WATER HYDROLOGY--
A SELF-PACED TRAINING COURSE

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CONVERSION FACTORS AND ABBREVIATIONS

Multiply inch-pound units	by	To obtain SI (metric units)
inch (in.)	25.4	millimeter (mm)
inch per year (in/yr)	25.4	millimeter per year (mm/yr)
foot (ft)	0.3048	meter (m)
mile (mi)	1.609	kilometer (km)
square mile (mi^2)	2.59	square kilometer (km^2)
foot squared per day (ft^2/d)	0.0929	meter squared per day (m^2/d)
cubic foot per second (ft^3/s)	0.02832	cubic meter per second (m^3/s)
gallon per minute (gal/min)	0.06309	liter per second (L/s)
gallon per day (gal/d)	0.003785	cubic meter per day (m^3/d)
million gallons per day (Mgal/d)	0.04381	cubic meter per second (m^3/s)
gallon per day per square foot ($gal/d/ft^2$)	40.72	liter per day per square meter ($L/d/m^2$)

PREFACE

The instructional materials in this report were developed from those used in the course "Ground-Water Concepts," which has been given since the 1970's at the U.S. Geological Survey's National Training Center in Denver, Colorado. The materials were generated by a melding of the ideas and work of many individuals, especially those who have served as instructors of the course at the National Training Center over the years. To these instructors we wish to express our appreciation for their efforts. The authors wish to acknowledge particularly Edwin P. Weeks for his involvement with and contributions to the development of the course materials. Eugene P. Patten, Jr. also was involved in the early development of the course, and later as the Chief of the Office of Ground, Water Resources Division (WRD), directed the development of the self-paced version of "Ground-Water Concepts." Lastly, recognition is due the many hydrologists in the U.S. Geological Survey who have served over the past eight years as advisors to participants taking the self-paced version of the course. The success of the self-paced version of the course is in large part due to the efforts of these hydrologists.

The purposes of this course are (1) to review selected fundamental aspects of ground-water flow mechanics, (2) to introduce the basic differential equations of ground-water flow and the techniques for their numerical solution, and (3) to discuss and illustrate basic concepts and techniques that are essential for the successful implementation of ground-water investigations at any level of inquiry including, particularly, concepts and techniques related to modeling ground-water systems. In addition, the course provides a sound technical background for more advanced courses in modeling and the study of other aspects of the ground-water discipline.

Quantitative concepts of ground-water mechanics that are particularly relevant to modeling ground-water systems are emphasized in this course. Many ground-water projects today are designed to (1) provide the necessary data for a future modeling effort, (2) include a modeling component in their initial design, or (3) focus on the development of a ground-water model as an integral part of the project, either to better understand the existing hydrologic system or to predict the response of the hydrologic system to stress. In the concepts course, we seek to emphasize the importance of computer simulation (modeling), not only as a predictive tool, but also as a valuable tool for better understanding the operation of ground-water flow systems. Although the relevance of the concepts presented in this course to modeling will be apparent, these same concepts and approaches are basic to the conceptualization and analysis of any ground-water problem, whether or not computer simulation will be used or applied to the solutions of the problem.

In organizing this course and these study materials, we tried to take a small, constructive step towards encouraging the individual study of ground-water hydrology.

CHAPTER 1--COURSE GUIDE

INTRODUCTION

During the 1970's, most formal technical training within the U.S. Geological Survey (USGS) Water Resources Division (WRD) was given at the WRD National Training Center in Denver. In the early 1980's, concerns about increasing costs for conducting courses at the National Training Center led to the recognition of a need for a more flexible approach to technical training within the Division. A multi-tier approach to technical training has expanded and evolved within the past few years. The various levels and types of training courses include (1) a continued offering of courses at the National Training Center; (2) "short" courses, workshops, and seminars offered at Regional or District Offices; and (3) individual self-paced study programs. It is with the latter that we are concerned primarily in this study guide.

A two-week course of Ground-Water Concepts has been offered for a number of years at the National Training Center. The course materials in this report were developed and expanded from those that were used in the 2-week course.

Purpose of the Concepts Course

The purpose of the concepts course is (1) to review selected fundamental aspects of ground-water flow mechanics, (2) to introduce the basic differential equations of ground-water flow and techniques for their numerical solution, and (3) to discuss and illustrate basic concepts and techniques that are essential for the successful implementation of ground-water investigations at any level of inquiry including, particularly, concepts and techniques related to modeling ground-water systems.

Since its inception several years ago, the concepts course has been the most basic training course in the ground-water discipline that has been offered at the National Training Center. However, the course design assumes that the participants have some initial familiarity with ground-water mechanics. The technical background required for successful participation in and completion of this course will be discussed further in a later paragraph.

An important goal of the concepts course, in addition to the overall purpose stated above, is to provide a sound technical background for more advanced courses in modeling and other aspects of the ground-water discipline. Ideally, a ground-water hydrologist should complete the concepts course before taking other ground-water courses. However, our experience indicates that taking the concepts course after other courses also proves to be beneficial to most hydrologists.

Quantitative concepts of ground-water mechanics that are particularly relevant to modeling ground-water systems are emphasized in this course. Many ground-water projects today either include a modeling component in their initial design; focus on the development of a ground-water model as an integral part of the project, either to better understand the hydrologic system or to predict the response of the hydrologic system to stress; or are designed to provide the data necessary for a future modeling effort. In the concepts course, we seek to emphasize the importance of modeling, not only as a predictive tool, but also as a valuable tool for better understanding the operation of ground-water flow systems. Continuation of this idea suggests that ground-water projects aimed primarily at understanding the ground-water flow system through a standard field investigation usually will benefit from inclusion of a modeling component in the project plan. Although the relevance of the concepts presented in this course to modeling will be apparent, these same concepts and approaches are basic for the conceptualization and analysis of any ground-water problem, whether or not the problem solution involves modeling.

In summary, because of its emphasis on those aspects of ground-water mechanics most relevant to modeling ground-water systems and the obvious and ever-growing importance of modeling in ground-water studies, we believe that a mastery of the content of the concepts course is essential for the successful practice of the ground-water discipline by professional hydrologists, including those not directly involved in modeling projects.

Purpose of This Course Guide

The primary purpose of this course guide is to function as a "road map" for study by individuals. However, this guide should also be useful in providing topics, references, and problems for use in short courses in ground-water discipline.

Technical Background for Concepts Course

In addition to some formal training and/or practical experience in ground-water hydrology, the most important formal prerequisites for successful participation in the concepts course are completion of (1) a standard two-semester course in basic college physics, and (2) a two-semester college course in calculus (both differential and integral). All professional personnel who have not had these courses should try to acquire them at local schools, not only to participate in this course, but also to enhance their overall professional development. Additional courses that would provide a useful background for this course, as well as having significant value for professional hydrologists, are (1) additional courses in mathematics, particularly one in differential equations; (2) some background in geology, particularly courses in structure and stratigraphy; and (3) any courses related directly to hydrology such as fluid mechanics, meteorology.

Technical background and degree of motivation vary widely among individuals. Lacking one or more of the background courses listed above, a potential participant may administer a rather simple self-evaluation. This evaluation consists of studying carefully the first three chapters of the self-paced text "Introduction to ground-water hydraulics" (Bennett, 1976). If this material can be thoroughly understood after diligent study, a person should feel some confidence about proceeding with the concepts course. If, on the other hand, understanding this material proves difficult, we recommend further background study before proceeding with the concepts course. Specific study materials might include some or all of the following references:

(1) "Groundwater," by Freeze and Cherry (1979). Concentrate on chapter 2 which contains a great deal of information on ground-water hydraulics; it may be advisable to skip the section on unsaturated flow until the sections on saturated flow are well understood. Also, work the problems at the end of the chapter.

(2) "Groundwater hydrology," by Todd (1980). Concentrate on chapters 2, 3, and 4.

(3) "Applied hydrogeology," by Fetter (1988). Concentrate on chapters 4, 5, 6, and 7.

(4) "Introduction to ground-water hydrology," by Heath and Trainer (1968). Concentrate on the first three parts; the basic concepts are well presented. This reference is probably best used in conjunction with other references.

(5) "Definitions of selected ground-water terms," by Lohman and others (1972). All the definitions in this reference should be mastered by a practicing ground-water hydrologist; one should check the definitions in this reference often as one studies the other references.

List of Major Topics in Course

The following list of major topics provides a brief overview of the course content. Many subheadings and minor topics are not included. To better indicate the depth of coverage of these topics within the framework of this course, it would be appropriate to preface most listings by "introduction to".

1. Systems and models in ground-water studies (the system concept, information necessary to describe a ground-water system, and steps in modeling a ground-water system.

2. Review of fundamental principles, laws, and definitions (principle of continuity, head, gradient, velocity, streamlines and potential lines, Darcy's law, hydraulic conductivity, concept of storage, and storage parameters).

3. Basic differential equations of confined ground-water flow.

4. Boundaries of ground-water systems and their simulation in models.

5. Discretization of continuous systems (definition of hydraulic conductance).

6. Techniques for numerical solution of differential equations.

7. Flow nets.

8. The source of water to a discharging well.

9. The principle of superposition and its application in ground-water modeling.

10. Ground-water/surface-water interactions and the simulation of streams in ground-water models.

11. Definition of initial conditions.

12. Radial flow and its simulation in ground-water models.

13. Problems involving a free surface boundary (Dupuit assumptions).

14. Evapotranspiration in the hydrologic system and its simulation in ground-water models.

15. Physical principles in solute transport.

16. Application of dimensional analysis to ground-water studies.

17. Workshop in developing models of ground-water systems.

General Comments on the Course and Instructions on Using the Course Guide

You will note in the following outline that no single "text" is listed for this course. The multiplicity of references is by design. We wish to encourage the idea of examining more than one reference in order to study a technical problem.

The course outline in the next section consists of a detailed list of topics that constitutes the curriculum of the course.

These topics are not of equal importance. For ease in organization, it is sometimes convenient to list a less important topic separately. Exhaustive study of some of these topics would be a life's work. In this course, we seek to impart an initial working knowledge of these topics.

Our contribution to an individual study version of the concepts course is to provide (1) a list of references and selected readings, (2) specially prepared notes on selected topics, and (3) specially prepared problems. In organizing these study materials, we believe that we have taken a small, constructive step in encouraging the individual study of ground-water hydrology. However, the most important component of any individual study program is the self-discipline and preservance of the student. No matter how complete the notes and problem explanations are, questions will inevitably arise, and unanswered questions cause frustration. To overcome some of these potential difficulties, the course, as designed for use within the U.S. Geological Survey, includes an advisor who is assigned to each participant. The advisors are experienced hydrologists who help the participants when they have difficulty with the course materials and examine and discuss the results of the problem sets as they are completed. In addition, the participants are encouraged to discuss their questions about the course with their more experienced colleagues, especially if all contacts with the advisor are by telephone.

A rationale exists for the order of topics in the course outline. The advisability of "skipping around" and your success in doing so probably will depend upon your previous technical background. However, the study of some topics near the end of the outline does not depend upon previous topics.

Mastery of some of the unfamiliar concepts in this course will require considerable mental effort. The importance of luck in winning at poker can be compared with the importance of persistence in acquiring new knowledge. There is no substitute for either. Some of the notes and references will not be easy to understand and will require several readings. The important element for learning is to maintain an aggressive attitude toward the material; that is, keep thinking, asking questions, and building slowly upon your present level of technical expertise.

DETAILED OUTLINE OF COURSE WITH READING ASSIGNMENTS, PROBLEMS, AND COMMENTS

A detailed outline of the individual study version of the concepts course follows. In general, associated with each topic, there are one or more reading assignments. In addition, a problem assignment and/or some brief comments on the specific topic may be included. Usually, the comments will not be detailed enough to serve as an explanation for any given aspect of the topic under discussion, but are designed to put the topic in some perspective--for example, its purpose, importance, and relation to other topics in the course; provide information in abbreviated form for thought and study; and sometimes to provoke questions.

The sequence of study suggested for each topic is as follows: (1) read the brief comments, (2) study carefully the reading assignments, (3) complete the assigned problems, and (4) read the brief comments again, if they contain technical information, as a review. In the absence of specific instructions to the contrary, the various assignments should be completed in the order listed.

Systems and Models in Ground-Water Studies

The purpose of this section is to provide an initial overview and perspective on the concept of a ground-water system and its relation to the development of a ground-water model.

What is a System, a Model, and Related Definitions

Study assignments

* Read Domenico (1972), "Concepts and models in groundwater hydrology," pages 1-38.

* Read note 1 (section 2.1), "Some general concepts and definitions related to systems and models in ground-water studies."

The following diagram, which is given in the reading material, has great conceptual usefulness, simple though it is.

```
                             __________
Input or Stress  |          | Output or Response
---------------> |  System  |    --------->
                 |__________|
```

What are the inputs or stresses to ground-water systems, both natural or man-induced? How do we usually measure the response of the ground-water system to stress? Are there other possibilities for measuring the effects of stress?

The word "model" is used in many ways and in many contexts in science and ground-water hydrology. Make a list of the different uses and meanings of this word as you proceed with the course.

Hydrogeologic Information Necessary to Describe a Ground-Water System and to Develop a Ground-Water Model

Study assignment

* Read Bear (1979), "Hydraulics of groundwater," pages 94-95 and 116-117.

Boundary-value problems are commonly encountered in quantitative analysis in ground-water hydrology. A boundary-value problem is a mathematical problem type that has been extensively studied and used to solve problems in many areas of science and technology. Quantitative results from a numerical model of a ground-water system represent a specific solution to a specific boundary-value problem. The information needed to quantitatively describe a ground-water system that is listed in Bear (1979) is the same information needed to define a mathematical boundary-value problem that represents the system.

The types of information necessary to describe an unstressed ground-water system include (1) the external and internal geometry of the system (the geologic framework), (2) the character and physical extent of the boundaries of the system, and (3) the material and fluid parameters of the system which include the transmitting parameters (transmissivity or hydraulic conductivity) and the storage parameters (storage coefficient or specific storage). The question of defining initial conditions will be discussed in a later section of this outline. Memorize the very important list of the types of information necessary to describe a ground-water system even though you may not understand the exact meaning and importance of the various items at this time. At the completion of this course, your questions and uncertainties in this regard should be dispelled.

Steps in Developing a Model of a Ground-Water System

Study assignment

* Read Mercer and Faust (1982), "Ground-water modeling," pages 1-8.

The discussion in this reference provides an overview on how to proceed in logical sequence to develop a model of a ground-water system from start to finish. In this course, we emphasize only a part of this sequence--namely, how to conceptualize and describe a ground-water system and translate this concept of how the system works into the framework of a numerical ground-water model.

Review of Fundamental Principles, Laws, and Definitions

As you work through this section, study carefully, in Lohman and others (1972), "Definitions of selected ground-water terms--revisions and conceptual refinements," the definitions of all the terms relating to ground water that you encounter.

Principle of Continuity--The Hydrologic Equation

Study assignment

* Read Domenico (1972), "Concepts and models in groundwater hydrology," pages 145-147 (all the texts will have some discussion of this topic).

The so-called "hydrologic equation"

$$\text{Inflow} = \text{Outflow} \pm \text{Change in Storage},$$

is really a very simple statement of the principle of continuity. We use this equation when we prepare a water budget of a hydrologic system. Most commonly, we prepare an average water budget for a period of years during which the system was at equilibrium (heads and flows in the system oscillate about a mean condition) without considering changes in storage. In this case, the hydrologic equation is simply

$$\text{Inflow} = \text{Outflow}.$$

The units of the hydrologic equation generally express rates of water flow, for example, millions of gallons per day (Mgal/d) or cubic feet per second (ft^3/s). If volume units are used, a time period is usually implied.

In using the equation of continuity for any purpose, it is important to specify carefully the <u>volume</u> of earth material for which the budget is being prepared. In budgets of river basins, one speaks of the area drained by a stream above a certain point on the channel. In many river basins, the ground-water component of the budget is negligible, and the area of the basin is the most important measure of its extent. However, if the ground-water component in the budget is considered seriously, it is necessary to define not only the areal extent of the basin (the top of the reference volume) but also the sides and bottom of the reference volume within the ground-water reservoir.

The principle of continuity is applied at many different scales and in many different contexts. Later in this course this principle will be applied to a reference block of aquifer material in the process of deriving the basic ground-water flow equations.

Head and Related Concepts

Study assignments

* Work through Bennett (1976), "Introduction to ground-water hydraulics--as programmed text for self-instruction," Part I, starting on page 3. (A note on the use of Bennett's self-programed text--Read page 1, "Instructions to the Reader," before starting the study assignment. Most of the text is not meant to be read continuously like other textbooks. Instead, discussions of parts of topics or concepts are interrupted by multiple-choice questions of varying degrees of difficulty. The value of the text as a learning tool is decreased if the student attempts to read the text in a continuous manner without answering the questions.)

* Read Freeze and Cherry (1979), "Groundwater," pages 18-36. Work problems 1, 2, and 3, pages 77-78 in Freeze and Cherry.

The hydrostatic pressure at a point in a fluid below a free surface that is in contact with the atmosphere (such as a stationary or slowly moving body of surface water) is defined by the equation

$$p = \gamma \ell = \rho g \ell$$

where p is the pressure at the point, γ is the weight density of the fluid, ρ is the mass density of the fluid, g is the acceleration due to gravity, and ℓ is the vertical distance of the point below the free surface. Make a sketch illustrating the application of this formula.

A piezometer is a pressure measuring device consisting of a tube, one end of which taps the fluid system, and the other end open to the atmosphere. In ground-water hydraulics, a tightly cased well that is open or screened to one interval of an aquifer can be considered a piezometer. Remember that a pressure measurement (also head measurement) in a well represents conditions in the aquifer at the location of the well screen or well opening. If the well screen or well opening is several feet long, the midpoint of the screen or opening is used as the measuring point. However, the measurement represents a weighted average of conditions along the well opening. Assuming that a well can be used as a piezometer, make a sketch that indicates clearly to you that the measurement of the water-level elevation in the well defines the total head at the measuring point.

Velocity, Stream Lines, and Potential Lines

Study assignments

* Read Milne-Thomson (1955), "Theoretical hydrodynamics," pages 1-9.

* Read Freeze and Cherry (1979), "Groundwater," pages 69-71.

Velocity is a vector; that is, it has both direction and magnitude. We can define velocity at a point or an average velocity for a specified area. In ground-water hydrology, the definition of average velocity is best understood by considering the flow of a liquid through a completely full pipe. We will define the average velocity (in feet per second) of the fluid in the pipe by

$$\text{---------->}\ V_{average} = Q/A$$

where Q is the fluid discharge (in ft^3/s) across any reference cross section of the pipe and A (in ft^2) is the area of the reference cross section. Note that A, the area of the reference cross section, is defined as being perpendicular to the average direction of fluid flow and that the specific discharge, Q/A, of the pipe and the average fluid velocity are the same. Note also that the velocities at different points in the flow cross section will, in general, not be equal. One of the simplifications commonly used in hydraulics is to work with average velocities in order to avoid the complication of considering the velocit distribution in a given cross section.

Freeze and Cherry (1972), pages 69-71, discuss the different velocities that we define in ground-water hydraulics. For a porous medium, such as aquifer material, it is very important to distinguish between the specific discharge (or Darcy velocity) which is defined directly from Darcy's law and the average linear velocity (sometimes termed the "actual" velocity), which includes the porosity in its definition and is always larger than the Darcy velocity. The reading assignment in Freeze and Cherry may be more meaningful after the completion of the following section on Darcy's law.

In ground-water hydraulics, potential lines (potential surfaces in three dimensions) represent the locus of points at which the ground-water head is a specified constant. Contour lines on potentiometric-surface maps represent the investigator's best estimate of the location of potential lines (actually projections of potential surfaces) in the natural ground-water system based on head measurements in wells.

Under steady-state conditions (flows that do not vary with time), stream lines trace out the approximate paths along which the water particles move. If one injects dye at a specific point in a laboratory sand model of a steady-state ground-water flow system, the moving dye will trace out the average stream line passing through the injection point. Thus, streamlines in ground-water systems have a physical reality.

We will encounter the concepts introduced here many times in later sections of the course--for example, in connection with boundary conditions and flow nets.

Darcy's Law, Hydraulic Conductivity, and Transmissivity

Study assignments

* Work through Bennett (1976), "Introduction to ground-water hydraulics," Part II, starting on page 14.

* Read Freeze and Cherry (1979), "Groundwater," pages 15-18.

Darcy's law is the basic rule that we use to describe the macroscopic flow of ground water. The law is deceptively simple in its expression, so simple, in fact, that we sometimes overlook some of its physical implications. Darcy's law expresses a linear relationship between Q, the volumetric flow across a reference cross section, and the hydraulic gradient $(h_2-h_1)/L$.

For a pipe packed with sand, Darcy's law may be written

$$Q = \frac{-KA\ (h_2 - h_1)}{L}$$

where Q is the volumetric flow across a cross section of the pipe normal to its length--that is, the quantity of water passing the cross section in a unit time, A is the area of the cross section, L is the length of the pipe, K is the hydraulic conductivity of the sand, and h_1 and h_2 are the heads near the ends of the pipe. Make a sketch and designate all the factors in the Darcy equation.

The formula above describes one-dimensional flow (all the velocity vectors are equal and parallel in any given cross section of the flow) in a sand-filled pipe or prism of uniform cross-sectional area. It is important to remember that the area A in Darcy's law is perpendicular to the direction of flow. (We will refer to this fact later when we define hydraulic conductance.) The head measurements h_2 and h_1 must be made on the same streamline, and the length L is the distance along the streamline between points 1 and 2 where head is measured. These observations are particularly important when studying parts of flow systems in which velocity vectors have discernible vertical components. Vertical flow components are commonly encountered near recharge areas and discharge areas in natural flow systems.

It is possible for ground water to flow from areas of lower pressure to areas of higher pressure. Can you sketch a hypothetical situation in which this would be the case? Of course, the head must decrease in the direction of flow.

Study carefully the discussion in Freeze and Cherry (1979, pages 15-18) in that hydraulic conductivity is shown to be a composite parameter that represents both the properties of the porous medium and the properties of the flowing fluid.

Concept of Storage and Definition of Storage Parameters

Study assignments

* Work through Bennett (1976), "Introduction to ground-water hydraulics," Part IV, starting on page 53.
* Read Todd (1980), "Groundwater hydrology," pages 26-37 and 42-46.

Because Darcy's law describes the steady flow of a fluid though a sand prism of uniform cross-sectional area, neither time nor storage factors are included in the expression (equation) of the law. However, in any type of system that undergoes transient flow (a flow whose properties--for example, velocity distribution--change with time) a storage element and a time factor must be considered. Distinguish carefully between unconfined storage and confined storage. When changes in unconfined storage take place, water displaces air or air displaces water in the pore spaces between the sand grains. For example, a decrease in unconfined ground-water storage means that air replaces water that drains under the influence of gravity. When changes in confined storage take place, the aquifer material remains saturated and the water and aquifer material are compressed.

Estimation of Transmissivity and Storage Coefficient Values for Application in Ground-Water Models

Study assignment

* Read Lohman (1972), "Ground-water hydraulics," pages 52-54.

Optional references

* McClymonds and Franke (1972), "Water-transmitting properties of aquifers on Long Island, New York," pages 6-14.
* Olmstead, and others (1973), "Geohydrology of the Yuma area, Arizona and California," pages 72-82.

For the purpose of this course, you need to study the first reference only. The optional references may be of some value if you are actually engaged in trying to estimate transmissivity values from available well information.

Darcy's Law as a Differential Equation and its Application to a Field Problem

Study assignment

* Work through Bennett (1976), "Introduction to ground-water hydraulics," Part III, starting on page 34.

Up to this point in the course, we have studied two major topics in the course outline. The first topic dealt with the concept of a system, particularly as this concept relates to ground-water systems. We also introduced the specific types of information necessary to describe a ground-water system, recognizing that a great deal more work is required before these concepts will be sufficiently understood so that you can apply them to specific problems with confidence. At this time, the conclusion from the first section is that the system concept applies very well to a certain physical reality in nature in which we are interested--namely, natural ground-water systems, and any ground-water problem can and should be formulated in terms of the system concept, even though the areal or volumetric scope of the problem does not include an entire natural ground-water system.

The second major topic introduces (or reviews) those essential physical concepts and laws that form the basis of ground-water mechanics. The present subsection forms a kind of bridge between the introductory material presented thus far and the immediately succeeding topics, which are an introduction to the differential equations of ground-water flow and boundary conditions.

In Part III of Bennett (1976), Darcy's law is expressed as a simple differential equation--that is, the differential form of the head gradient at a point, dh/dx, is substituted for the average head gradient between two points, $(h_1-h_2)/L$. This differential equation is used to solve a simple field problem.

You may note that the aquifer in the field problem is really a field scale Darcy prism. Solution of the field problem posed in Part III requires solving a simple boundary value problem. This process raises important questions such as (a) What constitutes a solution to a differential equation? (b) What are the boundary conditions of this problem? and (c) What boundary information must be incorporated in order to obtain a unique solution to the posed problem? These questions will be studied in considerably more detail in the following two sections of the course.

Basic Differential Equations of Confined Ground-Water Flow

Study assignments

* Work through Bennett (1976), "Introduction to ground-water hydraulics, Part V, starting on page 69.

* Read Mercer and Faust (1982), "Ground-water modeling," pages 9-24.

Optional references

* Freeze and Cherry (1979), "Groundwater," pages 63-67.

* Todd (1980), "Groundwater hydrology," pages 99-101.

All of these readings cover approximately the same topic. Start with the first two references. If you feel that further repetition of the material will be useful to you, continue to the last two references, which also will include the derivation of the basic flow equations if you wish to study this topic further.

An idea of fundamental importance in understanding the derivation of the basic flow equations is that they are formed by combining two basic "components," namely (1) the principle of continuity and (2) Darcy's law, a specific rule that describes the flow of ground water. You will note that the principle of continuity,

$$\text{Inflow} = \text{Outflow} \pm \text{Change in Storage},$$

is applied to a reference block of aquifer material in deriving the equations.

Boundaries of Ground-Water Systems and Their Simulation in Models

Study assignment

* Read the section "Boundary conditions" in Franke and others (1987), "Definition of boundary and initial conditions in the analysis of saturated ground-water flow systems--an introduction," pages 1-10.

* Read Franke and Reilly (1987), "The effects of boundary conditions on the steady-state response of three hypothetical ground-water systems--Results and implications of numerical experiments."

Problem assignment

* Work the exercises at the end of the first section on "Boundary Conditions" in the reference above. These exercises are specifically listed in the table of contents.

* Bear (1979), "Hydraulics of groundwater," pages 94-102 and 116-123.

The study and understanding of boundary conditions is probably the single most important topic in this course, if not in the entire field of ground-water hydrology. Selection of improper boundary conditions is the principal cause of failure in modeling studies.

Study the first and second references with the greatest care. The basic types of boundary conditions and their characteristics, which are listed and discussed in these references, should be stored in the memory bank of all ground-water hydrologists for instantaneous recall. Most of the information in Bear (1979) is contained in the first reference. In general, Bear expresses the concepts related to boundary conditions in more abstract, mathematical form than does the first reference.

Acquiring expertise in the proper application of boundary conditions requires practice and experience in many types of problems. Make a habit of defining the boundary conditions in any figures you see that depict various kinds of ground-water problems or systems. The first step in thinking about any ground-water problem is to try to decide what boundary conditions would describe most suitably the hydraulic situation in the natural flow system.

Leave the section on initial conditions in the first reference until later in the course, at least until you have studied carefully and understand the principle of superposition.

Discretization of Continuous Systems and the Electrical Analogy

Study assignments

* Read note 2 (section 2.2), "The electrical analog, simulation, and discretization" and section 2.3 (note 3), "Hydraulic storage capacity."

* Read Karplus (1958), "Analog simulation," pages 78-88, 171-184, and 204-209

* Read Prickett and Lonnquist (1971), "Selected digital computer techniques for groundwater resource evaluation," subsection "Mathematical derivation of finite-difference equations," pages 3 to 5, and subsection "Models with variable grid spacing," pages 17 to 19.

* Work problem 1, "Calculation of lumped hydraulic conductances and storage capacities in rectangular grids."

Optional reference

* Walton (1970), "Groundwater resource evaluation," pages 518-533.

Concentrate on notes 2 and 3 and on Karplus (1958), and consult Walton (1970) only if you are particularly interested in the electrical analogy.

Use of the electrical analogy and electric-analog models can have great conceptual value in solving ground-water problems if one is readily able to translate back and forth between electrical terminology and hydraulic terminology. Because of their value as teaching tools, electric-analog models are used in the course to solve several problems and to illustrate several important concepts in ground-water hydrology. We have included an electrical-analog problem set in section 3.5.

We must emphasize that we do not recommend the construction of electric-analog models for operational use. The advanced technological development and easy accessibility of computers makes the construction of large-scale electric-analog models impractical in the United States. However, the electrical analogy still has great conceptual value within the curriculum of this course. It can be particularly useful in understanding the process of discretization and the definition of hydraulic conductance.

Concepts associated with electrical networks are very useful in understanding discretization of continuous aquifer systems for the purpose of developing models. The discussion of discretization by Karplus (1958) is thorough. However, a general familiarity with the electrical analogy is necessary to translate the equations in electrical notation (in Karplus) to corresponding hydraulic equations. It is important to recognize at the outset that the equations in the two notations represent exactly the same concepts. The translation from one notation to the other is not at all difficult.

The first step in discretization is to develop a suitable network of nodes and branches. A known voltage is applied across the network and voltage is measured at the nodes. Head is calculated from the voltage measurements through appropriate conversion factors. Flow in the discretized system can occur only along branches between nodes. Pay particular attention to the concepts of vector area and vector volume in Karplus. These concepts define the block of aquifer material that is associated with each branch. The geometry of the block and the transmission characteristics (for example, hydraulic conductivity) of the block materials determine the transmitting capability of a given branch between two nodes. The branch conductance is the coefficient that defines quantitatively this transmitting capability.

In Karplus, you will encounter equations for current flow in branches between nodes of the form

$$I_{branch} = -\frac{1}{R}(V_2 - V_1),$$

where I is the current flowing through the branch in amperes, R is the branch resistance in ohms, and V_2 and V_1 are the voltage values at the two ends (network nodes) of the branch resistance. This equation is a form of Ohm's law. For our purposes, it is convenient to substitute C_e for 1/R , where C_e is the electrical conductance of the branch in mhos. Thus, the formula for branch current above can be written

$$I_{branch} = -C_e (V_2 - V_1).$$

(The minus sign indicates that current flows in the direction of decreasing voltage.)

Darcy's law may be written

$$Q = \frac{-KA (h_2 - h_1)}{L},$$

and we will define the hydraulic conductance, C_h, as

$$C_h = \frac{KA}{L}.$$

C_h is expressed in units such as gallons per day per foot. Thus, we may express the water flow in a branch of a network representing a discretized aquifer system as

$$Q_{branch} = -C_h (h_2 - h_1).$$

Compare this expression, which is merely a simplified method of writing Darcy's law, with the formula for a branch current above.

We may regard a branch in a discretized aquifer system as transmitting the flow through a Darcy prism located between the nodes at the ends of the branch. The hydraulic conductance of the branch is a coefficient that defines quantitatively the transmitting capability of this Darcy prism.

Remember that the area A in Darcy's law, and, therefore, also in the definition of hydraulic conductance, refers to the cross-sectional area of the flow perpendicular to the average flow direction. Thus, because flow occurs parallel to/or along branches in discretized systems, the area A in the hydraulic conductance is the area of the hypothetical Darcy prism perpendicular to the branch.

In transient simulations of discretized systems, areas and volumes of the aquifer must be specified to define the storage properties of the system. These "storage areas" and "storage volumes" are associated with the nodes of the discretization network, not the branches, as is the case for hydraulic conductance coefficients. Determine the general rule for defining storage areas associated with nodes from Karplus (1758, p. 204-209) and study figure 3 in Prickett and Lonnquist (1971) and section 2.3 (note 3), "Hydraulic Storage Capacity."[1] The figures in Prickett and Lonnquist depict the same concepts regarding discretization as the discussion in Karplus. However, hydraulic terminology is used in Prickett and Lonnquist. As you read

[1] Please note that hydraulic storage capacity, S_c, is a mathematically defined quantity that differs in its definition from other fairly common usages of this term.

Karplus, write the translation of all the formulas that you encounter from electrical notation into hydraulic notation.

Techniques or Numerical Solution of Differential Equations

Finite Differences--Steady-State Flow

Study assignments

* Work through Bennett (1976), "Introduction to ground-water hydraulics," Part VII, pages 119-135.

* Read Mercer and Faust (1982), "Ground-water modeling," pages 25-39.

* Read Wang and Anderson (1982), "Introduction to groundwater modeling," chapters 2 and 3, pages 19-66.

* Work problem 2 (section 3.2), "Numerical analysis, steady state."

Finite Differences--Transient Flow

Study assignments

* Read Bennett (1976), "Introduction to ground-water hydraulics," Part VII, pages 136-140.

* Read note 4 (section 2.4), "Discretization of time--methods of formulating nonequilibrium finite-difference equations."

* Read Wang and Anderson (1982), "Introduction to groundwater modeling," chapter 5, pages 67-88.

* Work problem 3, "Numerical analysis, transient state."

The purpose of this section is to introduce techniques for the numerical solution of differential equations. In this course, we are primarily interested in the numerical solution of the ground-water flow equations, which, in general, are second-order partial differential equations.

The key concept in this section is this: a set of simultaneous linear algebraic equations can be substituted for the differential equations of ground-water flow to solve the dependent variable (head) at points (nodes) in discretized space. The differential equation describes the variation of the dependent variable in continuous space. An algebraic equation must be solved for every node in discretized space at which head is not specified. Thus, the problem of solving the ground-water flow equations numerically becomes the problem of solving a large number of simultaneous linear algebraic equations. There are two principal approaches, each of which includes a number of individual techniques for solving such equation sets--(1) direct methods and (2) iteration techniques. Substitution of unknowns from one equation to another until only one unknown remains, as you learned in high school algebra, is an example of a direct method of solution. Solutions using iteration techniques involve solving the set of equations by successively approximating the unknown values, using a specific numerical scheme (algorithm) until a solution is obtained.

The problem assignment involves the numerical solution by hand calculator of a simple steady-state problem by both a direct method and an iteration technique, and the numerical solution of a nonequilibrium problem by hand. If you are becoming seriously involved in digital modeling, we highly recommend that you also take a course in numerical analysis techniques.

Flow Nets

Study assignments

* Read Freeze and Cherry (1979), "Groundwater," pages 168-189.

* Read Todd (1980), "Groundwater hydrology," pages 83-93.

* Read and work problem 4, "Impermeable wall problem."

Although flow nets can be prepared in transient problems for any "instant" of time, they are most commonly used in the solution of steady-state problems. A carefully prepared flow net provides a very valuable pictorial representation of a flow system. Commonly, an accurate flow net yields a solution of a ground-water problem because the information of interest in the problem, for example, head values at specific locations, can be obtained from it. If hydraulic conductivity is known or can be estimated, volumes of flow can be estimated using a flow net for the entire ground-water system or any part of it.

The impermeable-wall problem illustrates the preparation and applications of flow nets to solving different kinds of problems. One of the most important parts of the problem that may be new to you is the calculation of stream functions. This topic is covered in some detail in the problem explanation.

The impermeable-wall problem is an example of an intensively studied class of problems involving seepage beneath and/or through engineering structures. An important characteristic of these two-dimensional problems is that they represent the ground-water flow field in vertical cross section. The geometric scale of these problems is orders of magnitude smaller than the regional scale of many ground-water studies. Because of the large areal extent of many ground-water systems, the models of these systems often tend to emphasize simulation in map view as opposed to simulation in cross section. However, in thick layered systems in which there are large contrasts in hydraulic conductivity between adjacent layers, proper understanding and simulation of the vertical dimension of flow may be essential in determining how the system operates. In such situations, vertical cross-section studies may be highly advisable, if not essential. Unfortunately, vertical cross-section studies are a sadly neglected aspect of many ground-water investigations that involve modeling. To be most fruitful, such studies in vertical cross section should be undertaken concurrently, with areal-model simulations.

The Source of Water to a Pumped Well

Study assignments

* Read Theis (1940), "The source of water derived from wells--essential factor controlling the response of an aquifer to development," pages 277-280.

* Read note 5 (section 2.5), "Theis' concepts--response of an aquifer to development."

This paper by Theis is one of the classic papers in ground-water hydrology. We are specifically interested in the first part of the paper. With reference to the hydrologic equation

$$\text{Inflow} = \text{Outflow} \pm \text{Change in Storage},$$

there are three possible sources of water to a pumped well. Expressed in rather abstract terms, these include (1) removal of ground water from storage in the system, (2) increased inflow to the ground-water system due to the removal of water by the pumped well, and (3) decreased outflow from the ground-water system due to the removal of water.

Consider a shallow well pumping near a gaining stream. Initially, the water removed from the well is obtained from ground-water storage. As pumping progresses, ground water that formally would have discharged into the stream is diverted to the pumping well (decreased outflow of water from the ground-water system). Depending on the discharge of the well and the distance between the well and the stream, a situation may be reached in which a ground-water gradient from the stream to the pumping well is established. If this occurs, flow from the stream to the aquifer takes place (increased inflow to the ground-water system).

As you read Theis, list all the possibilities that occur to you of increasing inflow to and decreasing outflow from ground-water systems.

Principle of Superposition and its Application in Ground-Water Modeling

Study assignment

* Read Reilly, Franke, and Bennett (1987), "The principle of superposition and its application in ground-water hydraulics." Work all problems.

Optional reference

* Bear (1979), "Hydraulics of groundwater," pages 152-159.

The principle of superposition has wide application in ground-water hydraulics. We employ superposition, perhaps without realizing it, whenever we analyze an aquifer test in a confined aquifer. The method of images is based on superposition. Employing superposition is an important option in modeling ground-water systems.

The first reference contains most of the information that you should know about superposition. The application of superposition to modeling ground-water systems is emphasized. Several simple problems illustrating superposition are included. These should be completed as you proceed through the reading assignment. In the optional reference, the same basic ideas are couched in a somewhat more mathematical language.

Hydrologists encountering the principle of superposition for the first time often experience considerable difficulty with the concept. It takes considerable practice to learn to think in terms of superposition. The key word in understanding the principle is changes--for example, changes in head and changes in flow, which can be illustrated by analysis of data from an aquifer test. Absolute head measurements are converted to drawdowns, which represent changes in head superimposed on the ground-water system in response to pumping. This is done because the analytical solutions to linear well-hydraulics problems are expressed in terms of head changes; that is, the solutions use superposition.

The principle of superposition is one of the key concepts in ground-water hydrology. It is virtually mandatory that all ground-water hydrologists make the necessary effort to become comfortable with this concept.

Steady-State Ground-Water/Surface-Water Relations and the Simulation of Streams in Ground-Water Models

Study assignments

* Read Harbaugh and Getzen (1977), "Stream simulation in analog model of ground-water system of Long Island, New York."

* Read Prickett and Lonnquist (1971), "Selected digital computer techniques for groundwater resource evaluation," pages 33-35.

* Read note 6 (section 2.6), "Stream-aquifer interaction."

* Review Franke and Reilly (1987), "The effects of boundary conditions on the steady-state response of three hypothetical ground-water systems--results and implications of numerical experiments."

* Work problem 5 (section 3.5), "Electrical analog problem set."

* Work problem 6 (section 3.6), "Digital stream-aquifer interaction problem set."

Additional references that will be helpful in answering question 20 of problem 6 concerning the correction of drawdown at a pumping node in a model to approximate the drawdown in a real well located at the node are:

Prickett and Lonnquist (1971), "Selected digital computer techniques for groundwater resource evaluation," page 61.

Note 7 (section 2.7), "Well drawdown correction at a pumping node."

Because the interaction between streams and shallow aquifers in natural systems is important in understanding ground-water flow, the simulation of these relations in models is of corresponding importance. The purpose of the first three readings in this section is to become acquainted with the several different possible relations between streams and aquifers and how these different relations are treated in ground-water models. Commonly, the type of relations between the stream and the aquifer changes during the course of a problem. Usually, streams are treated as specified head, head-dependent flux, or specified-flux boundaries. The characteristics and effects of using specific types of boundaries should be reviewed in Franke, Reilly, and Bennett (1987) and Franke and Reilly (1987). The different physical situations in the natural system that these different boundary conditions seek to represent should be clearly differentiated.

The report by Franke and Reilly (1987) provides a comprehensive review of boundary conditions and superposition. More importantly, however, the effects of different boundary conditions on the response of three otherwise similar ground-water systems to a pumping stress are illustrated in a series of numerical examples, and the pitfalls in assigning boundary conditions to stressed ground-water systems are discussed. Try to relate the concepts discussed in this paper to the problem sets in this section.

The two longest and most important problems in the concepts course, the "Electrical-analog model problem," (problem 5) and the "Digital stream-aquifer problem," (problem 6) are placed here in the course outline because to work through these problems requires the application of many of the concepts studied previously--specifically, boundary conditions, hydraulic conductance, flow nets, the source of water to a pumping well, superposition, and ground-water/surface-water interactions. Try to think very specifically in terms of these concepts as you work these problems. Working through these problems should provide a valuable learning experience, perhaps the most valuable in the entire course.

Definition of Initial Conditions in Modeling Ground-Water Sytems

Study assignments

* Read the section, "Initial conditions" in Franke, Reilly, and Bennett (1987), "Definition of boundary and initial conditions in the analysis of saturated ground-water flow systems--an introduction."

* Read Rushton and Redshaw (1979), "Seepage and groundwater flow," pages 153-156 and 182-184.

The second part of the first reference discusses the definition of initial conditions in ground-water models. You should undertake the study of initial conditions only after you have studied carefully the two sections on boundary conditions and superposition. The readings in the second reference are short and provide a good review of the discussion of initial conditions in the first reference.

Radial Flow and its Simulation in Ground-Water Models

Study assignments

* Work through Bennett (1976), "Introduction to ground-water hydraulics," Part VI, starting on page 88.

* Read Freeze and Cherry (1979), "Groundwater," pages 314-319, 343-347, 349-350.

* Read Todd (1980), "Ground water hydrology," pages 115-119.

* Read Lohman (1972), "Ground-water hydraulics," pages 11-15.

* Work problems 2, 7, and 12 on page 379 in Freeze and Cherry (1979), "Groundwater."

Because of the importance of wells in ground-water hydrology, radial flow is an important topic. Many analytical solutions to different problems related to wells are available. Numerical simulation of radial flow problems has become a valuable tool in recent years and permits the approximate solution of problems which are too difficult to be solved analytically.

In the classroom version of the concepts course, we prepare a flow net for a radial-flow problem as we did for the impermeable-wall problem. This problem is now available for self-study (Bennett and others, 1990) and is an optional exercise if you are particularly interested in radial flow.

The final task in this section is to become very familiar with three solutions to radial flow to a well:

(1) the Theis nonequilibrium equation

$$s = \frac{Q}{4\pi T}\int_u^\infty \frac{e^{-u}du}{u};$$

(2) steady flow to a completely penetrating well in a confined aquifer,

$$Q = \frac{2\pi Km\ (h_2 - h_1)}{\ln r_2 / r_1};$$ and

(3) steady flow to a completely penetrating well in an unconfined aquifer,

$$Q = \frac{\pi K\ (h_2^2 - h_2^2)}{\ln r_2 / r_1}.$$

Check the different texts for the boundary conditions assumed in each solution and how to use the solutions to obtain numerical results in field problems.

Problems Involving a Free Surface

Study assignments

* Read Lohman and others (1972), "Definitions of selected ground-water terms," review the following definitions--ground-water, unconfined; potentiometric surface; specific retention; specific yield; and water table.

* Read Franke, Reilly, and Bennett (1987), "Definition of boundary and initial conditions in the analysis of saturated ground-water flow sytems," review the following boundary conditions--streamline, free surface, seepage face; and section on water table as a boundary.

* Read Freeze and Cherry (1979), "Groundwater," pages 48, 61, 186-189, 324-32 and 375-377.

* Read Todd (1980), "Groundwater hydrology," pages 111-123 and 494-516.

* Read Jacob (1950), "Flow of ground water," pages 378-385.

* Read Bennett and Giusti (1971), "Coastal ground-water flow near Ponce, Puerto Rico," pages 206-211.

Although the list of study assignments shown here is longer than previous lists, the individual readings are short and in some measure repetitive.

The most common free-surface boundary is the water table. In general, free-surface problems are difficult to solve because they are governed by nonlinear differential equations. An additional complication in the solution of these problems is that the transmissivity of the aquifer is a function of the position of the free surface and must be redetermined for each iteration of the problem solution.

Our goals in this section are actually quite modest and include getting acquainted with the general problem of free surfaces, understanding the free surface as a boundary condition, familiarity with the Dupuit assumptions and the two or three most common solutions based on these assumptions, and an elementary beginning in the study of freshwater/saltwater relations.

In the classroom version of the concepts course, one of the laboratory exercises is a paper-cutting experiment utilizing the electrical analogy in which a freshwater/saltwater interface (free surface) is determined for the simulated isotropic and homogeneous aquifer. In this experiment, obtaining a solution (the position of this interface) involves "cutting away" part of the aquifer in a series of steps that represent successive approximations to the solution--each "cut" being closer to the solution than the previous cut. This concept of cutting away part of the aquifer to achieve a solution to a free surface problem is important. In digital modeling the exact counterpart to this cutting process in digital modeling might be termed "block chopping." In principle, the two processes are exactly the same. A note of warning is necessary here. Block chopping in a digital model is fairly easy to do in a cross section for a single aquifer. However, difficulties arise with this procedure in two-dimensional cross-section problems in which aquifers and confining layers alternate in vertical section. In three-dimensional problems, block chopping may prove to be exceedingly difficult.

The purpose of the final reference (Bennett and Giusti, 1971) is to acquaint you with an investigation in which interface determinations in paper models were used to gain insight into a ground-water problem in a coastal aquifer. Pay particular attention to the boundary conditions of the paper models and the exact procedure for cutting them to form the freshwater/saltwater interfaces.

Evapotranspiration in the Hydrologic System and its Simulation in Ground-Water Models

Study assignments

* Read Veihmeyer (1964), "Evapotranspiration," pages 11-1 to 11-33 in Chow, "Handbook of hydrology."

* Read Prickett and Lonnquist (1971), "Selected digital computer techniques for groundwater resource evaluation," pages 37-38.

The purpose of the first reading is to attain some perspective on evapotranspiration in the hydrologic system. Many of the details are not essential to the main thrust of the concepts course.

In some areas, particularily where the depth to the water table is a few feet or less, evapotranspiration can involve significant quantities of ground water. In such areas, salvaging ground-water outflow by pumping may be an important water-management alternative. Changes in evapotranspiration in an area due to land-use changes or major construction projects such as a new reservoir may significantly alter the local hydrologic regimen. The preceding examples are instances in which it may be necessary to include the effects of evapotranspiration in a model of the ground-water system.

Usually, as discussed in the second reference (Prickett and Lonnquist), areas of shallow water table in which ground-water evapotranspiration is significant are treated as a linear head-dependent flux boundary in ground-water models. The main purpose of this section is to make you aware of the need to consider evapotranspiration in models in some areas and this highly simplified treatment of it in models.

Physical Principles in Solute Transport

Study assignments

* Read Freeze and Cherry (1979), "Groundwater," pages 69-73, 75-76, 103-104, and 388-401.

* Read Ogata (1970), "Theory of dispersion in a granular medium," pages 1-6.

Movement of contaminants in ground-water systems is a topic of considerable societal interest at this time. Quantitative analysis of contaminant transport in ground-water systems is a major technical challenge. The primary purpose of this section is to introduce the physical processes involved in transport and, in addition, begin study of the simplest equations that are used to describe contaminant transport. New concepts involve new terminology. Perhaps the most valuable approach at this stage is to learn the transport vocabulary--for example, dispersion, diffusion, advection, coefficient of diffusion, coefficient of hydrodynamic dispersion, and Fick's first and second laws. Make a list and write a careful definition of all the transport terminology that you encounter in the readings.

Application of Dimensional Analysis to Ground-Water Studies

Study assignment

* Read Shames (1962), "Mechanics of fluids," pages 188-194.

Optional references

* Read Bridgman (1931), "Dimensional analysis," pages 1-55 (read first page of preface before starting text).

Most engineering handbooks have a section on dimensions, dimension systems, and dimensional analysis. Dimensional analysis is a valuable tool in planning a physical experiment or a numerical investigation and in classifying types of problems. More specifically, the advantages of dimensional analysis are:

1. Dimensional analysis provides the smallest possible number of independent problem parameters. Thus, the number of experiments or calculations necessary to investigate a given range of parameter values may be reduced to a minimum.

2. The use of dimensionless parameters permits the results of any type of investigation to be expressed in their most general form, independent of unit systems.

With reference to (2), note that dimensionless parameters are used to express all the type curves that we employ in the analysis of aquifer tests. To utilize the type curves, we must calculate values of the dimensionless parameters using consistent units for the individual variables contained in the dimensionless parameters.

Several forms of the basic ground-water equations contain parameter combinations for recharge (W), and transmissivity (T), and storage coefficient (S) in the form of W/T and S/T. These are not dimensionless parameter combinations. The following discussion is related in concept to (1) above. Assume we are calibrating a steady-state model for transmissivity. In addition, some uncertainty exists in the proper value of steady-state recharge that should be used in the model. It might seem logical to investigate numerically in the model a series of cases in which both W and T values are varied in some consistent way, perhaps by varying all W and T values by certain percentages of an initial base value. However, this procedure would be highly inefficient. The mathematical equation that describes this system is really a function of the ratio W/T, and not either W or T alone. In making this numerical investigation, one should investigate a series of ratios of W/T, making sure that this series of ratios includes the entire range of interest and possible range for values for the parameters W and T. Using this procedure, the same (or greater) return of information is obtained with fewer tests, which expresses the potential value, in general, of using dimensional analysis to obtain the minimum number of independent, dimensionless parameters that define a problem.

The application of dimensional analysis to a specific problem is not always simple. The purpose of this section is to make you aware of this powerful and useful tool. If you see that this approach is applicable, you can obtain assistance in applying dimensional analysis to your specific problem.

Workshop in Developing Models of Ground-Water Systems

In the classroom version of the concepts course, the participants are divided into groups of three to five people. Each group is given a descriptive hydrologic report of an area that does not include a model investigation and a hypothetical problem in that area, and is asked to prepare a presentation on how to solve the problem by developing a model of the ground-water system.

The focus of the presentations is (1) the conceptualization of the natural flow system and (2) a discussion on how to develop an appropriate model of the system that will solve the problem posed. A key element in the presentation is the careful identification of the boundary conditions in both the natural system and the model.

The format described above can be used in any training session in which several people are involved. However, an individual will also benefit from studying a hydrologic report for the purpose of developing a conceptual model for that system.

The following outline can be used in preparing workshop presentations:

1. Location of study area and geography.

2. Geologic framework--pertinent features but not lengthy stratigraphic details.

3. Natural hydrologic system--how the system operates; locations and quantity of recharge; areas of discharge; water levels for pertinent surfaces; careful designation of boundaries of the natural hydrologic system; and interesting details, data available, and methods to estimate distribution of transmissibility and storage parameters.

4. Effects of the man on hydrologic system--brief historical survey.

5. Definition of problem to be solved by the model.

6. Description of the model--areal extent; areal discretization scheme (mesh spacing, equal or unequal mesh); number of model layers; careful designation of model boundaries; compare with boundaries in (3) and justify any differences, definition of initial conditions, time discretization scheme if unsteady model; superposition versus absolute heads; listing of preliminary model runs and what one would hope to learn from these runs; calibration procedures; and subjective evaluation of the reliability of final model results to solve the problem that was posed.

REFERENCES

Most of the references in the following list are referred to specifically in connection with some topic in the course outline. The list consists primarily of standard textbooks in ground-water hydrology, U.S. Geological Survey publications, and references outside of hydrology that deal with specialized topics.

These references (and many more) should be readily available to every ground-water hydrologist, and many, particularly the standard textbooks, should be in every hydrologist's personal library. As our knowledge of ground-water hydrology and the number of technical "tools" available to study and apply to the solution of ground-water problems continue to expand, it becomes virtually mandatory that we continually consult all levels and types of technical literature.

As noted previously, by design, no specific textbook has been chosen as the "text" for this course. Rather, we wish to encourage the habit of consulting several sources. Many of the topics in this course are discussed in several of the references listed below. In the course outline, we have, rather arbitrarily, selected one or two references so that the reader is not overwhelmed by a lengthy reading list. However, it is a good principle to always consult several references, if possible, on topics of particular importance or interest.

The initial set of references for this course is:

Bear, Jacob, 1979, Hydraulics of groundwater: McGraw-Hill, 567 p.

Bennett, G. D., 1976, Introduction to ground-water hydraulics--a programed text for self-instruction: U.S. Geological Survey Techniques of Water-Resources Investigations, Book 3 (Applications of Hydraulics), Chapter B2, 172 p.

Bennett, G. D., and Giusti, E. V., 1971, Coastal ground-water flow near Ponce, Puerto Rico: U.S. Geological Survey Professional Paper 750-D, p. 206-211.

Bennett, G.D., Reilly, T.E., and Hill, M.C., 1990, Technical training notes in ground-water hydrology: Radial flow to a well: U.S. Geologcial Survey Water Resources Investigations Report 89-4134, 83 p.

Bridgman, P. W., 1931, Dimensional analysis: Yale University Press, 113 p., (Yale Paperbound Series, Y-82).

Domenico, P. A., 1972, Concepts and models in groundwater hydrology: McGraw-Hill, 405 p.

Fetter, C. W., 1988, Applied hydrogeology: Columbus, Ohio, Merrill Publishing Company, 592 p.

Franke, O. L., Reilly, T. E., and Bennett, G.D., 1987, Definition of boundary and initial conditions in the analysis of saturated ground-water flow systems--an introduction: U.S. Geological Survey Techniques of Water-Resources

Investigations, Book 3 (Applications of Hydraulics), Chapter B5, 15 p.

Franke, O. L., and Reilly, T. E., 1987, The effects of boundary conditions on the steady-state response of three hypothetical ground-water systems--results and implications of numerical experiments: U.S. Geological Survey Water-Supply Paper 2315, 19 p.

Freeze, R. A., and Cherry, J. A., 1979, Groundwater: Prentice-Hall, 604 p.

Harbaugh, A. W., and Getzen, R. T., 1977, Stream simulation in analog model of ground-water system of Long Island, New York: U.S. Geological Survey Water-Resources Investigations 77-58, 15 p.

Heath, R. C., and Trainer, F. W., 1968, Introduction to ground-water hydrology: John Wiley, 284 p.

Jacob, C. E., 1950, Flow of ground water, chapter 5, in Engineering Hydraulics: edited by H. Rouse, John Wiley, p. 321-386.

Karplus, W. J., 1958, Analog simulation: McGraw-Hill, 434 p.

Lohman, S. W., 1972, Ground-water hydraulics: U.S. Geological Survey Professional Paper 708, 70 p.

Lohman, S. W., and others, 1972, Definitions of selected ground-water terms--revisions and conceptual refinements: U.S. Geological Survey Water-Supply Paper 1988, 21 p.

Mercer, J. W., and Faust, C. R., 1982, Ground-water modeling: National Water Well Association, Columbus, Ohio, 60 p. (This reference material was published as a series of five articles in "Ground Water," in 1980: v. 18, no. 2, p. 108-115; v. 18, no. 3, p. 212-227; v. 18, no. 4, p. 395- 409; v. 18, no. 5, p. 486-497; and v. 18, no. 6, p. 569-577.

McClymonds, N. E., and Franke, O. L., 1972, Water-transmitting properties of aquifers on Long Island, New York: U.S. Geological Survey Professional Paper 627E, 24 p.

Milne-Thompson, L. M., 1955, Theoretical hydrodynamics: Macmillan, 632 p.

Ogata, Akio, 1970, Theory of dispersion in a granular medium: U.S. Geological Survey Professional Paper 411-I, 34 p.

Olmsted, F. H., Loeltz, O.J., and Ireland, Burge, 1973, Geohydrology of the Yuma area, Arizona and California: U.S. Geological Survey Professional Paper 486H, 227 p.

Prickett, T. A., and Lonnquist, C. G., 1971, Selected digital computer techniques for groundwater resource evaluation: Illinois State Water Survey, Bulletin 55, 62 p.

Prince, K. R., 1980, Preliminary investigation of a shallow ground-water flow system associated with Connetquot Brook, Long Island, New York: U.S. Geological Survey Water Resources Investigations 80-47, 37 p.

Reilly, T. E., Franke, O. L., and Bennett, G. D., 1987, The principle of superposition and its application in ground-water hydraulics: U.S. Geological Survey Techniques of Water-Resources Investigations, Book 3 (Applications of Hydraulics), Chapter B6, 28 p.

Remson, Irwin, Hornberger, G.M., and Molz, F.J., 1971, Numerical methods in sub-surface hydrology, Wiley-Interscience, 389 p.

Rushton, K. R., and Redshaw, S. C., 1979, Seepage and groundwater flow: John Wiley, 339 p.

Shames, I. H., 1962, Mechanics of fluids: McGraw-Hill, New York, p. 188-194.

Spiegel, M. R., 1961, Theory and problems of statistics: Schaum Outline Series, McGraw-Hill, 359 p.

Theis, C. V., 1940, The source of water derived from wells--essential factors controlling the response of an aquifer to development: Civil Engineering Magazine, p. 277-280, May 1940.

Todd, D. K., 1980, Groundwater hydrology, 2nd edition: John Wiley, 535 pages.

Trescott, P. C., Pinder, G. F., and Larson, S. P., 1976, Finite-difference model for aquifer simulation in two dimensions with results of numerical experiments: U.S. Geological Survey Techniques of Water-Resources Investigations, Book 7, Chapter C1, 116p.

Veihmeyer, F. J., 1964, Evapotranspiration: Chapter 11, in Handbook of Applied Hydrology, edited by V. T. Chow, McGraw-Hill, New York.

Wang, H. F., and Anderson, M. P., 1982, Introduction to groundwater modeling--finite difference and finite element modeling, W. H. Freeman, 237 p.

Walton, W. C., 1970, Groundwater resource evaluation, McGraw-Hill, 644 p.

CHAPTER 2--NOTES ON GROUND-WATER RELATED DEFINITIONS AND CONCEPTS

NOTE 1, SOME GENERAL CONCEPTS AND DEFINITIONS RELATED TO SYSTEMS AND MODELS IN GROUND-WATER STUDIES

Systems

The word system occurs frequently in ground-water literature in combinations such as hydrologic system and ground-water system. The following introduction to the system concept, much of which is based on a discussion by Domenico (1972, pages 1-38), and a distillation of those aspects of the concept relevant to a definition of a ground-water system, is important in establishing a general framework for ground-water resource evaluation.

A general dictionary definition of a system is "an orderly combination or arrangement of parts or elements into a whole, especially such combination according to some rational principle giving it unity and completeness." In thermodynamics, a system is a portion of the universe defined by a closed mathematical surface. The rest of the universe is referred to as the surroundings or the environment of the system. To be useful, this definition must be supplemented by additional information describing the physical properties of the enclosing surface (the walls or boundaries of the system) such as to whether these boundaries are impermeable, permeable, or selectively permeable to the flow of matter and/or energy across them. These additional considerations lead to the following definitions:

Open system--a system that constantly exchanges both matter and energy with its environment and is maintained by this exchange.

Relatively closed system--a system that constantly exchanges energy but not matter with its environment.

Absolutely closed (isolated) system--a system that exchanges neither energy nor matter with its environment.

In general, ground-water systems are open systems because they exchange both matter (water) and heat energy obtained from the sun or the interior of the earth with their surrounding environment. A simple, but very useful, schematic representation of a system with its accompanying input and output is shown in figure 2.1.1.

Various definitions of natural ground-water flow systems are possible, depending on one's objectives and point of view. In this discussion, the term flow system refers to the part of the ground-water regime that has been isolated for study, and implies the following:

1. a three-dimensional body of earth material saturated with flowing water;

2. the moving water is bounded by a closed surface--the <u>boundary surface</u> of the flow system;

```
                Flow of matter
                and/or energy
              and/or information
                 _____________
                |             |
  Input  -----> |   System    | -----> Output
(or stress)     |_____________|     (or response)
```

Figure 2.1.1--Schematic representation of a system.

3. under natural conditions, heads and flows associated with specific locations or parts of the system show some variation with time, normally oscillating around a mean condition;

4. for the flow system to operate continuously through time, water input (continuous or intermittent) to the system and water output (continuous) from the system must occur through at least part of the boundary surface.

Systems may be further classified as being either the distributed parameter or lumped parameter type. A distributed parameter system is a system in which spatial dimensions comprise an integral part of the formulation and solution of the problem (Karplus, 1958). A distributed parameter problem is the same as a mathematical field problem. In such problems, the space coordinates (x, y, and z) and time (t) are independent variables. Most modeling problems in ground-water hydrology are distributed parameter or field problems because we wish to determine ground-water heads, drawdowns, or other system responses (the dependent variables) as a function of location and time (the independent variables). In a lumped parameter system, time is the only independent variable. A water budget of a hydrologic system is a lumped representation of that system. The entire system is treated as a single entity. Problems involving a lumped system do not require a space coordinate system for their formulation or solution

Models

A general definition of model is a representation of some or all of the properties of a device, system, or object; the more properties represented by the model, the more complex the model becomes. Ground-water system models are always much simpler than the natural ground-water system, and, are therefore, only a partial representation of the natural system. When discussing various kinds of models, it is often convenient to refer to the natural system (the original on which a model is based) as the prototype.

The general purposes of models are (1) to predict the response of the prototype system to a specified stress and (2) to provide information and insight into how the prototype system functions. The word "model" with an appropriate modifier is used in a number of different ways in science and, more specifically, in ground-water hydrology.

A physical model usually employs the same physical system as the prototype but at a different scale. Laboratory-scale sand models of a well problem or of ground-water flow beneath an engineering structure are typical examples. In these examples, the mechanics of flow are exactly the same in both the model and prototype. To insure that the flow pattern in the model and prototype are the same, it is necessary only that they be geometrically similar. To perform flow calculations for the prototype on the basis of model results, a scaling factor for time is also required. Thus, the relation between model and prototype is simple and easily defined. In hydraulic modeling, an area that has been quite highly developed, the criteria for insuring similarity between model and prototype are much more complicated. Here, more extensive study of the theory of models and dimensional analysis is required.

A mathematical model represents the prototype by utilizing mathematical equations and procedures (algorithms). The models most commonly used today in ground-water hydrology can be called mathematical-numerical models. A differential equation that approximately ("adequately") describes a physical process is a mathematical model of that process. The various differential equations that describe ground-water flow, which will be discussed in a later section, are examples. The solution to these differential equations in a specific problem usually requires numerical procedures (algorithms).

Before the advent of powerful computers, which are used for developing mathematical-numerical models, it was often convenient to study a problem in one kind of physical system by means of a model utilizing a different physical system. Such models are called analog models. For many years, electric-analog models provided the only feasible means of simulating complex ground-water systems. Through proper connection of resistors, capacitors, other electrical components, and measurement of voltages and currents, the same differential equations are solved in electric analog models as are solved in numerical models.

A <u>Conceptual model</u> of the ground-water system should be developed in the early stages of a ground-water investigation, particularly in studies involving mathematical-numerical modeling. In this context, a conceptual model is a clear, qualitative mental picture of how the prototype system operates. This mental picture includes the external configuration of the system, location and amount of recharge and discharge, location and hydraulic characteristics of natural boundaries of the system (for example, an almost impermeable surface or fresh/salt-water interface), and the approximate pattern of ground-water flow through the system. Although a first approximation of the ground-water flow pattern may be inferred from the spatial dimensions of the system and the location and hydraulic characteristics of the system's natural boundaries, the spatial distribution and physical characteristics of aquifers and confining units may influence the flow pattern significantly. Field-measured head distributions in aquifers in the form of potentiometric-surface maps provide the best indication of the detailed flow pattern in ground-water systems.

NOTE 2, THE ELECTRIC ANALOG, SIMULATION, AND DISCRETIZATION

Introduction

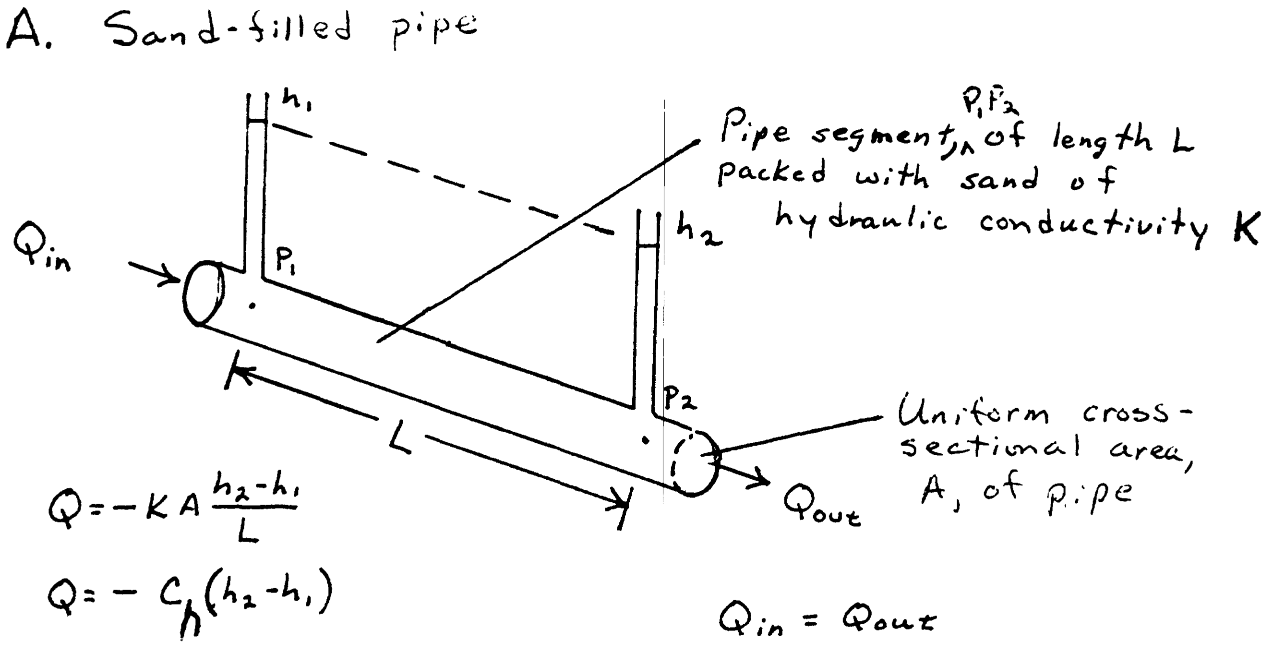
The following notes introduce some concepts that are useful in understanding the problems presented in this course. These concepts include (1) the analogy between flow of electric current in a conductor and flow of water through a porous medium; (2) approximate solution of a partial differential equation by simulation, using a set of algebraic finite-difference equations; (3) spatial discretization, the process of dividing a continuous medium into discrete blocks in order to carry out a finite-difference simulation; and (4) hydraulic conductance, a parameter which is defined to clarify the discretization process and to assist in implementing the electrical analogy.

Some of the problems you will be asked to solve depend on all of these concepts, whereas, others involve only one or two. The concepts are introduced here in a single discussion in order to illustrate their interdependence and to present a unified explanation.

Electrical Analogy and Hydraulic Conductance

The movement of water through a porous medium is governed by Darcy's law. For flow through the pipe segment packed with sand shown in figure 2.2.1A, Darcy's law may be written

$$Q = -KA\frac{h_2 - h_1}{L} \qquad (1)$$



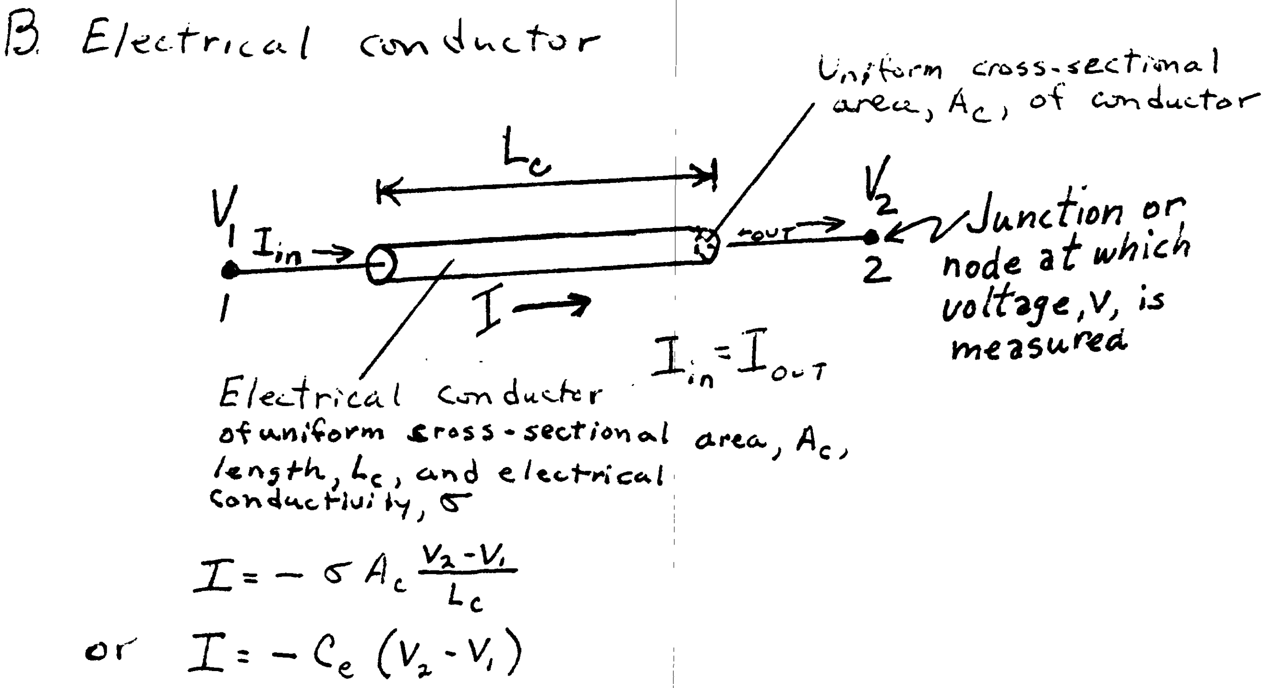


Figure 2.2.1.--Flow of water through a sand-filled pipe and flow of electrical current in a conductor.

where Q is the volumetric flow across a cross section of the pipe segment normal to its length, that is, the quantity of water passing the cross section in a unit time, A is the area of cross section, L is the length of pipe segment, K is the hydraulic conductivity of the sand, and h_1 and h_2 are the heads or the hydraulic potentials at the ends of the pipe segment at p_1 and p_2.[1] The flow of electric current through a conductor is governed by Ohm's law, which for the electrical circuit of figure 2.2.1B may be written

$$I = -\sigma A_c \frac{V_2 - V_1}{L_c} \qquad (2)$$

where I is the current or quantity of electrical charge (coulombs) passing a cross section normal to the length of the conductor in a unit time, A_c is the cross sectional area of the conductor, L_c is the length of the conductor, σ is the electrical conductivity of the material of which the conductor is made, and V_1 and V_2 are the voltages, or electrical potentials, at the ends of the conductor. The terms $(h_2 - h_1)/L$ and $(V_2 - V_1)/L_c$ are actually expressions for the gradients of hydraulic head and electrical potential, respectively, in these two examples.

Equations (1) and (2) illustrate the analogy between current flow in a conductor and flow of a fluid through a porous medium. The electrical current, I, which is measured in coulombs per

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[1] The difference in head is arbitrarily defined as $h_2 - h_1$, rather than $h_1 - h_2$. When $h_1 > h_2$, Q is positive and flow is from p_1 to p_2. When $h_1 < h_2$, Q is negative and flow is from p_2 to p_1.

second, or amperes, is analogous to the volumetric flow of water, Q; electrical potential, or voltage, V, is analogous to hydraulic head; electrical conductivity, σ, which is measured in coulombs per second per volt per meter, or mhos per meter, is analogous to hydraulic conductivity, K.

In electrical circuit analyses, it is common to use the electrical conductance, $\sigma A_c/L_c$, thus bringing the conductivity and the dimensions of the conductor into a single constant. Using the symbol C_e for electrical conductance, equation (2) can be written

$$I = -C_e(V_2 - V_1) \tag{3}$$

C_e has the units of amperes per volt, or mhos.[2] We can define a similar parameter, hydraulic conductance, C_h, (not to be confused with hydraulic conductivity, K) for the hydraulic element of figure 2.2.1A, as

$$C_h = \frac{KA}{L}, \tag{4}$$

which will permit us to write Darcy's law for flow through the element as

$$Q = -C_h(h_2 - h_1). \tag{5}$$

Hydraulic conductance is expressed in units such as gallons per day per foot or feet-squared per day.

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[2] Electrical circuits are usually described in terms of resistance, R, rather than conductance. Electrical resistance is simply the inverse of conductance, i.e., $R = 1/C_e$ where resistance is measured in volts per ampere, or ohms.

In using the electrical analogy, it is convenient to define "scale factors" relating the analogous quantities in the two systems. For example, if we wished to let 1 milliampere (1/1000 ampere) represent a flow of 1 million gallons per day, we would use a current scale factor, k_I, of 10^9 gallons per day per ampere; if we intended to let a voltage difference of 1 volt represent a head difference of 100 feet, we would use a voltage scale factor, k_V, of 100 feet per volt; and if we wished to let an electrical conductance of 10^{-3} mhos represent a hydraulic conductance of 10^4 gallons per day per foot, we would need a conductance scale factor, k_C, of 10^7 gallons per day per foot per mho.

Finite-Difference Simulation, Discretization, and Analog Models

Simulation and Discretization

Steady-state horizontal flow in an aquifer is governed by a differential equation,

$$\frac{\partial}{\partial x}\left(K_x\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y\frac{\partial h}{\partial y}\right) = 0 \qquad (6)$$

where h is the hydraulic head and x and y are the coordinates at which h is defined. The head distribution in the aquifer is given by a solution to this differential equation--that is, by an algebraic expression giving head as a function of x and y, such that when the derivatives of the function are substituted into equation (6), the equation is satisfied. An infinite number of functions will do this, and the particular solution for a given problem is one that will also satisfy the conditions prescribed on the boundaries of the problem. Because it is usually difficult to determine the functions that will satisfy a differential equation, approximate methods of solution are frequently applied. In these methods, the continuous ground-water system is conceptually divided into discrete segments--that is, it is discretized--and algebraic equations are formulated for each segment. These algebraic equations are then solved as a simultaneous system, and the head values determined in this procedure are assumed to approximate those that would be given by the function satisfying the differential equation. The system of algebraic equations actually is used to simulate the differential equation, and the entire process is referred to as simulation. The particular kind of simulation we will consider here is finite-difference simulation; it is described in the following paragraphs.

Many methods of solving the system of algebraic equations associated with the simulation of a differential equation could be employed; in these notes we will discuss two--the use of a hydraulic model and the use of an electric analog model, based on the theory given in the preceding section.

Finite-Difference Simulation

Five points, or nodes, are indicated on figure 2.2.2A: a central point; designated 0; and four surrounding points--two along the x axis, designated 1 and 3, and spaced a distance Δx from the central point, and two along the y axis, designated 2 and

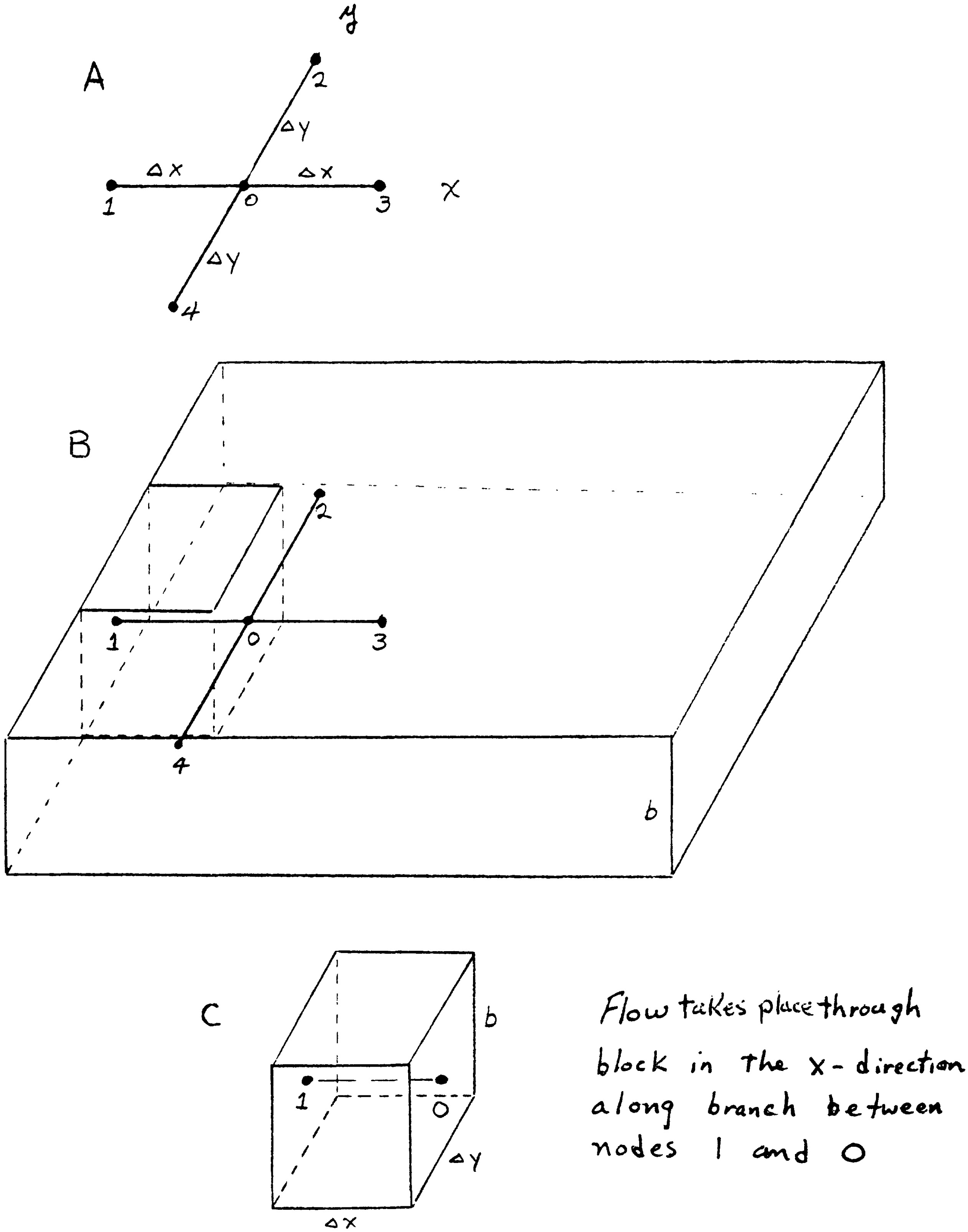


Figure 2.2.2.--Discretization of a continuous aquifer of thickness b into square blocks for simulation of two-dimensional flow.

4, and spaced a distance Δy from the central point. Figure 2.2.2B also shows the outline of a block of aquifer material lying along the x axis between point 1 and point 0, and figure 2.2.2C shows this block in detail. Flow takes place through the block from nodes 1 to 0. The block extends laterally outward halfway to points 2 and 4 along the y axis, and thus has a width Δy. The flow in the x direction through this block, from the face at 1 to the face at 0, is given approximately by Darcy's law as

$$Q_{1-0} = -K_{x_{1-0}} b\Delta y \frac{h_0 - h_1}{\Delta x} \tag{7}$$

where $K_{x_{1-0}}$ is the hydraulic conductivity of the material in the x direction between 1 and 0, b is the aquifer thickness, and h_0 and h_1 are the heads at points 0 and 1 respectively. The term $K_{x_{1-0}} b\Delta y/\Delta x$ is the hydraulic conductance of the block in the x direction, which we designate $C_{x_{1-0}}$. Equation 7 can thus be written

$$Q_{1-0} = -C_{x_{1-0}}(h_0 - h_1) . \tag{8}$$

The flow away from the central point through a similarly designated block between 0 and 3 is

$$Q_{0-3} = -K_{x_{0-3}} b\Delta y \frac{h_3 - h_0}{\Delta x} = -C_{x_{0-3}}(h_3 - h_0) . \tag{9}$$

Flow from 2 to 0 through a block of width Δx oriented along the y axis would be

$$Q_{2-0} = -K_{y_{2-0}} b\Delta x \frac{h_0 - h_2}{\Delta y} = -C_{y_{2-0}}(h_0 - h_2) \tag{10}$$

while flow from 0 to 4 through a similarly designated block would be

$$Q_{0-4} = -K_{y_{0-4}} b\Delta x \frac{h_4 - h_0}{\Delta y} = -C_{y_{0-4}}(h_4 - h_0)\,. \tag{11}$$

Equations (8) through (11) give approximate expressions for flow toward and away from the vicinity of the central point, 0. Steady-state conditions are assumed in this problem, and the equation of continuity therefore states that if no fluid sources are present, inflow to the region of point 0 must equal outflow from that region--that is, inflow minus outflow must be zero. Thus we may write

$$Q_1 + Q_2 - Q_3 - Q_4 = 0 \tag{12}$$

or

$$-K_{x_{1-0}} b\Delta y \frac{h_0 - h_1}{\Delta x} - K_{y_{2-0}} b\Delta x \frac{h_0 - h_2}{\Delta y} + K_{x_{0-3}} b\Delta y \frac{h_3 - h_0}{\Delta x} + K_{y_{0-4}} b\Delta x \frac{h_4 - h_0}{\Delta y} = 0 \tag{13}$$

and in terms of hydraulic conductances

$$-C_{x_{1-0}}(h_0 - h_1) - C_{y_{2-0}}(h_0 - h_2) + C_{x_{0-3}}(h_3 - h_0) + C_{y_{0-4}}(h_4 - h_0) = 0. \tag{14}$$

Equations (13) and (14) are algebraic-difference equations in contrast to the differential equation (6). That is, equations (13) and (14) involve head differences rather than derivatives, and their solution consists of a set of specific numerical values for h_0, h_1, h_2, etc., rather than an algebraic expression for head as a function of x and y. Equations (13) and (14) are, in fact, finite-difference approximations to the differential equation (6) in the sense that we can expect numerical values determined for the heads h_0, h_1, h_2, etc., to approximate values that would be given by the solution to (6) at the corresponding points. Obviously, for a full representation of the system, we would have to designate nodes throughout the aquifer and write an equation of the form of (13) or (14) for each node. The result would be a set of algebraic equations which would be solved simultaneously for the heads at the various nodes. To clarify this process, we utilize the concept of a hydraulic network model.

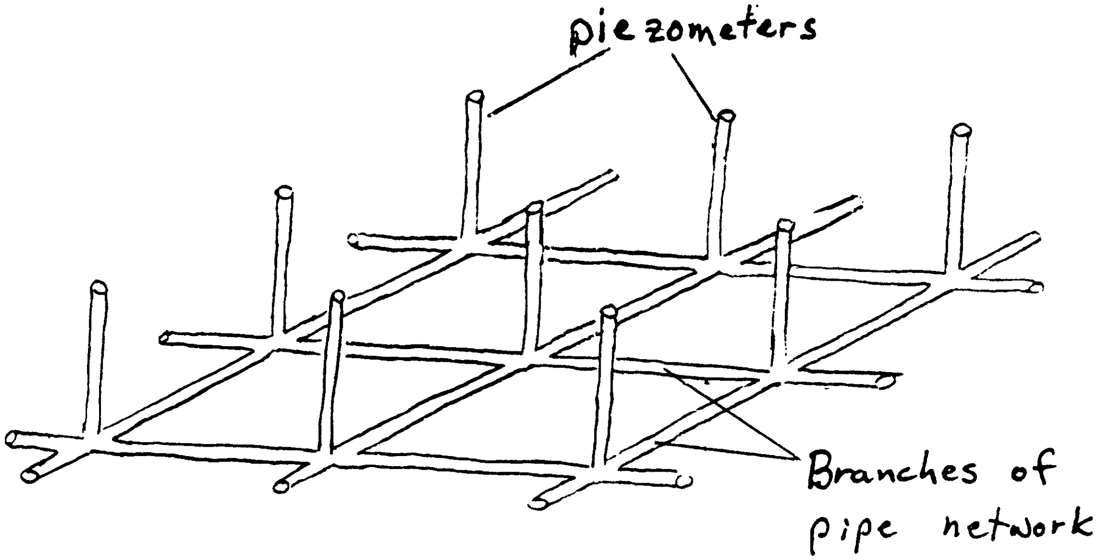
Hydraulic Network Modeling

Figure 2.2.3A shows a network made up of sand-packed pipes of the type shown in figure 2.2.1A; piezometers have been placed at each junction, or node, of the network. Flow through an individual pipe of the network is given by

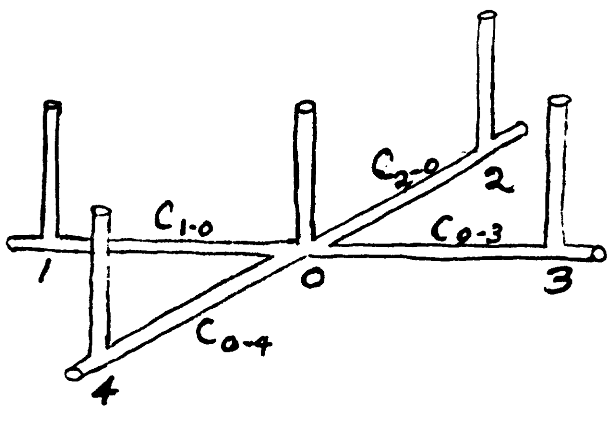
$$Q = -C_p(h_2 - h_1) \tag{15}$$

where C_p is the hydraulic conductance of the sand-filled pipe, and h_1 and h_2 are the heads at the ends of the pipe.

Our aim is to use the pipe network of 2.2.3A as a model of the continuous aquifer of 2.2.2B. Intuitively, we would expect heads at the various nodes to approximate those at corresponding



A. Part of continuous pipe network with piezometers



B. Representative segment of pipe network consisting of four pipes and five junctions. C is hydraulic conductance

Figure 2.2.3--Sand-filled pipe network representing two-dimensional horizontal flow in a continuous aquifer.

points in the aquifer if (1) the conductances of the various pipe segments in the mesh are made equal to the conductances of corresponding blocks of porous material in the aquifer, (2) the mesh is made sufficiently fine, and (3) boundary conditions in the aquifer are reproduced on the mesh. In fact, as we shall see, heads in the mesh are governed by a set of finite-difference equations, analogous to equations (13) or (14) above; and our pipe network model actually simulates the differential equation (6) using this set of finite-difference equations.

Figure 2.2.3B shows a segment of the pipe network consisting of four pipes and five junctions. To obtain an equation for the head at the central junction or node, designated 0, we proceed as in the development of equations (13) and (14). That is, we first write expressions for the flows between the central node and the four surrounding nodes. The flow toward the central node in the x direction--that is, from node 1 to node 0--is given by

$$Q_{1-0} = -C_{1-0}(h_0 - h_1) \tag{16}$$

where C_{1-0} is the hydraulic conductance of the pipe between 1 and 0, and h_0 and h_1 are the heads at the respective junctions. The flow away from the center in the x direction, toward node 3, is similarly given by

$$Q_{0-3} = -C_{0-3}(h_3 - h_0) . \tag{17}$$

Similarly, for flow toward the center in the y direction, we have

$$Q_{2-0} = -C_{2-0}(h_0 - h_2) \tag{18}$$

while for flow away from the central junction in the y direction, we have

$$Q_{0-4} = -C_{0-4}(h_4 - h_0). \tag{19}$$

Again, we are dealing with a steady-state system--as the elements in the pipe network are purely transmissive--they have no appreciable capacity to store water. Thus inflows to the central junction must equal outflows from it--i.e., inflow minus outflow must be zero. We therefore have

$$Q_{1-0} + Q_{2-0} - Q_{0-3} - Q_{0-4} = 0$$

or

$$-C_{1-0}(h_0 - h_1) - C_{2-0}(h_0 - h_2) + C_{0-3}(h_3 - h_0) + C_{0-4}(h_4 - h_0) = 0 \tag{20}$$

If the conductances of the pipes C_{1-0}, C_{2-0}, C_{0-3}, and C_{0-4} are made equal to the conductances $C_{x_{1-0}}$, $C_{y_{2-0}}$, $C_{x_{0-3}}$, and $C_{y_{0-4}}$ of equation (14), equations (14) and (20) are identical.

Now suppose the pipe network of figure 2.2.3 is extended so as to represent the full aquifer, with conductance in each pipe segment equal to the conductance of the corresponding aquifer block and with each junction of the network corresponding to a node in the aquifer. Figure 2.2.4 shows a portion of the pipe network and the corresponding portion of the aquifer, and

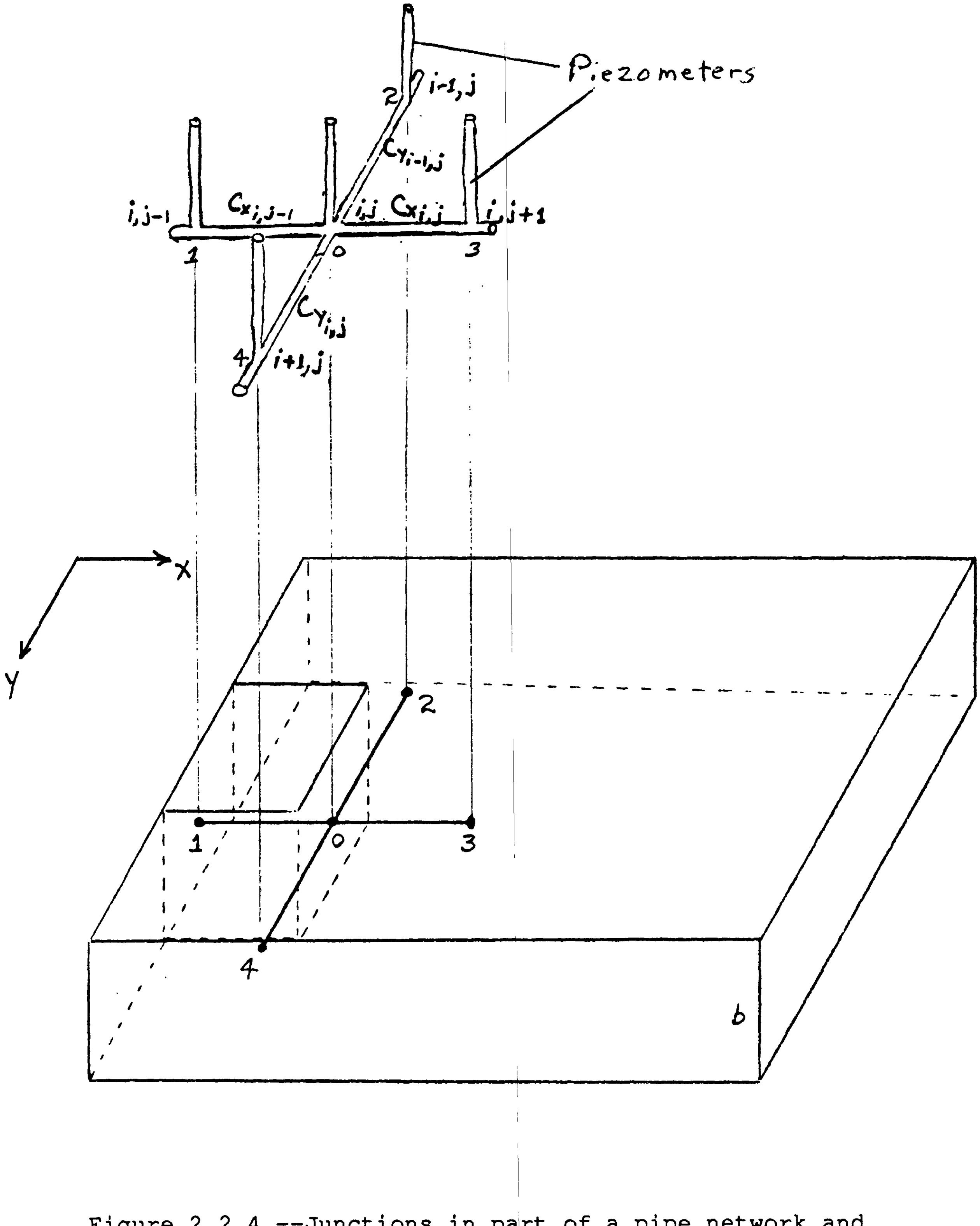


Figure 2.2.4.--Junctions in part of a pipe network and nodes in the corresponding part of a discretized aquifer.

illustrates the correspondence between aquifer nodes and pipe network junctions. If we use an i,j subscripting system to designate the junctions of the pipe network or the nodes of the aquifer, either equation (14) or equation (20) can be rewritten for a general junction or node, i,j as

$$-C_{x_{i,j-1}}(h_{i,j}-h_{i,j-1}) \; -C_{y_{i-1,j}}(h_{i,j}-h_{i-1,j})$$

$$+C_{x_{i,j}}(h_{i,j+1}-h_{i,j}) \; +C_{y_{i,j}}(h_{i+1,j}-h_{i,j}) = 0, \qquad (21)$$

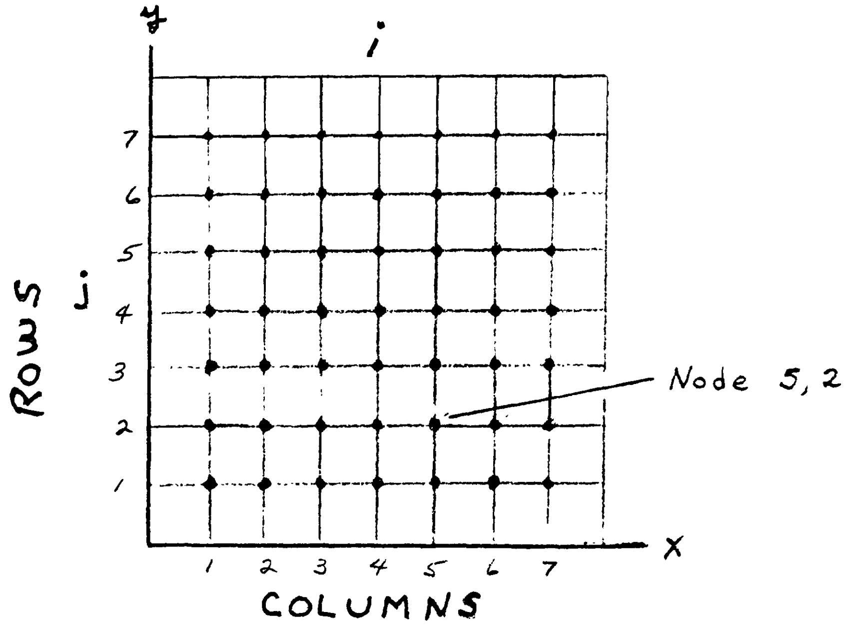
where $C_{x_{i,j-1}}$ is the hydraulic conductance of the aquifer block or pipe segment extending in the x direction between i,j-1 and i,j; $C_{x_{i,j}}$ is that for the interval between i,j and i,j+1; and so on.

Note that the subscripting convention we are using here is the reverse of that used in Part VII of Bennett's (1976) Programed Text (fig. 2.2.5). That is, we are using i as the row number and j as the column number where rows extend parallel to the x axis and columns extend parallel to the y axis. Thus, if we move parallel to the x direction (along a row) so that y remains constant and x increases, i will remain constant while j will increase by 1 at each node. If we move parallel to the y direction (along a column) so that x remains constant and y increases, j will remain the same while i will increase by 1 at each node. As we move outward parallel to the x axis, successive values of Cx are designated $C_{x_{i,j-1}}$; $C_{x_{i,j}}$; $C_{x_{i,j+1}}$; etc. As we move outward parallel to the y axis, subscripts will follow the pattern i-1,j; i,j; i+1,j; etc. (fig. 2.2.4). Note that the origin for numbering the nodes is different also. In this course, we have numbering the nodes is different also. In this course, we have used the grid and subscripting convention that is shown in figure 2.2.5B because it is the one that is used in ground-water flow models currently in use by the U.S. Geological Survey.

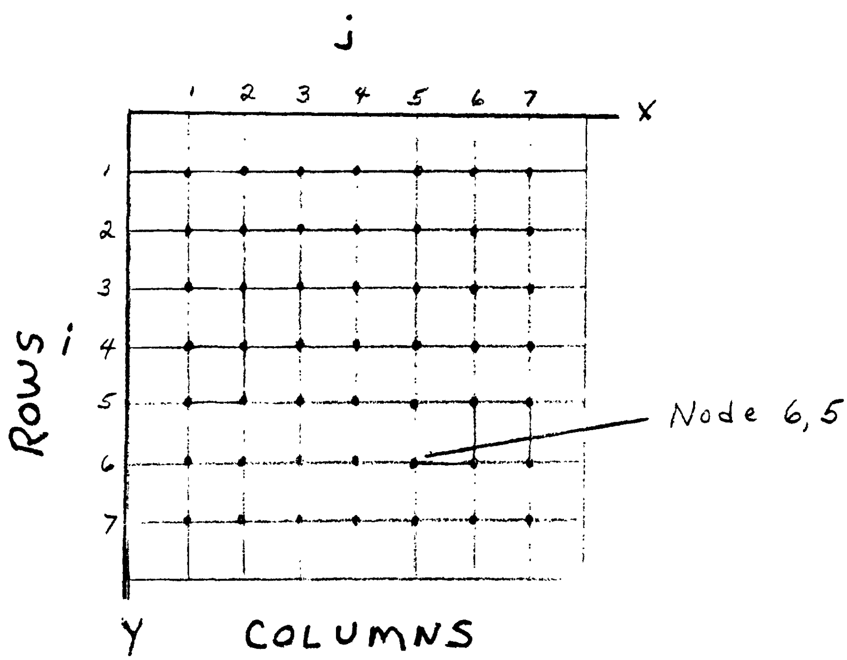
For the pipe network, if the number of junctions in the mesh is n, we have a system of n simultaneous algebraic equations of the form of (21); and, of course, we have n unknowns--the head values at the n junctions. Suppose we had hydraulic equipment with which we could duplicate boundary flows, pumpages, and other features of the aquifer system. By imposing hydraulic boundary conditions on the pipe network, and then measuring the head in each piezometer, we could obtain an experimental solution to our set of finite-difference equations. This, in turn, would serve as an approximation to the solution of our partial-differential equation of flow in the aquifer, equation (6).

Electric Analog Modeling

The disadvantage of a hydraulic model constructed of pipes, sand and water is that it would be cumbersome, time-consuming, and difficult to operate. However, an alternative experimental approach can be devised using the electrical analogy. We have seen that the flow of water through the sand-packed pipe of figure 2.2.1A, as described by equations (1) or (5), is analogous to the flow of electric charge through the conductor of figure 2.2.1B as described by equations (2) or (3). Thus in place of the network



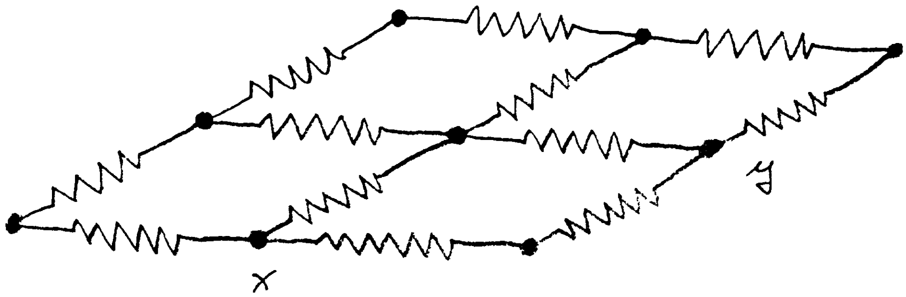
A. Grid and subscripting convention used by Bennett (1976).



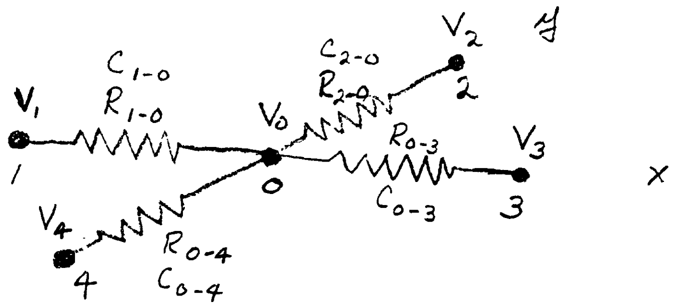
B. Grid and subscripting convention used in this course and in current USGS models

Figure 2.2.5.--Two examples of grids and subscripting conventions.

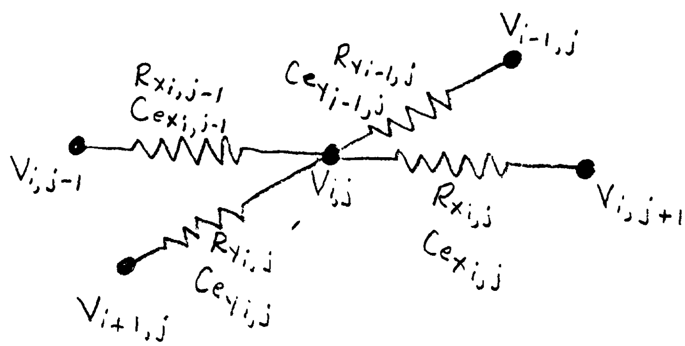
of sandpacked pipes of figure 2.2.3A, we could construct a network of electrical conductors, or resistors, as shown in figure 2.2.6A. We would want to choose the resistors so that the electrical conductance of each was proportional to the hydraulic conductance of the corresponding sand-packed pipe in the hydraulic model. Then we would need electrical equipment through which we could control voltage along specified model boundaries, input or withdraw current in specified amounts, and measure both current and voltage. According to the characteristics of this equipment and of the hydrologic problem to be solved, we would decide on the required scale factors--that is, the feet of head to be represented by each volt and the gallons per minute of flow to be represented by each ampere of current. We could then arrange our equipment to maintain specified voltages at those junctions of the network corresponding to specified-head boundaries in the aquifer, and to withdraw or add current at junctions where withdrawal or inflow of water occurs in the aquifer. We could then measure voltages at each junction of the network; and using our voltage-head scale factor, convert these measured voltages to head values to obtain a solution to the set of equations (21). The solution is again an experimental one; however, because electrical equipment is less cumbersome than hydraulic equipment, and electrical measurement far more convenient than hydraulic measurement, the solution is obtained much more easily than it could have been with a physical hydraulic model.



A. Resistor (conductor) network corresponding to pipe network in fig. 2.2.3A



B. Four resistors (conductors) and five junctions of network in A



C. Four resistors (conductors) and five junctions in i, j notation

Figure 2.2.6.--Resistor (conductor) network corresponding to pipe network.

To clarify this procedure still further, we will develop the electrical counterpart to equation (21). Equations (20) and (21) were developed by considering the five-junction pipe array of figure 2.2.3B; we will consider the corresponding five-junction segment of the electrical network as shown in figures 2.2.6B and 2.2.6C.

The flow of current in the x direction, from junction 1 to junction 0 is given by

$$I_{1-0} = -C_{e_{1-0}}(V_0 - V_1) = \frac{-1}{R_{1-0}}(V_0 - V_1) \tag{22}$$

where $C_{e_{1-0}}$ is the electrical conductance and R_{1-0} is the electrical resistance of the resistor between junction 1 and junction 0, V_0 is the voltage at the central junction, and V_1 is the voltage at junction 1 (fig. 2.2.6B). The flow of current from junction 0 to junction 3 is similarly given by

$$I_{0-3} = -C_{e_{0-3}}(V_3 - V_0) = \frac{-1}{R_{0-3}}(V_3 - V_0)\,. \tag{23}$$

In the y direction, the flow toward the center from junction 2 is given by

$$I_{2-0} = -C_{e_{2-0}}(V_0 - V_2) = \frac{-1}{R_{2-0}}(V_0 - V_2)\,. \tag{24}$$

Similarly, the current from 0 to 4 in the y direction is

$$I_{0-4} = -C_{e_{0-4}}(V_4 - V_0) = \frac{-1}{R_{0-4}}(V_4 - V_0)\,. \tag{25}$$

There is no mechanism by which electrical charge can be accumulated at the central junction; thus the quantity of charge entering the junction in a given time must equal that leaving the junction--that is, the sum of the currents entering the junction must equal the sum of the currents leaving the junction. This relation is known as Kirchoff's junction rule; expressed as an equation, for the central junction of figure 2.2.6B and 2.2.6C, it takes the form

$$\text{Inflow} - \text{outflow} = 0$$

$$I_{1-0} + I_{2-0} - I_{0-3} - I_{0-4} = 0. \quad (26)$$

Substituting the expressions for the individual currents from equations (22), (23), (24), and (25) into equation (26) gives

$$-C_{e_{1-0}}(V_0 - V_1) - C_{e_{2-0}}(V_0 - V_2) + C_{e_{0-3}}(V_3 - V_0) + C_{e_{0-4}}(V_4 - V_0) = 0. \quad (27)$$

Using the i,j convention of equation (21), (fig. 2.2.6C), equation (27) becomes

$$-C_{e_{x_{i,j-1}}}(V_{i,j} - V_{i,j-1}) - C_{e_{y_{i-1,j}}}(V_{i,j} - V_{i-1,j}) + C_{e_{x_{i,j}}}(V_{i,j+1} - V_{i,j})$$

$$+C_{e_{y_{i,j}}}(V_{i+1,j} - V_{i,j}) = 0, \quad (28)$$

where $C_{e_{x_{i,j-1}}}$ is the electrical conductance of the resistor between junction i,j-1 and junction i,j; $C_{e_{x_{i,j}}}$ is the conductance of the resistor between i,j and i,j+1; and so on.

In terms of resistances (fig. 2.2.6C), equation (28) becomes

$$- \frac{1}{R_{x_{i,j-1}}}(V_{i,j} - V_{i,j-1}) - \frac{1}{R_{y_{i-1,j}}}(V_{i,j} - V_{i-1,j}) + \frac{1}{R_{x_{i,j}}}(V_{i,j+1} - V_{i,j})$$

$$+ \frac{1}{R_{y_{i,j}}}(V_{i+1,j} - V_{i,j}) = 0. \qquad (29)$$

Equations (28) and (29) are analogous to equation (21), with the terms C_e or 1/R taking the place of the hydraulic conductance term, C, and voltage taking the place of the hydraulic head, h. As with the hydraulic pipe system, we can write an equation of the form of (28) for every junction in the electrical network; if there are n junctions, there are exactly n unknown values of voltage, and we are actually dealing with a system of n simultaneous algebraic equations of the form of (28). Rather than attempting to solve this system algebraically, we seek an experimental solution by actually building the electrical network, imposing current sources or sinks and voltage as required to represent boundary conditions for the problem, and measuring the voltage at the individual junctions. The solution to the set of electrical equations can then be converted to a solution to the corresponding set of hydraulic equations of the form of (21) by applying the scale factors; and this solution to the set of hydraulic equations can be taken as an approximation to the solution of the hydraulic differential equation, (6).

Additional Notes on Discretization

In following this procedure of experimental solution, it is important to visualize clearly the segments or blocks of the aquifer that are represented by each resistor of the network. Figure 2.2.7 shows a segment of the aquifer and its array of nodes and discretized blocks for flow in the x and y directions. The figure also shows the corresponding segment of the pipe network and the corresponding segment of the electrical network. To this point in the discussion of discretization, we have used grids with uniform spacing, that is, the spacing in the x- and y-directions (Δx and Δy) are equal and uniform throughout the grid. Circumstances will arise in modeling ground-water flow in which more detail is required in part of the area being simulated, such as near a well field. In these situations the spacing between nodes may be much closer than elsewhere in the modeled area. In this section we will determine how to calculate conductances in a grid that has non-uniform spacing between nodes. A smaller segment of the aquifer and of each network is shown in figure 2.2.8. The upper 2 sketches in figure 2.2.8 show only two conducting elements of each network, one extending in the x direction from junction i,j to junction i,j+1 and one extending in the y direction from junction i,j to junction i+1,j. The block on figure 2.2.8 corresponds to block 1 on figure 2.2.7 and shows the segment of aquifer represented by the pipe $C_{x_{i,j}}$ and the resistor $R_{x_{i,j}}$. This segment extends along the x-axis from node i,j to node i,j+1, and extends transverse to the x axis a distance halfway to the adjacent row of nodes in the y direction. We let Δy_{i-1} represent the distance along the y axis between the row of nodes at i-1 and the row at i; and let Δy_i represent the distance along the y axis between the row of nodes at i and that at i+1; and we

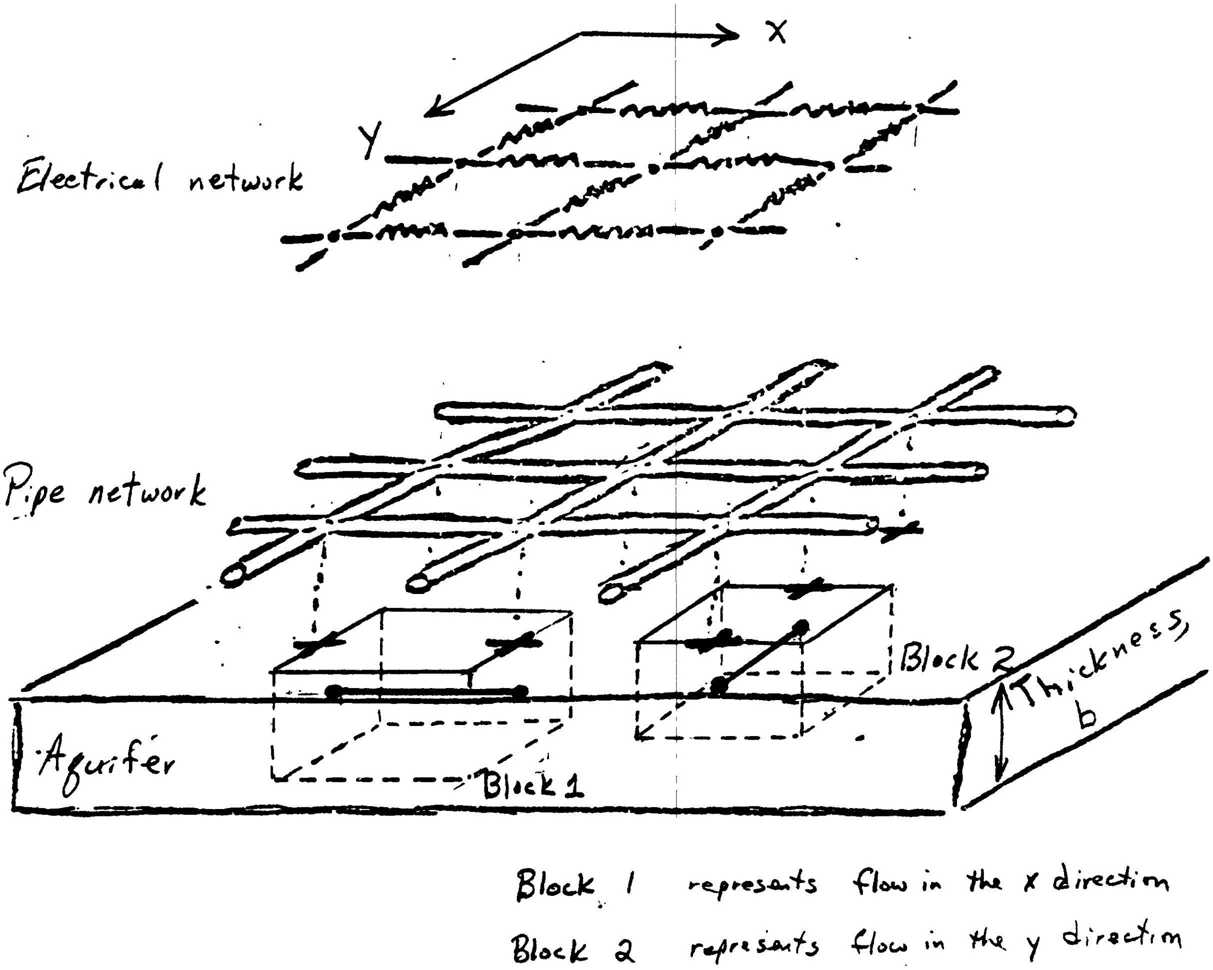
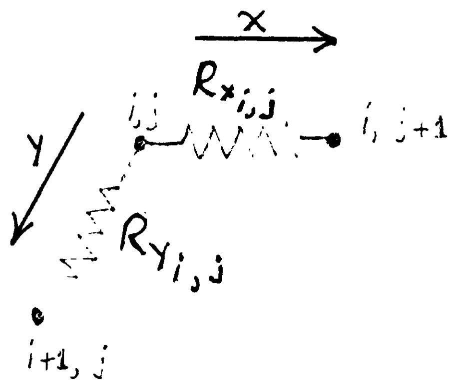
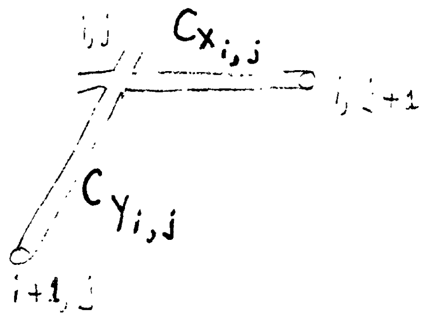


Figure 2.2.7.--A nodal array and discretized blocks of an aquifer and the corresponding junctions and branches in a pipe network and nodes and resistors in an electrical network.





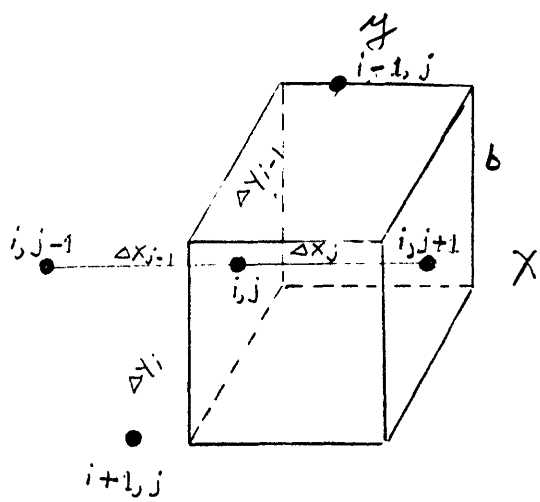


Figure 2.2.8.--Small segment of the discretized aquifer (block 1 fig. 2.2.7), the corresponding segments of the pipe network, and electrical network.

let b represent aquifer thickness (fig. 2.2.8). The block extends a distance $\Delta y_{i-1}/2$ toward the i-1 row and a distance $\Delta y_i/2$ toward the i+1 row. Thus the cross-sectional area of flow, for movement parallel to the x axis through the block is $b(\Delta y_{i-1}/2 + \Delta y_i/2)$. If the same spacing, Δy, is used between all rows of the mesh, the term $(\Delta y_{i-1}/2 + \Delta y_i/2)$ is equal to Δy, and the cross sectional area of flow is simply $b\Delta y$. Similarly, if we let Δx_j represent the distance along the x axis between node i,j and node i,j+1, the distance of flow through the block parallel to the x axis is Δx_j. If a uniform spacing, Δx, is used between all columns of nodes in the mesh, the subscripts are no longer necessary, and the distance of flow is simply Δx.

Finally, we let $K_{x_{i,j}}$ represent hydraulic conductivity in the x direction in the block, that is, in the interval between i,j and i,j+1 (note that we are defining $K_{x_{i,j}}$ in terms of this internode region, and not for a block of aquifer surrounding node i,j). The hydraulic conductance of the block, for this general case, is given by

$$C_{x_{i,j}} = \frac{K_{x_{i,j}}\, b\left(\frac{\Delta y_{i-1}}{2} + \frac{\Delta y_i}{2}\right)}{\Delta x_j}, \tag{30}$$

where for conductance as well as conductivity, the subscripts i,j are used here to refer to the interval between node i,j and node i,j+1. For the case in which hydraulic conductivity is the same throughout the aquifer and the row spacing (Δy) and column spacing

(Δx) are uniform, subscripts are not needed and the hydraulic conductance of the block is simply

$$C_x = \frac{K_x b \Delta y}{\Delta x}. \tag{31}$$

In using the electrical network to simulate flow in the aquifer, we require that the electrical conductance of the resistor $R_{x_{i,j}}$ (fig. 2.2.8) be proportional to the hydraulic conductance. $C_{x_{i,j}}$ (equation 30). Following the same procedures as for the flow in the x direction, we can show that the hydraulic conductance, for this general case, for flow in the y direction between i,j and i+1,j (corresponding to block 2 on figure 2.2.7) is given by

$$C_{y_{i,j}} = \frac{K_{y_{i,j}}\, b\left(\frac{\Delta x_{j-1}}{2} + \frac{\Delta x_j}{2}\right)}{\Delta y_i} \tag{32}$$

where $K_{y_{i,j}}$ is the hydraulic conductivity for flow in the y direction between i,j and i+1,j; Δx_{j-1} is the spacing between the column of nodes at j-1 and those at j; Δx_j is the spacing between the column of nodes at j and those at j+1; and b is the thickness of the aquifer. If K_y has the same value throughout the aquifer and if a uniform column spacing Δx and a uniform row spacing Δy are used in the mesh, the general expression for hydraulic conductance (equation 32) simplifies to

$$C_y = \frac{K_y b \Delta x}{\Delta y} \tag{33}$$

Again, the electrical conductance of the resistor $R_{y_{i,j}}$ must be proportional to the hydraulic conductance, $C_{y_{i,j}}$, in order to use the electrical network to simulate flow in the aquifer.

Figure 2.2.9A is a top view of a section of the aquifer showing the nodal array, and illustrating the i,j system of designating rows and columns; figure 2.2.9B shows the corresponding segment of the electrical network.

In using an electrical analog, we choose our nodes to correspond to junctions of the network, and define hydraulic conductance, represented by the individual resistors, for the internode regions; this leads to nodes located on the boundaries of the mesh as shown in figure 2.2.9A. Four aquifer blocks are indicated on figure 2.2.9B; two of these are in the interior of the array, and two are located along boundaries of the array. Block A is represented by the resistor between nodes 2,1 and 2,2. The hydraulic conductance of block A is calculated in the same manner as was that for block 1 in figure 2.2.7. Block B is represented by the resistor between nodes 1,2 and 2,2; its hydraulic conductance is calculated as was that for block 2 of figure 2.2.7. Block C is represented by the resistor between nodes 3,1 and 3,2 and extends along a boundary of the array; the block represents conductance in the x-direction parallel to the boundary. Unlike block A, block C does not extend an equal distance to either side of its row of nodes. The nodes coincide with the boundary itself. Thus, for uniform spacings Δx along the x axis and Δy along the y axis, the distance of flow remains Δx

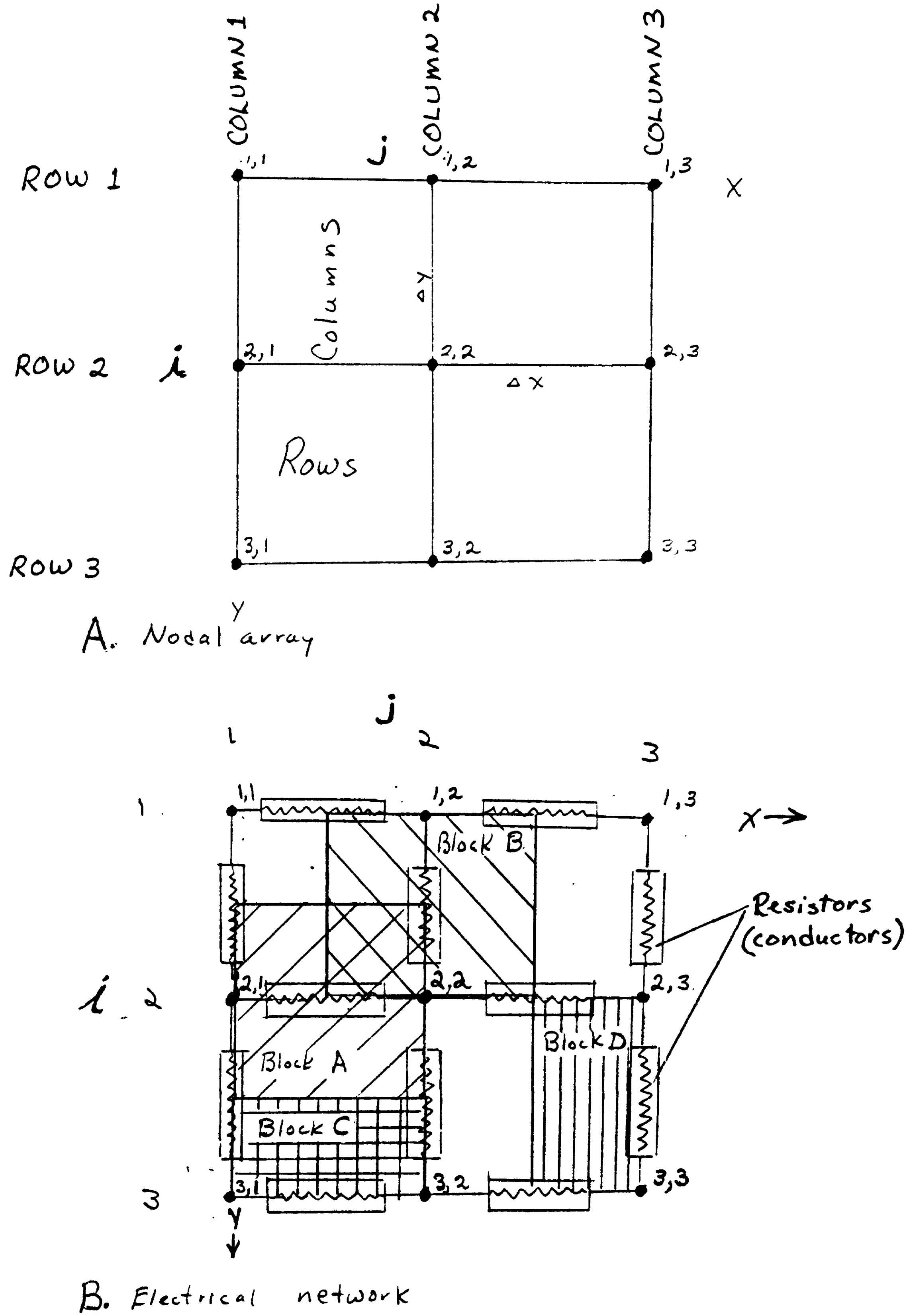


Figure 2.2.9.--A part of a nodal array for a discretized aquifer and the corresponding part of the electrical network.

but the cross-sectional area of flow is reduced to $b\Delta y/2$ so that conductance is reduced to half that of block A. Similarly, block D extends along a boundary in the y direction; the length of flow remains Δy, but the cross-sectional area of flow is $b\Delta x/2$ so that conductance is only half that of block B.

In working through the problems of this course, you will have an occasion to calculate aquifer hydraulic conductances in the horizontal directions and in the vertical, both in the interior of flow systems and along the boundaries. You also will calculate vertical hydraulic conductance between an aquifer and the bottom of an overlying stream.

Equivalent Conductances--Hydraulic Conductances in Series and Parallel

Ground water can move through aquifer material of different hydraulic conductances in series so that flow is directed through one and then the next; or it can move in parallel so that the same head difference exists across each segment of aquifer material and an incoming flow is divided between parallel conductors. In the process of discretization, it is often useful to view a segment of the continuous aquifer system as made up of hydraulic conductances arranged either in series or in parallel, and to calculate an equivalent hydraulic conductance for this series or parallel arrangement. As shown in figure 2.2.10B, the equivalent conductance for conductances linked in parallel is simply the sum of the individual conductances. (Remember, conductance is the inverse of resistance.) For conductances in series, on the other hand, the inverse of the equivalent conductance is equal to the sum of the inverses of the individual conductances. This is demonstrated in figure 2.2.10C. Application of these rules can greatly simplify the calculation of conductances for individual blocks of a model, particularly where the model blocks represent combinations of different geologic materials or units.

A Note on Discretization in Computer-Based Finite-Difference Simulation

Before leaving this discussion, we should note that there are two basic types of finite-difference discretizations. We have been dealing with the "point-centered" (or "face-centered") finite difference discretization. In the point-centered scheme, the points are placed in a grid and we calculate the conductances between nodes. This is the method used in electric analogs and many numerical computer programs. The other method of discretization is the "block-centered" method in which the aquifer is divided into blocks of uniform hydraulic conductivity and transmissivity and a node is assigned to the center of each block (fig. 2.2.11). This method is used in many of the U.S. Geological Survey programs at the present time.

Some subtle numerical differences exist between the two methods, which will not be discussed here. The most important aspect is that regardless of the form of discretization used, a conductance value must be calculated between nodes to be used in

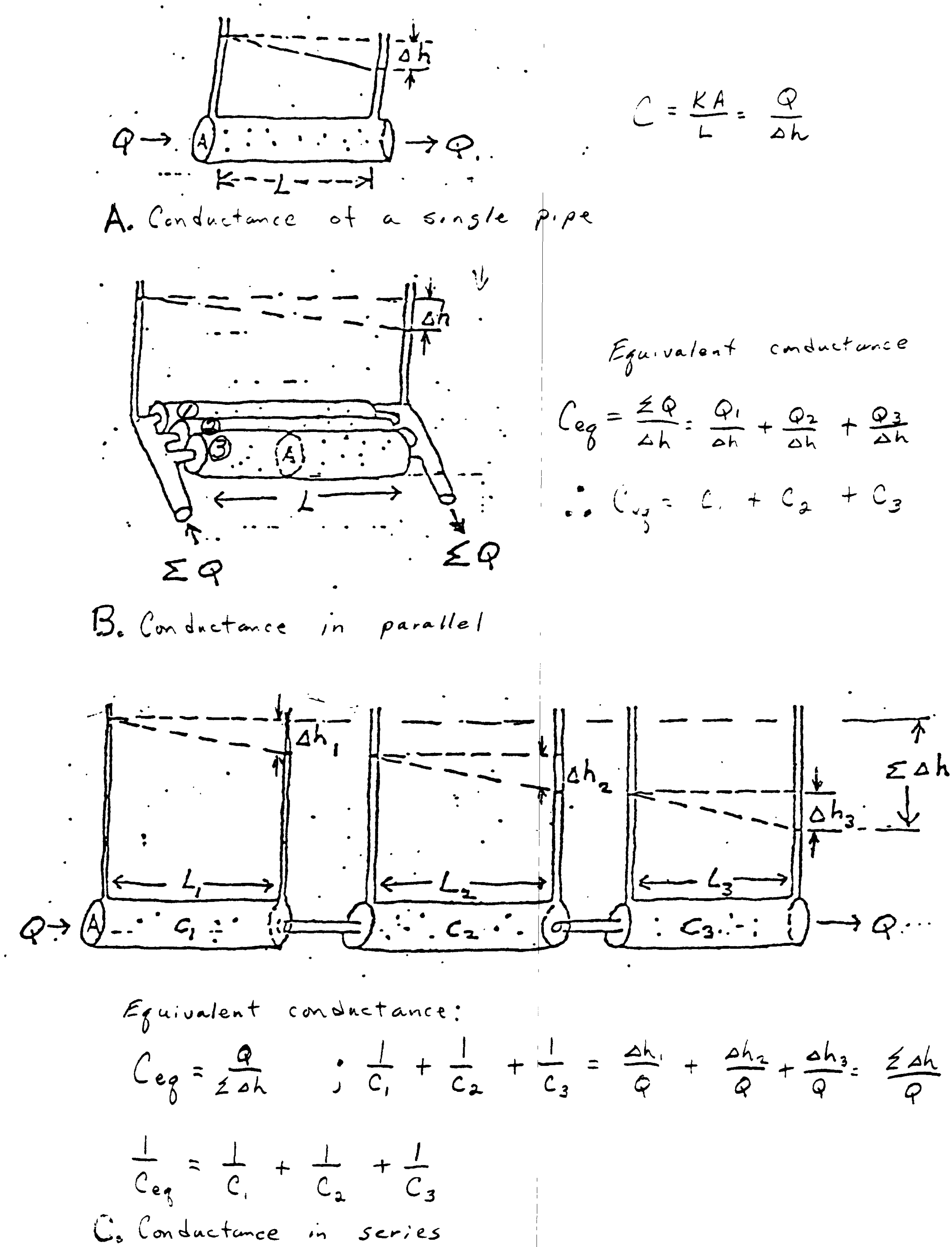


Figure 2.2.10.--Conductances in parallel and in series.

formulating a set of simultaneous algebraic equations. In "block-centered" schemes this is done, in effect, by applying the formula for conductances in series to the two "half-blocks" making up the interval between the adjacent nodes.

The point-centered scheme is used in this set of notes for three reasons:

1) convenience,

2) to keep the analogy with the electric analog more exact, and

3) to illustrate the calculation of coefficients and solution of the simultaneous equations more clearly.

Thus, the student should be aware of the type of discretization being used in these notes and in any simulation computer program.

Additional Notes on the Electrical Analogy

"Two systems are said to be analogs if there is a one-to-one correspondence between each element in the two systems as well as between the excitation and response functions of these elements and the system as a whole . . . . The analogy between systems is frequently derived or demonstrated by noting the similarities between the characteristic equations of the two systems." (Karplus, 1958, page 8).

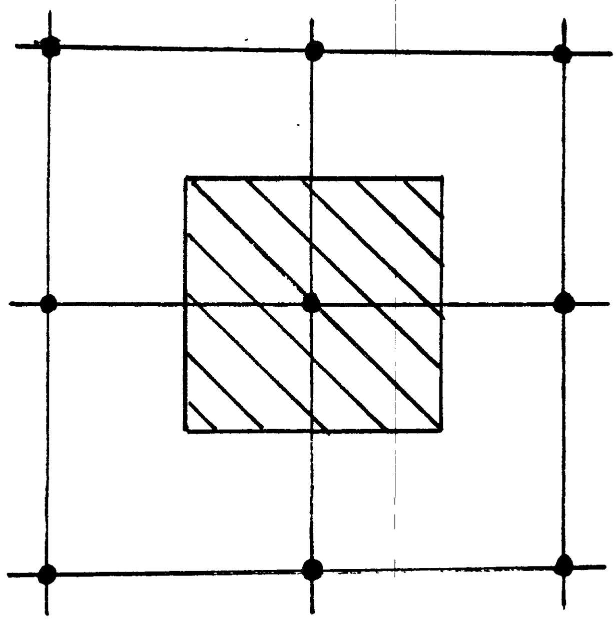


Figure 2.2.11.--Block-centered formulation for finite-difference analysis.

The Laplace equation, which combines the concept of continuity and a specific rule relating flow and a gradient, is applicable to other physical systems in addition to steady ground-water flow, (e.g., electrical current and heat flow). The Laplace equation is written for these three types of flows as follows:

Ground water	Electricity	Heat
$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$	$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$	$\frac{\partial^2 Temp}{\partial x^2} + \frac{\partial^2 Temp}{\partial y^2} = 0$

Three types of electrical media are:

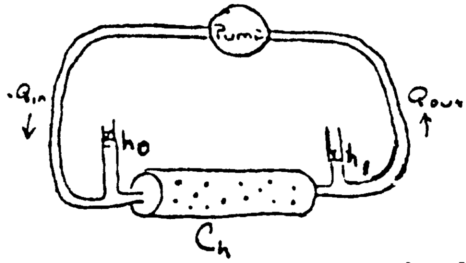
1. Conductive liquids } continuous
2. Conductive sheets } continuous
3. Resistor networks - lumped or discrete

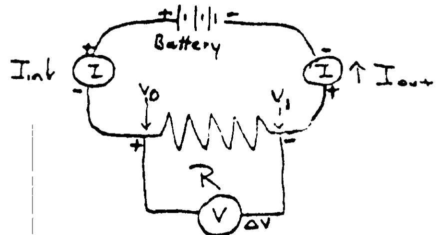
The similarity of the equations for the ground-water and electrical systems, however, is not the reason for the exploitation of the analogy between these two types of flow. The analogy is utilized because an electrical system analogous to a specific ground-water system can commonly be constructed, analogous electrical stresses applied to the electrical system, and the system response easily measured by readily available instruments. A comparison of ground-water systems and electrical systems is shown in figure 2.2.12.

The intrinsic structure of a resistor network lends itself to reproducing the finite-difference approximation of the ground-

BASIC LAWS:

Ground water	Electricity





FLOW:

Darcy's law

$$Q = C_h(h_0 - h_1)$$

$$C_h = \frac{KA}{L}$$

Ohm's law

$$I = \frac{V}{R} = C_e(V_0 - V_1)$$

$$C_e = \frac{\sigma A}{L}$$

CONTINUITY:

Conservation of flow volume

$$Q_{in} - Q_{out} = 0$$

$$\Sigma Q = 0$$

Conservation of charge(q). Kirchoff's first law: the sum of the currents at a junction or node is zero

$$\Sigma I = 0$$

Corresponding or Analogous Parameters

Ground-Water System	Electrical System
Q - volumetric flow rate [L^3T^{-1}]	I - current [amperes]
C_h - hydraulic conductance [L^2T^{-1}]	C_e - electrical conductance [siemans = ohms^{-1}] C_e = 1/R, R is resistance [ohms]
h - hydraulic head [L]	V - voltage [volts], measured as a voltage drop between two points (ΔV)
K - hydraulic conductivity [LT^{-1}]	σ- electrical conductivity [siemans L^{-1}]
A - cross-sectional area of flow [L^2]	A - cross-sectional area of resistor [L^2]
L - length of ground-water flow path [L]	L - length of resistor [L]

Figure 2.2.12--Ground-water systems and electrical systems.

water flow equation simulating flow in a discretized aquifer system. A comparison of finite-difference networks of resistors and Darcy cylinders in figure 2.2.13 illustrates this concept. A comparison of the formulation of basic laws of electricity for voltage versus the finite-difference formulation for head at a junction (node) is shown in table 2.2.1.

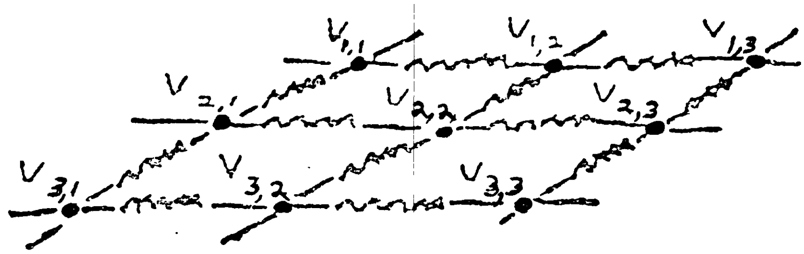
Three principal boundary types in a ground-water system have counterparts in the electrical system; these are:

Ground-water system		Electrical system
1. constant head	-	constant voltage
2. constant flow	-	constant current
3. streamline (no flow)	-	no current source (or sink)

Anisotropy in ground-water flow systems has an analog in electrical systems. In the resistor network, each resistor represents the resistance to flow in a block of aquifer in a specific direction (vector volume). The intrinsic nature of this network allows representation of homogeneous and anisotropic systems and certain nonhomogeneous (locally homogeneous) systems (see Bennett, 1976, p. 31-33). A simple representation of this concept is shows in figure 2.2.14.

The use of capacitors at junctions (nodes) with electrical systems enables transient problems to be investigated. The current through a capacitor is proportional to the time rate-of-change of voltage just as the time rate-of-change of storage

Finite-difference
network of resistors



Finite-difference
network of Darcy
cylinders

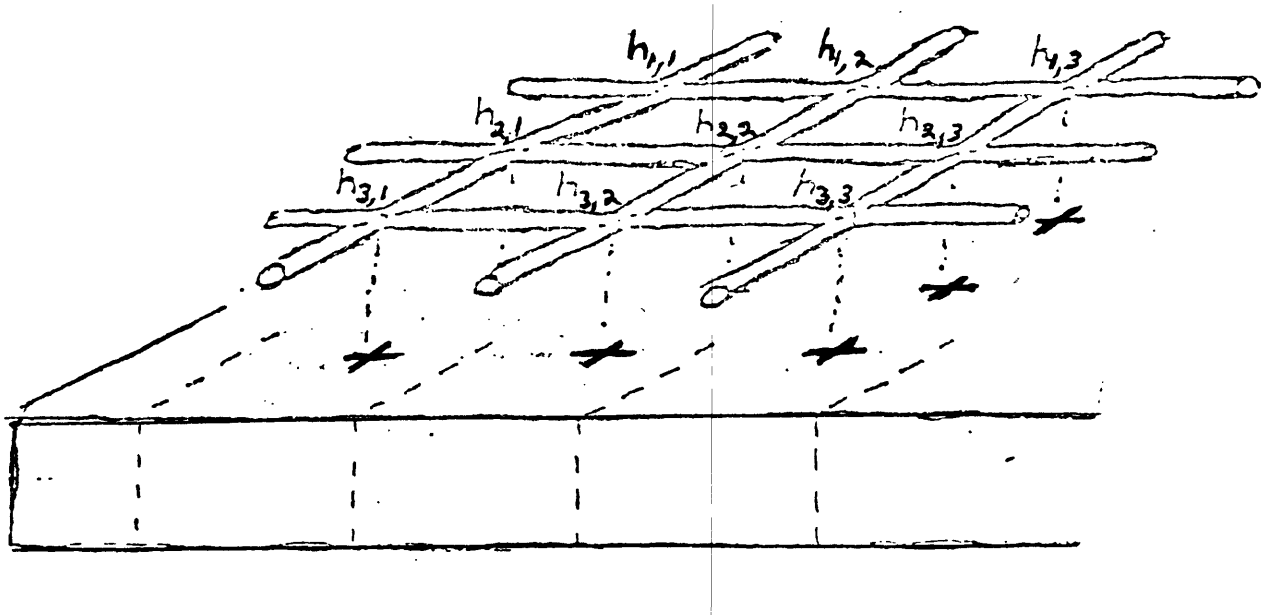
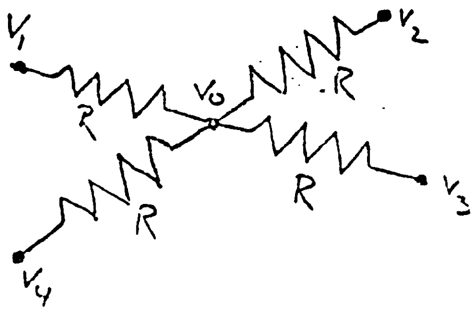


Figure 2.2.13--Finite-difference networks of resistors and Darcy cylinders.

Table 2.2.1--Formulation of basic laws of electricity for voltage versus the finite-difference formulation for head at a junction (node.)

Formulation of Basic Laws of Electricity for Voltage at a Junction (Node):



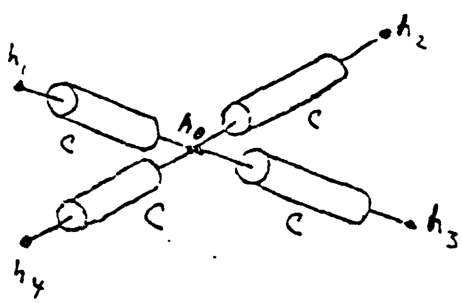
$$I = \frac{V}{R} \text{ and } \Sigma I = 0$$

$$\frac{1}{R}(V_1 - V_0) - \frac{1}{R}(V_0 - V_3) + \frac{1}{R}(V_4 - V_0) - \frac{1}{R}(V_0 - V_2) = 0$$

$$V_1 + V_2 + V_3 + V_4 - 4V_0 = 0$$

$$V_0 = \frac{V_1 + V_2 + V_3 + V_4}{4}$$

Finite-Difference Formulation for the Head at a Node:



$$Q = C\Delta h \text{ and } \sum Q = 0$$

$$C(h_1 - h_0) - C(h_0 - h_3) + C(h_4 - h_0) - C(h_0 - h_2) = 0$$

$$h_1 + h_2 + h_3 + h_4 + 4h_0 = 0$$

$$h_0 = \frac{h_1 + h_2 + h_3 + h_4}{4}$$

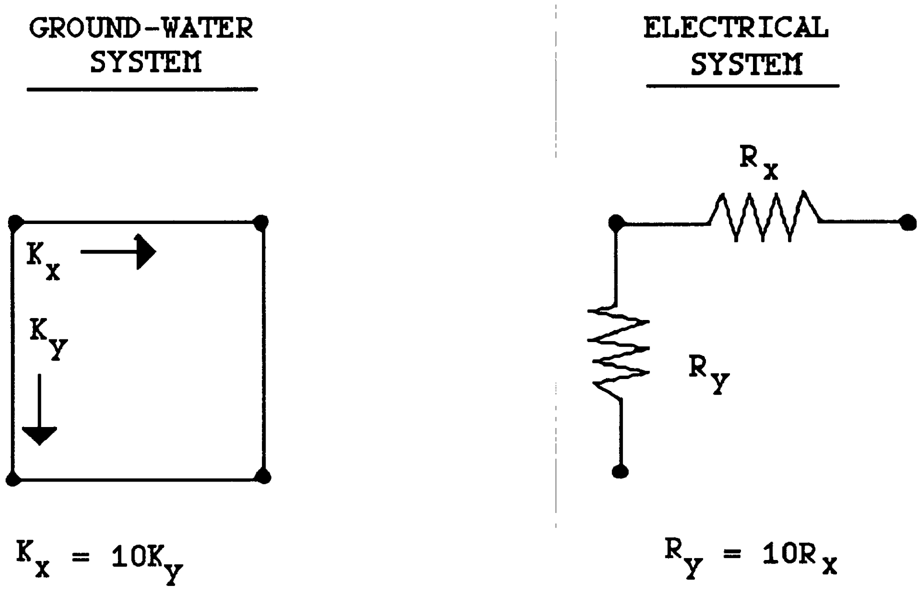


Figure 2.2.14--Electrical analogy of anisotropy in a ground-water flow system.

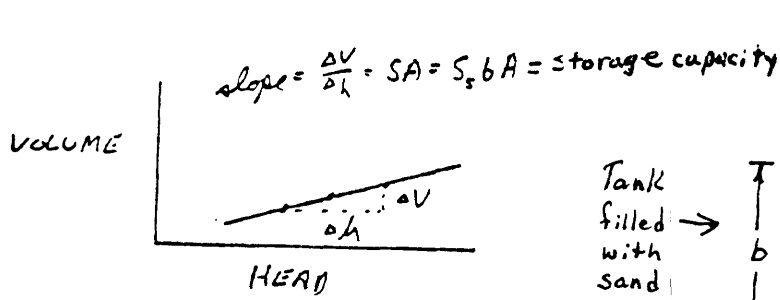
(volume of water) in an aquifer is proportional to the rate of change of head.

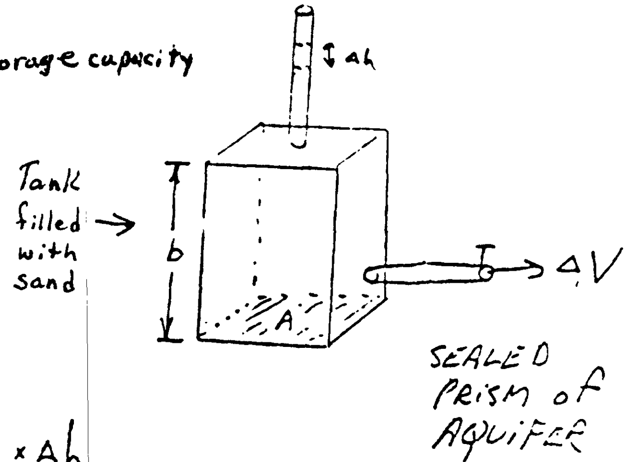
Ground water	Electricity
$\frac{\partial^2 h}{\partial x^2} = \frac{S}{T} \frac{\partial h}{\partial t}$	$\frac{\partial^2 V}{\partial x^2} = RC\frac{\partial V}{\partial t}$
	In this equation, C is capacitance in farads

NOTE 3, HYDRAULIC STORAGE CAPACITY

In extending the discretization process to nonequilibrium situations, we must consider the property of ground-water storage in the flow medium. In discretizing the steady-state flow process, we introduced the concept of hydraulic conductance, which combined a property of the porous medium, hydraulic conductivity, with the geometry of the represented block of aquifer material to yield a ratio of flow through the block to head difference across it. Following a similar pattern, we wish to obtain a relation between the volume of water, ΔV, released from storage (or taken into storage) in a block of aquifer and the change in head, Δh, accompanying that release (or gain). We define this expression as the storage capacity, S_c, of a block of aquifer material, and it represents the volume of water available from storage per unit change in head in the block.

First we consider a block of a confined aquifer extending completely through the aquifer from top to bottom, and having a base (map) area A and thickness b (fig. 2.3.1A). The storage coefficient, S, is the volume of water an aquifer releases from or takes into storage per unit surface area of the aquifer per unit change in head. In a confined water body, the water derived from storage with a decline in head comes mostly from the expansion of the water and compression of the aquifer; dewatering by gravity drainage from the pores does not occur as long as the head is above the top of the aquifer. The storage equation tells us that $\Delta V = SA\Delta h$. Thus, the ratio $\Delta V/\Delta h$ for the block of aquifer is





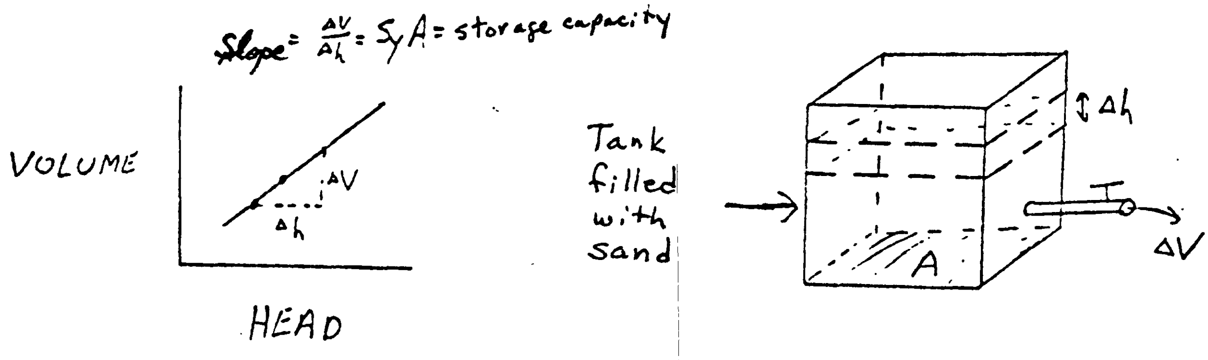
$$\Delta V = SA\Delta h$$

$$\Delta V = S_s \cdot b \times A \times \Delta h$$

$$(L^3) = (L^{-1})(L) \times (L^2) \times (L)$$

S_s = Specific STORAGE = Volume of water that drains out of the tank per unit volume per unit of head change.

A.--CONFINED AQUIFER



$$\Delta V = S_y A \Delta h$$

Change in Volume = Specific Yield × Area × change in water level
(L^3) (unitless) (L^2) (L)

S_y (specific yield) = Volume of water that drains out of the tank per unit area per unit change in head.

B.--UNCONFINED AQUIFER

Figure 2.3.1.--Concept of storage capacity in confined and unconfined aquifers.

S•A--the storage coefficient multiplied by the base area. Storage coefficient is itself the product of specific storage, S_s, and aquifer thickness b. The specific storage, S_s, is the volume of water released from or taken into storage per unit volume of the porous medium per unit change in head. Thus, the storage capacity, S_c, of a confined aquifer can be expressed as

$$S_c = \frac{\Delta v}{\Delta h} = S_s bA. \qquad (1)$$

We note that, like hydraulic conductance, storage capacity incorporates both a property of the medium, S_s, and a geometric factor describing the block b•A. If we were to consider a small block in the interior of the aquifer rather than a prism extending vertically through it, we could still define the storage capacity of that interior block as the ratio of the volume of water released from storage in the block to the accompanying head change within it. The volume of the block is $\Delta x \Delta y \Delta z$; thus again

$$S_c = \frac{\Delta v}{\Delta h} = S_s \Delta x \Delta y \Delta z$$

For unconfined aquifers, we consider a block of the aquifer extending from the water table to the bottom of the aquifer (figure 2.3.1B). The volume of water derived from an unconfined aquifer by drainage of pore spaces under the influence of gravity per unit area of the porous media per unit change in head is defined as the specific yield, S_y. The amount of water derived from expansion of the water and compression of the aquifer is very small and usually is ignored. Thus, the storage capacity for an unconfined aquifer is

$$S_c = \frac{\Delta v}{\Delta h} = S_y A. \qquad (2)$$

In discussing discretization of the flow process, we utilized the concept of a hydraulic network model built of sand-filled pipes. We can utilize a similar concept to clarify the discretization of storage. Figure 2.3.2 shows a similar network of sand-filled pipes except that now a storage element consisting of a tank has been connected to each junction of the network. Again, the junctions of the network correspond to nodes in the discretized representation of an aquifer. Assuming the network represents a confined aquifer, each of the tanks is characterized by a storage capacity, $\Delta V/\Delta h$, which is, in fact, just equal to the base area, A, times the storage coefficient of the tank in question. The model is designed so that the storage capacity of the tank connected to each junction is equal to that of a block of the aquifer surrounding the corresponding node and extending halfway to the adjacent node in each direction, as shown in figure 2.3.3.

Thus, whereas discretization of the flow process involved the delineation of aquifer blocks extending between nodes, discretization of storage involves division of the aquifer into blocks surrounding each node. When this process of division is completed, the volume of the block surrounding each node is multiplied by the specific storage in the vicinity of the node to

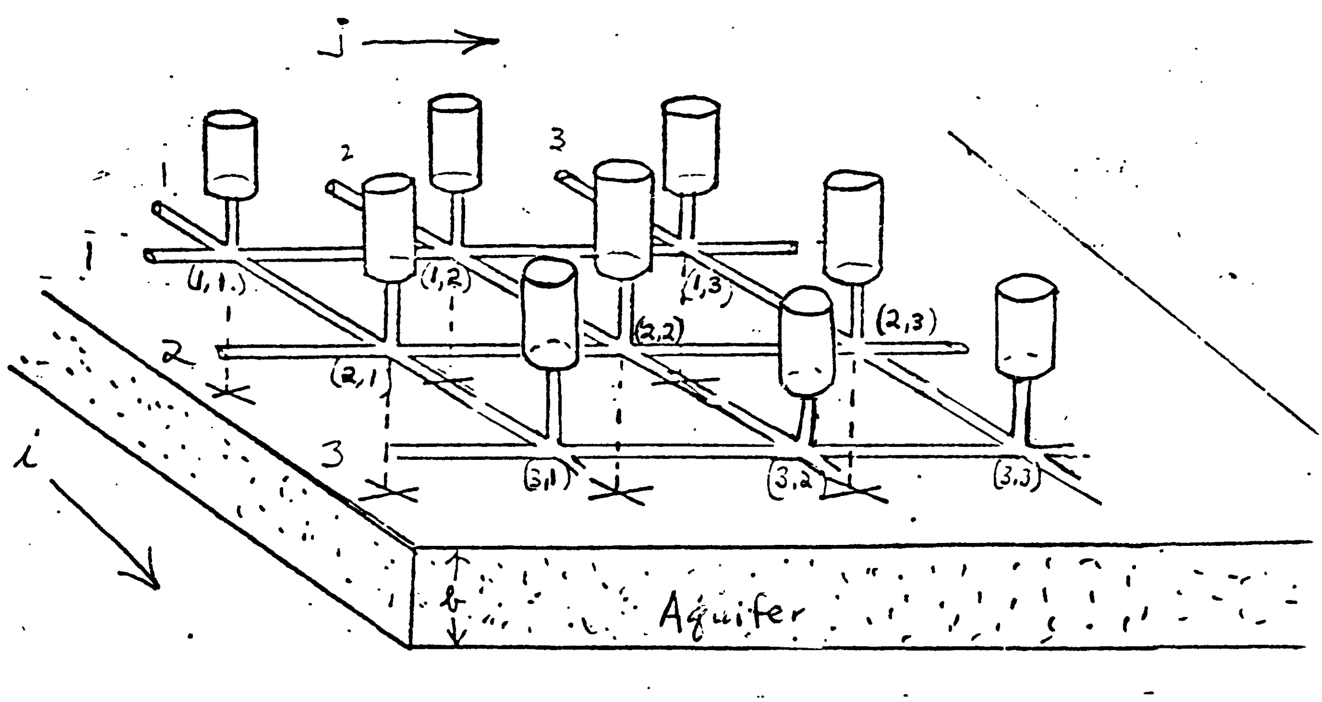


Figure 2.3.2.--Network of sand-filled pipes with storage tanks at each node representing an aquifer with a storage component.

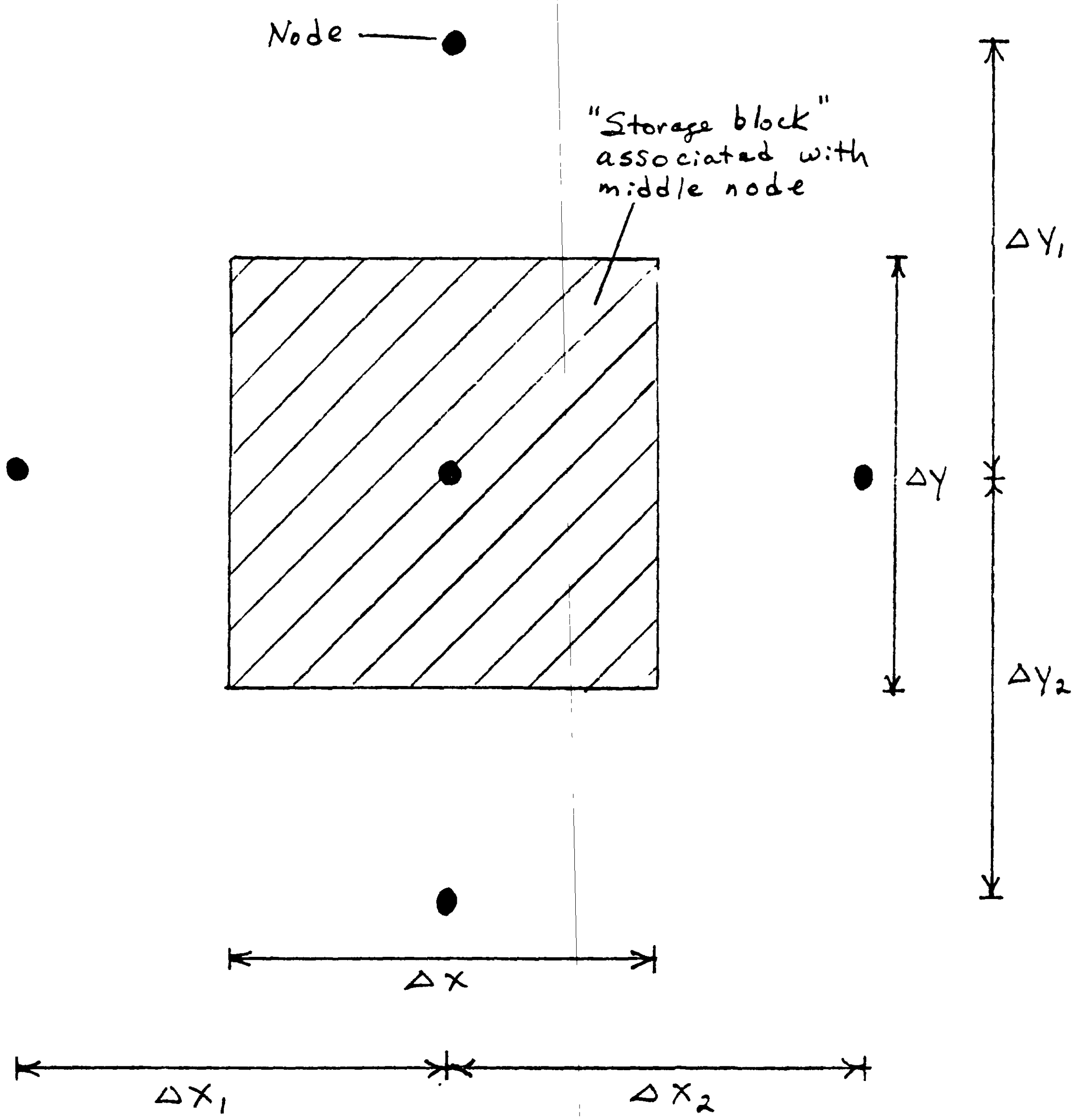
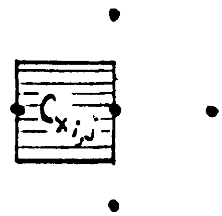


Figure 2.3.3.--Map view of storage tank associated with a node of a discretized aquifer.

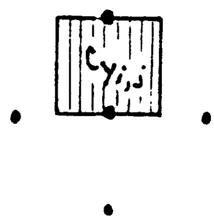
calculate the required storage capacity. In the hydraulic model postulated above, a tank would then be selected having the required storage capacity. In practice, of course, a hydraulic model of this sort would be cumbersome to use, and we are concerned instead with specifying storage capacities in a numerical model.

Figure 2.3.4 shows four views of the array of nodes associated with an aquifer model. Figure 2.3.4A shows the aquifer block whose properties must be designated to specify hydraulic conductance in the x direction $C_{x_{i,j}}$. Figure 2.3.4B shows the block whose properties must be designated to specify hydraulic conductance in the y direction, $C_{y_{i,j}}$. Figure 2.3.4C shows the block whose properties must be designated to specify storage capacity $S_{c_{i,j}}$. Three blocks are superimposed in figure 2.3.4D. Note that geometry enters the storage capacity calculation in a different way than it enters a hydraulic-conductance calculation. For hydraulic conductance, we multiply hydraulic conductivity by a ratio of flow area to flow length. For storage capacity, we multiply specific storage by block volume for a confined aquifer and specific yield by block base area for an unconfined aquifer.

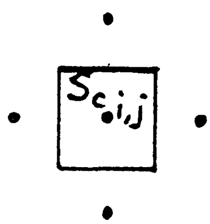
We continue to use the point-centered (or face-centered) system which leads to nodes along the boundaries of the model. For boundary nodes not located at a corner, the map area of the block used to define storage capacity turns out to be half that



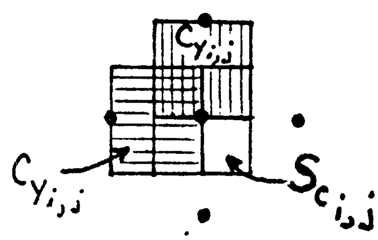
A. Aquifer block for hydraulic conductance in the x direction.



B. Aquifer block for hydraulic conductance in the y direction



C. Aquifer block for storage capacity around node



D. Blocks for hydraulic conductance in the x and y directions superimposed on block for storage capacity.

Figure 2.3.4.--Aquifer blocks associated with hydraulic conductance and storage capacity in a nodel array.

for an internal node; for a node located at a corner of the mesh, the map area turns out to be one quarter of that for an internal node. These reductions in map area generate corresponding reductions in storage capacity.

Most computer programs for ground-water flow simulation by numerical methods require either the storage coefficient or the specific yield associated with each node as input data. The program itself in effect, if not explicitly, calculates storage capacities in the process of solving the equations. In some of the problems that you will work in this course, you are asked to calculate storage capacities as a step in model design. The purpose of this exercise, like that of hydraulic conductance calculations, is to create a better appreciation for the discretization process and to bring the model design process a step closer to the equations actually formulated by the computer program.

In three-dimensional or cross-sectional simulation, storage varies in the vertical direction as well as in the map view, and the concept of a prism extending through an aquifer from top to bottom is no longer adequate. We must consider instead the storage capacities of individual blocks within the ground-water system. If we are dealing with a confined system, the storage process is dominated by compressive effects, and the storage capacity of each block is simply the representative specific storage of the aquifer material multiplied by the volume of the block. When we deal with an unconfined system, however, the storage capacity of an individual block depends upon whether or not the free surface--that is, the water table--passes through that block. If it does, the process of storage release within the block is dominated by drainage--that is, by the release of water from storage due to movement of the free surface. Thus, the storage capacity of the block is the specific yield of the material multiplied by the map area of the block. On the other hand, if the block is beneath the free surface in the interior of the system, compressive storage effects are generally still observed even though the aquifer, as a whole, is not confined. That is, at a point below the water table, a decrease in pressure must cause a slight expansion of the water and may cause a slight compression of the porous skeleton. The volume of water that is released from storage as a result of these processes is, of course, small compared with that released at the water table due to the lowering of the free surface (water table). In an areal simulation, where we are concerned with the total release from storage in a prism extending from the water table to the base of the aquifer, water derived from the compressive component is generally such a small fraction of the total amount of water removed from the prism that it is neglected. In a cross-sectional or three-dimensional approach; however, we are interested in representing the storage process in each individual block of the system. Thus, for blocks below the water table, we calculate storage capacity as we would in a confined system, multiplying the compressive specific storage of the material by the volume of the block.

In some three-dimensional or cross-sectional simulations, the water table may rise or fall enough so that it moves from the block in which it started to the adjacent block above or below. In these cases, a routine can be incorporated in the simulation that checks each block after each calculation, and changes the storage capacity if the free surface has moved into or out of the block.

NOTE 4, DISCRETIZATION OF TIME--METHODS OF FORMULATING NONEQUILIBRIUM FINITE-DIFFERENCE EQUATIONS

In nonequilibrium simulations by numerical methods, time as well as space must be discretized. This is done by dividing the time axis into discrete intervals that begin and end at specific times, t_1, t_2, t_3, t_4, . . . t_n, as shown in figure 2.4.1, and by considering heads only at those times--that is, h_1, h_2, h_3, h_4, . . . h_n. A ratio of finite differences, $\Delta h/\Delta t$ or for example, $(h_3 - h_2)/(t_3 - t_2)$, is then used to approximate the time derivative, $\partial h/\partial t$. As shown in figure 2.4.1, this is equivalent to approximating the slope of a hydrograph at a point using the average slope taken between t_2 and t_3. This approximation to the time derivative is then multiplied by storage capacity to obtain an approximation for rate of change in storage $\partial v/\partial t = (\partial v/\partial h)(\partial h/\partial t) = S_c(\partial h/\partial t)$. This expression is, in turn, equated to a finite-difference approximation for inflow minus outflow similar to that utilized in our earlier work with steady-state problems to obtain an approximation for the complete nonequilibrium flow equations.

Because we are now considering heads only at the specific times, t_1, t_2, . . . t_n, the expressions describing inflow minus outflow must be approximated using heads at these specific times--that is, we must formulate our expressions for the spatial derivatives and thus our expressions for inflow minus outflow at specific times, such as t_2 or t_3. On the other hand, we must formulate our expression for the time derivatives and, thus, for

$$\text{slope of hydrograph} = \frac{\Delta h}{\Delta t} \cong \frac{\partial h}{\partial t}$$

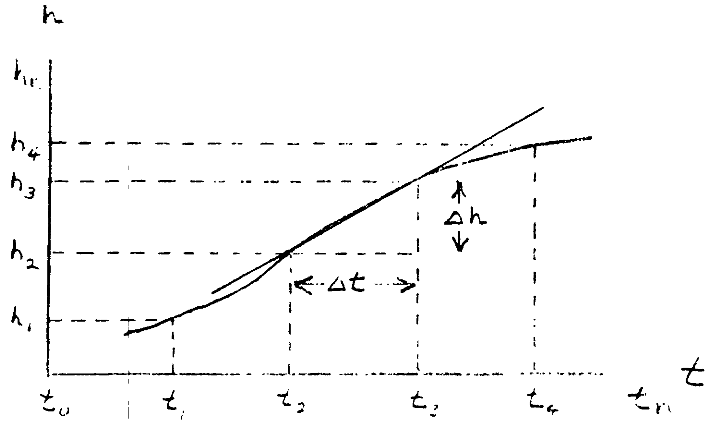


Figure 2.4.1.--Discretization of time on a hydrograph.

release of water from storage over intervals of time such as $t_3 - t_2$. We can envision several ways of combining these discretizations of space and time. We could, for example, (1) express inflow minus outflow at t_2, the beginning of the interval over which we approximate the time derivative; (2) express inflow minus outflow at t_3, the end of that interval; or (3) change the interval over which we simulate the time derivative to $t_4 - t_2$, and express inflow minus outflow at the intermediate time, t_3.

We will confine our comments here to the first two possibilities. Look first at the representation of inflow minus outflow at the beginning of the interval that we use in approximating $\partial h/\partial t$. Using the time derivative simulated over an interval Δt and the expression for inflow minus outflow approximated in terms of heads at the beginning of that interval is termed the forward-difference method. Figure 2.4.2 shows a graphical representation, sometimes termed the "finite-difference stencil" for this method. The time axis in figure 2.4.1 extends along the vertical. The array of five heads shown at time t_n is utilized in the approximation of inflow minus outflow; the time interval extending forward in time from t_n to t_n+1 is utilized in approximating the storage term. The finite-difference equation using the forward-difference method and written in terms of

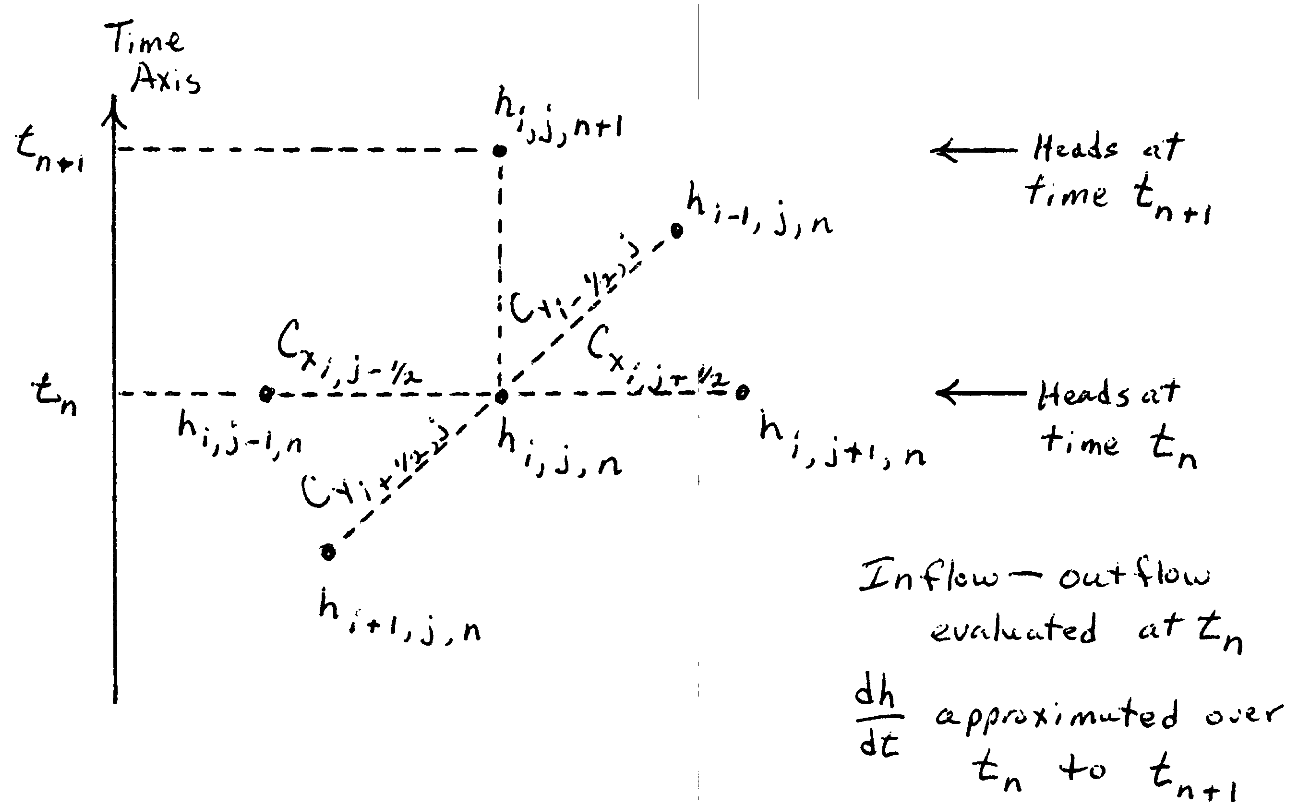


Figure 2.4.2. -- Discretization of time, forward-difference method.

conductance and storage capacities is

$$-C_{x_{i,j-1/2}}(h_{i,j,n} - h_{i,j-1,n}) + C_{x_{i,j+1/2}}(h_{i,j+1,n} - h_{i,j,n})$$

$$-C_{y_{i-1/2,j}}(h_{i,j,n} - h_{i-1,j,n}) + C_{y_{i+1/2,j}}(h_{i+1,j,n} - h_{i,j,n})$$

$$-Q_{i,j,n} = S_{c_{i,j}}\left[\frac{h_{i,j,n+1} - h_{i,j,n}}{t_{n+1} - t_n}\right], \tag{1}$$

where $C_{x_{i,j-1/2}}$ is the hydraulic conductance of a block of aquifer between node i,j-1 and node i,j; $C_{y_{i-1/2,j}}$ is the hydraulic conductance of a block of aquifer between node i-1,j and node i,j; and so on (see fig. 2.4.2). $S_{c_{i,j}}$ is the storage capacity of a block of aquifer surrounding node i,j; and $Q_{i,j,n}$ is a source/sink term that represents net additions to or withdrawals from the same block of aquifer surrounding node i,j, due to wells or other externally specified stresses, at time t_n. If no externally caused additions or withdrawals exist, $Q_{i,j,n}$ is zero and would not appear in the equation.

If we were using these equations in a simulation, we would be interested in calculating heads at the advanced time t_{n+1}, on the basis of heads at the earlier time, t_n, which would either have been calculated already or would represent known initial head values. Thus our equation would contain only one unknown term, $h_{i,j,n+1}$ and could easily be solved for this term. Similar equations could be written and solved at every other node in the mesh. Thus, the method appears at first glance to be straight-forward and simple. However, if the time increment is taken too large, any error that appears for any reason at any time step is guaranteed to grow in successive time steps until the finite-difference solution is eventually dominated by error and bears no relation at all to the analytical, or exact, solution of the differential equations of flow. This condition is termed numerical instability; because of it, forward-difference techniques are generally not used.

Figure 2.4.3 shows a finite-difference stencil for the second method in which the expression for inflow minus outflow is evaluated at the end of the time interval over which the storage term is approximated. This procedure is referred to as the "backward-difference" method. We will again say that the expression representing inflow minus outflow is evaluated using the five head array at time t_n; however, in this case, the storage term is evaluated over an interval extending backward in time from t_n to t_{n-1}.

The resulting form of the finite-difference equation is

$$-C_{x_{i,j-1/2}}(h_{i,j,n} - h_{i,j-1,n}) + C_{x_{i,j+1/2}}(h_{i,j+1,n} - h_{i,j,n})$$
$$-C_{y_{i-1/2,j}}(h_{i,j,n} - h_{i-1,j,n}) + C_{y_{i+1/2,j}}(h_{i+1,j} - h_{i,j,n})$$
$$-Q_{i,j,n} = S_{c_{i,j}}\left[\frac{h_{i,j,n} - h_{i,j,n-1}}{t_n - t_{n-1}}\right] \qquad (2)$$

$Q_{i,j,n}$ is the net withdrawal from the same block of aquifer surrounding node i,j at time t_n.

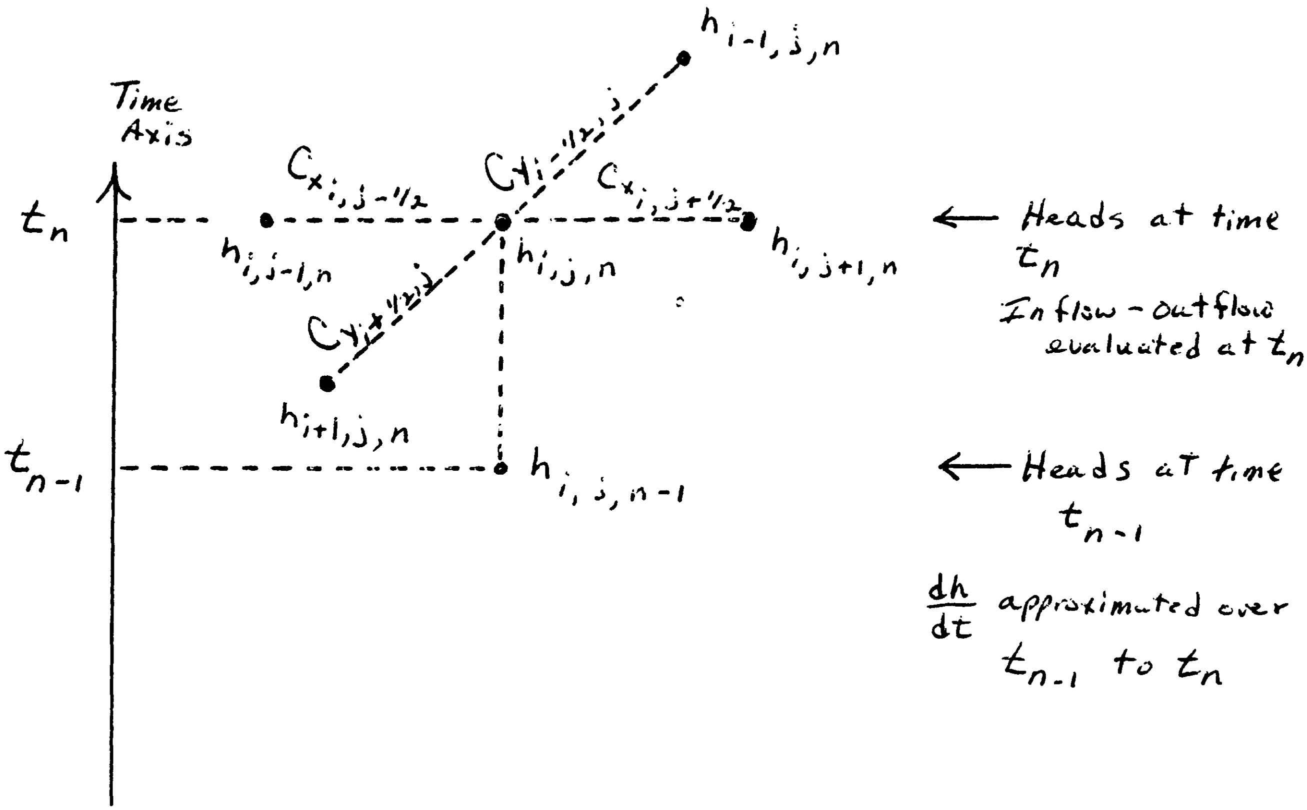


Figure 2.4.3. -- Discretization of time, backward-difference method.

As noted, nonequilibrium simulation, using the backward-difference formulation, involves the calculation of heads at an advanced time (t_n) on the basis of heads calculated (or otherwise known) for the preceding time (t_{n-1}). In using equations for this purpose, an immediate problem that was not present in the forward-difference formulation is evident. Equation 2 contains five values of head associated with the advanced time, t_n. All of these values are presumably unknown at the beginning of our calculation; only one known value of head, $h_{i,j,n-1}$, appears in the equation. Thus, whereas the forward-difference equation (1) contains only one unknown head, $h_{i,j,n}$, and can be solved for this unknown at once, there is no way to solve equation (2), considered independently, for the five unknown heads that it contains. However, an equation can be written for the unknown head, $h_{i,j,n}$ for time t_n for each internal node in the finite-difference mesh. The resulting set of equations can be solved simultaneously. By letting $C_x = C_y = C$ (distances between nodes are equal), equation (2) solved for $h_{i,j,n}$ is

$$h_{i,j,n} = \frac{h_{i-1,j,n} + h_{i+1,j,n} + h_{i,j-1,n} + h_{i,j+1,n} + \dfrac{S_{c_{i,j}}(h_{i,j,n-1})}{C(t_n - t_{n-1})} - \dfrac{Q_{i,j,n}}{C}}{4 + \dfrac{S_{c_{i,j}}}{C(t_n - t_{n-1})}} \quad (3)$$

Equations can also be developed for each node along the boundaries, utilizing the boundary conditions that apply to the problem. If the mesh consists of p nodes, there are p unknown values of head to be determined, and there are exactly p equations. Thus, we have a simultaneous set of p equations in p unknowns, and a solution can theoretically be obtained using standard algebraic techniques. In practice, the work involved in a direct algebraic approach is often too great, and we fall back instead on numerical (iterative) methods. But, however it is obtained, the solution to the set of simultaneous algebraic equations gives the head distribution throughout the mesh at the end of the time step. Knowing the heads at the end of this time step, we can then reformulate our equations for the succeeding time step--and again solve for heads at the end of the time step. The heads that we calculated for t_n now become the known heads used to calculate the $\Delta h/\Delta t$ term, while the heads at t_{n+1} make up the unknowns in each equation of the set.

The process of iterative solution has been considered earlier in chapter VII of Bennett's (1976) programed text and in the problems involving steady-state simulation by hand calculation. In an iterative method, a first estimate is made and assigned for each unknown head. A procedure is then initiated that recalculates these unknown heads and alters the first estimates producing a set of interim values closer to satisfying the equations. The procedure is repeated successively, each stage referred to as an iteration, producing a new set of interim heads that more nearly satisfies the system of equations. Ultimately, as the interim heads approach values which would exactly satisfy the equations, the changes produced by succeeding iterations become very small. In practice, the changes in head produced at each iteration are compared with some arbitrarily established quantity, usually termed the closure criterion. When the changes at all points in the mesh are less than this criterion, the interim head values are taken as sufficiently close to the exact solution of the system of equations and the iterative procedure is terminated.

Thus, during the iteration procedure for an individual time step, arrays of interim head values are recalculated in succession, each array containing one interim head value for each node in the mesh. In figure 2.4.4, these arrays are represented as iteration "planes" or levels, and a superscript (0,1,2,...m) is used to indicate the level of iteration. The figure shows an array $h^m_{i,j,n-1}$ containing final values of head for the preceding time, t_{n-1}. These values are the initial conditions from which heads change in the current time step (t_n). Another array, $h^0_{i,j,n}$ represents the first estimate chosen for the head at node i,j, at time t_n; $h^1_{i,j,n}$ represents the interim value after one iteration, and so on to the final iteration $h^m_{i,j,n}$. Since the closure criterion has been satisfied, the values $h^m_{i,j,n}$ are accepted as the solution of the system of equations for time t_n.

The iterative procedure involves the application of some form of equation 2 (e.g., backward-difference equation) at each level of iteration. Each time this is done during the calculations for

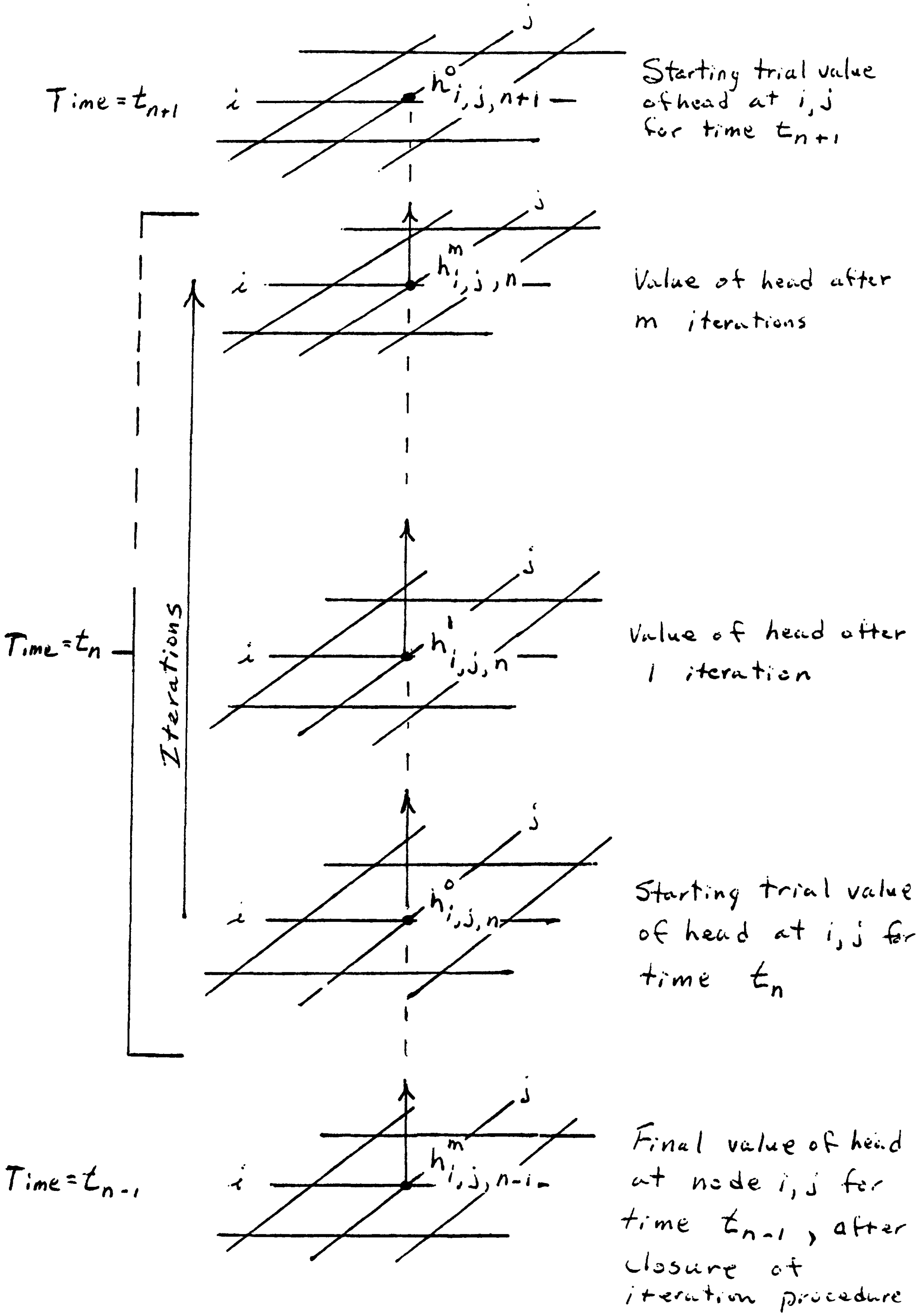


Figure 2.4.4--Iteration levels

time t_n, the values of $h_{i,j,n-1}$ (the final heads obtained for the end of the preceding time step) are utilized in the storage term. These head terms for the preceding time appear in the equation as constants or known terms; thus they retain the same values from one iteration to the next and are not modified in the iterative process. When the process is complete for time t_n, calculations for the next time step begin; head values at time t_n become the initial conditions and iterative calculations proceed to solve for head values at time t_{n+1}.

As the preceding discussion indicates, the iterative procedure yields only an approximation to the solution of the system of finite-difference equations for each time step; the accuracy of this approximation depends upon the closure criterion which is employed. However, it is important to note that even if an exact solution to the set of finite-difference equations were obtained at each time step, these exact solutions would themselves be only an approximation to the solution of the differential equation of flow. The discrepancy between the head $h_{i,j,n}$ given by the solution to the system of difference equations for a given node and time, and the head $h(x_i,y_j,t_n)$, which would be given by the formal solution of the differential equation for the corresponding point and time, is termed the truncation error. In general, the truncation error becomes greater as the mesh spacing and time-step length are increased. Finally, it must be recognized that even if a formal solution of the differential equation could be obtained, it would normally be only an approximation to conditions in the field in that hydraulic conductivity and specific storage are seldom known with great accuracy, and uncertainties with regard to representing the actual geometry or the natural hydrologic boundaries are generally present.

In any iterative method, one begins with an assumed distribution for the heads that one is trying to calculate and progressively modifies that distribution until agreement with the equations is achieved. In applying iterative methods to solve nonequilibrium problems, this is done at each time step; that is, at each time step, we make a first estimate of the head distribution and adjust that assumed distribution progressively until we obtain a distribution that satisfies our equation set. A question that arises immediately is how to make our first estimate of the head distribution in each successive time step. Clearly, if a good choice is made--that is, if the first estimate is close to the correct head distribution for the end of the step--the iterative procedure will go very quickly. If a poor choice is made--if the first estimate differs radically from the correct or acceptable solution, the iterative procedure will require many computational steps (iterations). A common practice is to take the head distribution calculated for the preceding time step as the first estimate for the current step. Referring to the diagram of figure 2.4.4, this is equivalent to setting $h^{0}_{i,j,n+1}$ equal to

$h^m_{i,j,n}$. This practice usually provides reasonable first estimates for the head distribution at t_{n+1}, but sometimes (particularly in problems involving iteration by hand calculation) can cause confusion in understanding exactly how the heads from t_n enter the calculation. A review of equation 2 shows that the head for the end of the preceding time step, $h_{i,j,n-1}$ appears in the equation only once in the term approximating the rate of withdrawal from storage. As such, it is treated as a constant in the iterative process--it is not changed continually as are the values of $h_{i,j,n}$. We must be careful to note that if we use the value of $h_{i,j,n-1}$ as the basis for our first estimate of $h_{i,j,n}$, this is not in any sense a reintroduction of $h_{i,j,n-1}$ into the equation; $h_{i,j,n-1}$ still appears only once in the equation in the storage term, and after a solution is achieved, $h_{i,j,n}$ is simply equal to the head at the end of the time step.

Finally, we should note that our first estimate of head at time t_n need not necessarily be taken equal to the values of t_{n-1}. There are many other ways of choosing a first estimate. Some programs employ a head predictor, which extrapolates trends in head over preceding time steps to obtain an estimate for the end of the current step; this extrapolated estimate is then used as the first estimate.

NOTE 5, THEIS CONCEPTS: RESPONSE OF AN AQUIFER TO WITHDRAWAL OF WATER

The points made by C.V. Theis (1940) in his paper, "The source of water derived from wells--essential factors controlling the response of an aquifer to development," may be summarized and extended as follows. Consideration of the hydrologic equation Inflow = Outflow $\pm$ ΔStorage suggests that, in principle, there are three possible sources of water to a pumping well--a decrease in ground-water storage, an increase in inflow to the ground-water system, or a decrease in outflow from the ground-water system. This abstract statement of principle can be clarified by application to a concrete example.

Consider a simple hydrologic system under predevelopment conditions in a state of dynamic equilibrium for which inflow = outflow (fig. 2.5.1A). When a well is added to the system and pumping starts at a rate Q_1, initially water is withdrawn only from storage near the well. As water levels continue to fall and hydraulic gradients are reduced in areas of natural discharge, natural discharge is reduced (fig. 2.5.1B). These processes reduce the amount of water that must come from storage--in effect, flow is rerouted from the original discharge area, the stream, to the pumped well. As the rate of storage depletion decreases, the rate of water-level decline slows and the system approaches a new equilibrium (fig. 2.5.1C).

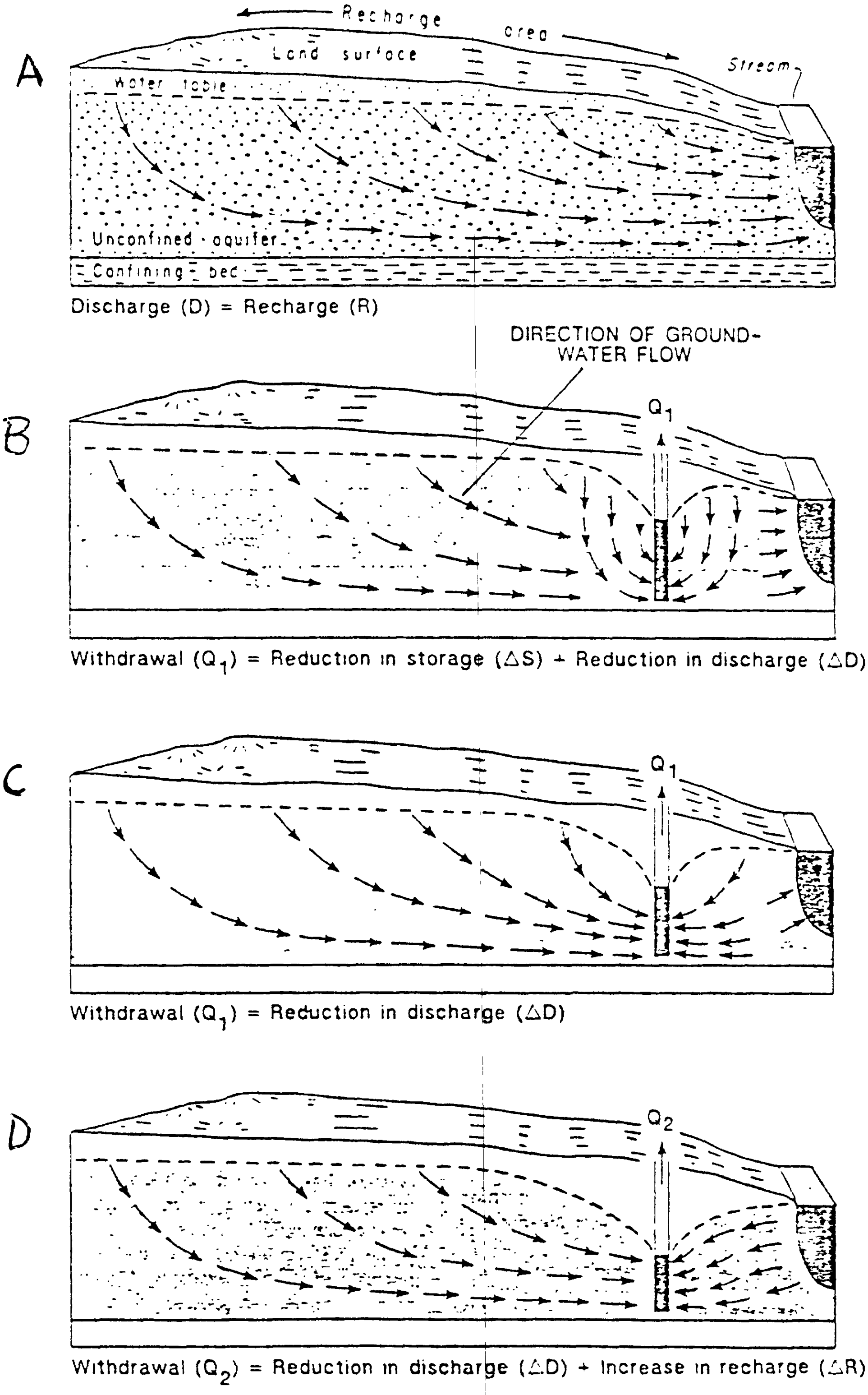


Figure 2.5.1.--Ground-water flow patterns in a hypothetical system under natural conditions (A) and in response to different levels of stress resulting from local pumping of ground water (B, C, and D). (Modified from Heath, 1983, p. 33).

At a later time the equilibrium condition depicted in figure 2.5.1C is further disturbed by a higher rate of pumping, Q_2, in figure 2.5.1D. After an initial removal of ground water from storage accompanied by a further decline in water levels, the new equilibrium condition exhibits no divide; that is, a hydraulic gradient between the stream and the pumped well has been established (fig 2.5.1D). Contrast this to the situation depicted in figure 2.5.1C in which a water-table divide exists between the well and the stream. The condition in figure 2.5.1D) induces more water from the stream into the aquifer. Thus, the stream, which formerly was a gaining stream under natural conditions and a lesser rate of pumping Q_1 (fig. 2.5.1C), is now locally a losing stream (fig. 2.5.1D) near the well.

In summary, the source of water to the well at the initial rate of pumping, Q_1, after a new equilibrium condition had been achieved, was reduced outflow of ground water to the stream. However, in contrast, the source of water to the well at the higher rate of pumping, Q_2, includes both reduced outflow to the stream and induced inflow from the stream to the aquifer.

In some cases, the pumping rate may exceed the increase in recharge and decrease in natural discharge that can be induced. In these cases, removal of water from storage continues until falling water levels or exhaustion of the supply forces a reduction in the pumping rate. A new equilibrium is then attained in which the reduced pumping rate equals the increases in recharge and decreases in discharge that have been achieved. If pumping rate is not held constant, but rather is increased from year to year, new periods of withdrawal from storage accompany each increase.

In the analog problem (Section 3.5), the hydrologic system is assumed to be at equilibrium, that is, the prepumping system is assumed to be at a natural steady state. When pumping is introduced, the assumption is made that we are viewing the system after the period of withdrawal from storage has passed--that is, after we have already reached a new equilibrium. Thus, for the analog problem, we will not be concerned with the period of storage release or with the changes in head with time. In Section 3.6 we will consider a nonequilibrium problem utilizing simulation by numerical finite-difference methods. In this problem the period of storage release and the changes in head and flow with time will be considered.

NOTE 6, STREAM-AQUIFER INTERACTION

Introduction

An understanding of stream-aquifer interaction is a key element in the formulation of an accurate conceptual model of ground-water systems. Streams in direct contact with or near a ground-water system can have a significant impact on steady-state ground-water levels and a complex effect on transient changes in ground-water levels.

The details of potentiometric contours on water-level surface maps near streams help demonstrate our concept of the ground-water flow system as related to streams. Consider three hypothetical cases of simple stream-aquifer interaction (1) a gaining stream, (2) a losing stream, and (3) a stream that is simultaneously gaining and losing water. In an aquifer system that contains a gaining stream (fig. 2.6.1A), ground water seeps from the aquifer into the stream channel. The stream channel intersects the water table, and the elevation of the stream surface is lower than the surrounding water table. (NOTE: The elevation of the stream surface represents the ground-water head on the streambed which is a boundary of the ground-water system.) In an aquifer system that contains a losing stream (fig. 2.6.1B), ground water seeps from the stream channel into the aquifer. The elevation of the stream surface is higher than the local water table. If the stream is hydraulically connected to the aquifer, the elevation of the

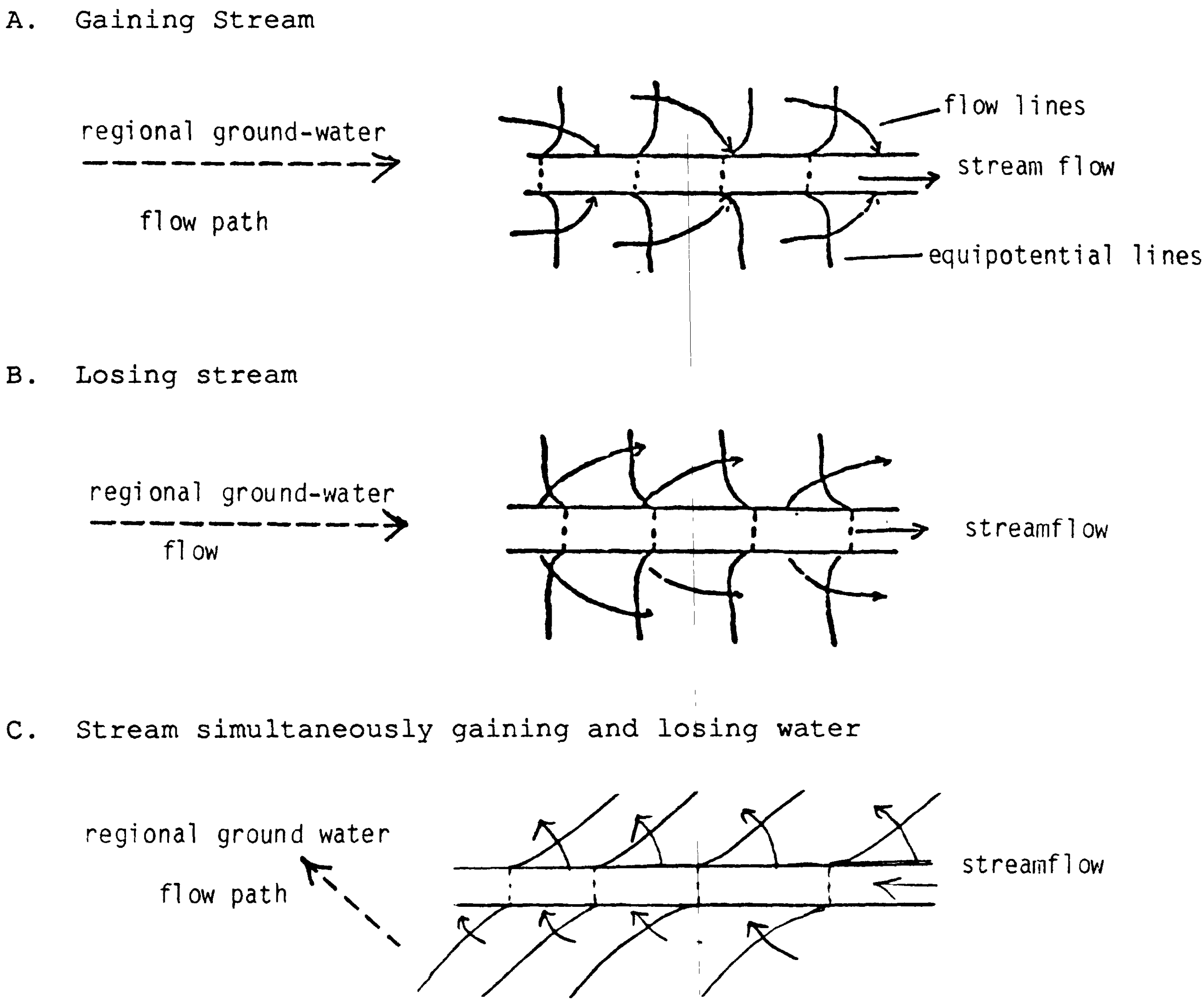


Figure 2.6.1.--Three classes of streams based on stream interaction with ground water.

stream surface represents the maximum head in the local ground-water system. In an aquifer system with a stream that both gains and loses water from a stream (fig. 2.6.1C), the regional ground-water flow path crosses the stream. Ground-water levels are higher than the stream-surface elevation on one side of the stream and lower on the other, resulting in ground-water flow both into and out of the stream channel in the same channel reach. The water leaving the stream continues to move in the general direction of the regional ground-water flow.

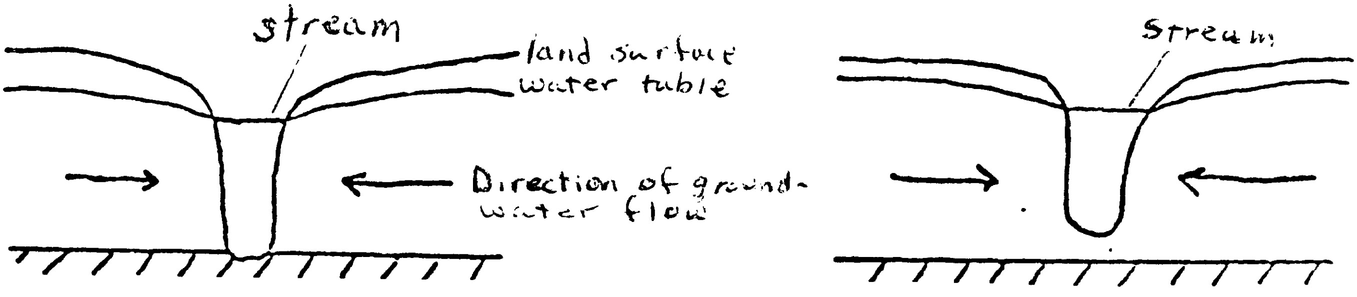
Simulating Large Streams With a Relatively Unlimited Supply of Water in Hydraulic Connection with the Aquifer

Many streams can be simulated relatively simply if we can assume that the quantities of water seeping (one way or the other) between the stream channel and the ground-water system are small compared to the stream discharge and that this seepage does not significantly affect the stream stage. Streams of this type can be simulated using a constant-head or specified-head boundary condition. However, utilization of this boundary condition implies that (1) the stream stage will remain constant (or will change in a pre-defined manner), and (2) the stream will act as an unlimited source or sink for any and all exchanges of water between the stream and the aquifer. The validity of these assumptions should be evaluated closely in each case. Two types of streams that will be considered under this unlimited supply of water classification are: fully penetrating streams, and partially penetrating streams.

Fully Penetrating Streams

Figure 2.6.2 shows two types of fully penetrating streams. In figure 2.6.2A the fully penetrating stream completely intersects the saturated aquifer material. Such a stream effectively isolates hydrologic response on one side of the stream from the other (assuming constant stream stage). However, a stream may behave as a fully penetrating stream even if it does not completely intersect the saturated aquifer material as in figure 2.6.2B. For this condition to exist, the aquifer material below the stream must be in good hydraulic connection with the stream, causing the head in the aquifer material underlying the stream to remain approximately equal to the stream stage. Under such conditions (again assuming constant stream stage), pumping a well on one side of the stream will have no effect on ground-water levels on the other side of the stream.

A fully penetrating stream can be simulated using a constant-head boundary condition for numerical finite-difference and analog simulations (fig. 2.6.3). Adjustment for the conductance of a streambed can be made in the conductance branch in a finite-difference grid or in the resistance in connecting resistors in an analog grid.

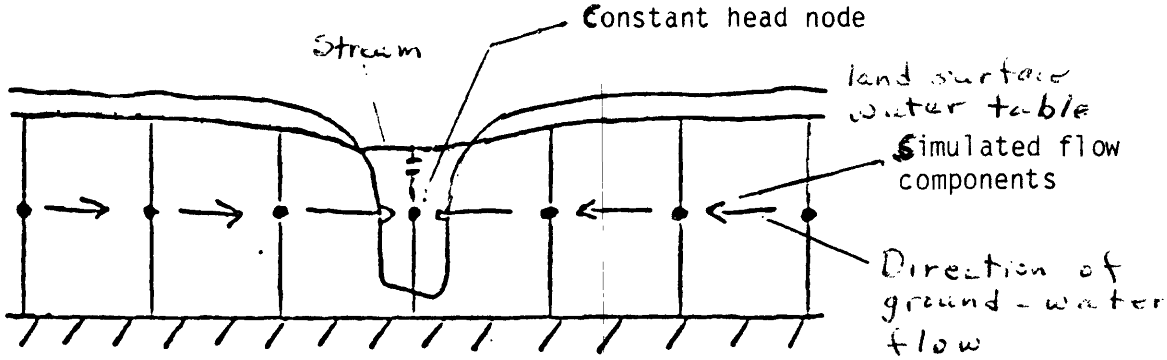


A.--Stream completely penetrates aquifer.

B.--Stream incompletely penetrates aquifer, but hydrologically has similar effect as completely penetrating stream.

Figure 2.6.2.--Fully penetrating streams.

A. Numerical Finite-Difference Simulation



B. Analog Simulation

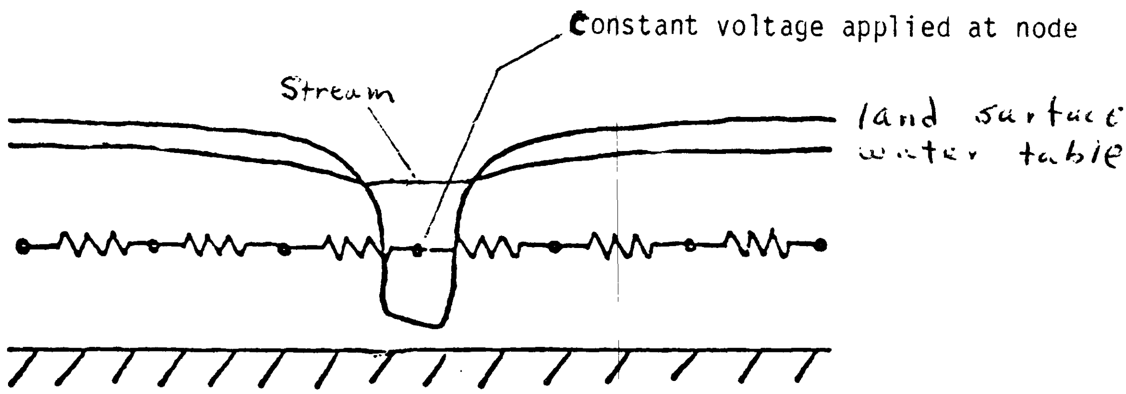


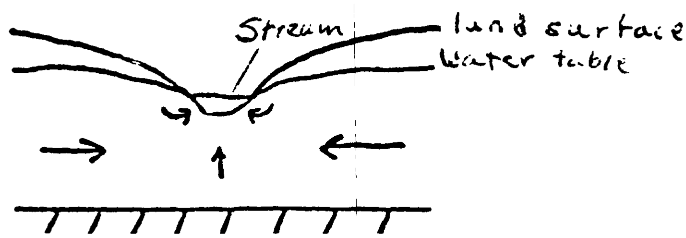
Figure 2.6.3.--Simulation of a fully penetrating stream in numerical finite-difference simulations and analog simulations.

Partially Penetrating Streams

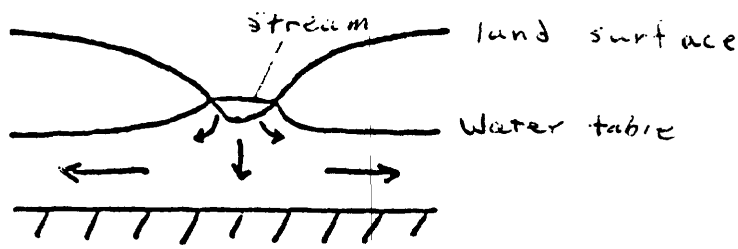
A partially penetrating stream intersects only part of the saturated aquifer material. The stream channel is hydraulically connected to the saturated aquifer material. However, the head in the aquifer directly beneath the stream may be greater (fig. 2.6.4A) or less (fig. 2.6.4B) than the stream stage. The head in the aquifer beneath the stream is part of the solution and can change in a simulation. Water can move from the aquifer on one side of the stream to the aquifer on the other side of the stream under the streambed. Therefore, the effects of a stress on one side of the stream can be observed on the other side of the stream.

In an aquifer with a penetrating stream, vertical flow components exist near the stream. This vertical flow can be simulated using a "leaky" conceptualization of the effect of the stream. This type of representation is shown in figure 2.6.5 for a numerical finite-difference and an analog simulation. This type of simulation approximates the effect of the stream as a "lumped" net leakage between the stream and the aquifer. In situations where the stream is gaining on one side and losing on the other (fig. 2.6.4C), the flow simulated is the net flow between the stream and the aquifer. In the simulated system (fig. 2.6.5), effects of stress in the aquifer on one side of the stream can extend to the other side because horizontal flow beneath the stream is simulated, and the partial penetrating stream affects the quantity of water removed or added vertically.

A--Gaining Stream



B--Losing Stream



C--Stream Simultaneously Gaining and Losing Water

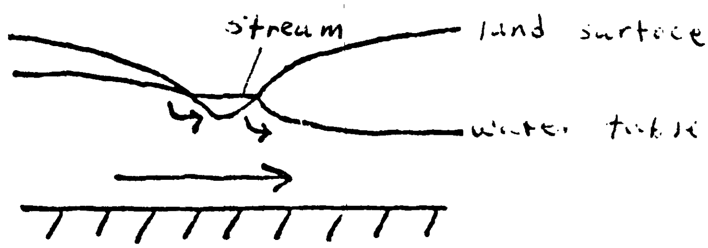
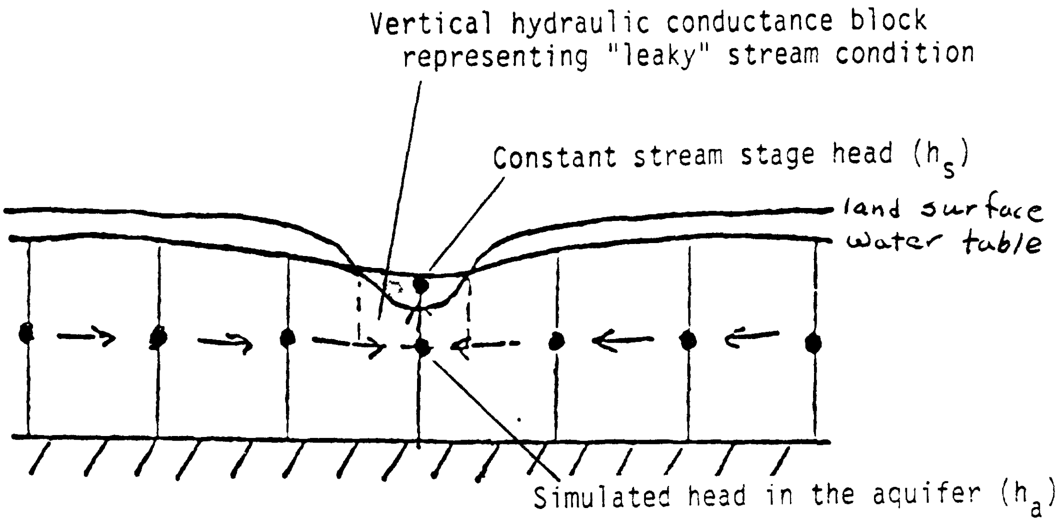


Figure 2.6.4.--Three general classes of partially penetrating streams.

a. Numerical finite-difference simulation



B. Analog simulation

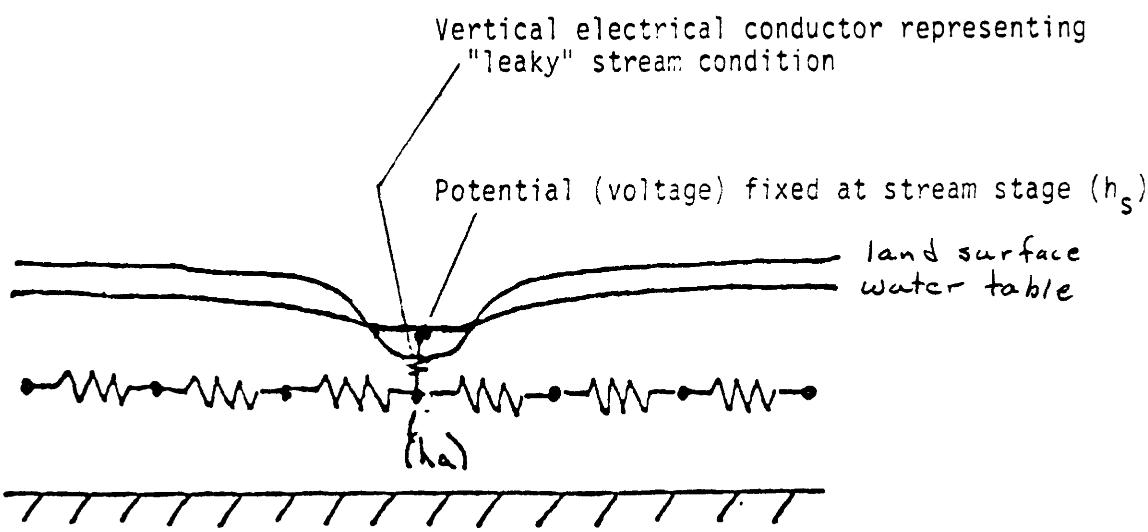


Figure 2.6.5.--Simulation of a partially penetrating stream in numerical finite-difference simulations and analog simulations.

The predicted quantity of seepage is dependent upon the stream surface elevation, h_s, the simulated head in the aquifer, h_a, and an expression for the vertical conductance branch, C_s connecting the stream and aquifer.

The seepage Q_s is then determined by the equation:

$$Q_s = (h_a - h_s)C_s.$$

This equation is graphed in figure 2.6.6. If the stream surface elevation, h_s, is assumed to be a constant then, seepage from the aquifer to the stream is represented in quadrant 1, whereas, seepage from the stream to the aquifer is represented in quadrant 3. The slope of the line in figure 2.6.6 is equal to $1/C_s$.

The vertical conductance between the stream and the aquifer is a lumped parameter, which takes into account the stream and aquifer geometry and the vertical hydraulic conductivity of the streambed and aquifer material. It is also important to note that in this type of simulation, the conductance, C_s, remains constant and does not change with the water level in aquifer.

Simulating Streams That Are Not Always In Hydraulic Connection With The Aquifer

The quantity of water, Q_s, that seeps to or from partially penetrating streams has been approximated by the linear equation:

$$Q_s = (h_a - h_s)C_s$$

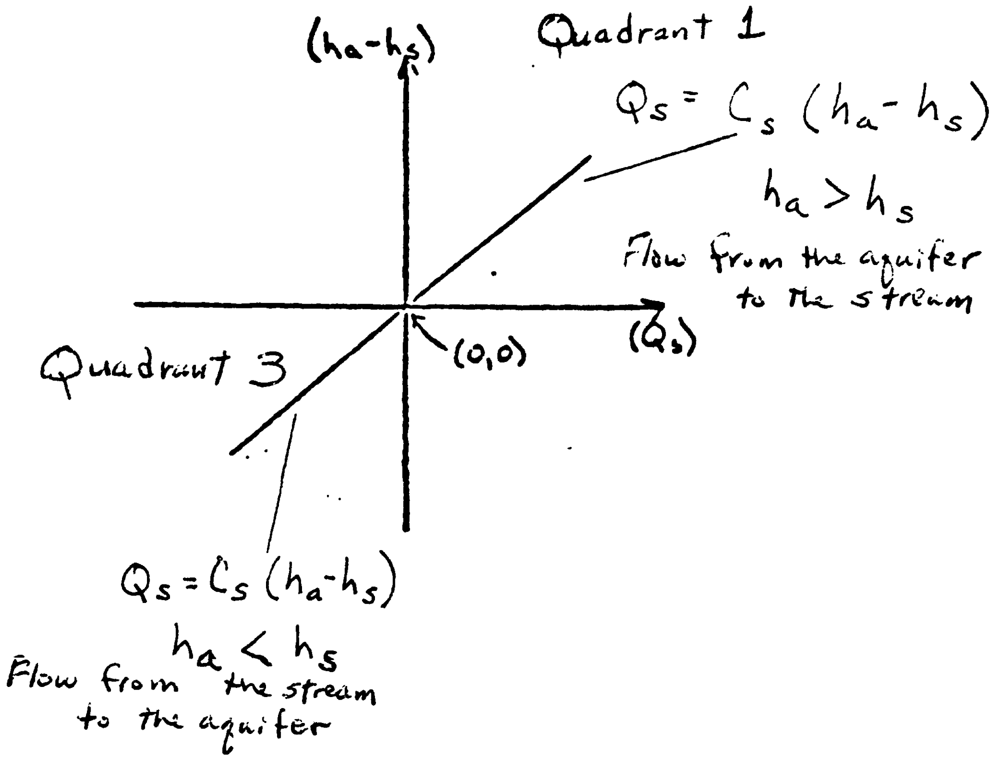


Figure 2.6.6.--Flow between a partially penetrating stream and an aquifer.

where h_a = head in the aquifer beneath the stream,

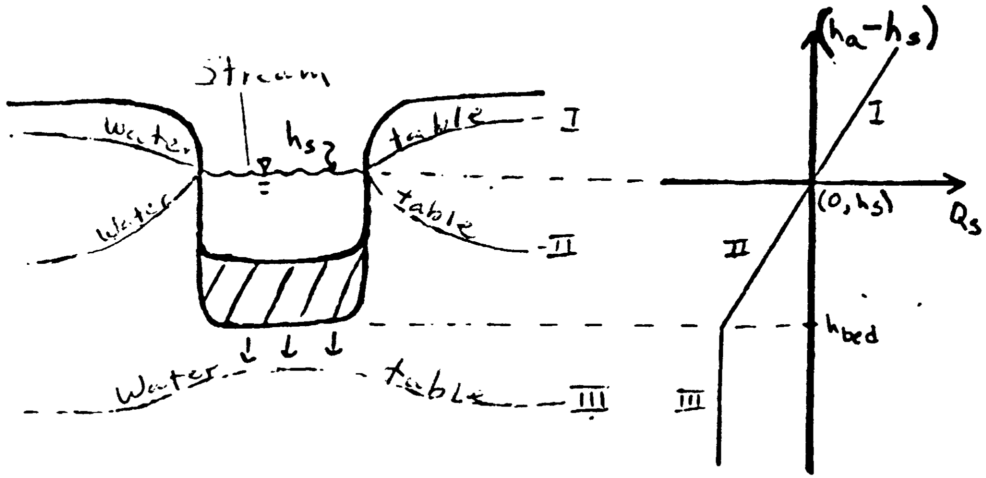
h_s = altitude (head) of the stream surface, and

C_s = conductance of the aquifer beneath the stream.

In a field situation, however, many factors can cause Q_s to behave as a nonlinear function of head. Some of these factors are (1) changes in the length of stream with changing ground-water levels, (2) changes in stream channel cross sectional geometry with changing ground-water levels, and (3) a break in the hydraulic connection between the streambed and the saturated aquifer below. In particular, some streams that are not fully penetrating change ground-water seepage characteristics according to the relative head in the aquifer. These streams can be modeled using a nonlinear seepage relation. Three situations situation are shown in figure 2.6.7 where ground-water levels decline from level I, where $h_a > h_s$ and ground water seeps to the stream; through level II, where $h_a < h_s$ and seepage flows from the stream to the aquifer; to level III, where $h_a < h_s$ and the stream is no longer hydraulically connected with the aquifer, and water seeps from the stream at a constant rate.

Another example of a nonlinear ground-water seepage relationship can be seen in streams on Long Island, New York (Harbaugh and Getzen, 1977). The streams are strictly gaining and dry up as the water table drops below the streambed. In figure 2.6.8, line segment A describes the linear seepage relation when the stream and aquifer are connected and seepage flows to the

FIELD CONDITION	GRAPHICAL REPRESENTATION	GOVERNING EQUATION



I. $Q_s = C_s (h_a - h_s)$

$h_a > h_s$

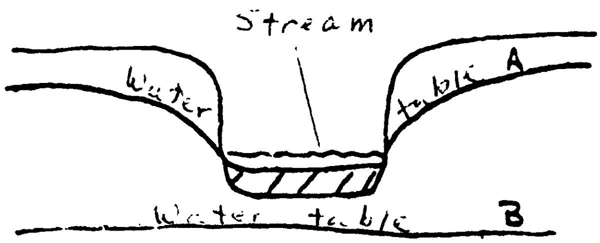
II. $Q_s = C_s (h_a - h_s)$

$h_a < h_s$

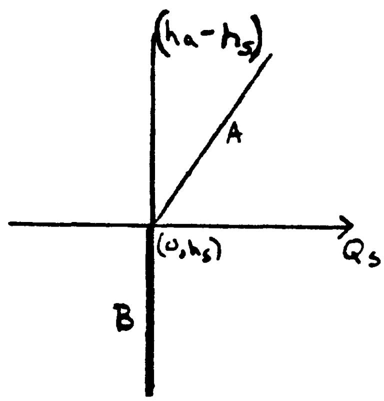
III. Q_s = constant

aquifer is unsaturated beneath stream

Figure 2.6.7.--Linear and non-linear relation between aquifer head and stream seepage.



FIELD CONDITION



GRAPHICAL REPRESENTATION

Figure 2.6.8.--Linear and non-linear ground-water seepage relations for streams on Long Island, New York (Harbaugh and Getzen, 1977).

stream, and line segment B defines the constant rate of zero (no) seepage after the stream has dried up.

Areal Discretization of a Stream Using a Finite-Difference Representation

In a finite-difference model, that is made up of many interconnected discrete model blocks, it is virtually impossible to have all model nodes positioned on the exact location of the actual river course (fig. 2.6.9). The nearest node usually is assigned as a stream node, and the most appropriate simplified governing equations are used to describe the interaction between the stream and the aquifer. In calculating the vertical stream conductances to represent the river, the actual reach of the river to be simulated by each model block should be taken into account. The effect (or error) associated with the inaccurate areal placement of the river is problem dependent and can be minimized during the model design phase of the project.

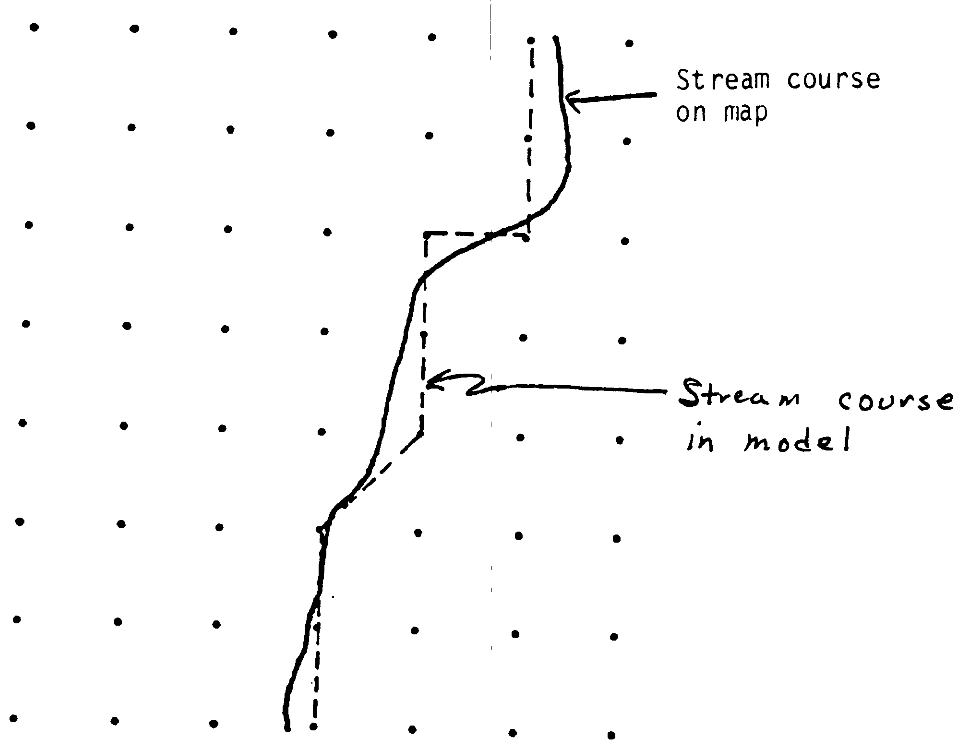


Figure 2.6.9.--Approximated areal location of stream channel as represented in a finite-difference simulation.

Summary and Conclusions

The purpose of simulating streams as presented is to reproduce the effect of the stream on the aquifer as realistically as possible. The simulation of the stream itself does not have to be very sophisticated, but the effect of the stream must be accurately represented. It is important to remember that all actual ground-water flow systems are three-dimensional in nature. The actual ground-water/surface-water interaction virtually is always three-dimensional (figure 2.6.10). Thus, all the simplifications that went into the simplified models presented must always be kept in mind and continually reassessed during a study.

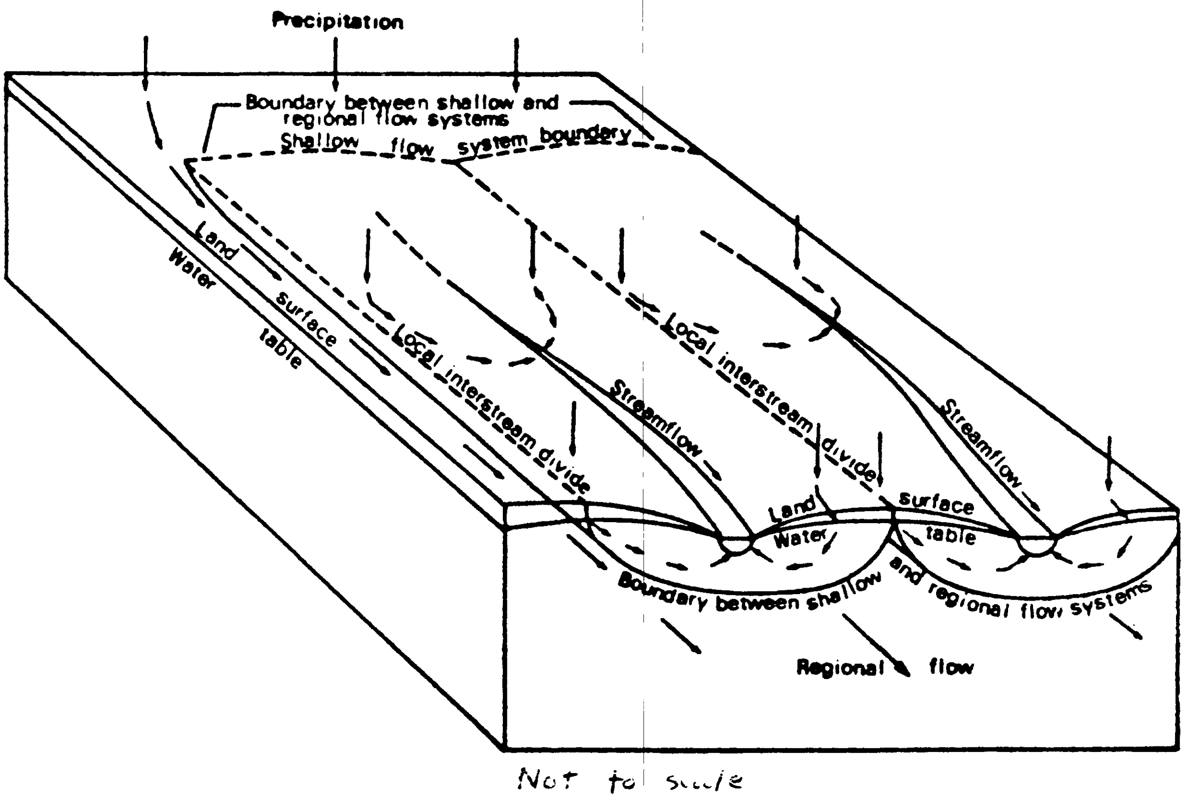


Figure 2.6.10.--Shallow ground-water flow systems associated with two streams and a regional ground-water flow system (Prince, 1980).

NOTE 7, WELL-DRAWDOWN CORRECTION AT A PUMPING NODE

The drawdown calculated by the model for a pumping node is an average for the area represented by the node; it does not include the drawdown in the "cone of depression" immediately around the pumped well. Situations may arise in which we want to know approximately what the actual drawdown would be in a real well at the node. Prickett and Lonnquist (1971, p. 61) and Trescott, Pinder, and Larson (1976, p. 8-10) give a formula for estimating the additional drawdown occurring in the cone around the well. The derivation of the formula follows.

The drawdown calculated at the model node can be envisioned as representative of a very large well of radius, r_a, positioned in the center of the model block (fig. 2.7.1A). If we can determine the radius, r_a, of this hypothetical large-diameter well, then the Thiem equation enables us to calculate the additional drawdown between the hypothetical large radius. r_a, and the actual well radius, r_w, under steady-state conditions. This additional drawdown can then be added to the node drawdown to obtain the actual drawdown in the pumping well (fig. 2.7.1B).

The well-known Thiem equation that describes steady-state radial flow to a fully-penetrating well in a confined aquifer is:

$$h_2 - h_1 = \frac{2.3Q}{2\pi T} \log \frac{r_2}{r_1}, \qquad (1)$$

where

r_1 and r_2 are the radial distances at point one and two respectively,

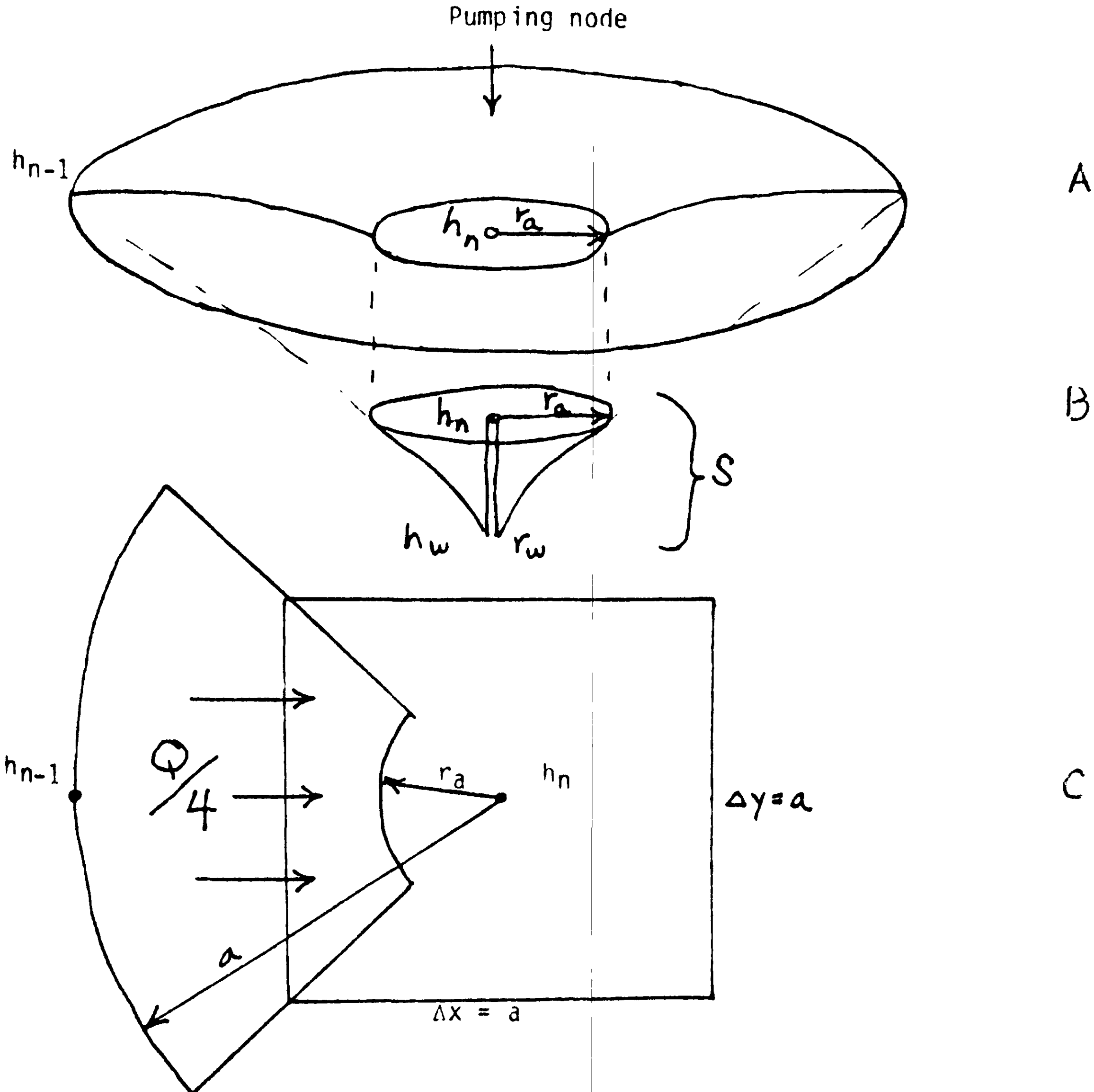


Figure 2.7.1.--Relations for determining drawdown in a real well from drawdown determined at a model node.

A. Hypothetical large-diameter well of radius r_a that accounts for drawdown at model node.

B. Relation of hypothetical large-diameter well of radius r_a to real well of radius r_w at model node.

C. Flow of water through one side of model block.

h_1 and h_2 are the hydraulic heads at point one and two respectively,

T is the aquifer transmissivity, and

Q is the well discharge.

If the node spacing is a, we can think of the flow toward this large hypothetical well as a steady radial discharge between an outer radius, a, and an inner radius, r_a (fig. 2.7.1C). The head at radius a is taken as that for the node adjacent to the pumping node, while the head at r_a is taken as that for the pumping node. Applying the Thiem equation gives

$$h_{n-1} - h_n = \frac{2.3Q}{2\pi T} \log \frac{a}{r_a}, \tag{2}$$

or

$$\log \frac{a}{r_a} = \frac{2\pi T(h_{n-1} - h_n)}{2.3Q}. \tag{3}$$

In the model configuration, the discharge into the pumping node can be thought of as entering through the four sides of a rectangular area around the node (fig. 2.7.1C). The flow through one side would be given by Darcy's law as:

$$\frac{Q}{4} = Ta \frac{h_{n-1} - h_n}{a}, \tag{4}$$

from which

$$\frac{T(h_{n-1} - h_n)}{Q} = \frac{1}{4}. \tag{5}$$

Substituting (5) into (3) gives

$$\log \frac{a}{r_a} = \frac{2\pi}{2.3}\,\frac{1}{4} = 0.68, \tag{6}$$

so that

$$\frac{a}{r_a} = 4.81, \text{ or } r_a = \frac{a}{4.81}. \tag{7}$$

That is, a well whose radius is 1/4.81 times the node spacing will have a drawdown or water level equal to that calculated by the model for the pumping node.

The additional drawdown in a real well of radius r_w, is determined by calculating the head loss for steady-state radial flow between r_a and r_w (fig. 2.7.1B), again using the Thiem equation. The additional drawdown(s) is thus equal to:

$$s = \frac{2.3Q}{2\pi T} \log \frac{r_a}{r_w} = \frac{2.3Q}{2\pi T} \log \frac{a}{4.81(r_w)}. \tag{8}$$

The total drawdown in the actual well is then equal to the drawdown at the node plus the additional drawdown calculated by equation 8.

This derivation assumes that (1) flow is within a square model block and can be described by a steady-state equation, (2) one fully-penetrating well is located in the center of the model block, (3) the aquifer is isotropic and homogeneous in the model block, (4) well losses are negligible, and (5) the aquifer is confined. Trescott and others (1976, p. 10) also give a form of the equation for unconfined aquifers.

CHAPTER 3--PROBLEMS

PROBLEM 1, CALCULATION OF LUMPED HYDRAULIC CONDUCTANCES AND STORAGE CAPACITIES IN RECTANGULAR GRIDS

Review of Concepts Related to Discretization

Before considering the details of calculating hydraulic conductance and storage capacities, we will review some of the important concepts related to discretization.

1. What is discretization?

Discretization is the dividing of a continuous system into a system made of lumped "discrete" elements. A map of a space discretized system consists of a network of lines (branches) which intersect at points (junctions or nodes)(figs. 3.1.1 and 3.1.2).

2. Why discretize?

Discretization (in our case, the finite-difference method of discretization) allows us to use a system of algebraic equations to represent the continuous differential equation governing the problem. For each node in our discretized system, there is one algebraic equation that expresses the principle of continuity in the vicinity of that node. For a system with n nodes, there are n simultaneous linear equations.

3. How is a system discretized?

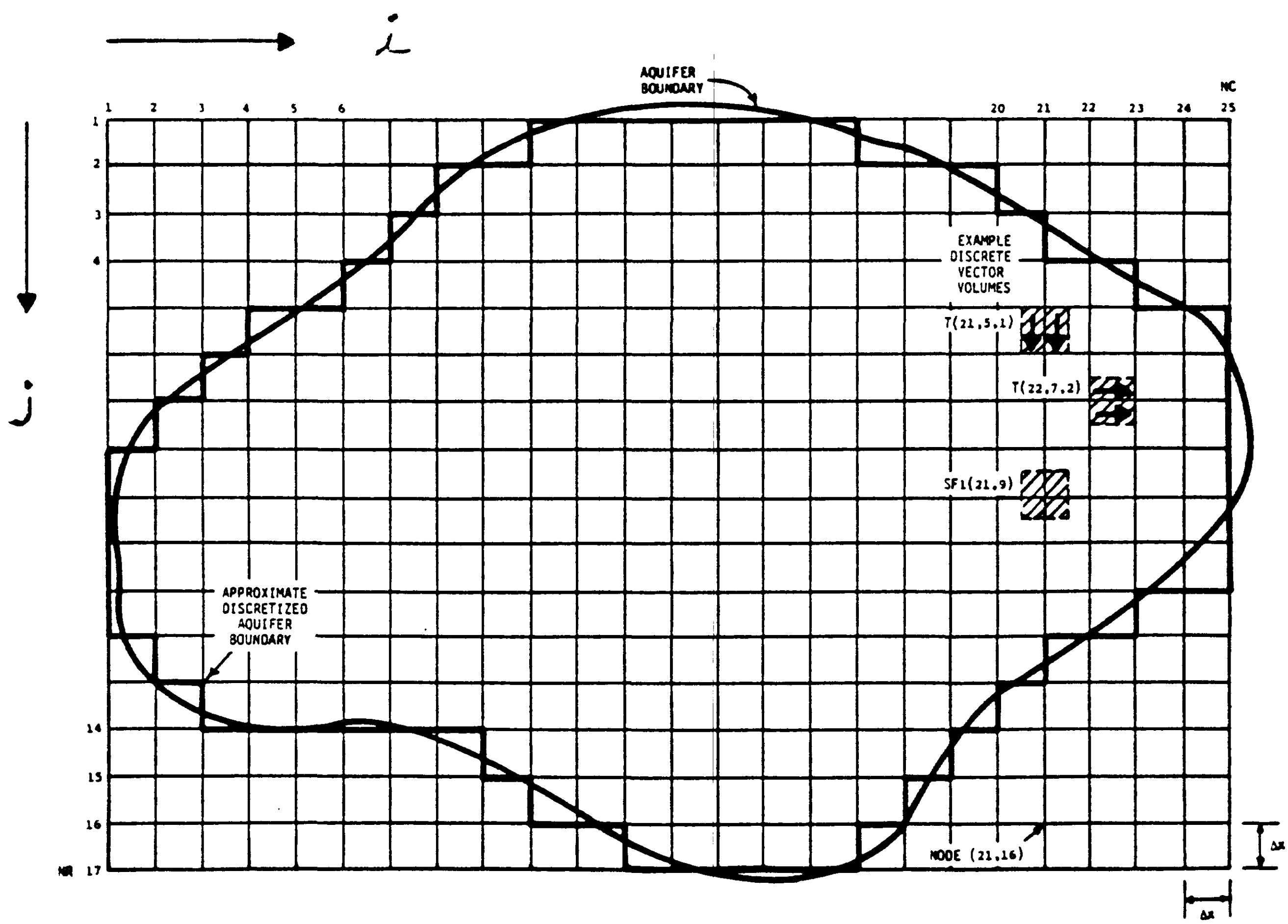


Figure 3.1.1.--Plan view of finite-difference grid over map of aquifer system (from Prickett and Lonnquist, 1971, figure 7).

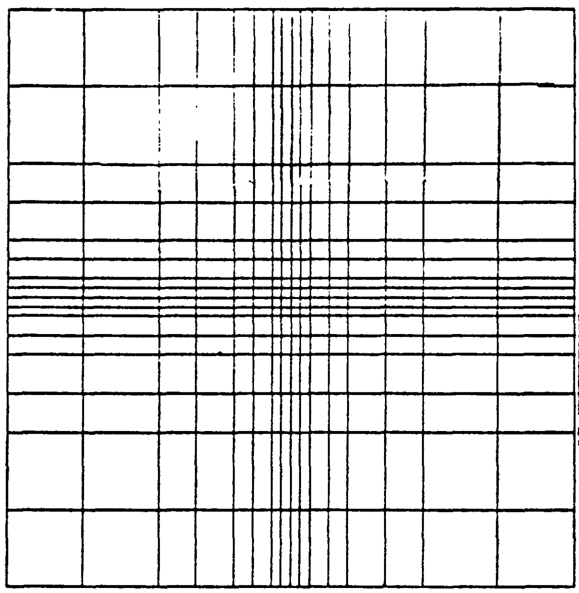


Figure 3.1.2.--Finite-difference grid using variable spacing (from Prickett and Lonnquist, 1971, figure 24).

a. A system can be discretized in space for the finite-difference method using either uniform grid spacing (network branches form squares, fig. 3.1.1) or variable grid spacing (network branches form rectangles, fig. 3.1.2).

b. Factors, which may be interrelated, that should be taken into account when determining the overall size and spacing of the finite-difference grid are as follows:

1) Scale and location of stresses to be applied to the model--point stresses, nonpoint stresses, and areal distribution of stresses. For example, if we know only total regional withdrawals from the system, a mesh size fine enough to allow simulation of individual wells may not be warranted.

2) Boundaries--system boundaries versus model boundaries. The model grid should extend to "natural" system boundaries if possible. If "artificial" model boundaries are used, they should be far enough away from the area or points of stress to have a negligible effect on model response.

3) Accuracy of input data describing system parameters in area of interest (regional estimates of system response versus site-specific estimates). For example, if we know only an average T value in an area, a fine mesh size requiring detailed mapping of T may not be warranted.

4) Information required from model (site specific versus regional). For example, are we interested in the drawdown in the vicinity of the well or is knowledge of the regional effects of the pumping sufficient?

5) Consideration of numerical error--the model solves a system of algebraic-difference equations that simulates the partial-differential equation of ground-water flow. The finite-difference approximations to the various derivatives contain "truncation" errors which, in general, are of the same order as the mesh spacing, the square of the mesh spacing, or the length of the time step. As the mesh is made more coarse or the time step is extended, these truncation errors increase, and the solution of the set of difference equations may deviate increasingly from the desired solution of the partial-differential equation.

6) Computer capabilities--size of core storage and computation time for a given problem size (the problem size is defined, in general, by the number of nodes in the discretized system), which in turn determines cost and operational feasibility of solving a problem.

Calculation of Areas and Volumes Associated with Branches in Rectangular Finite-Difference Grids

Our first goal in this problem assignment is to calculate hydraulic conductances for branches in rectangular finite-difference networks. The first step is to determine the top or map area (vector area of Karplus, 1958) associated with each branch, which along with the "thickness", defines the block of aquifer material (vector volume of Karplus, 1958) associated with each branch. As we will see later, this area associated with the branch is not the area needed to calculate hydraulic conductance.

Square networks (network branches form squares (figure 3.1.3) are a special case of rectangular networks (network branches form rectangles (figure 3.1.4). The following general procedure for determining the top or map area associated with branches in rectangular networks is applicable to both.

1. Starting at one (either) end of the branch (at a node) under consideration, draw a line that is perpendicular to the branch under consideration halfway to the next node.

2. From this point, draw a line equal in length and parallel to the branch under consideration.

3. Return to the starting node and repeat steps 1 and 2 in the opposite direction. (See examples in figures 3.1.3 and 3.1.4)

The resulting rectangle (or square) (figs. 3.1.3 and 3.1.4) represents the vector area associated with the branch. The vector volume of earth material associated with the branch is obtained by multiplying this area by the (in this case) aquifer thickness.

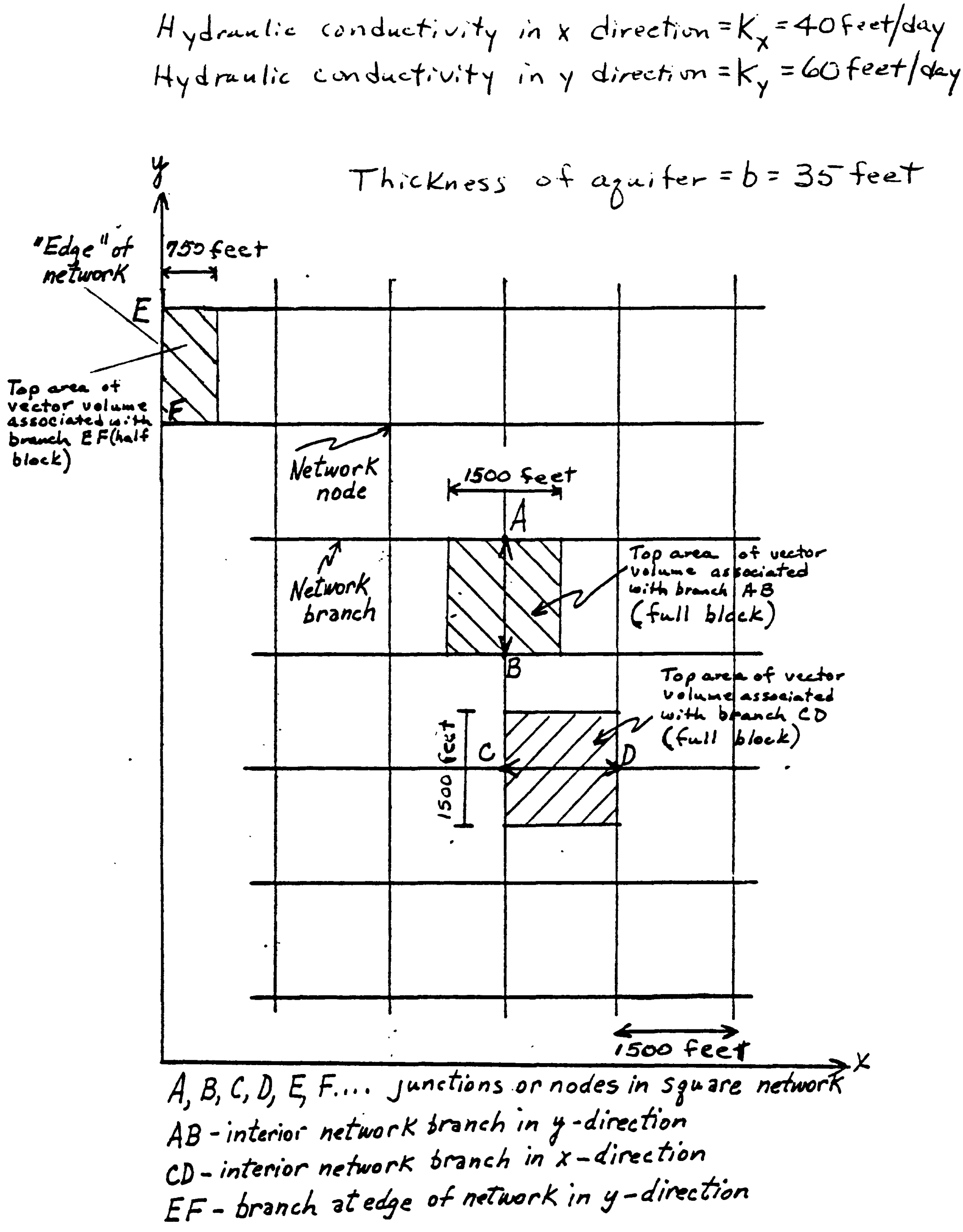


Figure 3.1.3.--Hydraulic conductance in a square finite-difference grid.

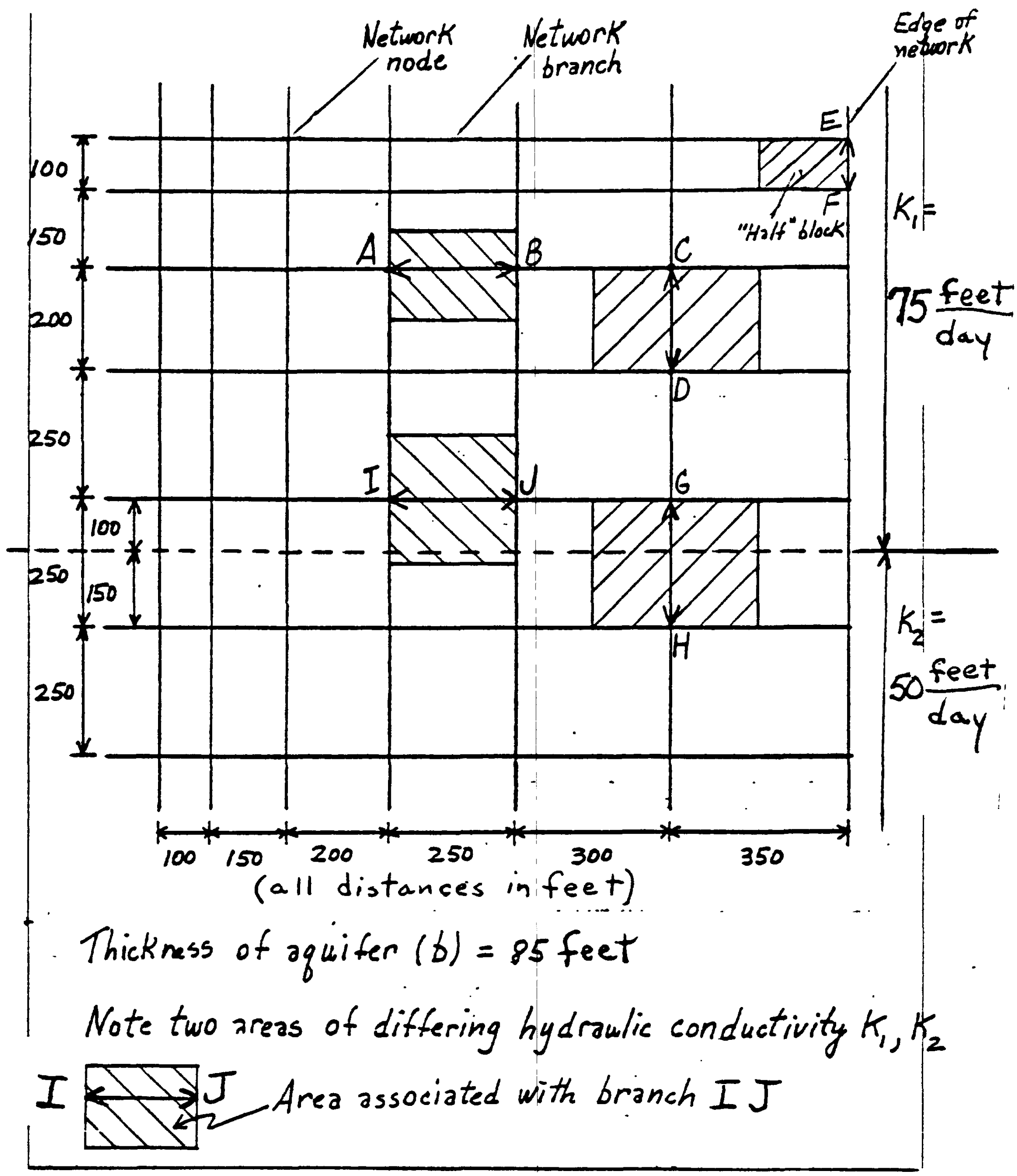


Figure 3.1.4.--Hydraulic conductance in a rectangular finite-difference grid.

Calculations of the areas and volumes associated with branches for the examples shown in figures 3.1.3 and 3.1.4 are given in table 3.1.1.

Definition of Hydraulic Conductance

The blocks of earth material whose volumes were calculated in table 3.1.1 can be thought of as Darcy prisms (represented by branches in a map of a finite-difference network) between junctions (nodes in network map) where head is measured. Darcy's law can be written

$$Q = KA\frac{\Delta h}{L},$$

where Δh is the difference in head at the two ends of the prism (fig. 3.1.5A. The cylinders shown in fig. 3.1.5 are hydrologically equivalent to vector volumes associated with a nodal network). Rearranging the formula above, we will define the hydraulic conductance as

$$C = \frac{KA}{L} = \frac{Q}{\Delta h}.$$

We see that the hydraulic conductance is a lumped coefficient, obtained directly from Darcy's law, that represents the transmitting capability of a block of earth material. This block of earth material is represented by a branch on a map of a discretized aquifer system.

Table 3.1.1.--Areas and volumes associated with network branches for examples in figures 3.1.3 and 3.1.4

Branch	Area associated with branch (square feet)	Volume associated with branch (cubic feet)
	Examples in Figure 3.1.3	
AB	(750 + 750)(1,500) = 2,250,000	(1,500)(1,500)(35) = 78,750,000
CD	(750 + 750)(1,500) = 2,250,000	(1,500)(1,500)(35) = 78,750,000
EF	(750)(1,500) = 1,125,000	(750)(1,500)(35) = 39,375,000
	Examples in Figure 3.1.4	
AB	(75 + 100)(250) = 43,750	(175)(250)(85) = 3,718,750
CD	(150 + 175)(200) = 65,000	(325)(200)(85) = 5,525,000
EF	(175)(100) = 17,500	(175)(100)(85) = 1,487,500
GH	(150 + 175)(250) = 81,250	(325)(250)(85) = 6,906,250
IJ	(125 + 125)(250) = 62,500	(250)(250)(85) = 5,312,500

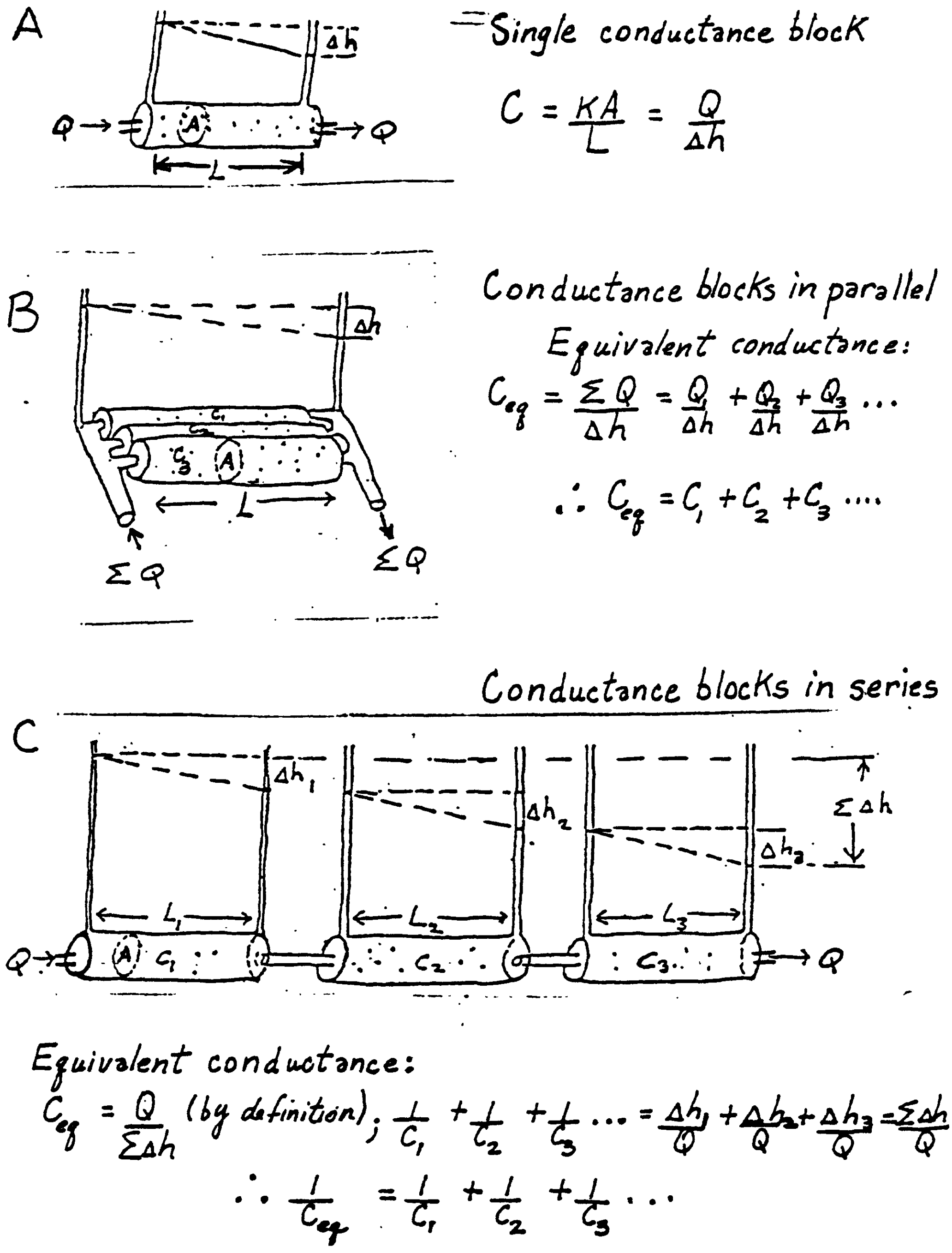


Figure 3.1.5.--Hydraulic conductances of blocks in parallel and in series.

Calculation of Hydraulic Conductance

The calculation of hydraulic conductances for homogeneous blocks or prisms of earth material requires only the application of the above formula defining hydraulic conductance. A complication arises, however, when the blocks are not homogeneous. The general procedure is to break the large block into smaller blocks, calculate the conductances for the smaller blocks, and then combine these conductances into one lumped conductance for the large block. The rules for combining conductances, which may be connected either in series or parallel, are summarized in figure 3.1.5. These rules should be carefully studied and memorized.

The structure of the formulas in figure 3.1.5 may recall the rules for combining resistances in series and parallel that one encounters in first-year physics. Remember that resistance and conductance, both electric and hydraulic, are reciprocals of one another. Thus, "opposite" rules apply in combining resistances and conductances. Resistances in series are combined by addition. Resistances in parallel are combined to an equivalent resistance by applying the reciprocal rule. Compare these rules for combining resistances with the rules for combining conductances in figure 3.1.5.

Calculations of hydraulic conductances for the selected network branches shown in figures 3.1.3 and 3.1.4 are given in table 3.1.2. The key to calculating hydraulic conductance is to have a clear picture of how the Darcy prisms are oriented relative to the finite-difference network. Conceptually, flow can occur only along branches, which can be regarded as an abbreviated graphical representation of a Darcy prism oriented parallel to the branch.

The area (A) in the hydraulic conductance equation is the cross-sectional area of the Darcy prism. Because the network branch is parallel to the axis of the Darcy prism, the cross-sectional area (A) of the Darcy prism is oriented perpendicular to the network branch.

Calculate the hydraulic conductances for the branches of the rectangular grid shown in the table on worksheet 3.1.1.

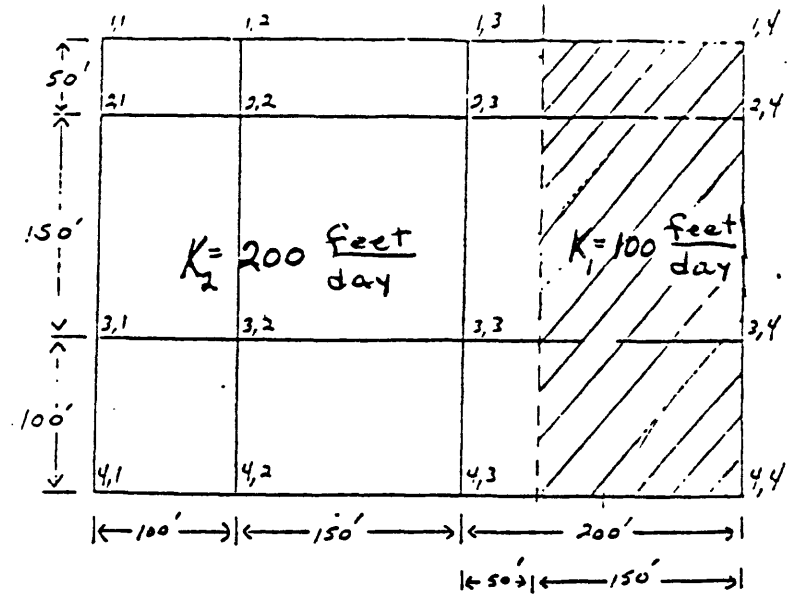
Calculation of Storage Areas Associated with Nodes in Rectangular Finite-Difference Grids

Our second goal in this problem assignment is the calculation of storage capacities in rectangular finite-difference grids. The first step in this process is the calculation of storage areas associated with nodes (area of field represented by a capacitor in Karplus, 1958). Because square finite-difference grids are a special case of rectangular finite-difference grids, the following procedure for determining storage areas is valid for both rectangular and square grids.

Table 3.1.2.--Calculations of hydraulic conductances for selected network branches in figures 3.1.3 and 3.1.4

Branch	Formula used	Hydraulic conductance (feet squared per day)
		Examples in Figure 3.1.3
AB	Single	$\frac{60[35(750 + 750)]}{1,500} = 2,100$
CD	Single	$\frac{40[35(750+750)]}{1,500} = 1,400$
EF	Single	$\frac{60[35(750)]}{1,500} = 1,050$
		Examples in Figure 3.1.4
AB	Single	$\frac{75[85(75+100)]}{250} = 4,462$
CE	Single	$\frac{75[85(150+175)]}{200} = 10,359$
EF	Single	$\frac{75[85(175)]}{100} = 11,156$
GH	Series	$\frac{1}{C_{eq}} = \frac{1}{\frac{75[85(150+175)]}{100}} + \frac{1}{\frac{50[85(150+175)]}{150}} = \frac{1}{20,719} + \frac{1}{9,208};\ C_{eq} = 6,378$
IJ	Parallel	$C_{eq} = \frac{75[85(125+100)]}{250} + \frac{50[85(25)]}{250} = 6,162$

Worksheet 3.1.1.--Calculation of hydraulic conductances.



Thickness = b = 100 feet

Calculate hydraulic conductances for the branches listed below:

BRANCH	FORMULA USED (series, parallel, or single)	C (feet squared / day)
1,1 - 1,2		
1,2 - 2,2		
1,3 - 1,4		
2,3 - 3,3		
3,1 - 4,1		
3,2 - 3,3		

Single: $C = \frac{KA}{L}$

Series: $\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$

Parallel: $C_{eq} = C_1 + C_2$

Four branches meet at each of the interior nodes, three branches meet at the "edge" nodes, and two to four branches meet at "corner" nodes depending on whether it is an inside or outside corner (fig. 3.1.6) (there may be many more than four "corner" nodes). Construct perpendicular bisectors for all branches intersecting at a given node and extend these lines until they intersect one another. The rectangular (or square) figure within the network boundaries formed by these intersecting perpendicular bisectors outlines the storage area for the node under consideration. Examples of storage areas are shown in figures 3.1.6 and 3.1.7, and numerical calculations of these same storage areas are given in table 3.1.3.

Definition of Storage Capacity

The definition of storage capacity is given here only to permit numerical solution of the following problem. The relation of storage capacity to the basic ground-water flow equations will be discussed elsewhere.

For unconfined flow, the storage capacity (S_c) may be defined as

$$S_c = S_y A,$$

where S_y is the specific yield and A is the storage area as determined in the previous section. For confined flow, the storage capacity, S_c, may be defined as

$$S_c = SA = S_s bA$$

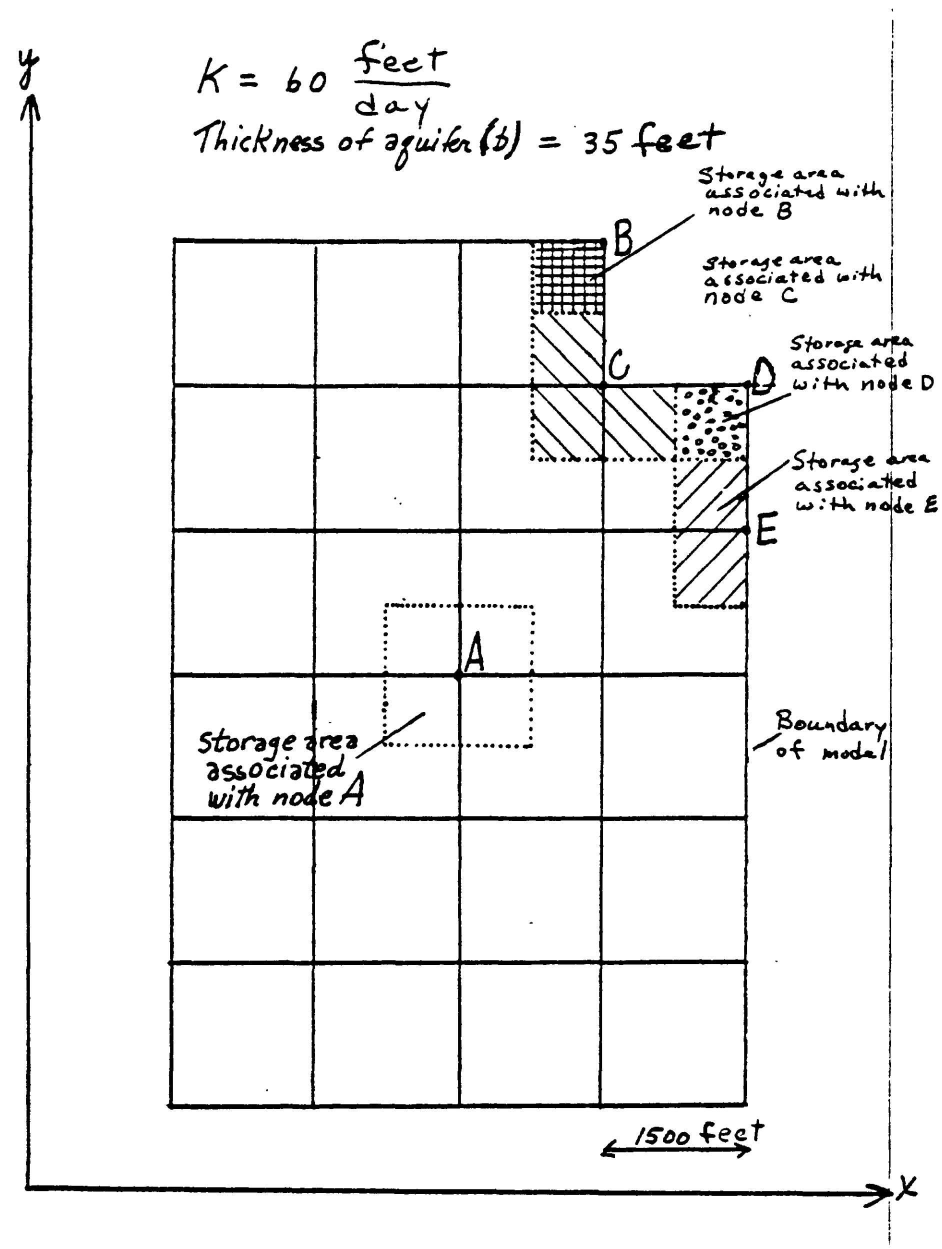


Figure 3.1.6.--Storage areas in a square finite-difference grid.

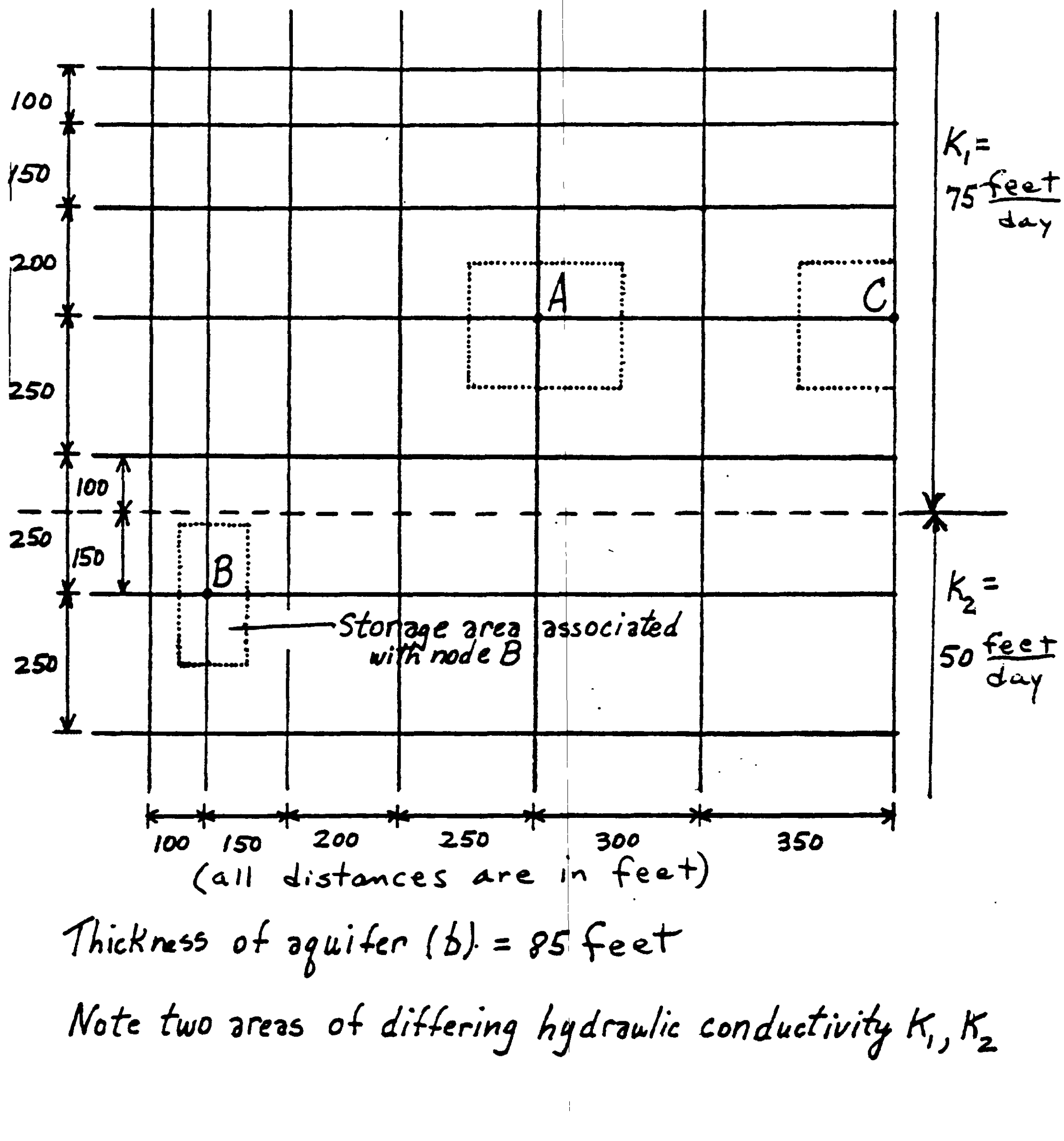


Figure 3.1.7.--Storage areas in a rectangular finite-difference grid.

Table 3.1.3.--Calculations of storage areas for selected nodes in figures 3.1.6 and 3.1.7.

Node	Storage area (feet squared)
	In Figure 3.1.6
A	4(750)(750) = 2,250,000
B	1(750)(750) = 562,500
C	3(750)(750) = 1,687,500
D	1(750)(750) = 562,500
E	2(750)(750) = 1,125,000
	In Figure 3.1.7
A	(125 + 150)(100 + 125) = 61,875
B	(50 + 75)(125 + 125) = 31,250
C	175(100 + 125) = 39,375

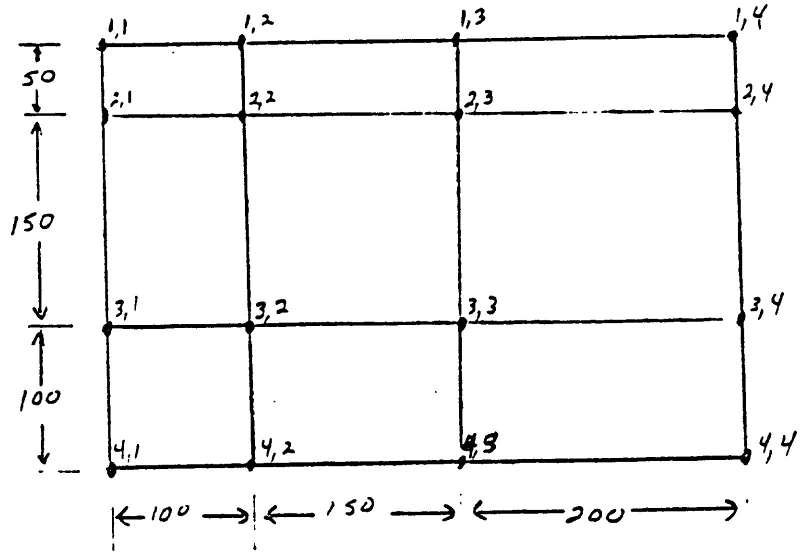
where S is the storage coefficient, S_s is the specific storage (ft^{-1}), and b is the aquifer thickness.

We see from the definitions above that calculation of storage capacity values involves multiplying the storage area by (presumably) known or assumed aquifer parameters.

Calculation of Storage Capacities

Calculate the storage capacities for the nodes of the rectangular grid that are listed on worksheet 3.1.2.

Worksheet 3.1.2.--Calculation of storage capacities



Thickness(b)=100 feet

S_s = $1. \times 10^{-5}$ feet^{-1}

CALCULATE STORAGE CAPACITIES FOR EACH NODE

NODE	STORAGE AREA	S_c	NODE	STORAGE AREA	S_c
1,1			3,1		
1,2			3,2		
1,3			3,3		
1,4			3,4		
2,1			4,1		
2,2			4,2		
2,3			4,3		
2,4			4,4		

PROBLEM 2, NUMERICAL ANALYSIS STEADY STATE

Introduction

The purpose of these notes and the associated problems is to (1) introduce the concept of representing the continuous partial-differential equation of ground-water flow by a set of simultaneous algebraic equations, the method of finite differences; and (2) use two methods to solve the set of simultaneous algebraic equations, an example of a direct method and an example of an iterative method.

Ground-water flow is described mathematically by partial-differential equations. These mathematical expressions (mathematical models) are based on the conservation of mass and energy and mathematically represent the cause and effect relation that are necessary to describe the physical process of flow through porous media. The specification of boundary conditions completes the mathematical description of a ground-water flow system. After the specific flow system under study has been defined mathematically, the general differential equations that describe ground-water flow can be solved for the specific system.

An analytical solution of these mathematical models (differential equations plus boundary conditions) specifies the dependent variable continuously in space and time. Usually, the dependent variable in the ground-water flow equations is hydraulic head, but sometimes it is fluid pressure.

Frequently, mathematical models can be simplified by making certain assumptions regarding the flow system. These assumptions entail a simplification in the representation of the physical system that translates into a less complex mathematical equation. Many such simplified mathematical equations have general, functional, or algebraic solutions (analytical solutions) that describe the dependent variable through space and time. In some instances however, the physical system does not lend itself to such simplification, and the mathematical models cannot be solved analytically. In these cases, some method of numerical simulation is used to solve the mathematical model and, in turn, to describe the physical processes.

The finite-difference method is one technique of numerical simulation. The finite-difference method involves representing continuous space and time by means of discrete blocks or elements in space and discrete increments of time. Thus, the flow system is represented in space by a set of discrete elements that allows it to be defined mathematically by algebraic equations, which approximate the continuous differential equations.

In the finite-difference method, the continuous partial derivative $\partial u/\partial x$ is approximated by an algebraic equation. The formal definition of the finite-difference method uses a trucated Taylor series expansion. A Taylor series for $u(x+\Delta x)$ can be written as:

$$u(x+\Delta x) = u(x) + \Delta x \frac{\partial}{\partial x}[u(x)] + \frac{1}{2!}(\Delta x)^2 \frac{\partial^2}{\partial^2 x}[u(x)] + \ldots \quad (1a)$$

Rearranging, we can solve for the derivative of the functions

$$\frac{\partial[u(x)]}{\partial x} = \frac{u(x + \Delta x) - u(x)}{\Delta x} - \frac{1}{2!\Delta x}(\Delta x)^2 \frac{\partial^2}{\partial x^2}[u(x)] - \ldots \quad (1b)$$

and truncating gives

$$\frac{\partial[u(x)]}{\partial x} \approx \frac{u(x + \Delta x) - u(x)}{\Delta x}, \quad (2)$$

which is an approximation of the derivative in terms of two specific values of the function. The error between the exact value of the derivative evaluated at x and the discrete approximation (equation 2) is defined by the magnitude of the truncated terms and is referred to as truncation error (see also chapter 2, section 2.4.

A more intuitive approach is to consider the curve described by the function u(x) and $\frac{\partial[u(x)]}{\partial x}$, the slope of the curve at point X. We can approximate this slope by the slope of a straight line segment connecting two points on the curve [u(x) and u(x+Δx)] that are separated by a discrete distance (Δx).

This approximation is given as

$$\frac{\partial u}{\partial x} \approx \frac{u(x + \Delta x) - u(x)}{(x + \Delta x) - x} = \frac{u(x + \Delta x) - u(x)}{\Delta x}. \quad (3)$$

The discrete system of blocks and the algebraic representation of the derivative between the blocks generate a system of simultaneous algebraic equations (one equation for each node) which can be solved to give the value of the dependent variable at points (nodes) in the system. Solving the set of simultaneous algebraic equations represents an approximate solution to the original continuous differential equation. For further discussion, see Bennett (1976, p. 136-152).

Definition of the Problem

The problem is to determine the steady-state head distribution within the aquifer shown in figure 3.2.1. The aquifer is surrounded by large surface-water bodies that impose constant-head boundaries on the aquifer system. The hydraulic head acting on the aquifer boundaries due to the depth of these surface-water bodies is known, and we will assume that the aquifer can be represented in two dimensions with uniform transmissivity and no recharge from above or below. Although the aquifer is unconfined, we will assume that the saturated thickness is constant. This allows us to use the governing two-dimensional steady-state ground-water flow equation for confined (or constant thickness) aquifers which is:

$$T_x \frac{\partial^2 h}{\partial x^2} + T_y \frac{\partial^2 h}{\partial y^2} = 0. \tag{4}$$

Assuming the aquifer system to be isotropic ($T_x = T_y$), the equation can be simplified to

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0. \tag{5}$$

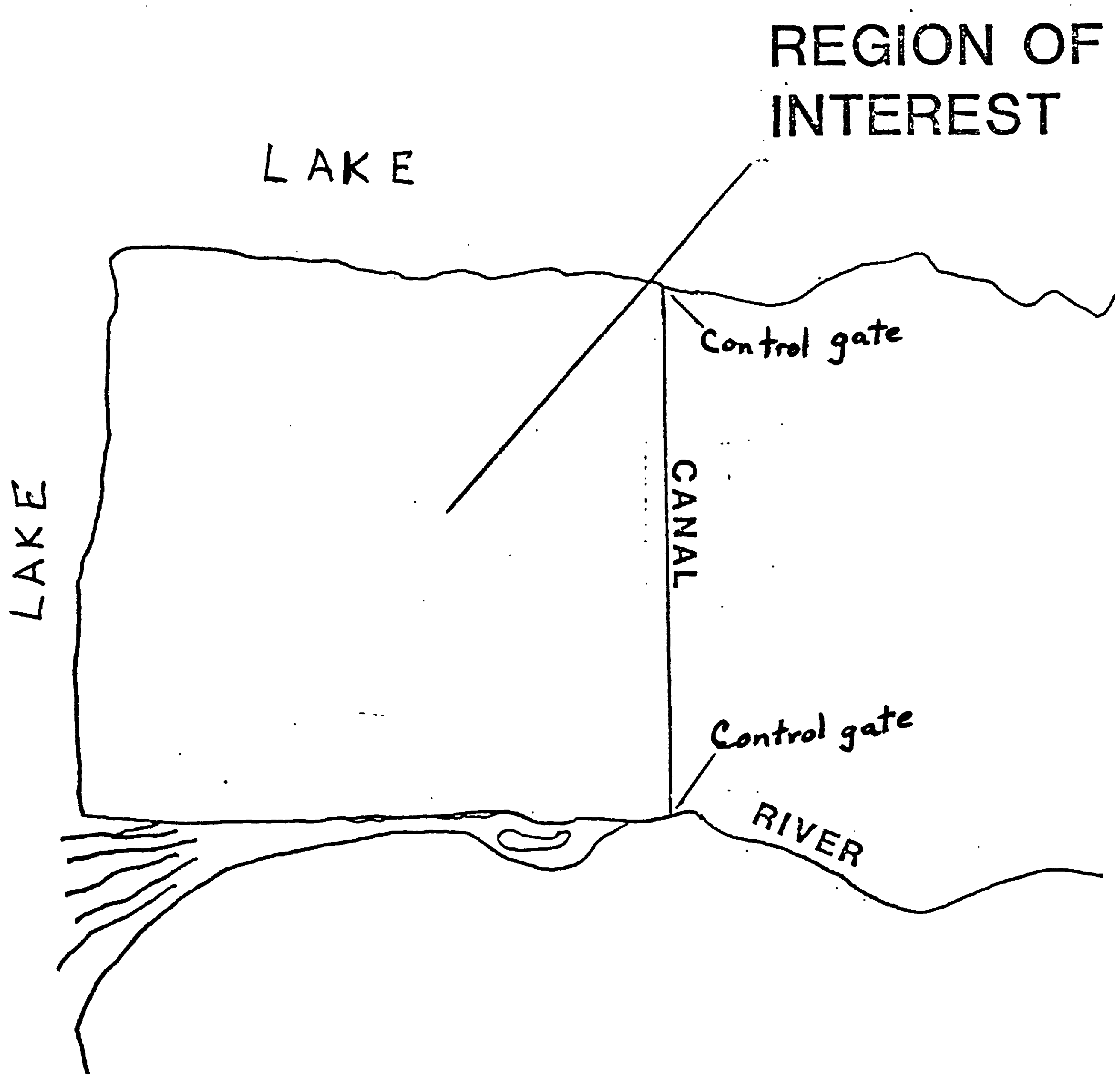


Figure 3.2.1--Location map for numerical analysis problem

Discretizing the Aquifer System

The continuous aquifer will be discretized (divided into blocks) into a 4 x 4 finite-difference grid using equal spacing between nodes (fig. 3.2.2). The values of constant head at the boundary nodes due to the surface-water bodies surrounding the aquifer system (the large lake, river, and canal) are also given in figure 3.2.2.

As noted previously, the purpose of discretization is to allow the system to be represented by a set of simultaneous algebraic equations. The general algebraic equation (the finite-difference equation) to be solved at each node (Bennett, 1976, p. 132) is

$$h_{i,j} = \frac{1}{4}(h_{i+1,j} + h_{i-1,j} + h_{i,j+1} + h_{i,j-1}) . \qquad (6)$$

The Set of Simultaneous Algebraic Equations

Applying the general equation (6) to each node in the model area, we can write the set of four simultaneous algebraic equations to be used to solve for the head at each node.

$$h_{2,2} = (1/4)(h_{3,2} + h_{1,2} + h_{2,3} + h_{2,1}); \qquad (7a)$$

$$h_{2,3} = (1/4)(h_{3,3} + h_{1,3} + h_{2,4} + h_{2,2}); \qquad (7b)$$

$$h_{3,2} = (1/4)(h_{4,2} + h_{2,2} + h_{3,3} + h_{3,1}); \qquad (7c)$$

$$h_{3,3} = (1/4)(h_{4,3} + h_{2,3} + h_{3,4} + h_{3,2}). \qquad (7d)$$

Because the aquifer is completely surrounded by known constant-head boundaries (which are represented by nodes of known

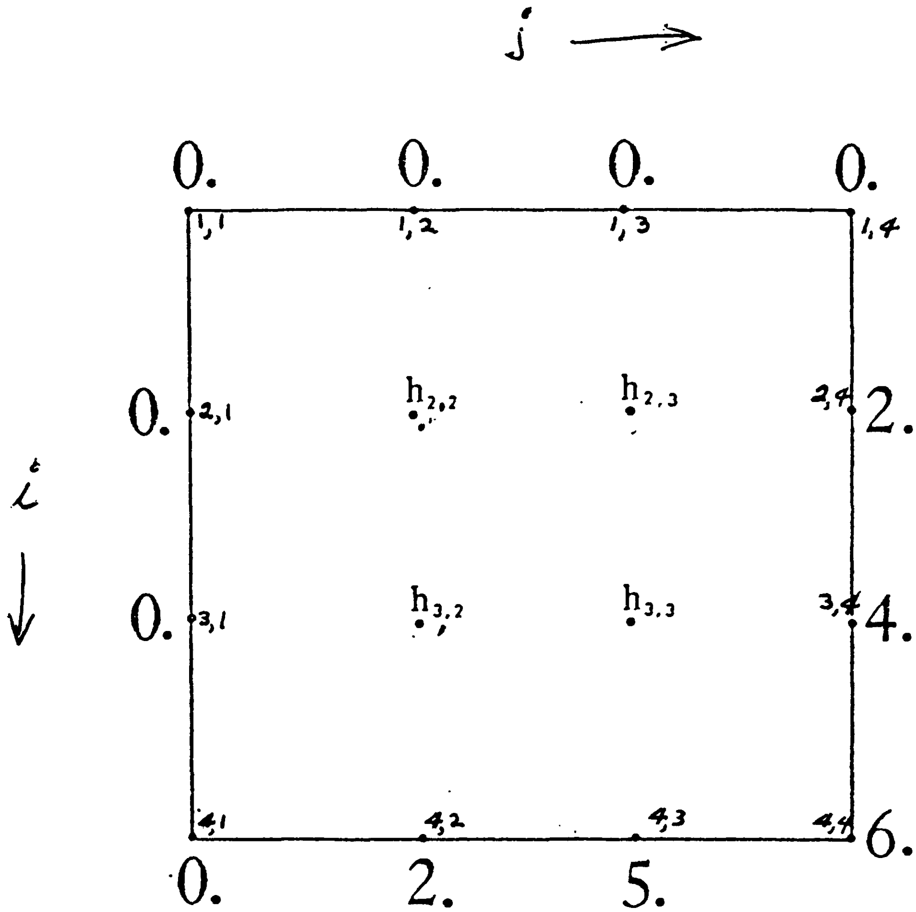


Figure 3.2.2--Discretized aquifer for numerical analysis problem

constant head), the only unknown head values are located at the four interior nodes.

Substituting the known values of constant head at boundaries nodes as shown in figure 3.2.2 in (7), the set of equations becomes

$$h_{2,2} = (1/4)(h_{3,2} + h_{2,3}) \quad (8a)$$

$$h_{2,3} = (1/4)(h_{3,3} + h_{2,2} + 2) \quad (8b)$$

$$h_{3,2} = (1/4)(h_{2,2} + h_{3,3} + 2) \quad (8c)$$

$$h_{3,3} = (1/4)(h_{2,3} + h_{3,2} + 9). \quad (8d)$$

Thus, we have a set of four equations and four unknowns. In order to obtain the values for these four unknown heads, the set of simultaneous equations must be solved.

Two Methods of Solving the Set of Simultaneous Algebraic Equations

Methods of solving simultaneous algebraic equations can be classified into three very broad groups (1) direct numerical methods, (2) iterative numerical methods, and (3) methods involving physical analogies (the electric analog computer, for example). There are many specific methods under each group. The two methods that follow are very simple examples of a direct solution technique and an iterative solution technique. These particular methods are conceptually simple, but are not the most efficient methods for solving ground-water flow problems and are being used for illustrative purposes only.

Direct Method

The conceptually most simple direct method of solving simultaneous equations is the elimination of variables by substitution.

Exercise:

Step 1: Subtract equation 8c from equation 8b.

This gives us $h_{2,3} = h_{3,2}$.

Step 2: Because $h_{2,3} = h_{3,2}$ substitute $h_{2,3}$ for $h_{3,2}$ in equations 8a and 8d. This then gives us the following set of three equations and three unknowns:

$$h_{2,2} = (1/4)(2h_{2,3}) \qquad (9a)$$

$$h_{3,3} = \qquad (9b)$$

$$h_{2,3} = (1/4)(h_{3,3} + h_{2,2} + 2.) \qquad (9c)$$

Fill in the rest of equation (9b) for $h_{3,3}$. The three unknown value of head are now $h_{2,2}$, $h_{3,3}$, and $h_{2,3}$.

Step 3: Substitute the right hand side of equations 9a and 9b into equation 9c for $h_{2,2}$ and $h_{3,3}$.

This gives us an equation all in terms of $h_{2,3}$.

Step 4: Solve this equation for $h_{2,3}$.

Step 5: Substitute the value of $h_{2,3}$ into equation 9a and solve for $h_{2,2}$.

Step 6: Substitute the value of $h_{2,3}$ into equation 9b and solve for $h_{3,3}$.

Step 7: The values of the four unknown heads are:

$h_{2,3}$ =

$h_{3,2}$ =

$h_{2,2}$ =

$h_{3,3}$ =

We have just solved the set of four simultaneous linear equations by a direct method--the elimination of variables by substitution.

We can represent the algebraic set of equations (8a, b, c, and d) using matrices, as in equation 10.

$$\begin{bmatrix} 1 & -1/4 & -1/4 & 0 \\ -1/4 & 1 & 0 & -1/4 \\ -1/4 & 0 & 1 & -1/4 \\ 0 & -1/4 & -1/4 & 1 \end{bmatrix} \begin{pmatrix} h_{2,2} \\ h_{2,3} \\ h_{3,2} \\ h_{3,3} \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \\ 9/4 \end{pmatrix} \tag{10}$$

To multiply the matrix and the vector, first multiply each element of the first row of the matrix by the corresponding element in the vector. Sum the four products. This, then is set equal to the first element of the vector on the right hand side, and gives equation 8a. Continuing for each row of the matrix, we get the three remaining algebraic equations. For a more detailed explanation of matrix multiplication see Wang and Andersen (1982, p. 94-95).

Many computer algorithms have been developed to directly solve sets of algebraic equations set up in matrix form. The procedure we have followed demonstrates one such algorithm. However, this method is very cumbersome for large numbers of simultaneous equations.

Iterative Method

Iterative methods can be thought of as a means of solving the set of simultaneous equations by successively approximating the unknown heads, using a specific numerical scheme (algorithm), until a solution is obtained; that is, until the approximations of head no longer change with additional "iterations." There are many iterative methods for the solution of the problem presented. One of these methods is the Gauss-Siedel Method (also called the Liebmann Method). For our problem and assumptions the Gauss-Siedel formula is:

$$h_{i,j}^{m+1} = \left(\frac{1}{4}\right)\left(h_{i-1,j}^{m+1} + h_{i+1,j}^{m} + h_{i,j+1}^{m} + h_{i,j-1}^{m+1}\right), \qquad (11)$$

where:

m = iteration level

i = row

j = column

Our basic procedure is to move through the grid in a set pattern and use equation 11 to estimate the head at every node at which head is unknown. We continue to traverse the grid until our estimates of the heads at each node no longer change or the change is small enough that we can consider any further changes as insignificant.

Exercise:

Note: Make all calculations on worksheet 3.2.1 to three places to the right of the decimal point.

Step 1: Starting at node 2,2, calculate a first approximation $h^1_{2,2}$ by summing the heads at the four surrounding nodes and dividing by 4:

$$h^1_{2,2} = (1/4)(0 + h^0_{3,2} + h^0_{2,3} + 0).$$

An estimate must be made for the initial values $h^0_{3,2}$ and $h^0_{2,3}$. For this problem, we will arbitrarily set the initial values (the zero iteration level) of all unknown heads at zero. This yields

$$h^1_{2,2} = 0.$$

This value should be entered on worksheet 3.2.1.

Step 2: Moving along the same row, calculate $h^1_{2,3}$.

$$h^1_{2,3} = (1/4)(0 + h^0_{3,3} + 2 + h^1_{2,2})$$

Note that 2 is the value of the constant head at node (2,4) and that we now use the value obtained by our initial value at node (2,2) which in our case did not change from our initial value. Thus,

$$h^1_{2,3} = 0.50.$$

Step 3: Moving to the next row, calculate $h^1_{3,2}$ and then $h^1_{3,3}$ and enter the results on work sheet 3.2.1.

Worksheet 3.2.1--Iterative method, steady state

$j \rightarrow$, $i \downarrow$

0.	0.	0.	0.
0.	$h^{0}_{2,2} = 0.$ $h^{1}_{2,2} =$ $h^{2}_{2,2} =$	$h^{0}_{2,3} = 0.$ $h^{1}_{2,3} =$	2.
0.	$h^{0}_{3,2} = 0.$ $h^{1}_{3,2} =$	$h^{0}_{3,3} = 0.$ $h^{1}_{3,3} =$	4.
0.	2.	5.	6.

Step 4: Start iteration number 2 by calculating $h^2_{2,2}$.

Step 5: Continue traversing through the grid until the unknown head values do not change in the hundreths place (this is our criterion for convergence). Seven or eight traverses should be sufficient.

FINAL EXAM

Multiple Choice Question:

<u>What Did We Do?</u>

We simulated a partial-differential equation of ground-water flow using a set of simultaneous algebraic equations. We solved the set of simultaneous algebraic equations using (check appropriate box or boxes):

- |_| Electric-Analog Solution Technique
- |_| Direct-Algebraic Solution Technique
- |_| Iterative-Numerical Solution Technique

PROBLEM 3, NUMERICAL ANALYSIS, TRANSIENT STATE

Algorithms for the Numerical Solution of the Nonequilibrium Ground-Water Flow Equation

Our next two problems involve the numerical simulation of unsteady flow--that is, we wish to calculate head for all nodes at which head is unknown for specified successive increments of time. The two-dimensional nonequilibrium ground-water flow equation, without a source/sink term (Q), is:

$$T_x \frac{\partial^2 h}{\partial x^2} + T_y \frac{\partial^2 h}{\partial y^2} = S\frac{\partial h}{\partial t}. \qquad (1)$$

To obtain an approximate numerical solution to this equation, we will use a forward-difference formulation and a variation of the backward-difference formulation as described in Bennett (1976), Part VII, p. 137-140. The iterative technique used in this exercise will be the Gauss-Siedel method that was also used in part 1 (steady-state problem).

Forward Difference Formulation

The equation for forward-difference formulation (Bennett, 1976, p. 137) is:

$$\frac{h_{i-1,j,n}+h_{i+1,j,n}+h_{i,j-1,n}+h_{i,j+1,n}-4h_{i,j,n}}{a} = \frac{S(h_{i,j,n+1}-h_{i,j,n})}{T\Delta t}, \qquad (2)$$

where

a = constant grid spacing

S = storage coefficient

T = transmissivity (note: $T_x = T_y$)

Δt = time step duration

n = time step number

Equation 2 can be rearranged and solved for $h_{i,j,n+1}$

$$h_{i,j,n+1} = \frac{T\Delta t}{Sa^2}\left[h_{i-1,j,n}+h_{i+1,j,n}+h_{i,j-1,n}+h_{i,j+1,n} + \left(\frac{Sa^2}{T\Delta t} - 4\right) h_{i,j,n}\right]. \tag{3}$$

Using our definitions for storage capacity

$$S_c = a^2 S$$

and hydraulic conductance

$$C = \frac{KA}{L} = \frac{Ta}{a} = T,$$

and rearranging terms, the forward-difference equation is

$$h_{i,j,n+1} = \frac{C\Delta t}{S_c}\left[h_{i-1,j,n}+h_{i+1,j,n}+h_{i,j-1,n}+h_{i,j+1,n} + \left(\frac{S_c}{C\Delta t} - 4\right) h_{i,j,n}\right]. \tag{4}$$

Backward Difference Formulation

The backward difference formulation of the finite difference equation using the Gauss-Siedel formulation (Bennett, 1976, p. 139) is:

$$h^{m+1}_{i,j,n} = \left(\frac{1}{\frac{4}{a^2} + \frac{S}{T\Delta t}}\right)\left[\frac{h^{m+1}_{i-1,j,n} + h^{m}_{i+1,j,n} + h^{m+1}_{i,j-1,n} + h^{m}_{i,j+1,n}}{a^2} + \frac{S}{T\Delta t}\, h_{i,j,n-1}\right] \qquad (5)$$

where m = iteration number.

Again, using our definition for storage capacity and hydraulic conductance and rearranging terms the Gauss-Siedel formulation of the backward-difference equation becomes:

$$h^{m+1}_{i,j,n} = \left(\frac{1}{4 + \frac{S_c}{C\Delta t}}\right)\left[h^{m+1}_{i-1,j,n} + h^{m}_{i+1,j,n} + h^{m+1}_{i,j-1,n} + h^{m}_{i,j+1,n} + \left(\frac{S_c}{C\Delta t}\right)h_{i,j,n-1}\right]. \qquad (6)$$

Definition of a Nonequilibrium Problem

We wish to examine the effects of opening the locks in the canal on heads in the aquifer (See section 3.2, problem 2, fig. 3.2.1). We will assume that the water level in the lock drops quickly to zero. Thus at time t = 0, the head values for nodes in the canal will be zero instead of 2 and 4, as previously shown. We are interested in defining the head distribution after one day. Additionally, we will assume that the transmissivity is constant and that effects of any changes in saturated thickness are negligable.

We will calculate the solution to this problem using three methods. The first two methods simulate the heads at one day by using only one time step, Δt = 1 day, one using the forward-difference and one using the backward difference equation. The third method simulates the heads at one day by two equal time steps (Δt = 1/2 day) using only the backward-difference equation.

The dimensions of the aquifer are 3,000 feet by 3,000 feet (a = 1,000 feet for the 4 x 4 grid). The aquifer parameters are

$S = .2$ and

$T = 2 \times 10^5$ feet^2/day.

Solution of the Nonequilibrium Problem

Calculate the coefficients (the parts of equations containing S_c, C, and Δt) for the forward- and backward-difference equations 4 and 6 using the given aquifer parameters, model time step, and node spacing. With these coefficients, write the forward-difference equation for time-step duration = 1 day. Similarly, write the backward-difference equation for time-step duration = 1 day:

Method 1--Forward Difference and time-step duration = 1 Day

Calculate the heads for the four central nodes at the end of one day using the forward-difference equation above. Use the values obtained at the end of the steady-state problem (section

3.2, problem 2, worksheet 3.2.1) for initial conditions, $h_{i,j,n}$ except along the canal where the heads will be zero. Values of $h_{i,j,n+1}$ at the end of one day are:

$h_{2,2,n+1}$ = ; $h_{2,3,n+1}$ = ; $h_{3,2,n+1}$; $h_{3,3,n+1}$ =

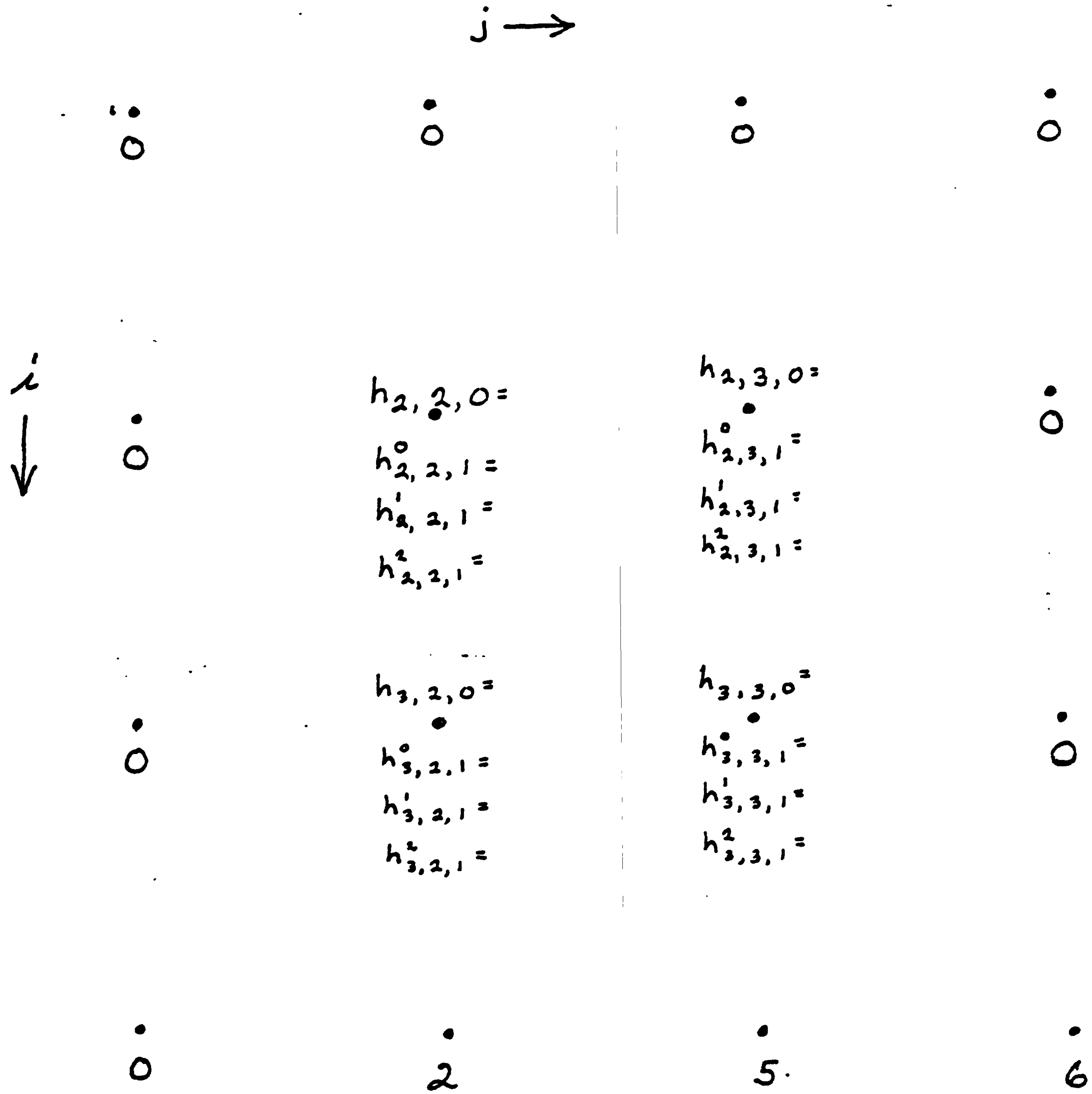
Method 2--Backward Difference and time-step duration = 1 Day

Using worksheet 3.3.1 and the backward-difference equation determined above, iterate through the finite-difference grid until a solution for t = 1 day is attained for each of the four central nodes. Remember that $h_{i,j,n-1}$ is the head at the previous time step and remains constant for all iterations in the same time step. The initial conditions ($h_{i,j,o}$ on work sheet 3.3.1) for the four central nodes are the values that were obtained from the steady-state problem (section 3.2, problem 2, worksheet 3.2.1). Arbitrarily set the initial value of head, $h^{o}_{i,j,1}$, used in the first iteration to <u>one half the steady state head</u>. Because water levels are expected to decline considerably, this should reduce the number of iterations needed to arrive at the solution. This will also avoid confusion between the initial conditions before the canal locks are opened, $h_{i,j,o}$, and the initial value of heads for the first iteration, $h^{o}_{i,j,1}$, (return to discussions of the backward-difference method, section 2.4, note 4, if this is not clear).

Method 3 - Backward Difference and time-step duration = 1/2 Day

Before employing method 3 we must recalculate the

Worksheet 3.3.1: Transient state, backward difference,
time-step duration = 1 day

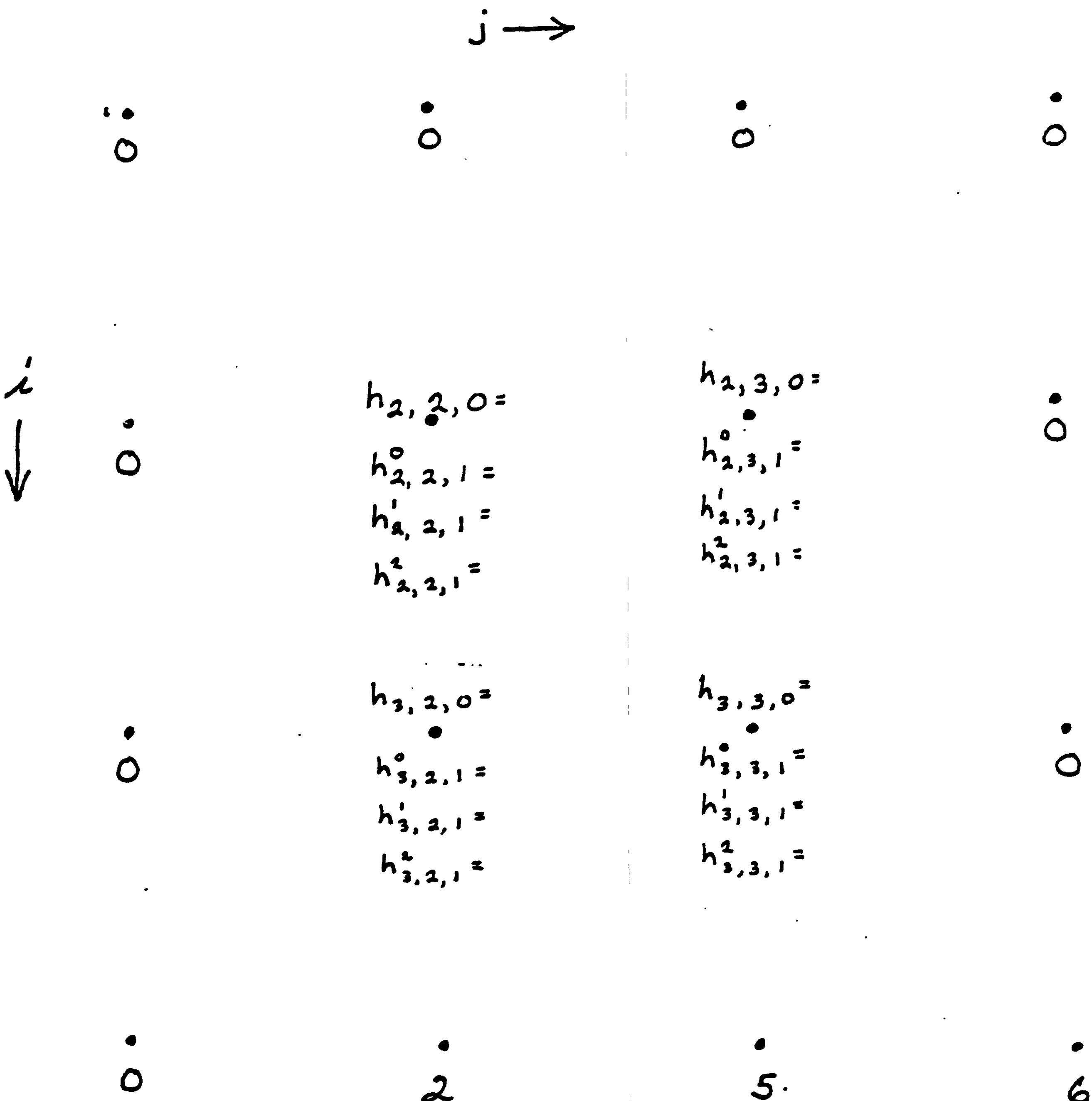


coefficients for the backward-difference equation (equation 6). Using these coefficients write equation 6 for time-step duration = 1/2 day:

To obtain a solution for the four central nodes at one day we must use two half-day time steps. Again the initial condition, $h_{i,j,o}$, is the solution to the steady state problem (section 3.2, problem 2, worksheet 3.2.1). Use worksheet 3.3.2 to obtain a solution at t = 1/2 day. Then, using these results as the initial conditions, $h_{i,j,o}$, for a second half-day time step, use worksheet 3.3.3 to obtain a solution at t = 1 day. As was done for the first 1/2-day time step, arbitrarily set the initial value of head, $h^o_{i,j,1}$, used in the first iteration to one half of the initial-condition heads, $h_{i,j,o}$.

On worksheet 3.3.4, draw hydrographs of head at node $h_{2,3}$ for both simulations. What does this graph indicate? Figure 3.3.1 shows hydrographs of head for node $h_{2,3}$ resulting from four and eight time steps. How do the hydrographs differ as the time step Δt becomes smaller and smaller? What conclusions can be drawn from figure 3.3.1?

Worksheet 3.3.2: Transient state, backward difference, time-step duration = 1/2 day (first time step; t = 1/2 day)



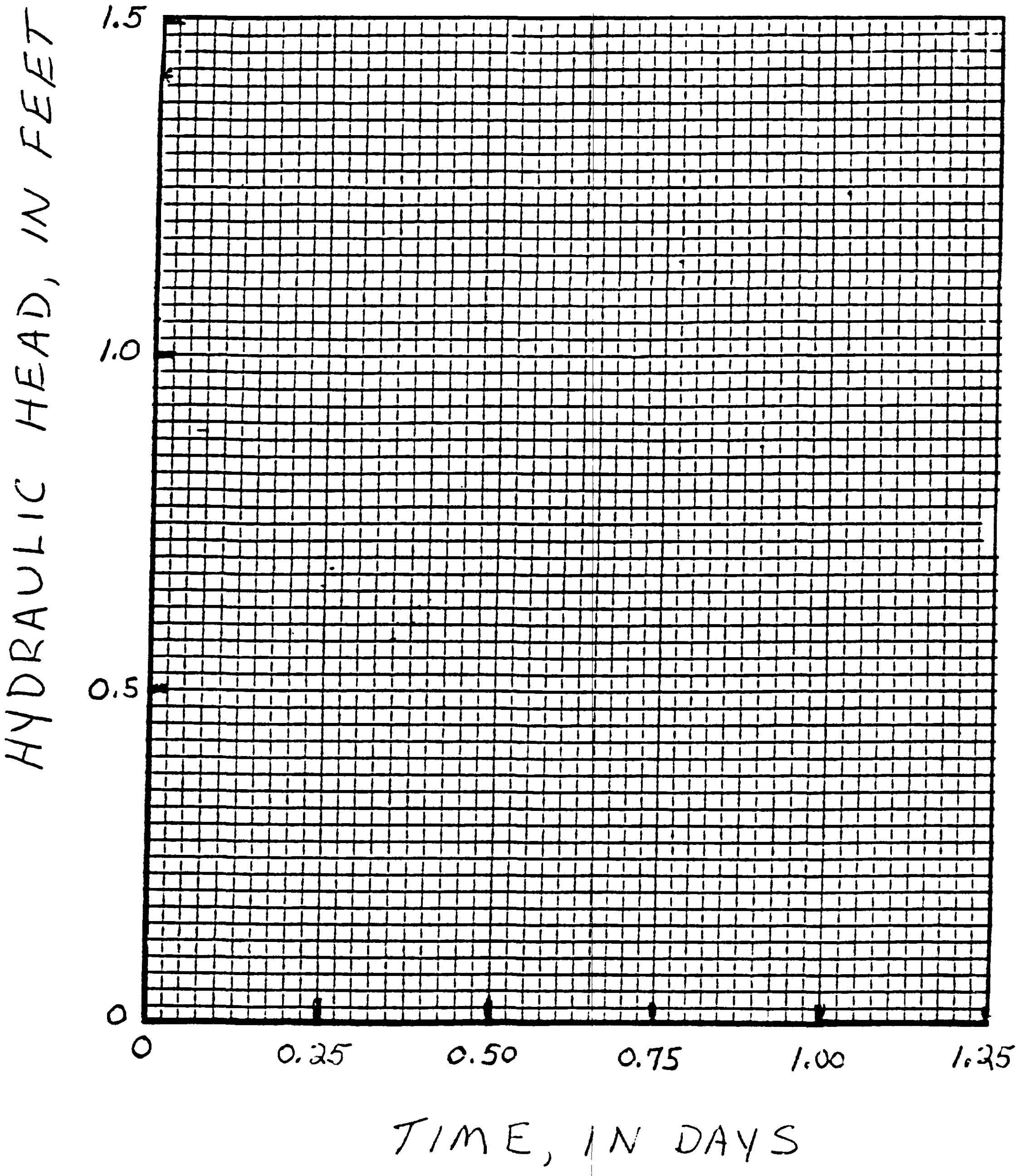
Worksheet 3.3.3: Transient state, backward difference,
time-step duration = 1/2 day (second time step; t = 1 day)

j →

i ↓

0	0	0	0
0	$h_{2,2,0}=$ $h^0_{2,2,1}=$ $h^1_{2,2,1}=$ $h^2_{2,2,1}=$	$h_{2,3,0}=$ $h^0_{2,3,1}=$ $h^1_{2,3,1}=$ $h^2_{2,3,1}=$	0
0	$h_{3,2,0}=$ $h^0_{3,2,1}=$ $h^1_{3,2,1}=$ $h^2_{3,2,1}=$	$h_{3,3,0}=$ $h^0_{3,3,1}=$ $h^1_{3,3,1}=$ $h^2_{3,3,1}=$	0
0	2	5	6

Worksheet 3.3.4--Transient state, head vs. time.



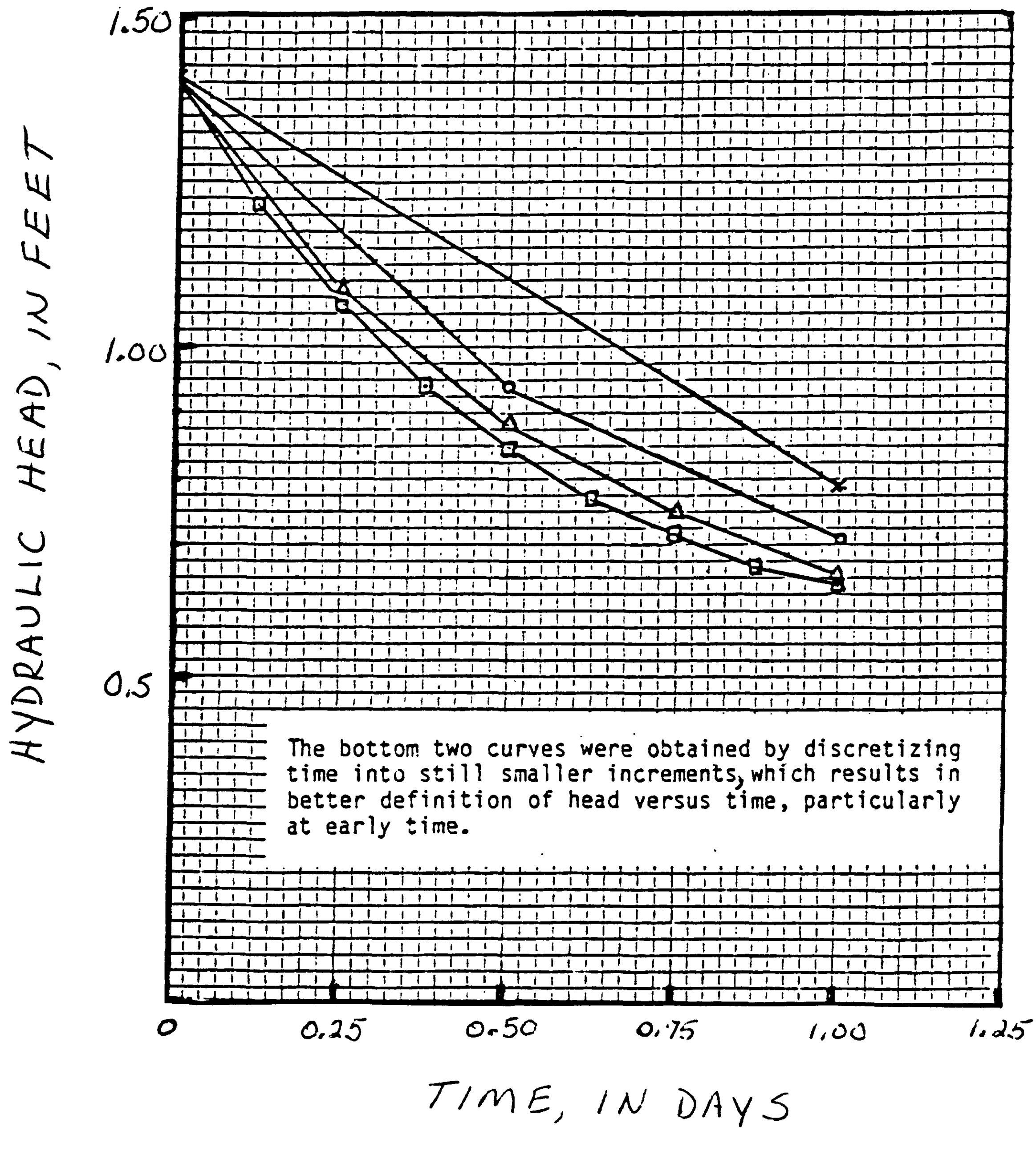


Figure 3.3.1.--Heads for node $h_{2,3}$ resulting from four simulations using different time steps.

PROBLEM 4, IMPERMEABLE WALL PROBLEM

Ground-Water Flow System

A cross section of a ground-water flow system near a partially penetrating impermeable wall is shown in figure 3.4.1. This section depicts a two-dimension flow field. Flow is assumed to be only in the plane of the figure; that is, no flow is perpendicular to the plane of the figure. The flow field has unit thickness--that is, the thickness of the flow system perpendicular to the page is 1 foot. The wall is impermeable, as are the bottom and lateral boundaries. The "top" of the ground-water flow system to the left of the impermeable wall lies 5 feet beneath a standing body of water whose surface elevation remains constant at 55 feet above the impermeable bottom boundary (datum). To the right of the impermeable wall the surface of the aquifer material is at an elevation of 25 feet above datum; ground water discharges at this surface to nearby surface drains and by evaporation. The earth material near the impermeable wall is fine sand, which is assumed to be isotropic and homogeneous.

The head distribution in this cross section, obtained by numerical simulation, is shown on figure 3.4.2 and worksheet 3.4.1. The "node" at which each head value applies is located at the decimal point of the head value. All head values are in feet above datum. The nodes form a square discretization grid with an equal 5-foot spacing between nodes.

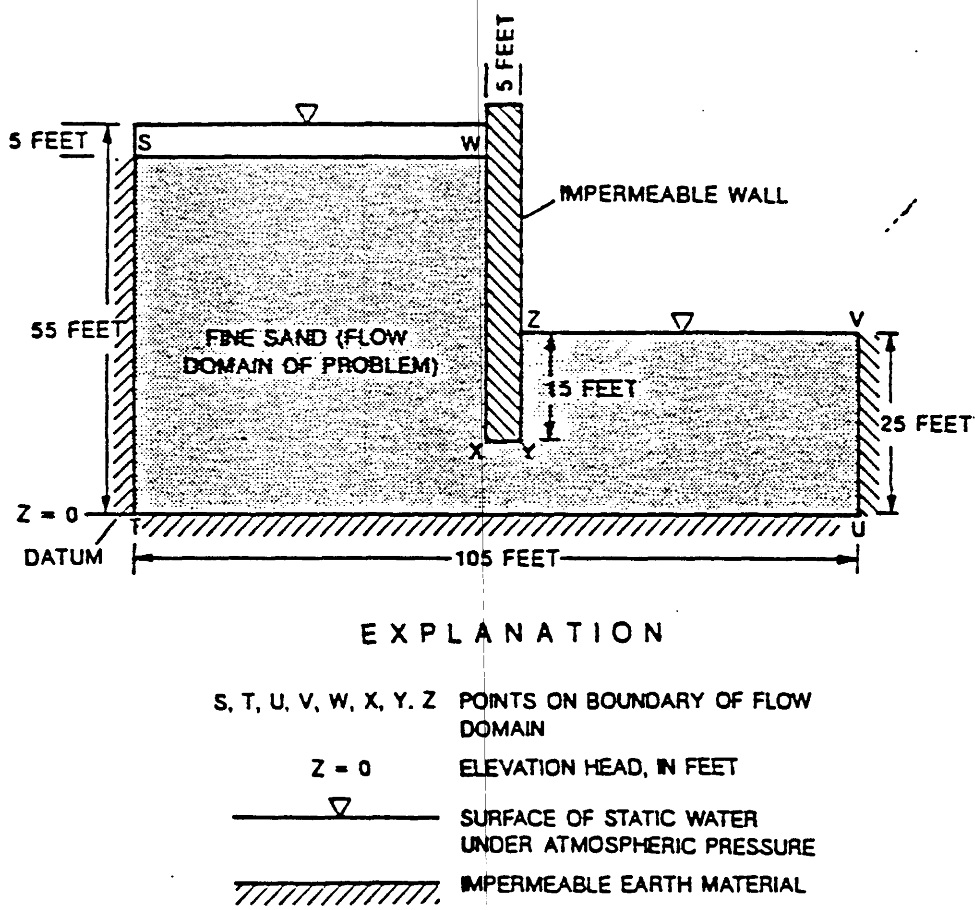


Figure 3.4.1.--Cross section of a ground-water flow system near a partially penetrating impermeable wall.

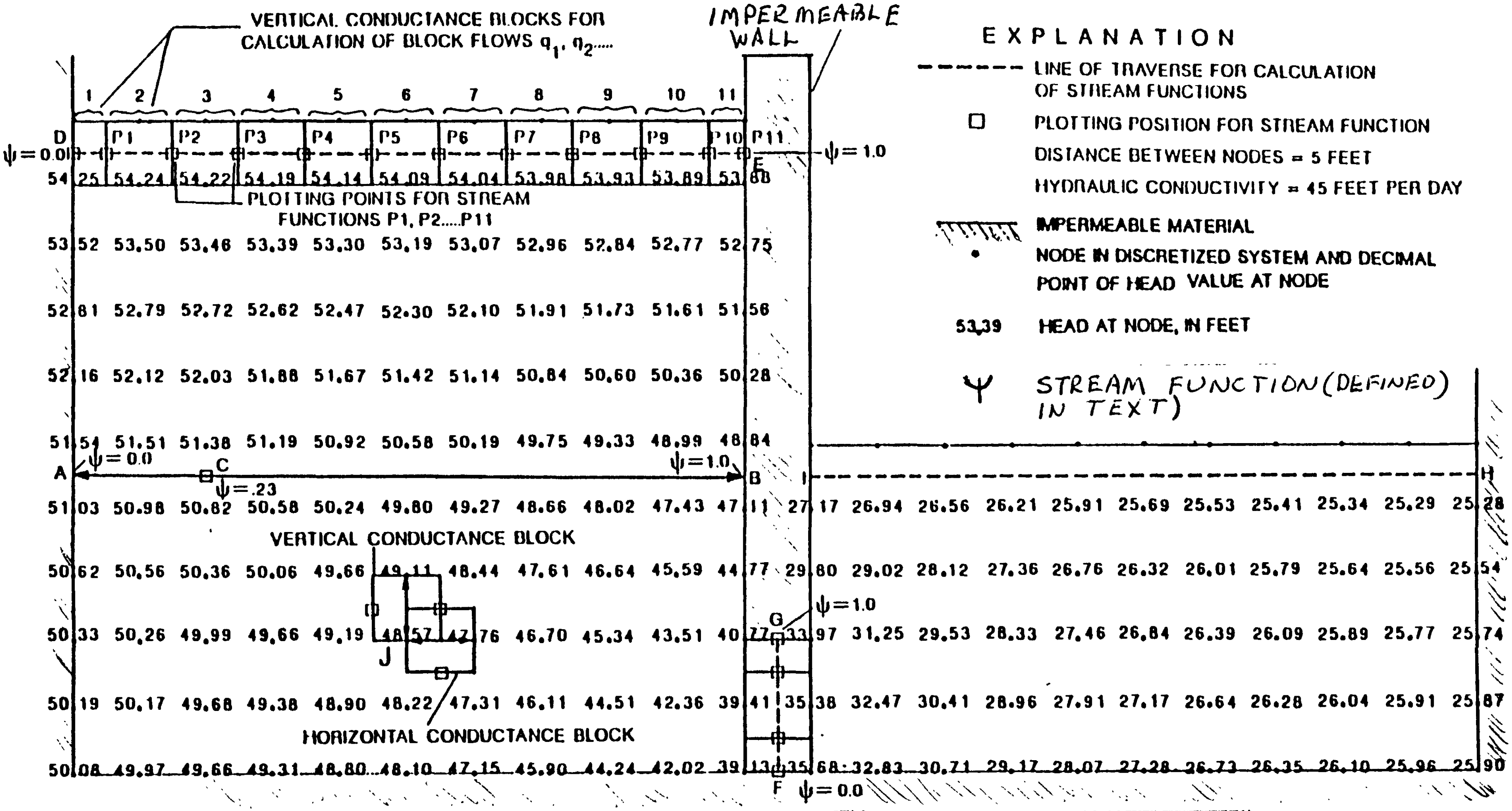


Figure 3.4.2.--Information for contouring head and determining stream functions and conductance.

The head values on figure 3.4.2 and worksheet 3.4.1 represent the "standard output" for a digital simulation of this problem. Commonly, or perhaps usually, these head data can be contoured to provide insight into the flow pattern. In this problem set, we will use these head data as the starting point for calculating additional potentially useful information--for example, position of streamlines, contours of constant pressure, and approximate times of travel or residence times within the flow system.

The first step in analyzing any ground-water problem is to develop a simple (compared to the complexity of the real system) conceptual "picture" or model of the operation of the ground-water system. To attain a reasonable conceptual model of the flow system, the minimum required information is (1) the shape, or geometry, of the flow system and (2) the boundary conditions. The geometry of the flow system has already been defined in figure 3.4.1. The next step is to define the boundary conditions of the problem.

Using colored pencils, delineate carefully the extent and type of the boundaries in the impermeable wall problem on figure 3.4.1. You will find four boundaries and two different boundary conditions. Remember that your designations of the boundaries by means of colored pencils must result in a loop or closed curve without gaps in color. A "gap" without color would represent a portion of the boundary surface for which you have not defined the governing boundary condition.

Where does ground water enter the system? Where does ground water discharge from the system? Sketch the approximate pattern of several flowlines and potential lines on figure 3.4.1. Does your conceptual model of the flow system "make sense"?

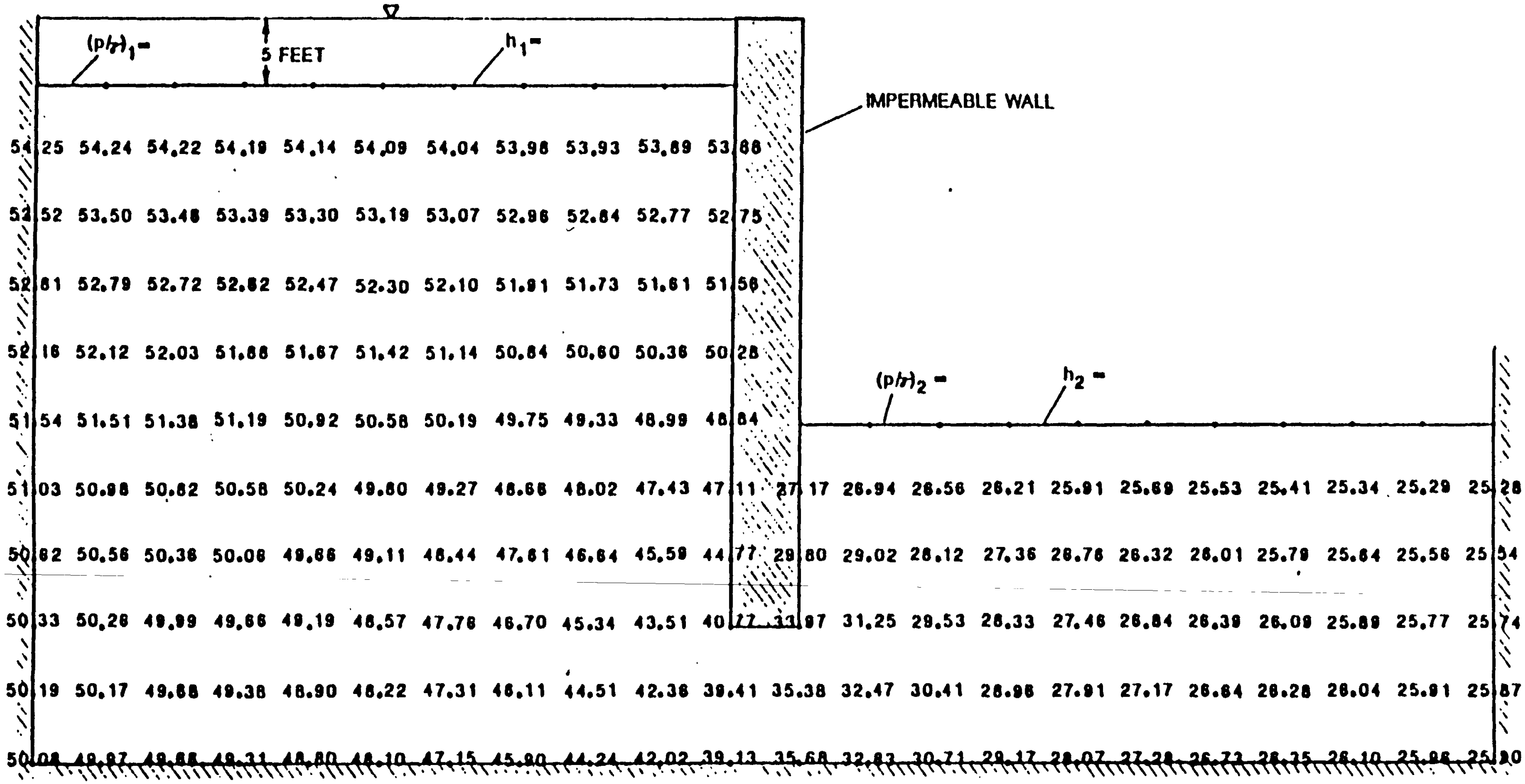
Make a table of elevation head, z, pressure head, p/γ, and hydraulic head, h, values for the upper left-hand and upper right-hand horizontal boundaries. What is the total head drop (Δh) for the ground-water system? Is this information consistent with your concept of the flow system?

Flow Net

Our first goal is to construct a fairly accurate flow net from the head data on worksheet 3.4.1. You may wish to make copies of this worksheet before you begin in case you make errors. Contour the head data using a contour interval of 2.5 feet--that is, draw contours for 52.5, 50, 47.5, 45, . . . 27.5 feet. The contour lines should be smooth curves that intersect streamline boundaries at right angles. Draw all contours in pencil so that corrections and improvements can be made easily. Draw these contours carefully because later work depends on their position.

The next step is to determine the position of several interior streamlines in the flow system. These streamlines intersect the head contours at right angles, and are generally

Worksheet 3.4.1.--Worksheet for construction of a flow net.



EXPLANATION

- SURFACE OF STATIC WATER UNDER ATMOSPHERIC PRESSURE
- IMPERMEABLE MATERIAL
- • NODE IN DISCRETIZED SYSTEM AND DECIMAL POINT OF HEAD VALUE AT NODE
- 53.39 HEAD AT NODE, IN FEET

Z IS ELEVATION HEAD, p/γ IS PRESSURE HEAD, h IS TOTAL HEAD, IN FEET

DISTANCE BETWEEN NODES = 5 FEET

HYDRAULIC CONDUCTIVITY = .45 FEET PER DAY constructed such that the flows between adjacent streamlines are equal. (Two adjacent streamlines define a flow tube.)

To begin, identify the two bounding streamlines in the system. We have decided arbitrarily to draw four interior streamlines, so that the system is divided into five flow tubes. Thus, the internal streamlines must be positioned such that one-fifth of the total flow Q beneath the impermeable wall is transmitted by each flow tube.

To assist in locating the four internal streamlines, we will calculate stream functions along selected traverses across the flow field. However, before considering the procedure for calculating stream functions, we will discuss what the stream function represents.

For the moment, let us assume that our flow system is the original continuous system composed of fine sand--that is, we have not yet discretized the system for the purpose of obtaining a numerical solution for head values at nodes. Also assume that we know the total flow through the system. Now, make an arbitrary traverse from one bounding streamline to the other bounding streamline. To do this, designate a point on one bounding streamline as the starting point of the traverse. All traverses across the system must begin on the same bounding streamline. For example, let the traverse start at point A on the outside bounding streamline and end at point B on the other bounding streamline as shown in figure 3.4.2. Even though the direction of ground-water flow may not be perpendicular to the traverse line at any given point, we must, nevertheless, intersect the total flow through the system along the traverse from A to B.

Let us assume further that we measure each increment of flow as we proceed along the traverse and thus, knowing the total flow, we can assign to any point on the traverse the proportion of the total flow that we have encountered to that point. This proportion is equal to the stream function ψ. For example, at point C, assuming that we started at point A, we have encountered 0.23Q, where Q is the total flow--that is, 0.23Q is behind us on the traverse and 0.77Q still remains in front of us on the traverse. At A we have intersected none of the flow and the stream function $\psi = 0$. At B, we have intersected the total or 100 percent of the flow and $\psi = 1.0$.[1]

The stream function is constant along a streamline. Consider a number of closely spaced traverses through the flow field similar to AB and assume that we know the value of the stream function at every point on the traverses. By connecting points of equal stream function--for example, $\psi_1 = 0.40$ and $\psi_2 = 0.60$, we are drawing a flow tube bounded by the streamlines $\psi_1 = 0.40$ and $\psi_2 = 0.60$ such that 20 percent (0.2Q) of the total flow occurs

[1] The stream function actually is the total flow traversed to a given point on a traverse line such as point C on traverse AB (fig. 3.4.2). We have defined a dimensionless, proportion-of-total-flow function, which is the stream function divided by a constant, the total flow in the system. For convenience, we will refer to this ratio simply as the stream function.

within this flow tube ($\psi_2 - \psi_1 = 0.60 - 0.40 = 0.20$). The stream function is a scalar[2] function of position just as head is a scalar function of position. A unique value of the stream function may be defined for every point in a continuous flow field. We could write the ground-water flow equations in terms of stream functions as the dependent variable instead of head, although for various reasons this is seldom done.

Next, we will develop a procedure for calculating stream functions in the discretized impermeable wall problem (fig. 3.4.2) along three traverses--DE (near the upper left-hand constant head boundary), FG (beneath the impermeable wall), and HI (near the upper right-hand constant head boundary). The calculation of stream functions is facilitated by using the format in table 3.4.1. We begin with traverse DE (fig. 3.4.2). Note that blocks 1 and 11 are "half" blocks. Calculate the conductance of the blocks on the traverse using the familiar formula C = KA/L. Determine the flow through each block using the head differences across the blocks. Next, calculate the cumulative flow for the blocks along the traverse from D to E (See format in table 3.4.1). Divide the cumulative flow at the right-hand edge of each block by the total flow. This calculated value is the stream function at the right-hand edge of that particular block, that is, the percent of the total flow across line DE that is traversed between D and that particular block edge. Note that the plotting positions of

[2] A scalar quantity can be identified by a single number, and it has no implied direction; a scalar may be contrasted to a vector quantity,which has direction and requires more than one number for its description.

the stream functions are at the right-hand edges of the blocks. For example, the stream function along traverse DE for block 1 is plotted at "P1" (fig. 3.4.2); the stream function for block 2 is plotted at "P2" and so on across to block 11 where the stream function ψ = 1.0 is plotted at "P11", the boundary of the flow system.

This choice of plotting positions permits a unique value of the stream function to be plotted on the discretized grid no matter how we make a traverse across the flow field. Compare the two plotting positions of the stream functions at two edges of a typical vertical conductance block and the two plotting positions at two edges of an overlapping horizontal conductance block near J in figure 3.4.2. One plotting position is shared by both blocks. By extension of this pattern, the stream function plotting positions form a square array of points throughout the flow domain that is offset from the square array of points that constitutes the head nodes. Complete the stream function calculations for traverses FG and HI in table 3.4.1.

The procedure for completing the flow net is the following. Plot the individual stream function values on worksheet 3.4.1 at the appropriate points. By interpolation, mark on each traverse line the position of the stream functions ψ= 0.20, ψ= 0.40, ψ= 0.60, and ψ= 0.80. Having completed this, you have established three points on the four streamlines that you wish to draw. Next, sketch the four streamlines on worksheet 3.4.1, being careful to

Table 3.4.1.--Format for calculation of stream functions in impermeable wall problem

[For locations of numbered blocks, traverse DE, and plotting positions for stream functions p1, p2, ..., see figure 3.4.2; C_{block} is hydraulic conductance of discretized block which equals KA/L, where K = hydraulic conductivity of earth material in block, A = cross-sectional area of block perpendicular to direction of ground-water flow, and L = length of block; h_1 and h_2 are head values at nodes located at ends of block; $\Delta h = h_1 - h_2$; q_{block} = flow through a single block; Σq_{block} = flow in a numbered block plus the flows through all lower numbered blocks (cumulative sum of block flows in traverse); Q_{total} = total flow through the ground-water system beneath the impermeable wall; ft^2/d, foot squared per day; ft, foot; ft^3/d, cubic foot per day]

TRAVERSE D E: BLOCK NUMBER	C_{block} = KA/L (ft^2/d)	h_1 (ft)	h_2 (ft)	Δh_{block} (ft)	q_{block} = $C\Delta h$ (ft^3/d)	Σq_{block} (ft^3/d)	$\Psi = \frac{\Sigma q_{block}}{Q_{total}}$	PLOTTING POSITION FOR STREAM FUNCTIONS
1	22.5	55.00	54.25	.75	16.88	16.88		P1
2								P2
3								P3
4								P4
5								P5
6								P6
7								P7
8								P8
9								P9
10								P10
11							1.00	P11

Table 3.4.1.--Format for calculation of stream functions in impermeable wall problem (cont.).

BLOCK NUMBER	$C_{block} = KA/L$ (ft^2/d)	h_1 (ft)	h_2 (ft)	Δh_{block} (ft)	$q_{block} = C\Delta h$ (ft^3/d)	Σq_{block} (ft^3/d)	$\Psi = \frac{\Sigma q_{block}}{Q_{total}}$

draw the streamlines perpendicular to the already existing potential lines. Starting on the left-hand end of the upper-left horizontal boundary (worksheet 3.4.1) label the streamlines "a" through "f" ("a" and "f" are the designations for the two bounding streamlines). The result should be an acceptable flow net. Of course, you can improve the flow net by calculating additional values of stream functions along additional traverses through the flow system and refining, thereby, the positions of the four internal streamlines.

Water Pressure

Our next goal is to construct a map showing contours of equal water pressure for the ground-water system near the impermeable wall. Calculate the water pressure at every second node using the formula and following the example calculation shown in figure 3.4.3. Plot calculated pressures at the corresponding nodes on worksheet 3.4.2. Because of large changes in head near the impermeable wall, you should calculate pressure at several additional nodes in this region. Use γ_w = 62.4 pounds per cubic foot (lbs/ft^3) and calculate pressure in pounds per square foot (lbs/ft^2). To facilitate the calculation write the (constant) value of z for each row of head values on worksheet 3.4.2 before beginning the calculations.

2. Contour the point values of water pressure using a contour interval of 500 lbs/ft^2. At what point in the system is

If $h = \frac{p}{\gamma_w} + z$,

where h = hydraulic head,

p = fluid pressure,

γ_w = weight density of water, and

z = elevation head,

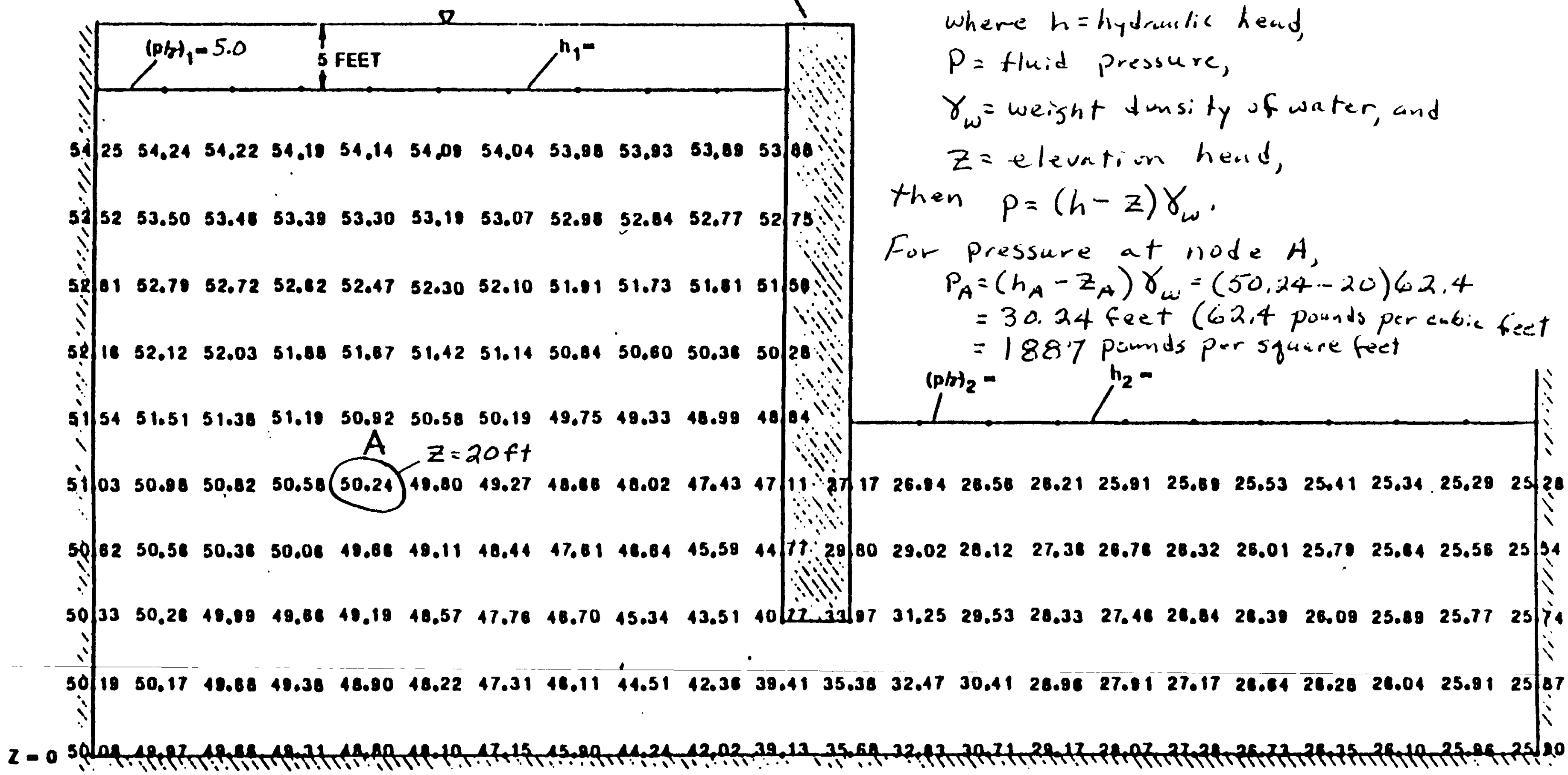
then $p = (h - z)\gamma_w$.

For pressure at node A,

$p_A = (h_A - z_A)\gamma_w = (50.24 - 20)62.4$

= 30.24 feet (62.4 pounds per cubic feet

= 1887 pounds per square feet



EXPLANATION

- ▽ SURFACE OF STATIC WATER UNDER ATMOSPHERIC PRESSURE
- IMPERMEABLE MATERIAL
- • NODE IN DISCRETIZED SYSTEM AND DECIMAL POINT OF HEAD VALUE AT NODE
- 53.39 HEAD AT NODE, IN FEET

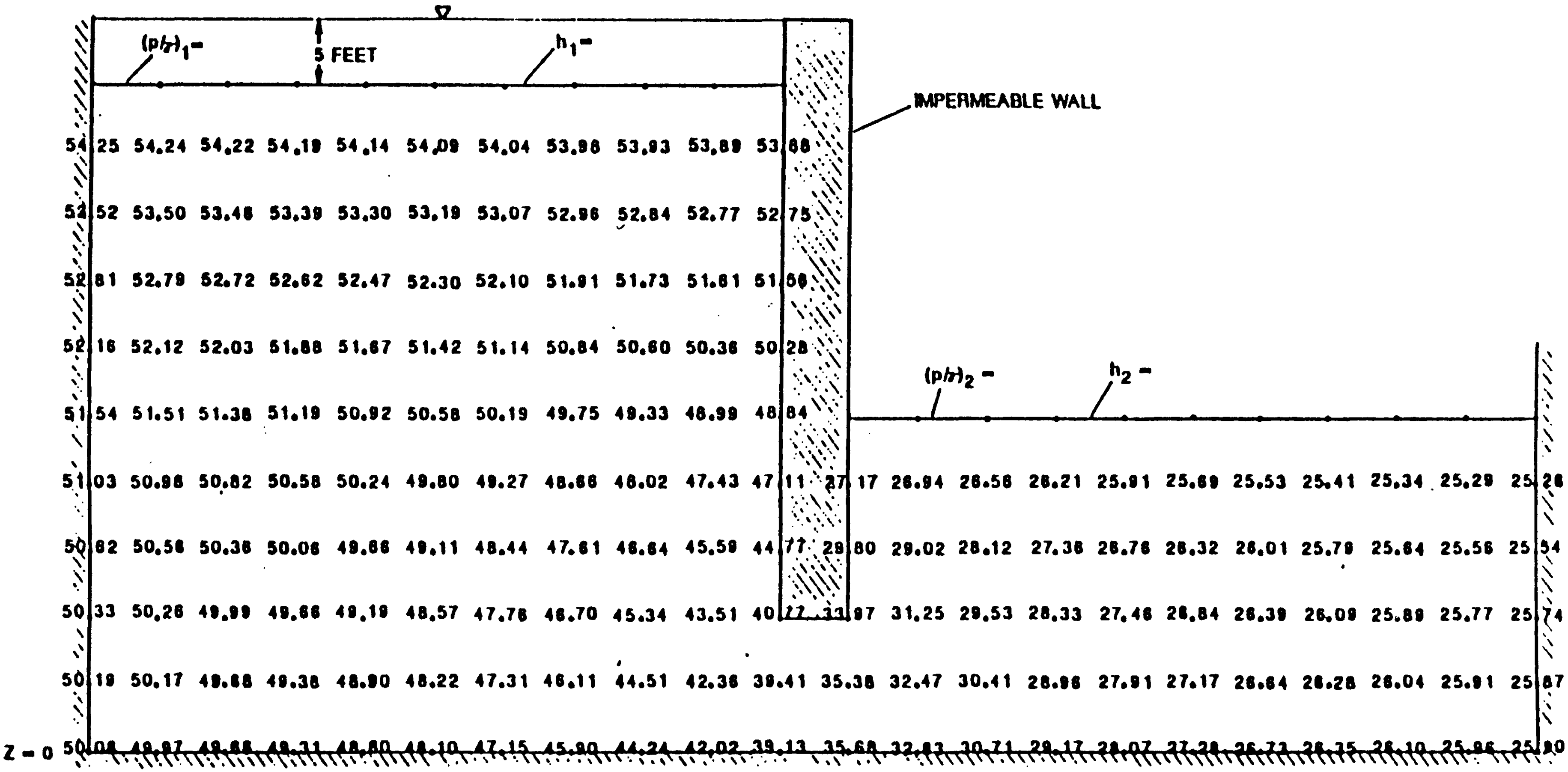
Z IS ELEVATION HEAD, p/γ IS PRESSURE HEAD, h IS TOTAL HEAD, IN FEET

DISTANCE BETWEEN NODES = 5 FEET

HYDRAULIC CONDUCTIVITY = .45 FEET PER DAY

Figure 3.4.3.--Guide for calculating water pressure at nodes.

Worksheet 3.4.2--Worksheet for plotting and contouring calculated point values of water pressure.



EXPLANATION

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- IMPERMEABLE MATERIAL
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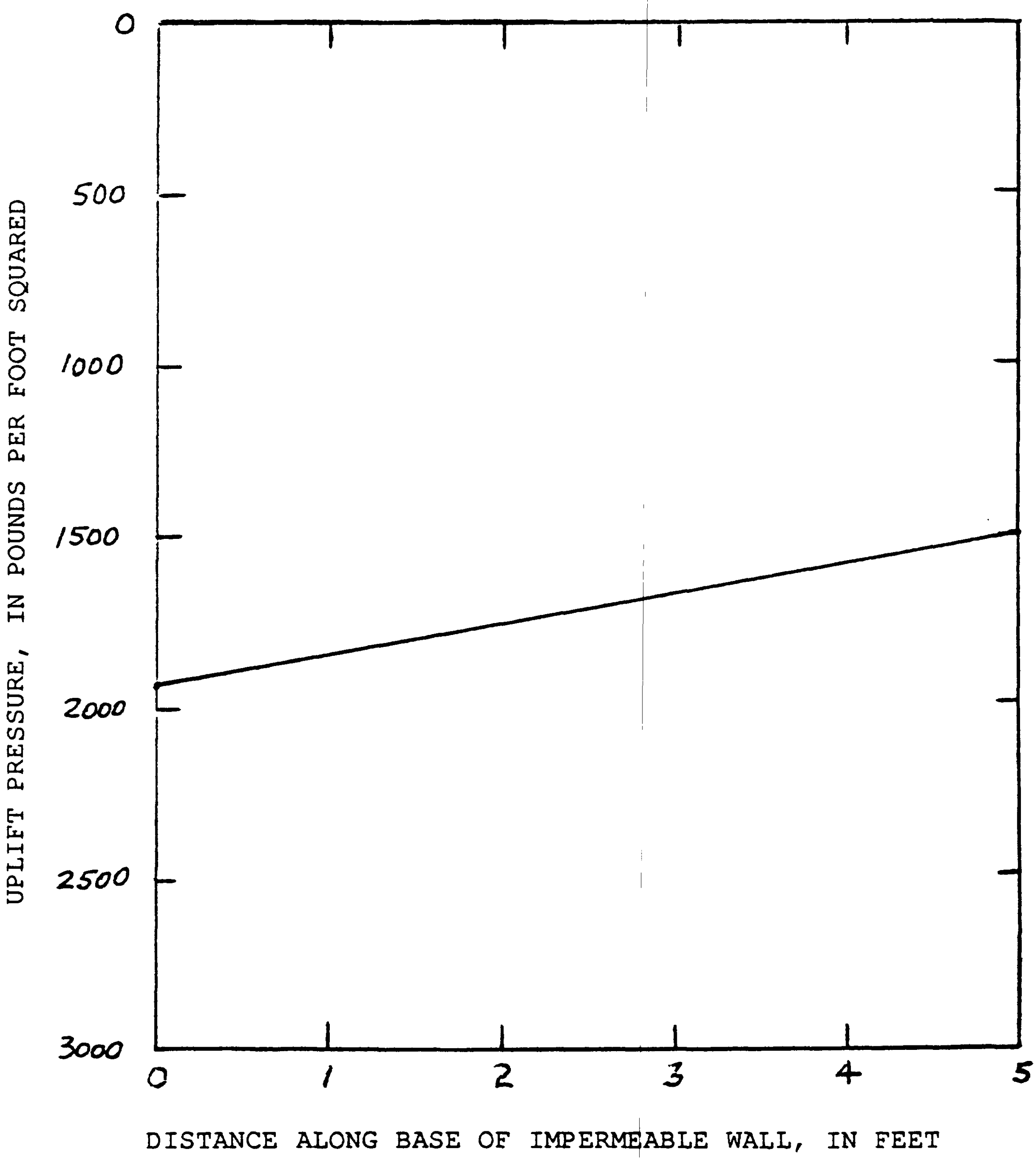


Figure 3.4.4.--Uplift pressure acting on the bottom of the impermeable wall.

the water pressure a maximum? Note the pattern of pressure contours when your map is completed. No obvious relation exists between the contours of equal pressure and the other two sets of lines we have drawn thus far, namely, the potential lines and flow lines of the flow net on worksheet 3.4.1

Next, we make a short digression to consider a problem that sometimes arises in practical engineering work--namely, calculation of the total uplift force acting on the bottom of an impermeable structure. A graph of the uplift pressure acting on the bottom of the impermeable wall is shown in figure 3.4.4. We have calculated already the water pressure at the two bottom edges of the impermeable wall (1,920 and 1,496 lbs/ft^2) (see worksheet 3.4.2). On the graph we assume a linear change in pressure along the bottom of the wall between the two edges.

We wish to calculate the total uplift force acting on a 1-foot section of the wall perpendicular to the plane of the graph. Thus, we are considering the uplift force on a 5 ft x 1 ft = 5 ft^2 area of wall bottom.

Calculate the total uplift force due to water pressure acting on the section of wall defined above. This calculation requires graphical integration of the uplift pressure (fig. 3.4.4), which is simple in this case. Our principal concern is that you have a clear picture of what you are summing and the units involved. What are the units of the answer you are seeking?

Next, calculate the weight of the wall acting on the same 5-square-foot section of wall bottom and compare with the value for total uplift force calculated above. Assume that the wall extends vertically for 50 feet, the wall is made of concrete, and the weight density of the concrete is 165 lbs/ft^3.

The same procedures that we employed in this simple problem to calculate uplift force can also be applied to more complicated problems.

Time of Travel

Our goal, after making an important assumption, is to graph on worksheet 3.4.3 the progress in time of "tracer water" through the impermeable-wall flow system. Assume that at some instant of time (t=0, or reference time in this problem), water of different quality enters the flow field at the upper left-hand inflow boundary and moves through the system. We assume that the "new" water moves by piston flow or plugflow. This means that the "new" water completely displaces the "old" water. Thus, we assume that no mixing of the two waters occurs (the processes of dispersion and diffusion are not acting), that is, a sharp boundary or "front" exists between the two fluids as the "new" water advances through the system.

From Darcy's law, the specific discharge, or Darcy velocity q, is given by

$$q = \frac{K\Delta h}{L},$$

where L is the distance between two points on the same streamline at which head values h_1 and h_2 are known and $h_2 - h_1 = \Delta h$. The "actual" or average linear velocity v is given by

$$v = \frac{q}{n} = \frac{K\Delta h}{nL},$$

where n is the porosity of the fine sand. Recalling that distance of travel (L) = velocity x time or L = vt, then v = L/t. Substituting for v, and rearranging, we obtain

$$t = \frac{L}{\frac{K\Delta h}{nL}} = \frac{nL^2}{K\Delta h}.$$

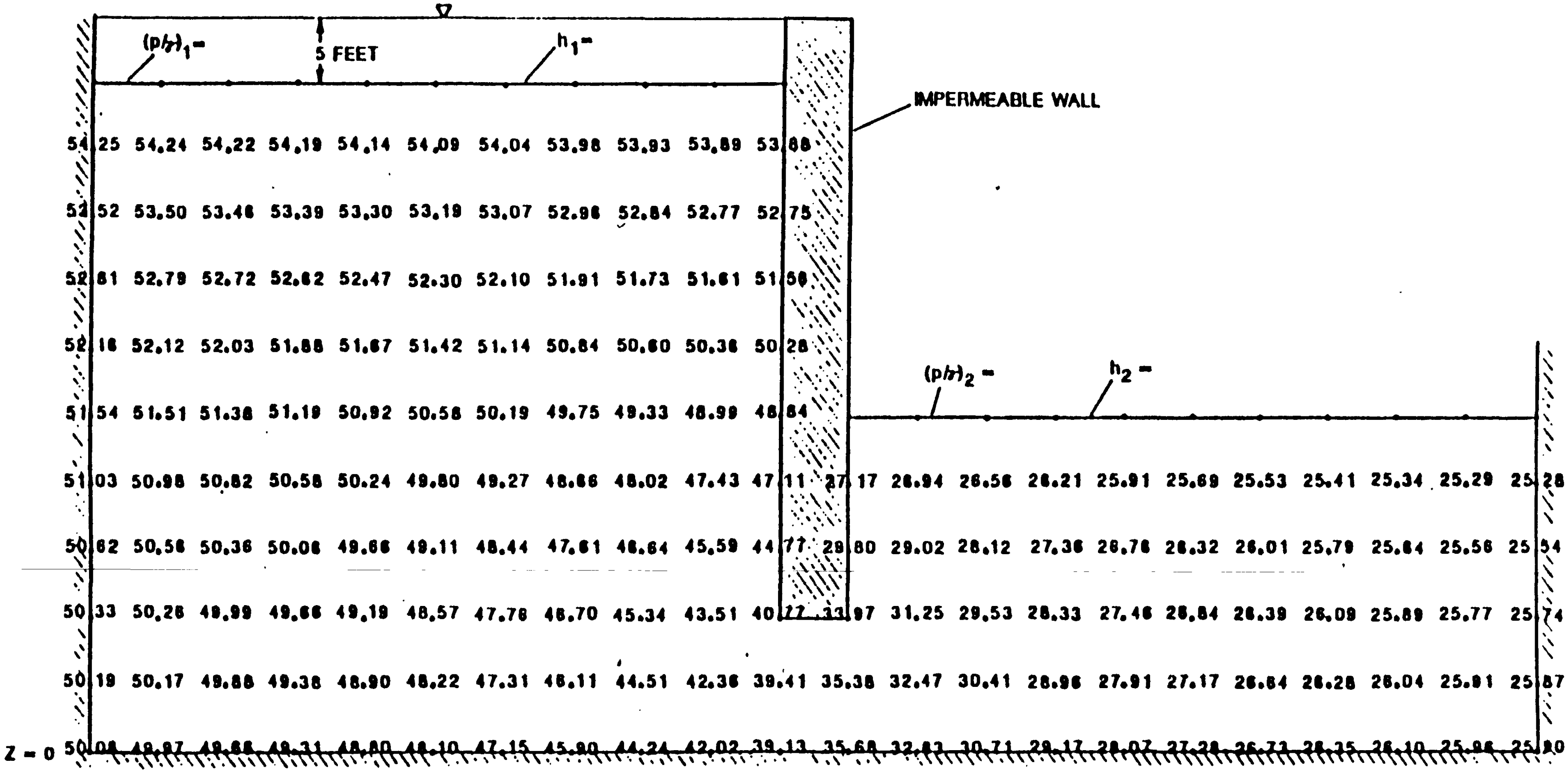
This is the basic formula for calculating the time of travel between two points on a streamline that are L distance apart.

Given that K = 45 ft/day and n = 0.30, the formula for time of travel between two points on a streamline in the impermeable wall problem becomes

$$t = \frac{6.67 \times 10^{-3} L^2}{\Delta h},$$

where t is in days.

Worksheet 3.4.3.--Worksheet for plotting and contouring calculated point values of time of travel.



EXPLANATION

SURFACE OF STATIC WATER UNDER ATMOSPHERIC PRESSURE

IMPERMEABLE MATERIAL

NODE IN DISCRETIZED SYSTEM AND DECIMAL POINT OF HEAD VALUE AT NODE

53.39 HEAD AT NODE, IN FEET

Z IS ELEVATION HEAD, p/γ IS PRESSURE HEAD, h IS TOTAL HEAD, IN FEET

DISTANCE BETWEEN NODES = 5 FEET

HYDRAULIC CONDUCTIVITY = .45 FEET PER DAY

Using the format in table 3.4.2, calculate times of travel from node to node along the two bounding streamlines (streamlines "a" and "f" on worksheet 3.4.1) of the flow system. For these two streamlines, because we are calculating travel times between nodes, L is constant and equals 5 feet. Thus, for these two streamlines only,

$$t = \frac{.167}{\Delta h}.$$

Our main interest in this problem is not the travel times between points on the streamlines, but the total time of travel from the upper left-hand boundary to the point in question. The value of Σt in table 3.4.2 represents this total calculated travel time along the given streamline from the inflow boundary to the given point on the streamline. Plot the values of Σt at the appropriate points on worksheet 3.4.3

Trace two internal streamlines (streamlines "b" and "d") from the flow net on worksheet 3.4.1 onto worksheet 3.4.3 and mark on these streamlines the points of intersection with potential lines. Next, calculate travel times along the two internal streamlines between points of intersection of potential lines. Note that in this case Δh is constant and L varies. Calculate and plot values of Σt at appropriate points on worksheet 3.4.3 as before.

Contour Σt values for Σt equal to 0.25, 0.50, 0.75, 1.00, 1.50, 2.00, 5.0, and 10.0 days. The contour lines represent

Table 3.4.2--Format for calculation of time of travel along selected flow lines in impermeable wall problem

[h is head at a node or other point in flow system; L is distance between two points on a flowline at which head is known; Δh is difference in head between two points on a flowline; t is time of travel between two points on a flowline; Σt is time of travel from inflow boundary to point on flowline]

h (feet)	L (feet)	Δh (feet)	t (days) = $\frac{6.67 \times 10^{-3} L^2}{\Delta h}$	Σt (days)

Table 3.4.2.--Format for calculation of time of travel along selected flow lines in impermeable wall problem (cont.)

h (feet)	L (feet)	Δh (feet)	t (days) = $\frac{6.67 \times 10^{-3} L^2}{\Delta h}$	Σt (days)

Table 3.4.2.--Format for calculation of time of travel along selected flow lines in impermeable wall problem (cont.)

h (feet)	L (feet)	Δh (feet)	t (days) = $\frac{6.67 \times 10^{-3} L^2}{\Delta h}$	Σt (days)

calculated positions of the sharp front between "new" and "old" water at successive times after introduction of the "new" water at the inflow boundary.

What time is required for "new" water to first reach the discharge boundary? What time is required for "new" water to completely fill the flow system? At the end of this analysis, remember that we assumed piston flow in our time calculations. Our calculations are only approximate, even for this assumption. However, this approach is useful for giving order of magnitude estimates of travel times in ground-water flow systems.

At the end of this problem set, take time to review what you have done and the specific procedures that you used. All the procedures in this problem set involve basic concepts in ground-water hydraulics that you need to understand completely.

PROBLEM 5, ELECTRICAL ANALOG PROBLEM

Power Supply and Digital Multimeter

The electrical-analog unit consists of 3 parts--a resistor-grid board, a power supply, and a multimeter. The power supply has two switches, three leads, and two potentiometers. The switch on the top left side of the power supply controls the power to both the power supply and multimeter. The multimeter has a separate on-off switch. The "voltage" potentiometer controls the voltage between the ground (black) and positive (red) leads of the power supply. The third lead (green) is a current lead; the "current" potentiometer controls the current that is withdrawn from the resistor-grid board through this lead. The "current" on-off switch should always be in the "on" position when the current lead is used. The power switch should be switched off when the unit is not in use. If a unit does not work, first check that the unit is plugged in and the power switch and the current switch are turned on.

The analog unit is equipped with either a Data Precision[1] multimeter or a Keithley multimeter (any multimeter that will measure and display voltage and millamperes to hundredths can be used). A sketch of the front panel of the Data Precision multimeter is shown in figure 3.5.1. The positions of the controls, digital readout, and jacks on the Keithley multimeter

[1] The use of product or firm names in the report is for identification purposes only and does not constitute endorsement by the U.S. Geological Survey

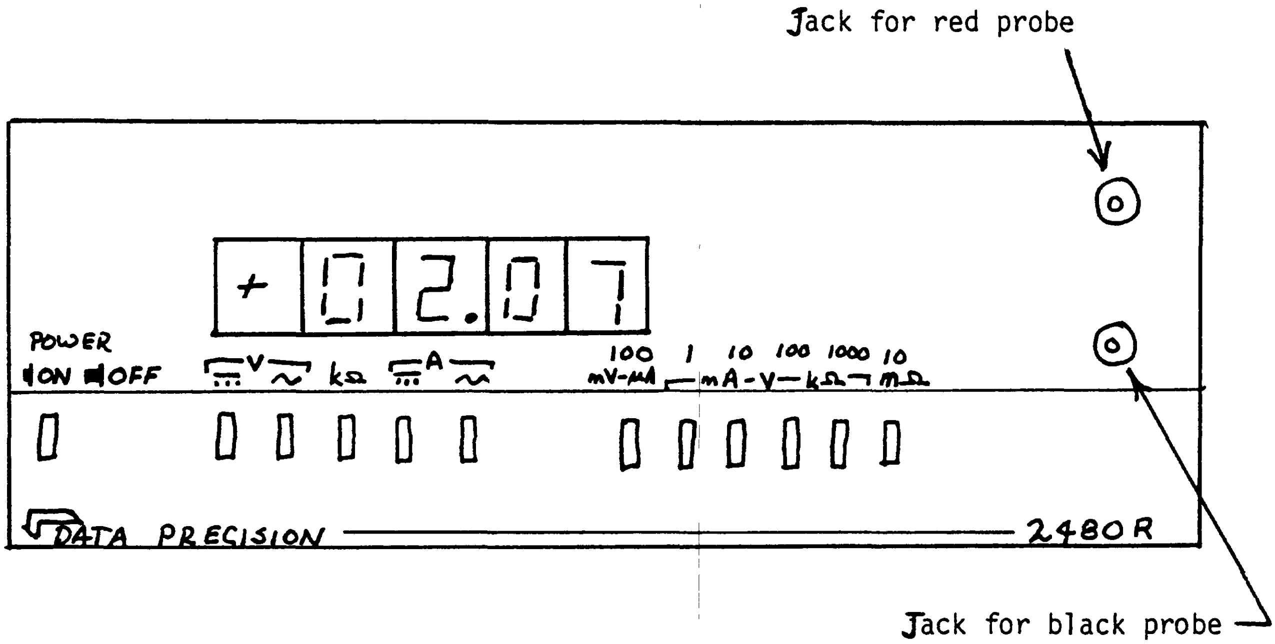


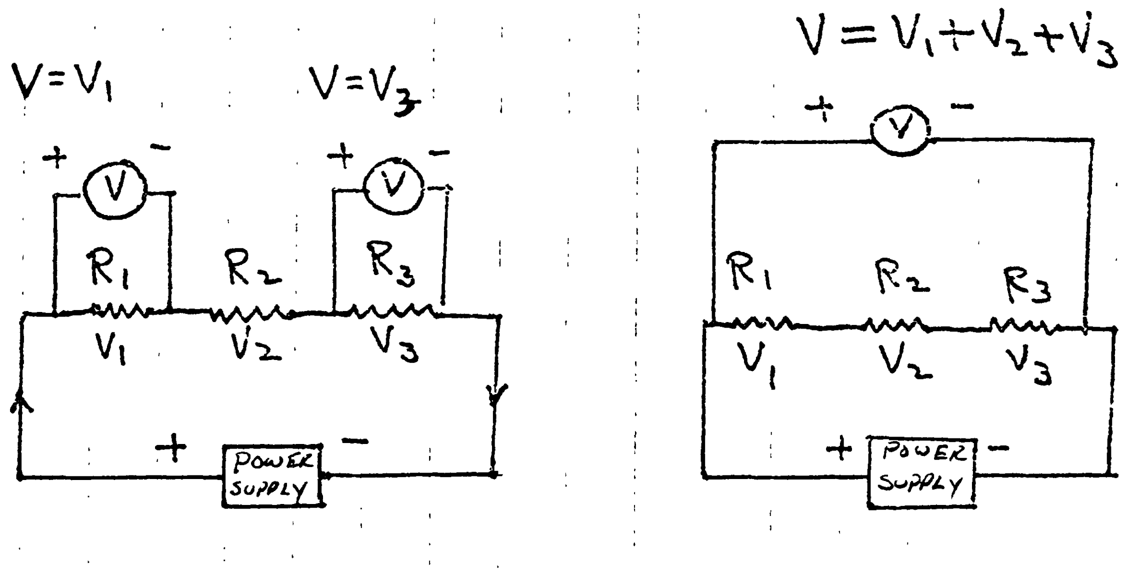
Figure 3.5.1--Front panel of Data Precision multimeter

are similar to those on the Data Precision multimeter. The multimeter is equipped with red and black measuring probes that should be plugged into the two corresponding colored jacks on the right side of the front panel of the meter. Check to insure that the probes are plugged into the proper jacks. To measure voltage with the Data Precision multimeter, push the power button in and push the left-most button under "V" that is marked "⎓" on the left side of the panel (same control on the Keithley meter is marked "v"). The symbol "⎓" is for direct current measurements; the symbol "~" is for alternating current measurements. On the Keithley multimeter, direct-current measurements are made with the AC-DC button (left side) in the out position and a "V" will appear in the upper right-hand corner of the digital readout. Only direct current is used in the electrical analog problem. Next, select the proper range for the measurements. On the Data Precision multimeter, measurements of voltage should be made in the 0- to 100-volt range. Push the button marked "100", under the "mA-V-kΩ" bar. On the Keithley multimeter, voltage should be measured with the third button from the right (under 200) pushed in.

To measure current on the Data Precision multimeter, push the button marked "⎓" under "A" on the left-most group of buttons on the front panel; push the button marked "100" under the "mA-V-kΩ" bar in the right-most group of buttons (this is the same range that is used for voltage measurements). The multimeter will now read in milliamperes. On the Keithley multimeter, push in the button under "A" (fourth button from the left under the digital readout) and the button under 200 (third from the right). A "mA" symbol will appear on the right side of the digital readout when the button under "A" is pushed in.

When using the multimeter as a voltmeter, it is connected in parallel with (across) resistance(s) (fig. 3.5.2). When using the multimeter as an ammeter, it should be connected in series with (through) resistance(s) (fig. 3.5.3). For those with little familiarity with the use of voltmeters and ammeters, figures 3.5.2 and 3.5.3 should be studied carefully before proceeding with this section. Karplus (1958, p. 19-22) describes voltage as an "across" variable and current as a "through" variable. This concept may aid in understanding how to connect the multimeter when measuring voltage and current in the analog model problems.

The functions of the electrical equipment perhaps can be understood best by considering an analogous hydraulic model in which water is supplied through pipes measured by flow meters. Each of the wires leading into or out of the model are analogous to pipes supplying water to the hydraulic model. The bus wires along the east and west sides of the electrical model are analogous to manifolds which might be used to distribute inflow or collect outflow uniformly along the sides of a hydraulic model. The ammeter is analogous to a flow meter and measures the quantity



Volt meter is used to measure voltage across resistors R_1 and R_2

Volt meter is used to measure total voltage in circuit that includes resistors R_1, R_2, and R_3

Figure 3.5.2.--Use of voltmeter to measure voltage.

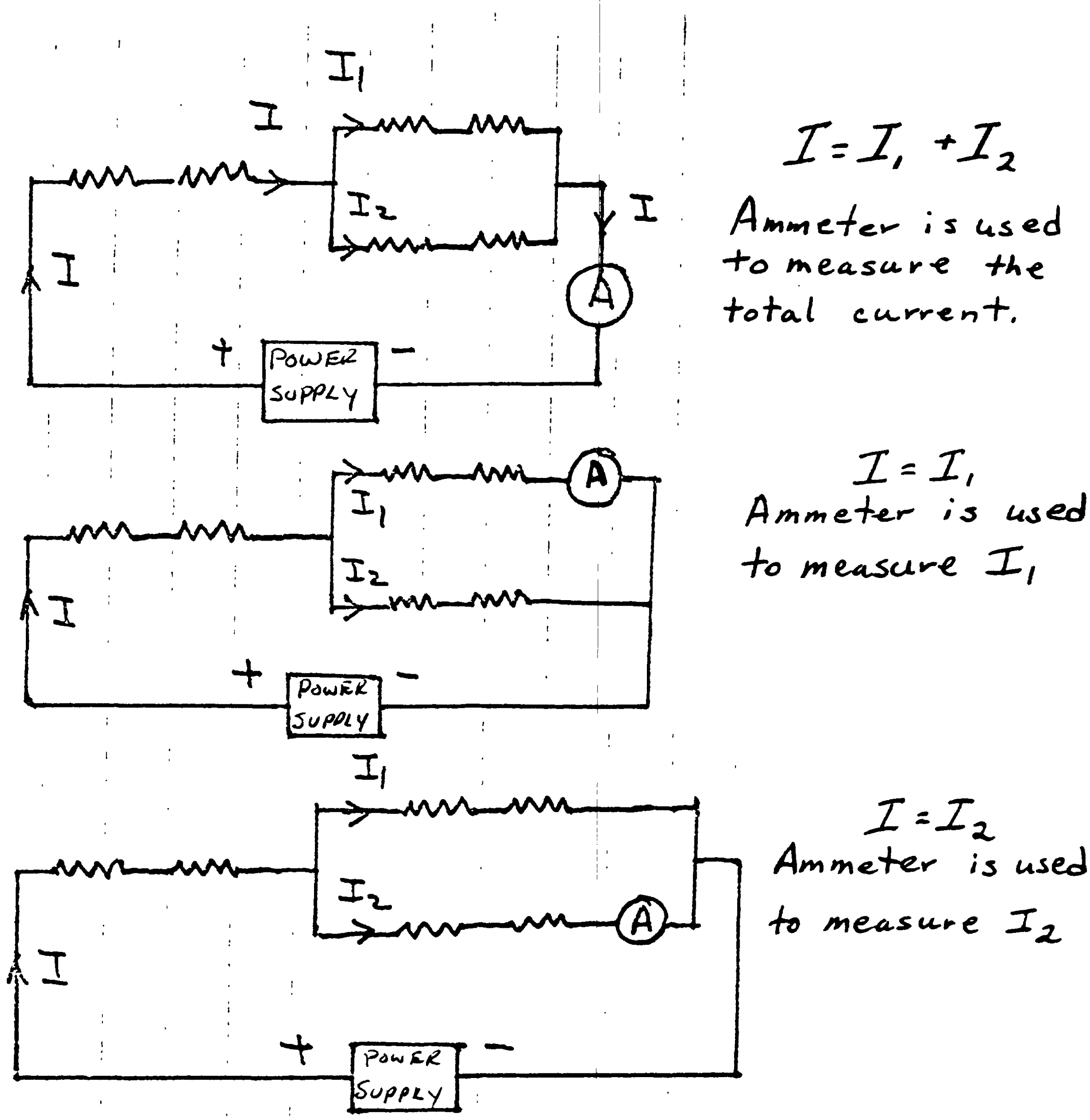


Figure 3.5.3.--Use of ammeter to measure current.

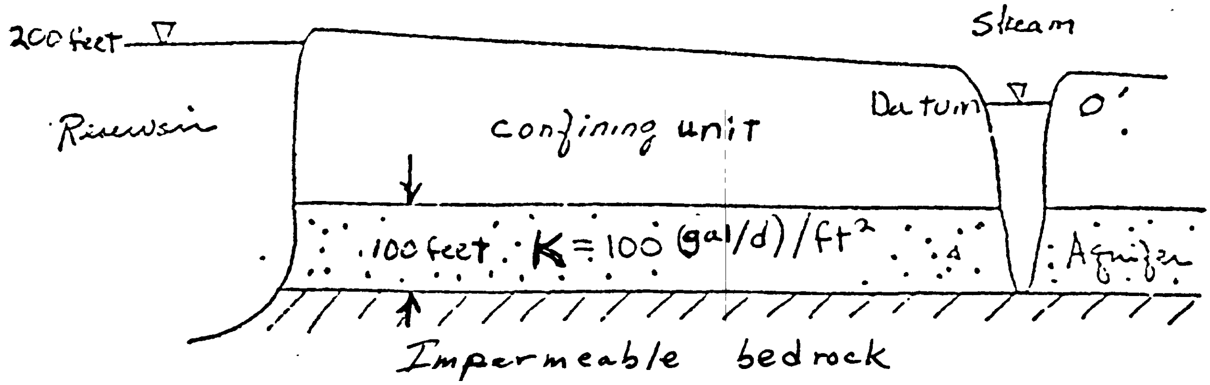
of electrical charge moving per unit time just as a flow meter measures quantity of water moving per unit time. Just as a flow meter must be inserted in a pipe section to measure the flow through that pipe, so the ammeter must be inserted in a section of wire to measure the current through the wire.

In a hydraulic model, we might measure pressure (or head) differences between two points in the model by attaching one side of a pressure gage (or head gage) to one of those points and the other side to the other point. If one of these points were on the pressure (or head) datum, our reading would give the pressure above datum. The voltmeter can be considered analogous to such a gage. To measure voltage above datum, we connect one side of the voltmeter to a point on the electrical datum (ground) and one side to the test point.

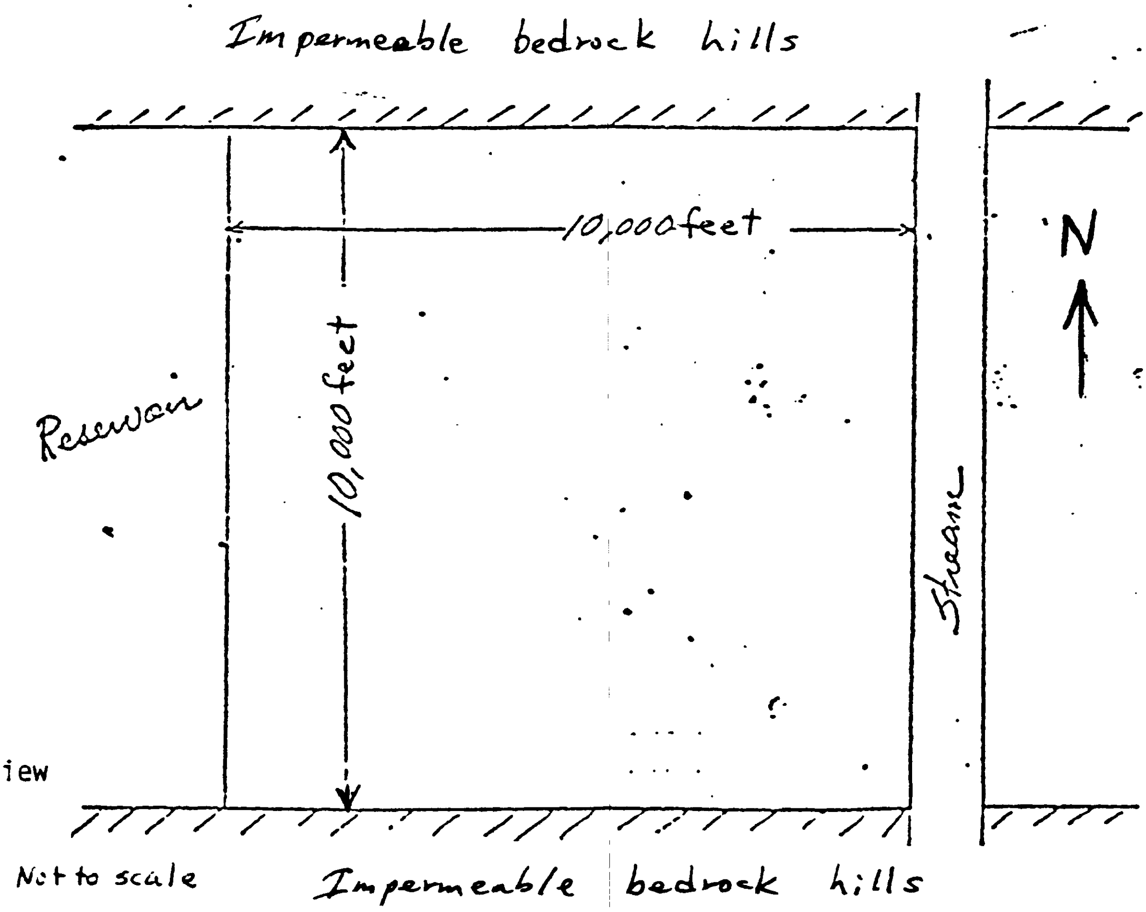
Electrical Analog Model Design

A.--Confined Aquifer Bounded by Impermeable Bedrock Hills and Fully Penetrating Stream and Reservoir

Figure 3.5.4 shows a cross section and map of a confined sand aquifer 10,000 feet on each side, bounded on the north and south by impermeable bedrock hills, on the west by a fully penetrating reservoir, and on the east by a fully penetrating stream. The reservoir surface is 200 feet above datum while the stream surface is at the datum. The hydraulic conductivity of the sand is 100 gallons per day per square foot and its thickness is 100 feet. In



A. Cross section



B. Plan view

Figure 3.5.4.--Cross section and plan view of confined aquifer.

this exercise, we will deal only with steady-state flow so that we need not specify a storage coefficient for the sand aquifer.

We wish to design a resistance network analog to simulate this ground-water system. We assume that we have a supply of 1,000 ohm resistors together with a limited supply of 2,000 ohm resistors, and that it would be convenient, using the laboratory equipment at our disposal, to represent the 200-foot head difference across the aquifer by a potential difference of 2 volts. We will use a uniform mesh spacing of 500 feet in each direction.

1. Describe all the boundaries of the hydrologic system. What is the electrical conductance, in Siemens, of a 1,000 ohm resistor? Of a 2,000 ohm resistor?

2. Sketch the block of aquifer represented by a single internal resistor of the network and by a resistor along one of the impermeable boundaries of the network; show all dimensions.

3. Calculate the hydraulic conductance of each of these blocks.

4. Suppose we use 1,000 ohm resistors in the interior of the network, and 2,000 ohm resistors along the impermeable boundaries. What is the ratio, in either case, of the hydraulic conductance of

the block to the electrical conductance of the resistor simulating the block? We designate this ratio as k_c. What are its units?

5. What is the ratio of head difference in the aquifer to the voltage difference simulating that head difference? We designate this ratio as k_v. What are its units?

We have defined ratios between hydraulic conductance of a block of aquifer and electrical conductance of a model element (resistor) and between head difference in the aquifer and voltage difference in the model. It remains to establish a ratio between flow in the aquifer and current or rate of flow of electrical charge in the model. Recall the analogy between Darcy's law and Ohm's law, as outlined in the course notes, and recall the definitions of hydraulic and electrical conductance. One Siemen of electrical conductance is one ampere of current per volt of potential difference.

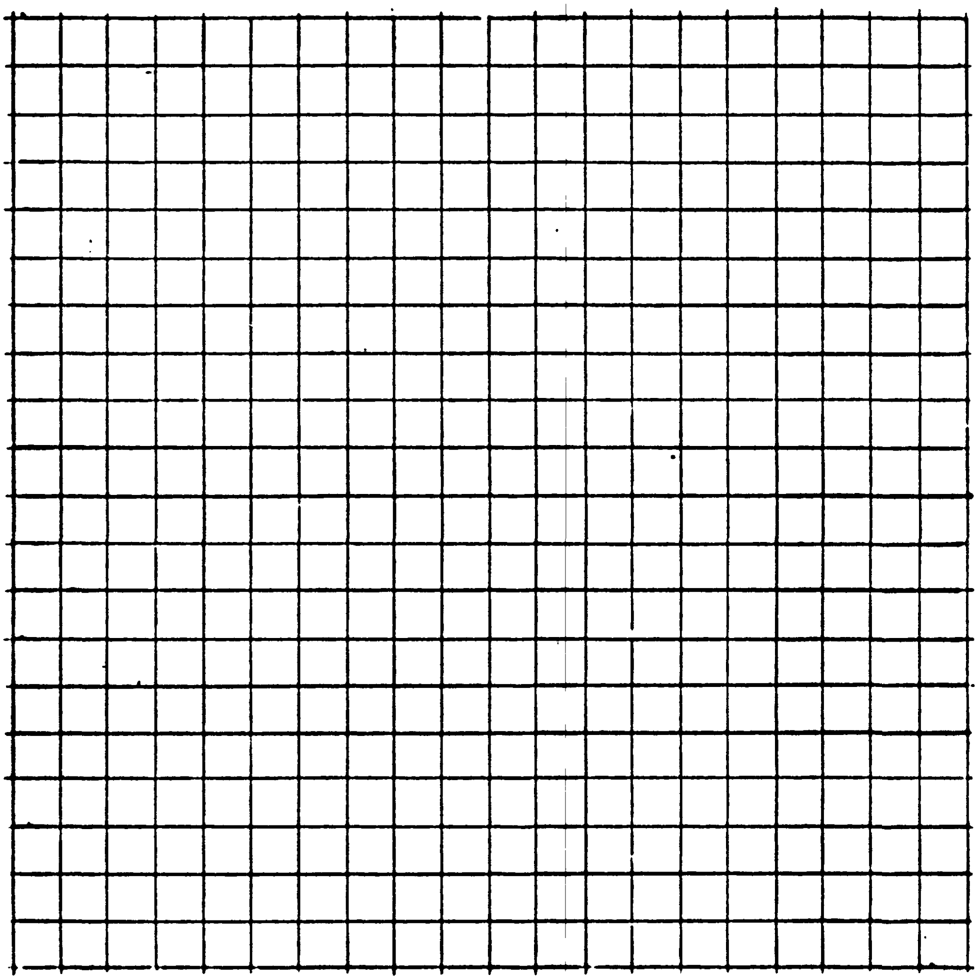
6. What will be the ratio of flow in the aquifer, in gallons per day, to current in the model? We designate this ratio as k_i. What are its units?

7. Of the three ratios which we have defined, k_c, k_v, and k_i, how many are necessary to describe the analog which we are using?

8. The grid on worksheet 3.5.1 represents the resistance network of our model. Show the location of the two constant-head boundaries on this network. Sketch, and indicate how these boundaries will be simulated electrically. Assume that we have a power supply that has a zero voltage (ground) lead, and a lead for which we can adjust the voltage. It is convenient to set the voltage at zero (ground) on the boundary having the lower head. Indicate in your sketch how the power supply should be connected to the network. What voltage must be applied along the boundary representing the reservoir? A voltmeter will measure the voltage difference between any two points. Indicate in your sketch how this meter should be connected to the resistance net in order to measure voltages. Give the relation by which these voltages can be converted into heads. Indicate in your sketch how an ammeter should be connected in order to measure the current entering the model through the boundary representing the reservoir. Give the relation by which the current measurement is converted to a flow value. What does this flow represent? Is it greater than, equal to, or less than the discharge from the aquifer to the stream at the eastern edge of the model? Explain your answer.

9. Suppose we wish to simulate pumping from a well at a rate of 2 million gallons per day. What current withdrawal from the electrical nodes would we use to represent this pumping rate?

Worksheet 3.5.1.--Worksheet of electric-analog model grid for showing boundaries and method for connecting power supply, ammeter, and voltmeter..

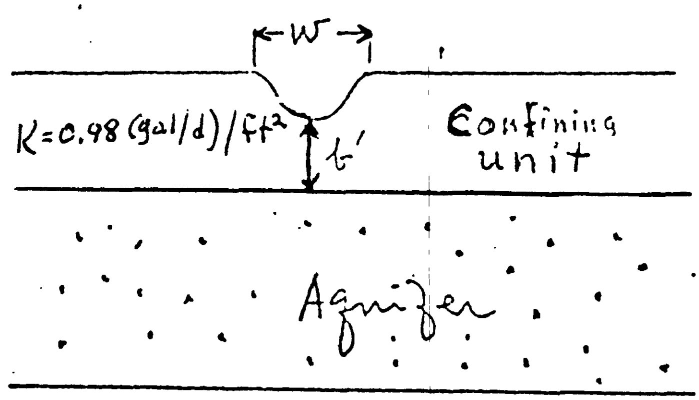


B.--Partially Penetrating Stream

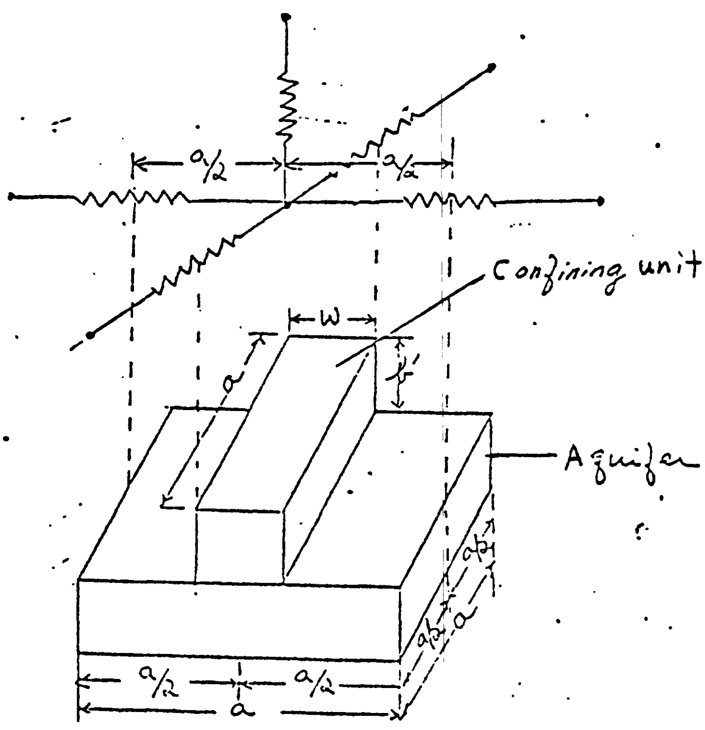
Suppose we wish to simulate a stream that does not fully penetrate the aquifer but rather is separated from it by a vertical interval, b', of the confining unit as shown in figure 3.5.5A. A junction, or node, of the network is associated with a block of aquifer extending half way to the adjacent nodes in all directions as shown in figure 3.5.5B. Referring to figure 3.5.5, b' represents the thickness of confining unit between the top of the aquifer and the bottom of the stream, w is the width of the streambed; the length of stream channel within the block represented by the node is a, the node spacing. Flow between the stream and the aquifer is assumed to be vertical, and restricted to the block ab'w located vertically below the streambed. We wish to simulate flow through this block by adding a vertical resistor to the net as shown in figure 3.5.5B. The mesh spacing of our analog is 500 feet. We will assume the thickness b' is 10 feet; w, the stream width, is 40 feet; and the vertical hydraulic conductivity of the confining unit, k', is 0.98 gallon per day per square foot.

1. Calculate the vertical hydraulic conductance of the block ab'w between the streambed and the aquifer.

2. Using the value of k_c computed previously, calculate the



A. Hydrogeologic cross section



B. Simulation of one block of streambed

Figure 3.5.5.--Simulation of nonpenetrating stream.

electrical conductance required to simulate this hydraulic conductance. What is the resistance corresponding to this conductance?

3. Why is it necessary to use the value of k_c obtained for the lateral network in calculating the vertical resistance?

4. A voltage must be applied to the system model in order to simulate the head, or stream-surface elevation, of the partially penetrating stream. Where should this voltage be applied?

Electrical Analog Experimental Procedures

These exercises refer to the aquifer described in the section "Electrical Analog Model Design." The individual exercises studied with the electrical model are designated by a title and capital letters A, B, C, etc. Individual questions are numbered under each heading.

A.--Confined Aquifer Bounded by Impermeable Bedrock Hills and Fully Penetrating Stream and Reservoir

NOTE: Before proceeding with this problem, go through the check-out procedures for the analog unit, if not already completed.

Initially, we consider the flow system shown in figure 3.5.4 of the section, Electrical Analog--Model Design, Part A: an aquifer bounded by a fully penetrating reservoir and a fully penetrating stream with no pumping in progress. To simulate constant heads in the reservoir and stream, connect the positive (red) lead of the <u>power supply</u> to the boundary representing the reservoir (bus wire on left side of analog board) and the negative ground (black) lead to the boundary representing the stream (bus wire on right side of analog board) (fig. 3.5.6)[2]. To measure voltage at junctions of the resistor network, plug the red and black leads into the red and black jacks on the front of the <u>multimeter</u>. On the Data Precision multimeter, push in the leftmost button under "V" that is marked "⎓" on the left side of the meter panel and push in the button marked "100" under the "mA-V-kΩ" bar. To measure voltage on the Keithley multimeter, push in the button under "V" on the left side of the front panel and push in the button under "200" on the right side of the front panel (third button from the right). Connect the negative ground (black) lead of the multimeter to the stream boundary and the positive (red) lead to the reservoir boundary. Turn on the power supply and multimeter and adjust the voltage potentiometer on the power supply (knob marked "voltage adjust") until the meter reads +2.00 volts. Make certain that the wires from the power unit do

[2] Note that the wiring sketches for this problem are not to scale, that is, the the number of rows and columns shown on the sketches are not an even multiple of those on the resistor grid of the analog unit. In the following parts of the problem set, carefully check the distances from the edge of the board when attaching the pumping well, the fully penetrating stream, and the nonpenetrating stream.

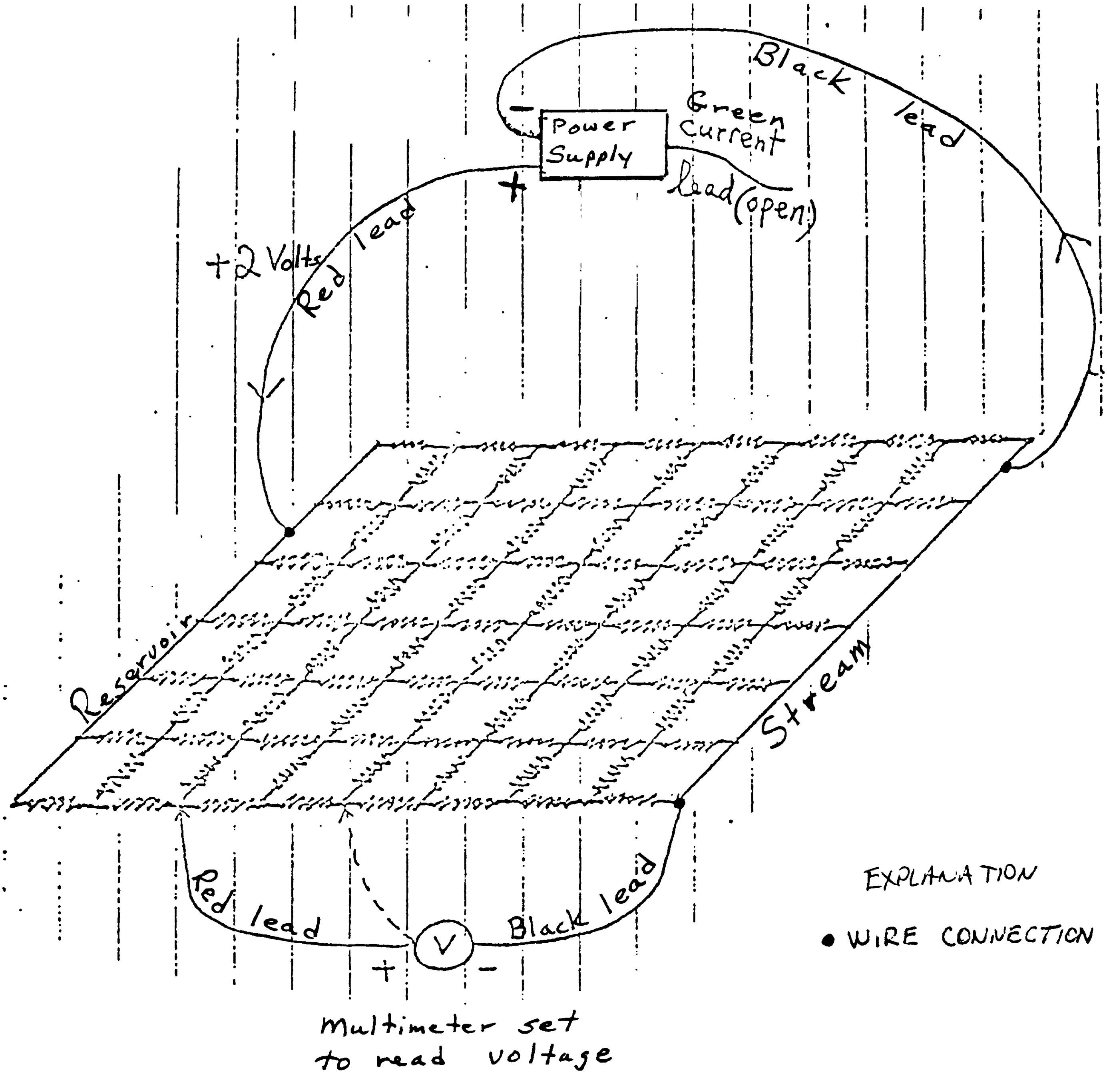


Figure 3.5.6--Connections to measure voltage at nodes for Problem A-1.

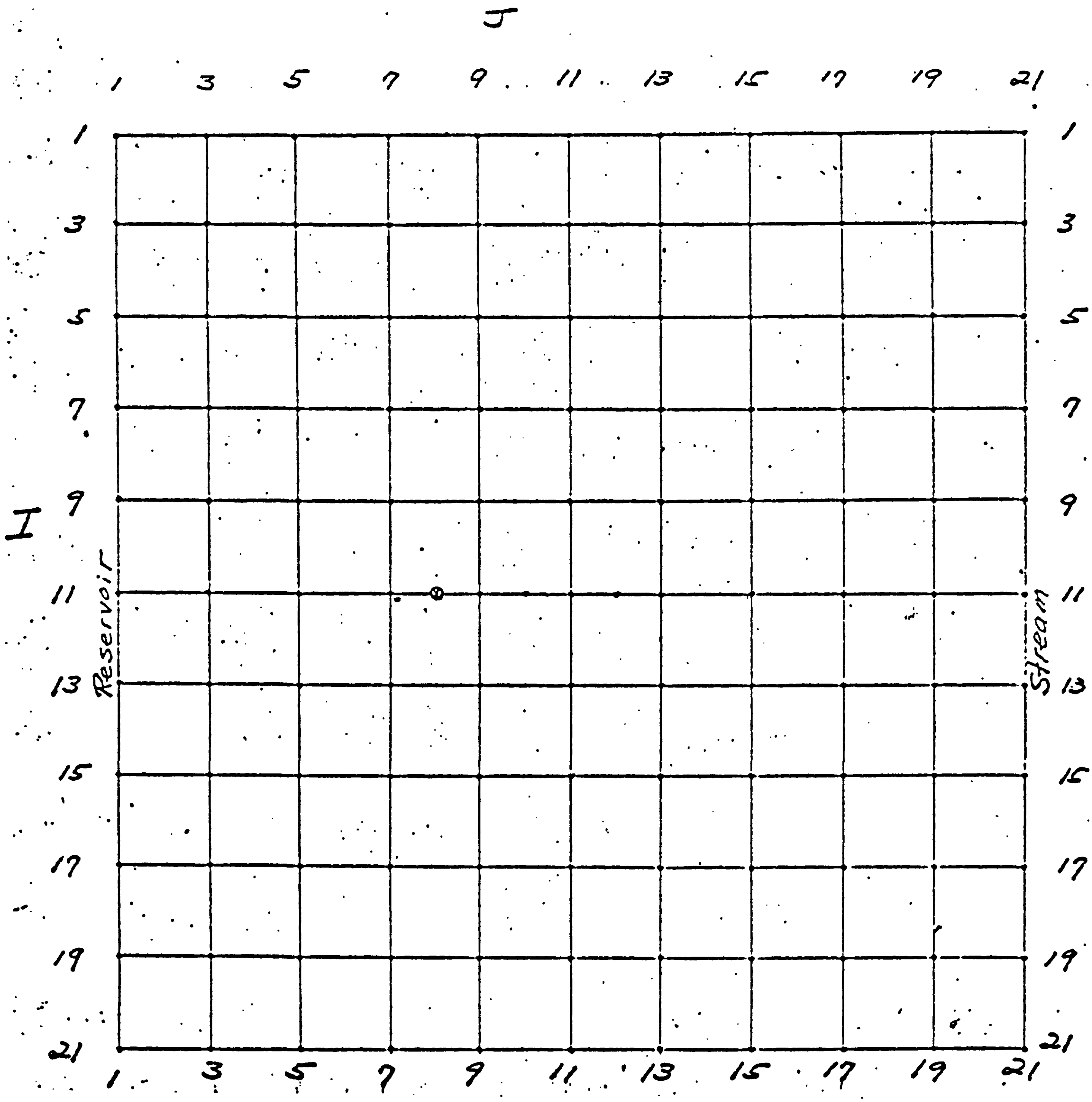
not brush against the knob of the voltage potentiometer. The voltage potentiometer is very sensitive to small movements of the shaft.

1. Measure voltage at every other node in the model by moving the red lead of the voltmeter from node to node. Recheck the voltage at the reservoir from time to time to make certain that it has remained at 2.00 volts. Make sure good contact is made between the probe and node. Solder flux can collect on the surface of the node, which creates resistance and causes a lower voltage reading. Convert these voltages to heads and record the results on worksheet 3.5.2. Keep in mind the natural symmetry of the system; therefore, it is not necessary to measure all nodes in the model.

2. Calculate the total ground-water discharge into the stream using the head gradient toward the stream, the transmissivity of the aquifer, and the length of the stream.

3. Set the multimeter for current measurements and measure the current through the stream boundary by placing the meter in series between the stream boundary (bus wire on right side of analog board) and the negative ground lead (black) of the power supply (fig. 3.5.7). By connecting the ground lead of the power supply to the ground lead of the meter (black to black) and the positive lead (red) of the meter to the bus wire on the right side

Worksheet 3.5.2.--Worksheet for plotting initial head distribution.



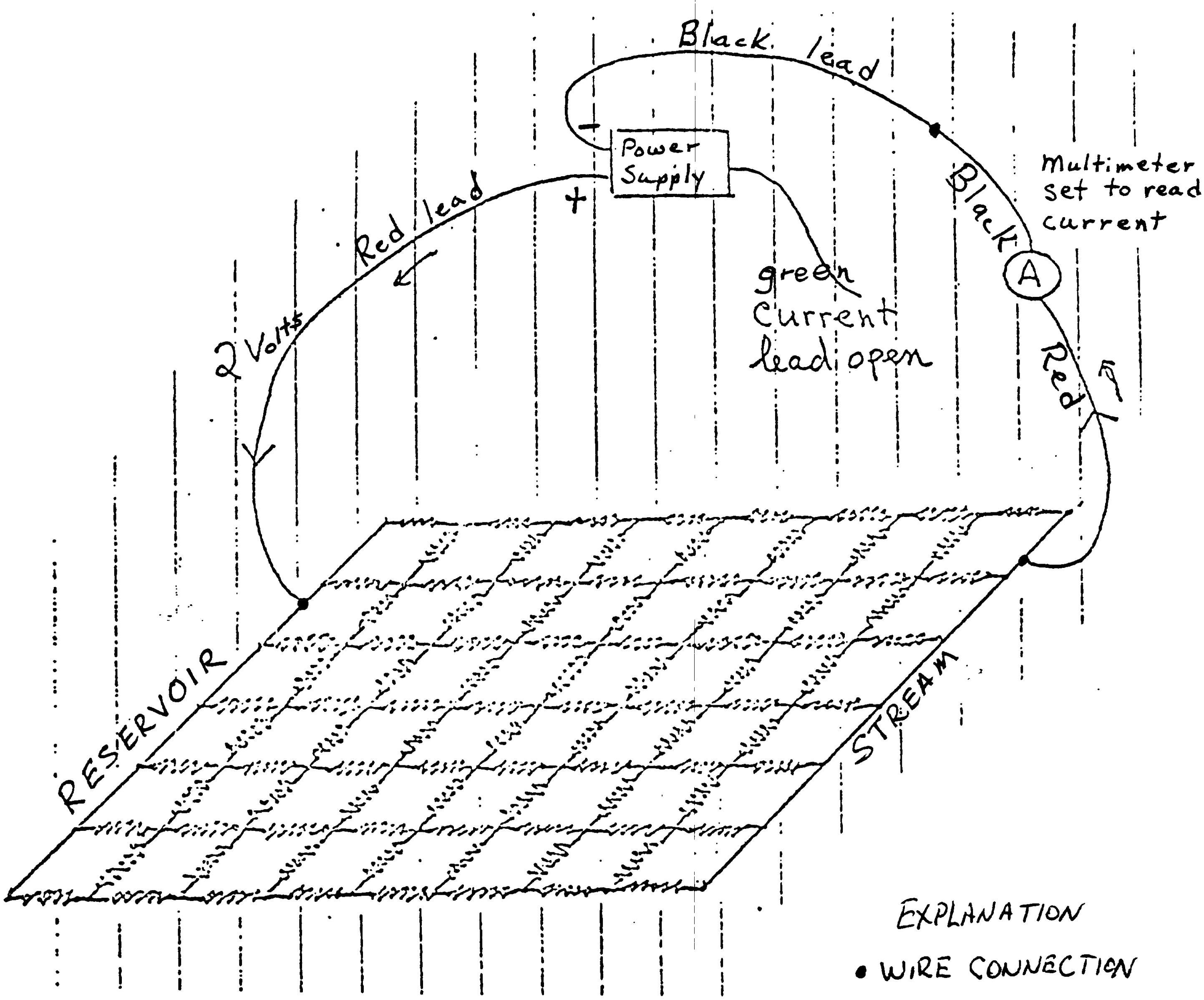


Figure 3.5.7--Ammeter in series with stream boundary and ground lead to measure current to stream boundary.

of the board the current meter will always read positive. The direction of current flow is arbitrarily defined as from positive to negative (ground). Current will be displayed in milliamps (thousandths of an ampere). Convert this current to flow in gallons per day, and compare it to the flow calculated from the head gradient and aquifer transmissivity.

B.--Confined Aquifer with Discharging Well

We now wish to simulate the effect of a well that is located midway between the bedrock hills and 3,500 feet east of the reservoir and is being pumped at a rate of 2 million gallons per day. During this simulation we will maintain the boundary voltages exactly as in Part A.

1. Determine which node should represent the well (<u>note that the sketches used in this problem are not to scale</u>). Connect the green current lead of the power supply to this node through the ammeter; that is, place the multimeter, set to measure current, in series between the current lead of the power supply and the node representing the well (see fig. 3.5.8). Adjust the current potentiometer on the power supply until the multimeter indicates that the current from the node is +2 milliamperes. Check the sign of the display to be sure that the current is away from, rather than toward, the network. By connecting the black lead of the ammeter (ground) to the green lead (current) of the power supply, the meter will always read a positive current. The direction of

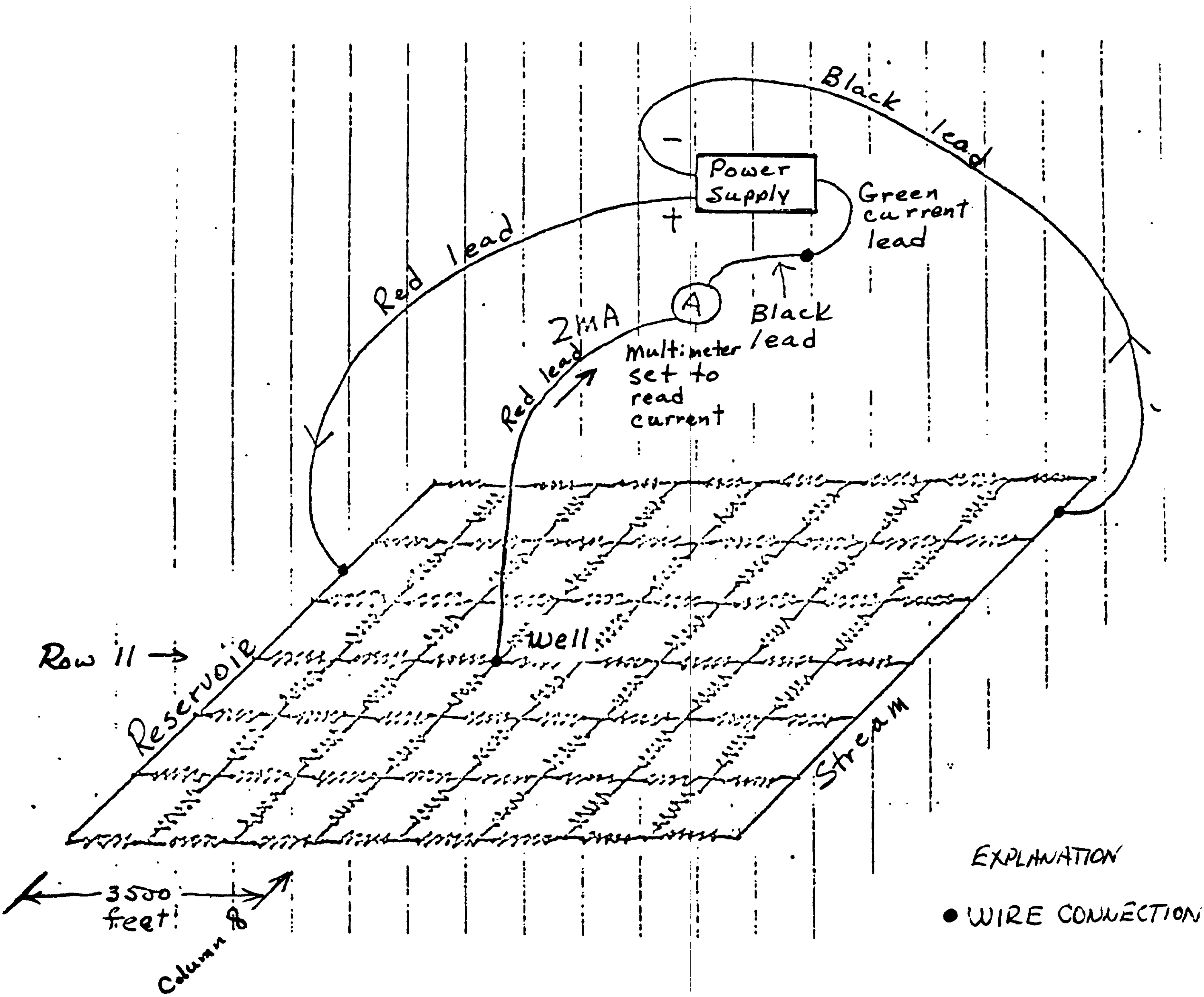


Figure 3.5.8.--Simulation of the discharging well. The ammeter is in series with the current lead to measure the well current.

current flow is from the positive lead (red) to the ground lead (black) through the meter. In this configuration the current is flowing out of the network and into the power unit. This can be verified by setting the meter for voltage measurement and checking voltages at the well and at surrounding nodes; voltages must be higher at the surrounding nodes than at the well for the current to be toward the well, and thus away from the net. After adjusting the current, remove the ammeter from the circuit and connect the green current lead of the power supply directly to the node representing the well.

2. Set the meter for voltage measurements and reconnect the black lead of the multimeter to the stream and the red lead to the reservoir. Recheck the voltage at the reservoir because it may have dropped slightly when the current lead was attached to the well node. Readjust the meter to 2.00 volts to maintain the 200 feet of head at the reservoir. Measure voltages in the model (see fig. 3.5.9), convert the voltages to heads, and plot the values on worksheet 3.5.3. Contour the heads and sketch the limiting flowlines separating flow to the well from flow to the stream. Compare the voltage readings with those obtained in A. How far does the "influence" of the well extend?

3. Construct an east-west profile of head values along a line extending from the reservoir to the stream through the well.

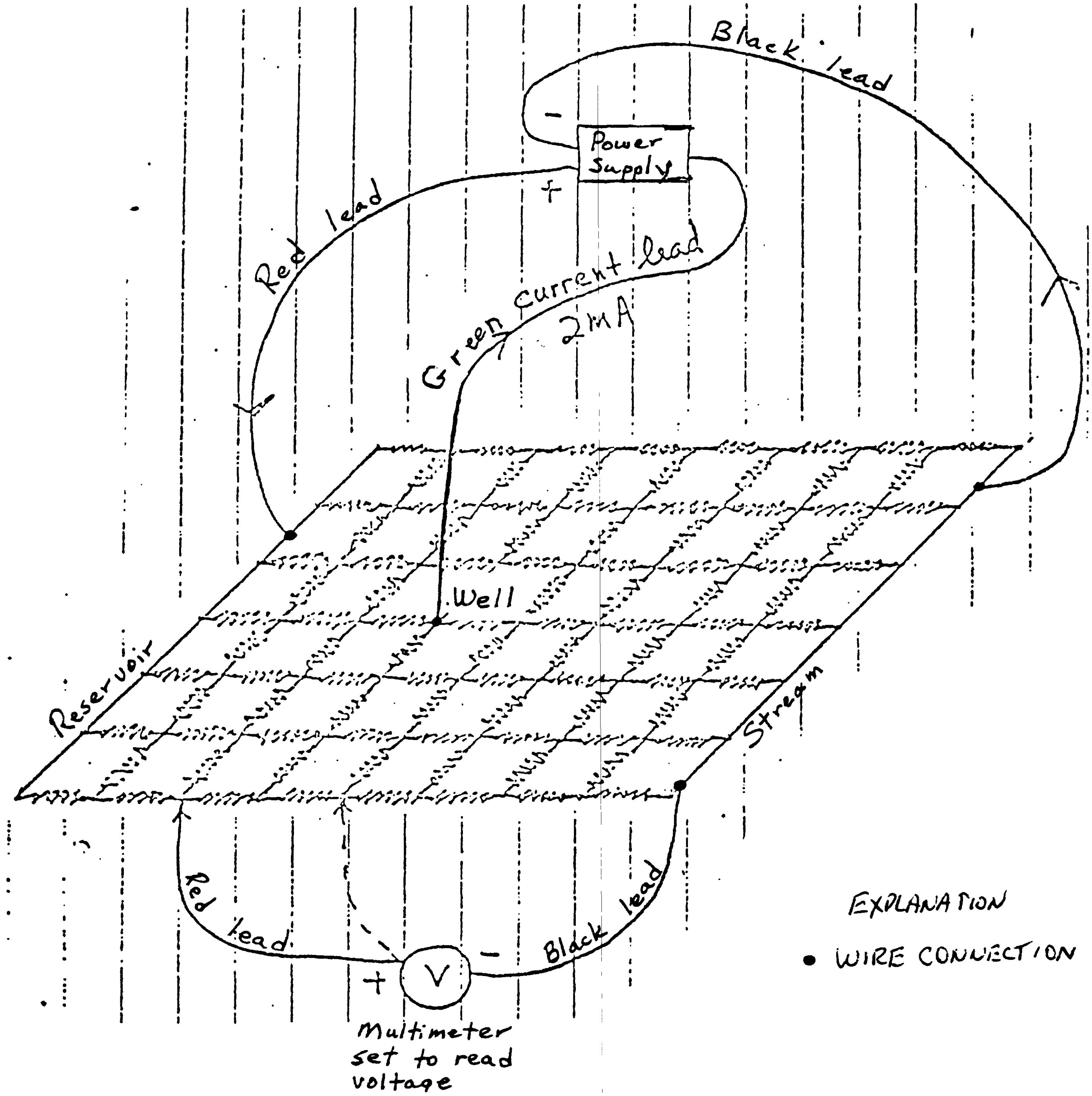
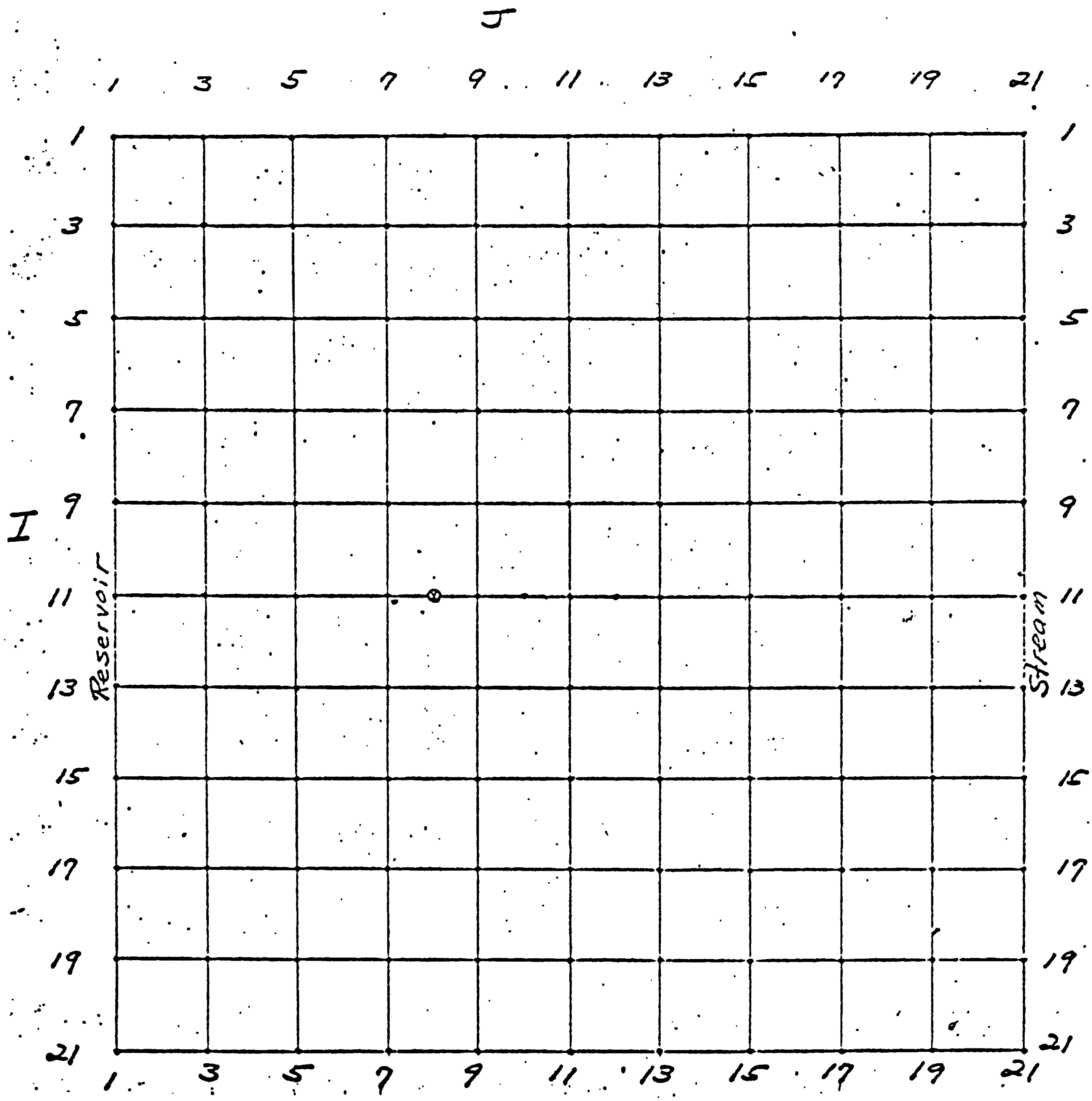


Figure 3.5-9.--Use of voltmeter to measure voltage throughout the system as the well is discharging.

Worksheet 3.5.3.--Worksheet for plotting and contouring head values that reflect the influence of a discharging well.



4. Measure the flows entering the model at the reservoir boundary and leaving at the stream (figs. 3.5.10 and 3.5.11).

5. What is the total flow into the model and the total flow from the model? What must be the relation between these two quantities?

6. The results of this experiment provide a new equilibrium involving the well, the aquifer, and the stream in contrast to that in Part A involving only the reservoir and the stream. Discuss the relation between these two equilibria in terms of the principles outlined by Theis (1940) in Ground-Water Note 34 "The source of water derived from wells."

7. In a field situation, the two equilibria would be separated by a period during which a gradually diminishing fraction of the well discharge is supplied by withdrawal from aquifer storage. Do the results we have obtained here give any information at all about this nonequilibrium period?

C.--Confined Aquifer with Discharging Well in Superposition

We now wish to illustrate the principle of superposition which states that two solutions corresponding to two separate flow conditions (but the same boundary conditions) can be added algebraically to obtain a third solution that applies when the two flow conditions are combined in the aquifer. Connect both the

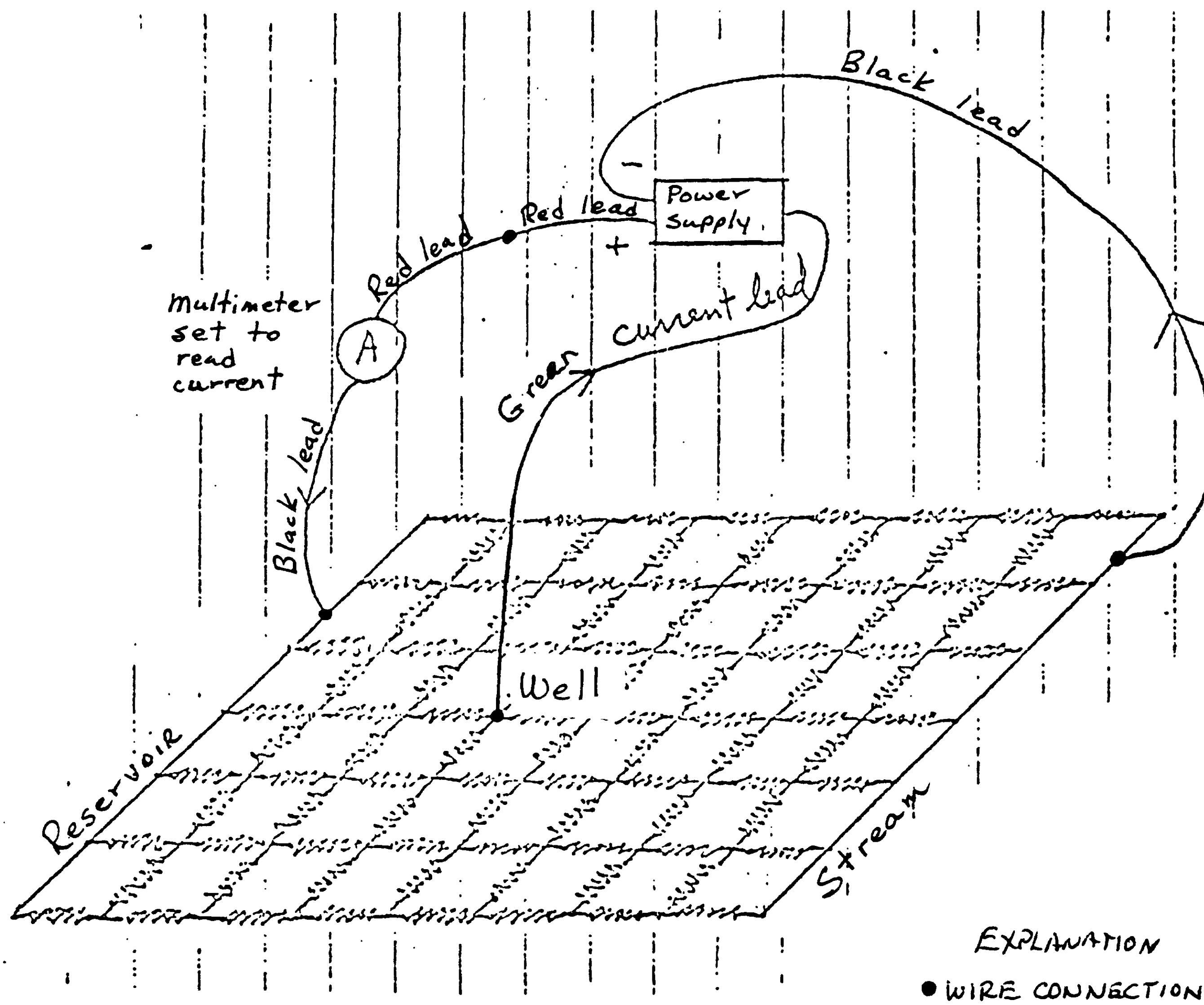


Figure 3.5.10.--Connection of ammeter in series with positive lead of power supply and reservoir boundary to measure current entering the reservoir boundary.

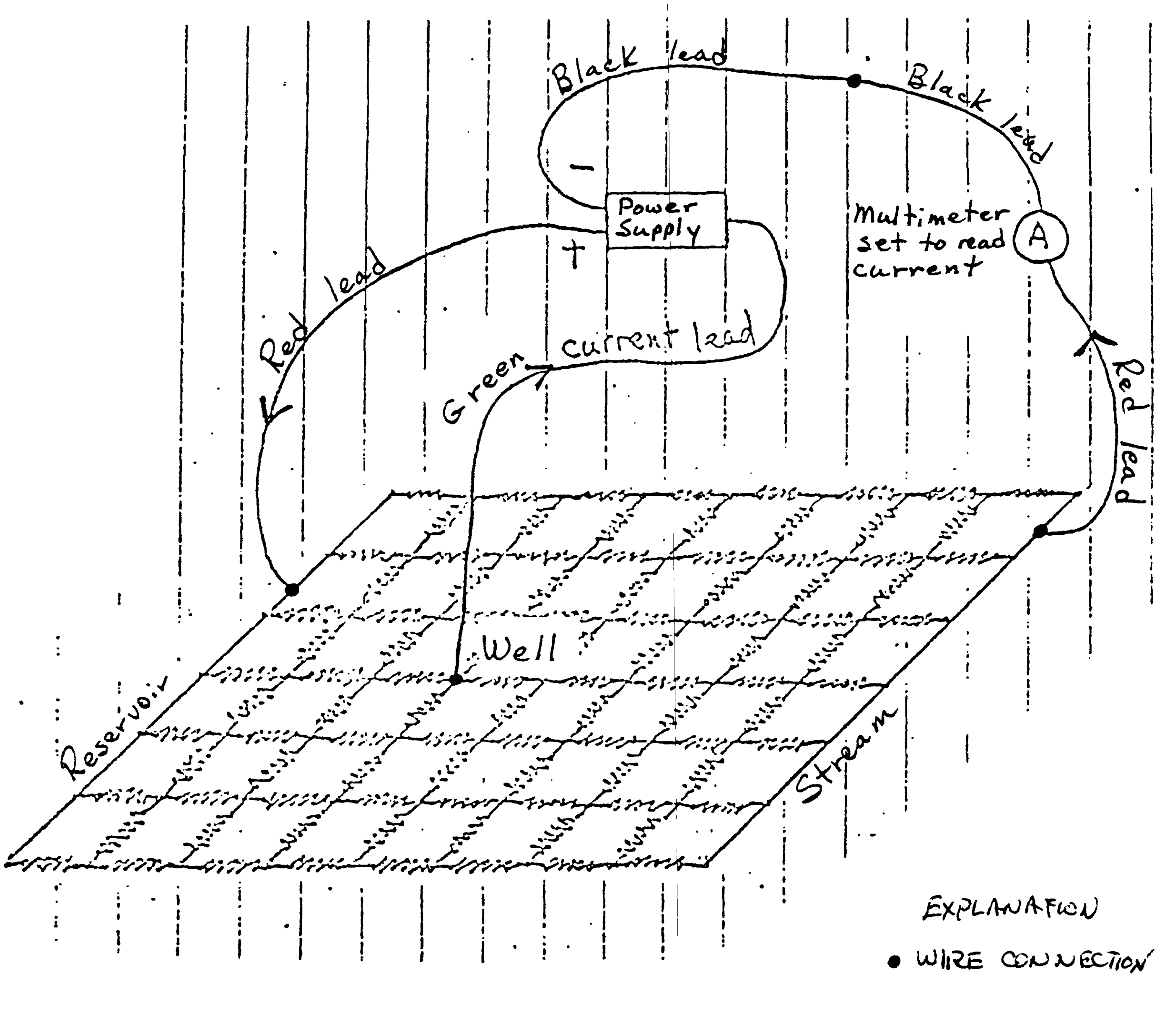


Figure 3.5.11.--Connections for measurement of current leaving aquifer from the stream boundary.

reservoir boundary and the stream boundary to the ground lead (black) of the power supply so that both are at zero potential (see fig. 3.5.12). With the well disconnected, check the potential at various points in the net. What is the potential distribution under these conditions? Reconnect the well and readjust the current to 2 milliamperes.

1. Measure voltages in the model, convert the voltages to drawdowns, plot values on worksheet 3.5.4, and contour.

2. Measure the two boundary currents in the model and convert to flows (see figs. 3.5.13 and 3.5.14).

3. Add the drawdowns measured in this experiment algebraically to the heads measured in part A (worksheet 3.5.2), plot the resultant values on worksheet 3.5.5 and contour. Compare the resulting head-change map with the heads measured in part B (worksheet 3.5.3).

4. Explain how this result illustrates the principle of superposition.

5. What does the measured inflow along the reservoir boundary in this experiment actually represent?

6. What does the measured inflow along the stream boundary actually represent?

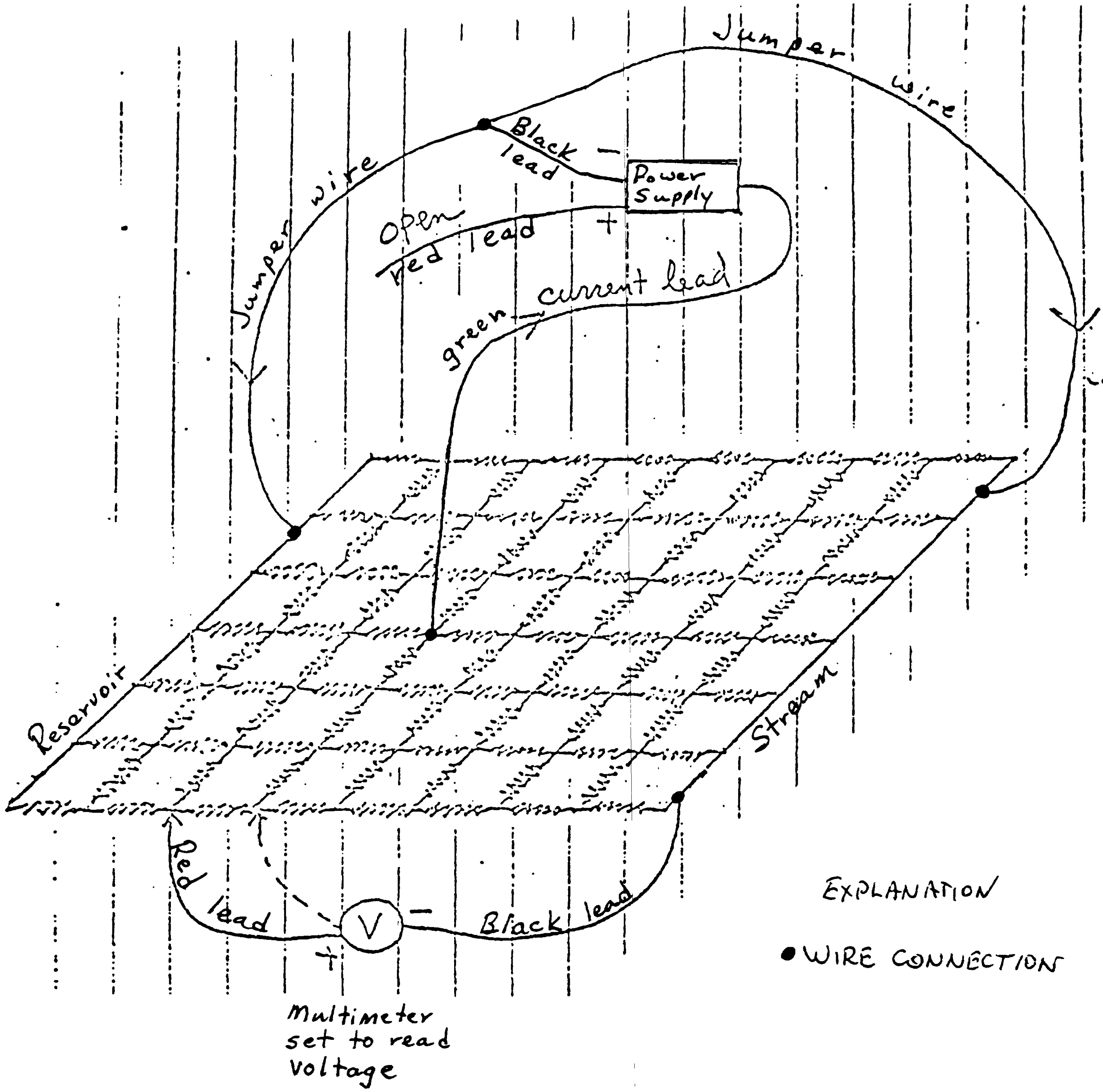
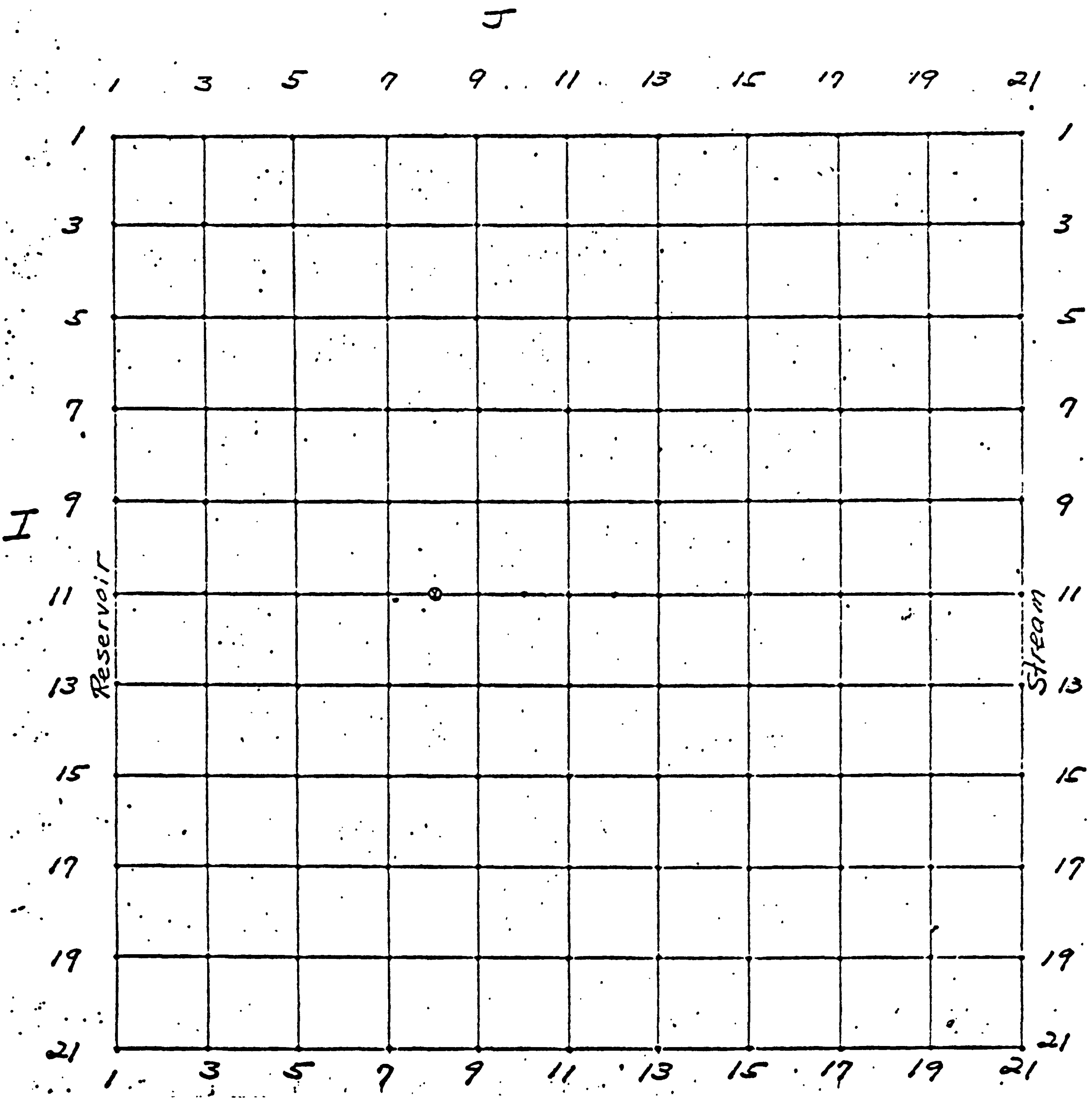


Figure 3.5.12.--Connections for measurement of voltages when well is discharging and potentials on both boundaries equal zero.

Worksheet 3.5.4--Worksheet for plotting and contouring drawdowns when well is discharging and potentials at reservoir and stream boundaries equal zero.



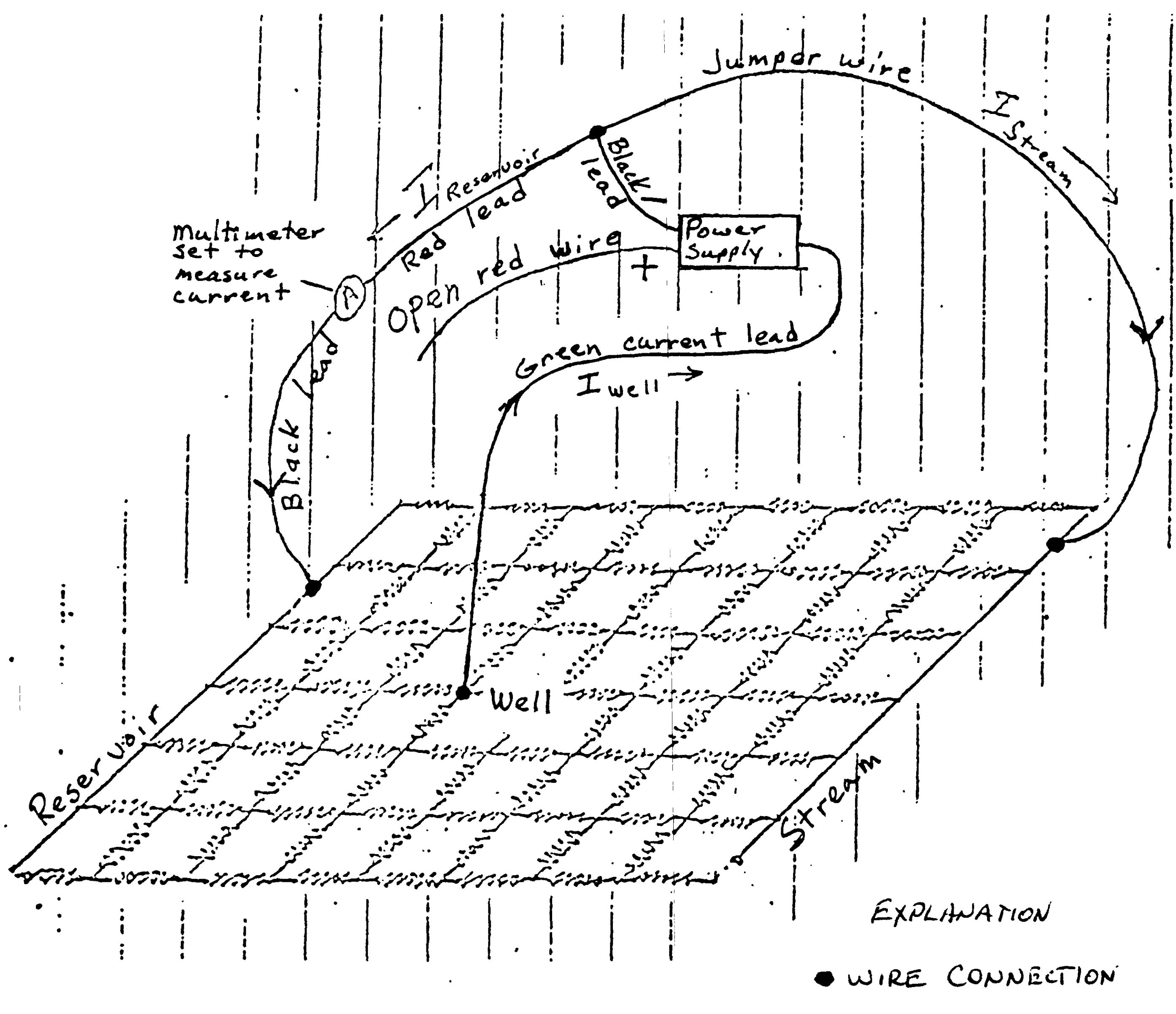


Figure 3.5.13.--Connections for measurement of current leaving the reservoir boundary when well is discharging and boundary potentials equal zero.

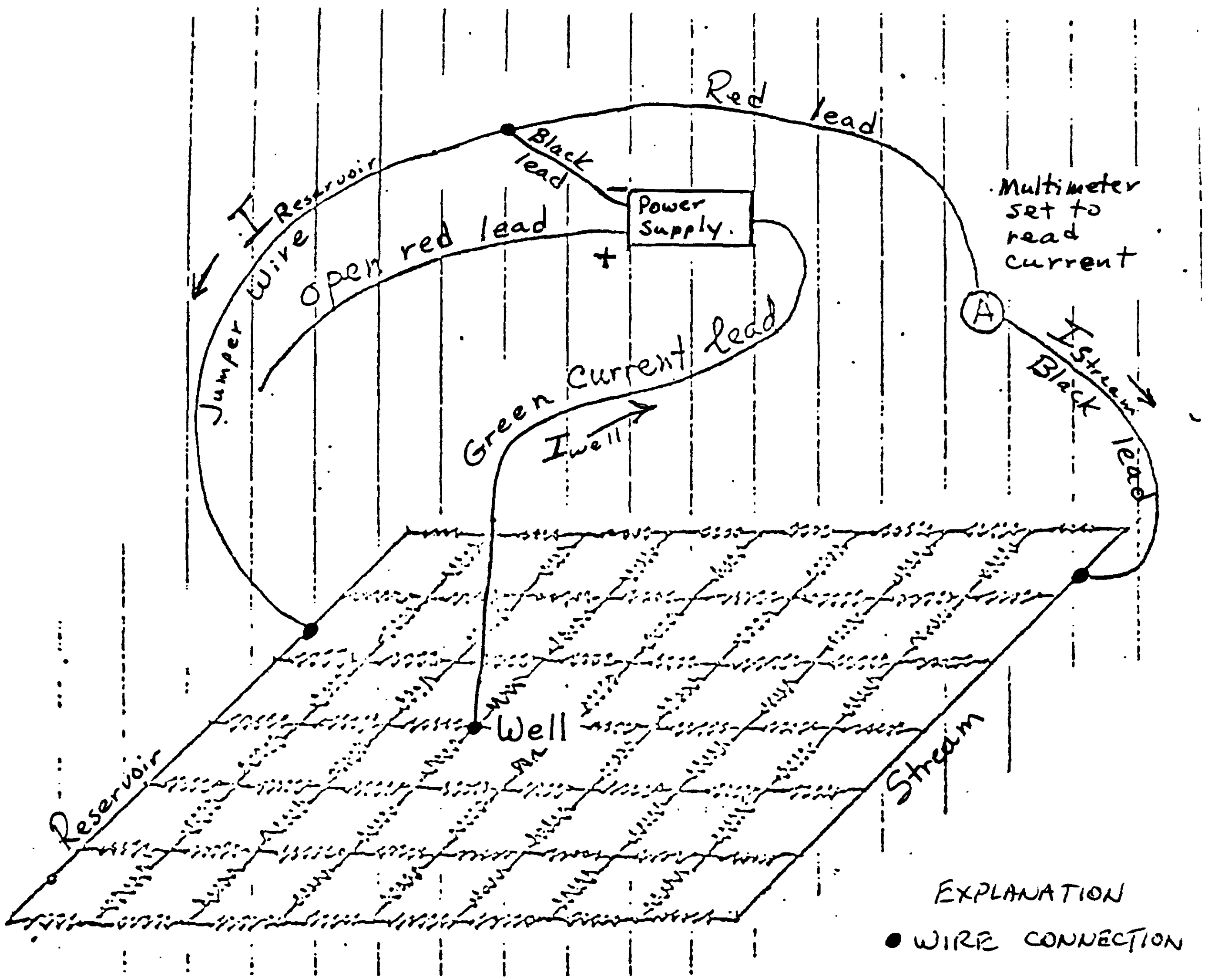
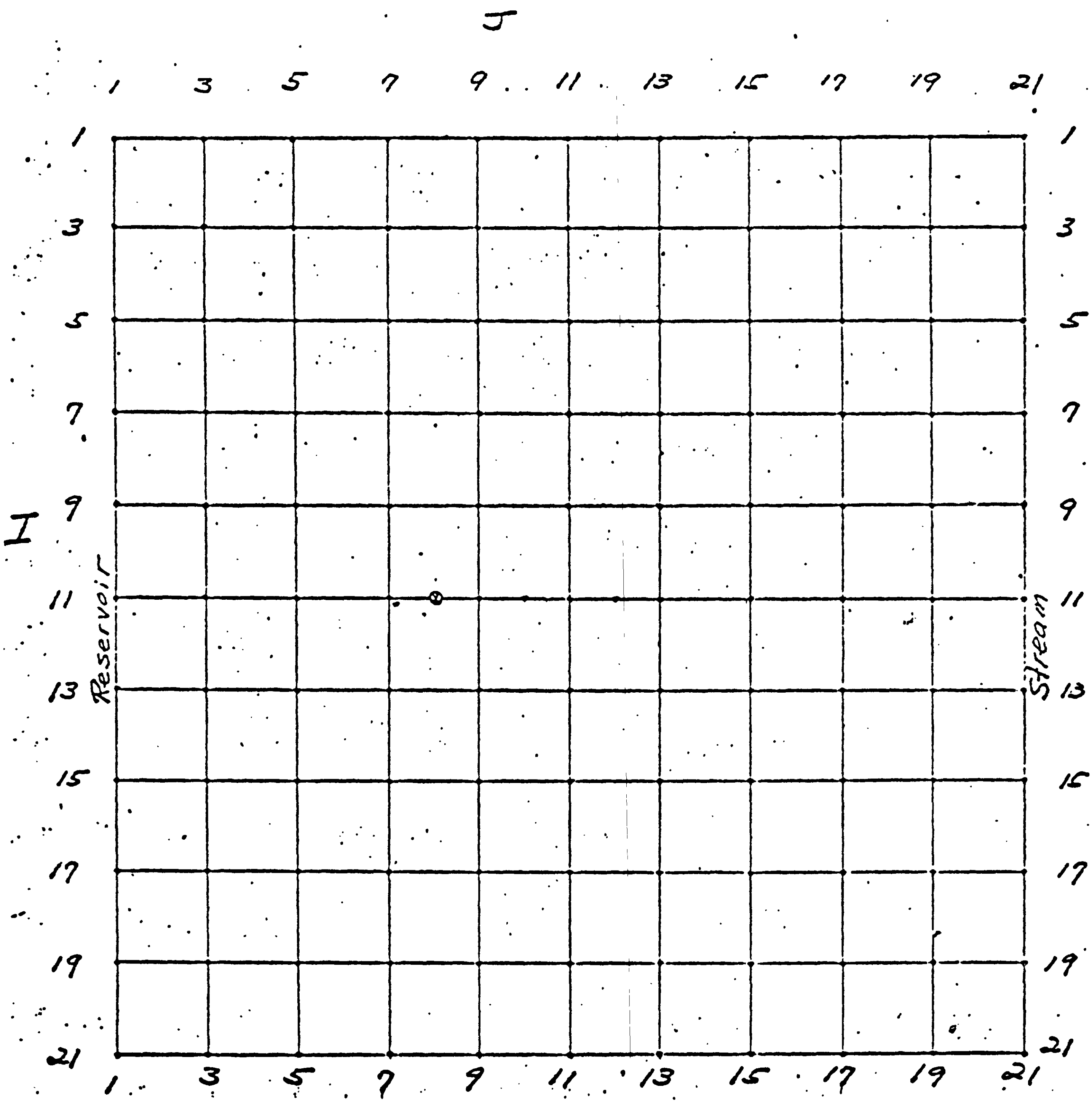


Figure 3.5.14.--Connections for measurement of current leaving stream boundary when well is discharging and boundary potential equals zero.

Worksheet 3.5.5.--Worksheet for plotting and contouring heads that equal the algebraic sum of drawdowns measured in C-1 and heads measured in A-1.



D.--Nonpenetrating stream

We now wish to add a new stream to the system that crosses the aquifer from north to south, parallel to the existing reservoir and stream boundaries, and 7,000 feet east of the reservoir boundary (<u>note again that the sketches used in this problem are not to scale</u>). This stream does not penetrate the aquifer; rather, it is separated from it by an interval of the confining unit as shown in figure 3.5.5 in the section, Electrical Analog Model Design, Part B. As in the model design problem, we take b' = 10 feet, w = 40 feet, the length of channel in each node as the mesh spacing, 500 feet, and k' = 0.98 gpd/ft^2. Using these values and a value of 10^7 for k_c, the resistance required to represent the block ab'w of figure 3.5.5 is calculated as 5,100 ohms. Connect the 5,100-ohm resistor rack to the row of nodes representing the course of the "nonpenetrating" stream (fig. 3.5.15). Make certain that solid contact is made between each alligator clip and a resistor wire at each node. The surface of this stream is assumed to be at the same elevation, 0 feet, as the surface of the fully penetrating stream at the the eastern edge of the aquifer. The bus wire along the upper ends of the resistor rack must, therefore, be held at zero voltage. This is accomplished by connecting both it and the bus wire representing the eastern boundary to the ground lead of the power supply. We wish first to simulate the prepumping condition with the new stream in the system so the lead representing the well is

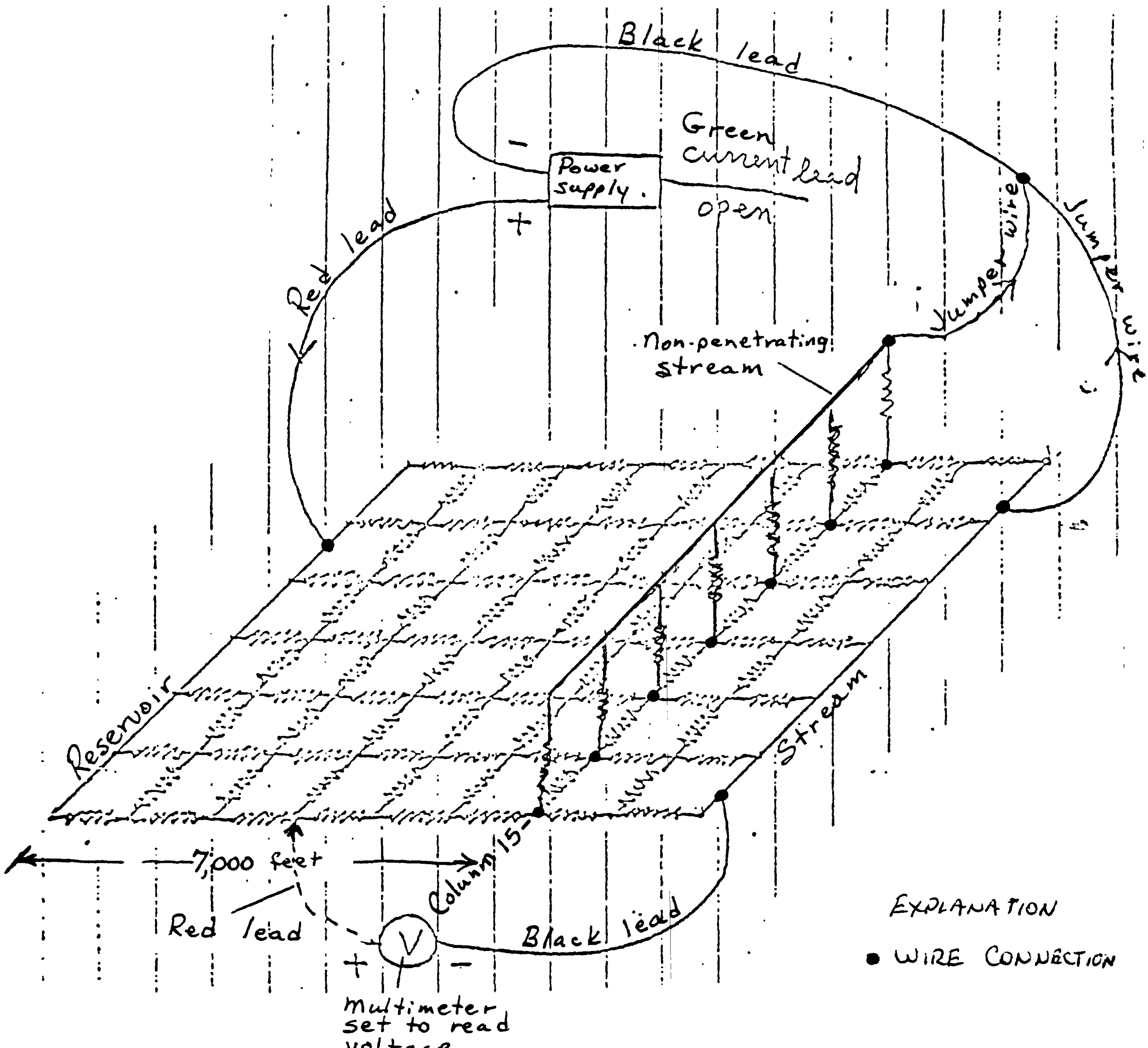


Figure 3.5-15.--Connections for measurement of voltages without discharging well but with addition of nonpenetrating stream with same potential as the stream boundary.

disconnected. The voltage on the reservoir boundary is again set to 2 volts.

1. With the model set up in this way, measure voltages at enough points to determine the new pattern; convert these voltages to heads, and plot and contour the values on worksheet 3.5.6.

2. Construct a head profile across the model from the reservoir to the fully penetrating stream.

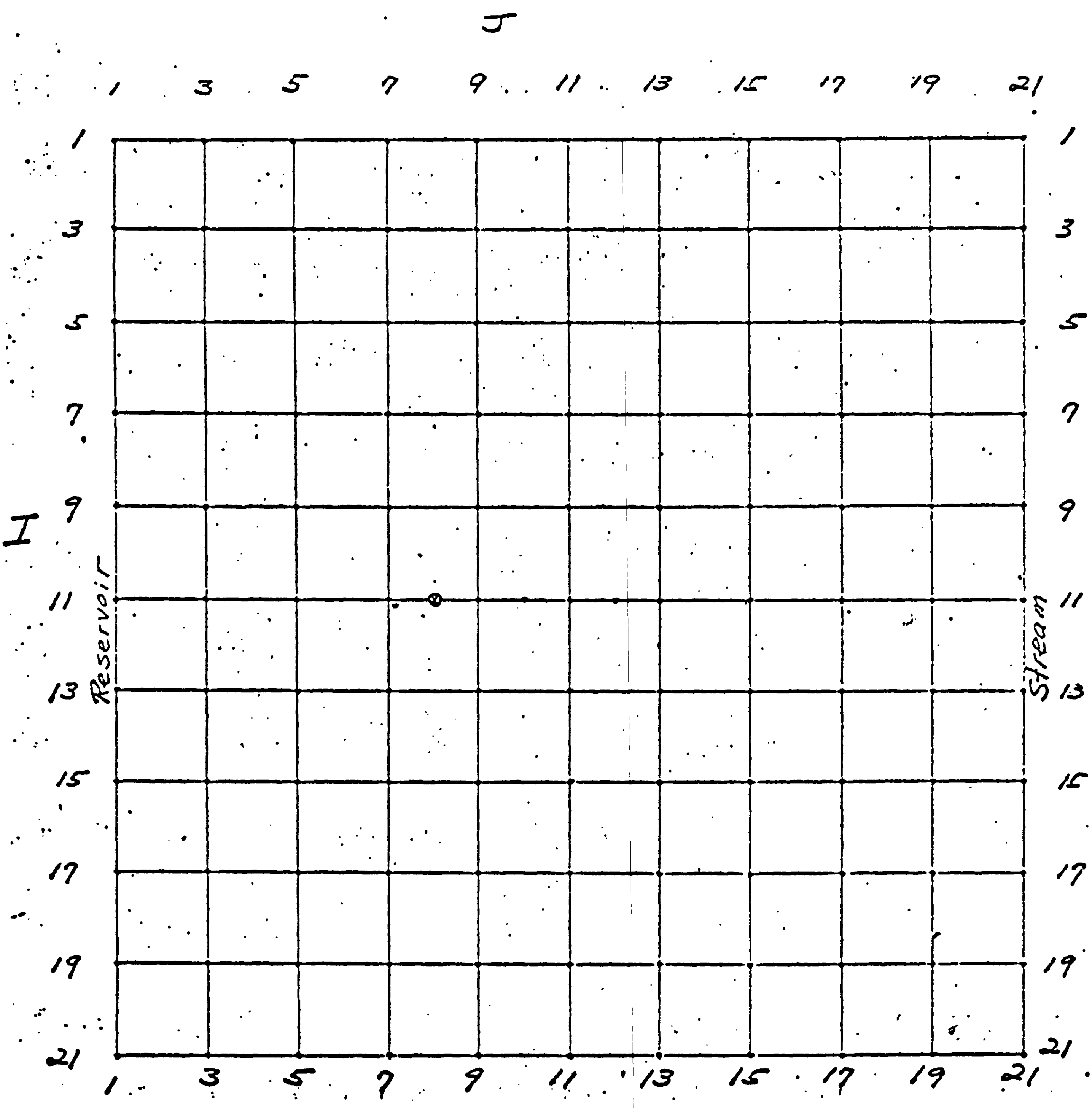
3. Measure the currents entering or leaving through the reservoir boundary, the nonpenetrating stream, and the fully penetrating stream at the eastern boundary; convert these currents to flows. What relations exist among these flow values? (figs. 3.5.16, 3.5.17, 3.5.18).

E.--Confined aquifer with nonpenetrating stream and discharging well

Add the well, as in part B, to the configuration of part D; adjust the current through the model to 2 milliamperes and readjust the reservoir potential to 2 volts. (fig. 3.5.19). Describe the boundaries of the hydrologic system.

1. Measure voltages, convert to heads, and plot and contour the values on worksheet 3.5.7.

Worksheet 3.5.6.--Worksheet for plotting and contouring heads when the simulation includes a nonpenetrating stream (no discharging well).



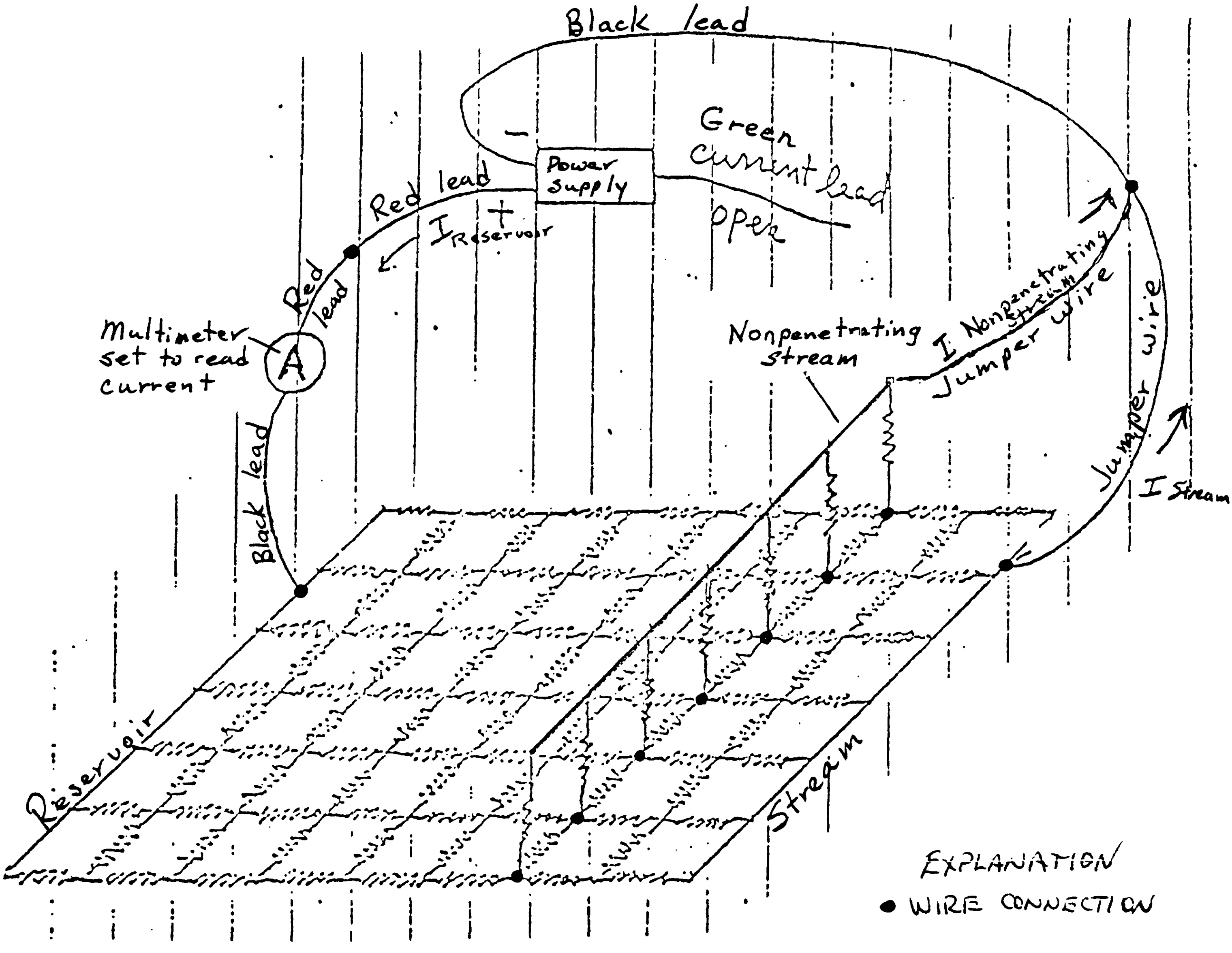


Figure 3.5.16.--Connections for measurement of current into the reservoir boundary when the simulation includes a nonpenetrating stream.

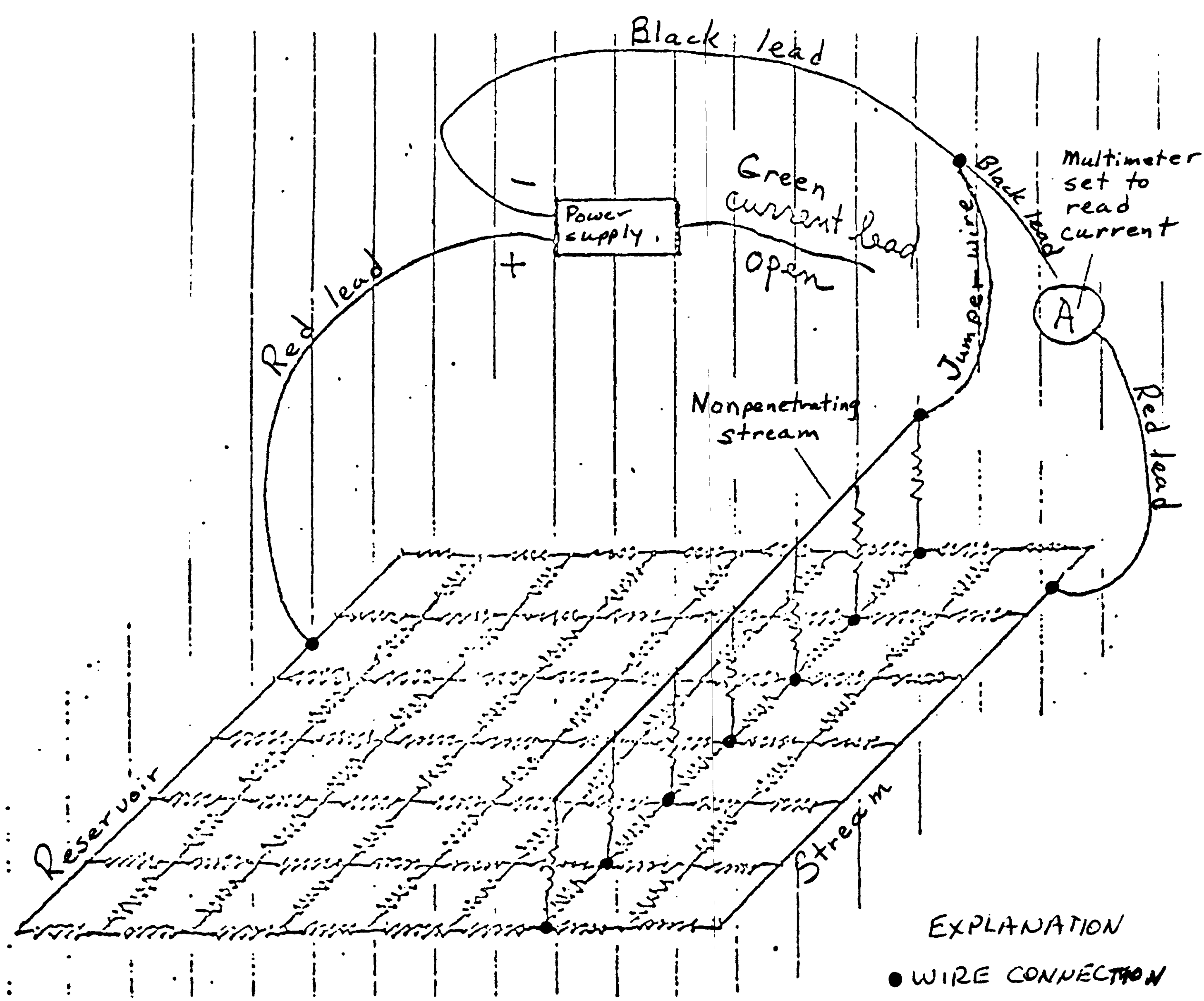


Figure 3.5.17.--Connections for measurement of current leaving the stream boundary when the simulation includes a nonpenetrating stream.

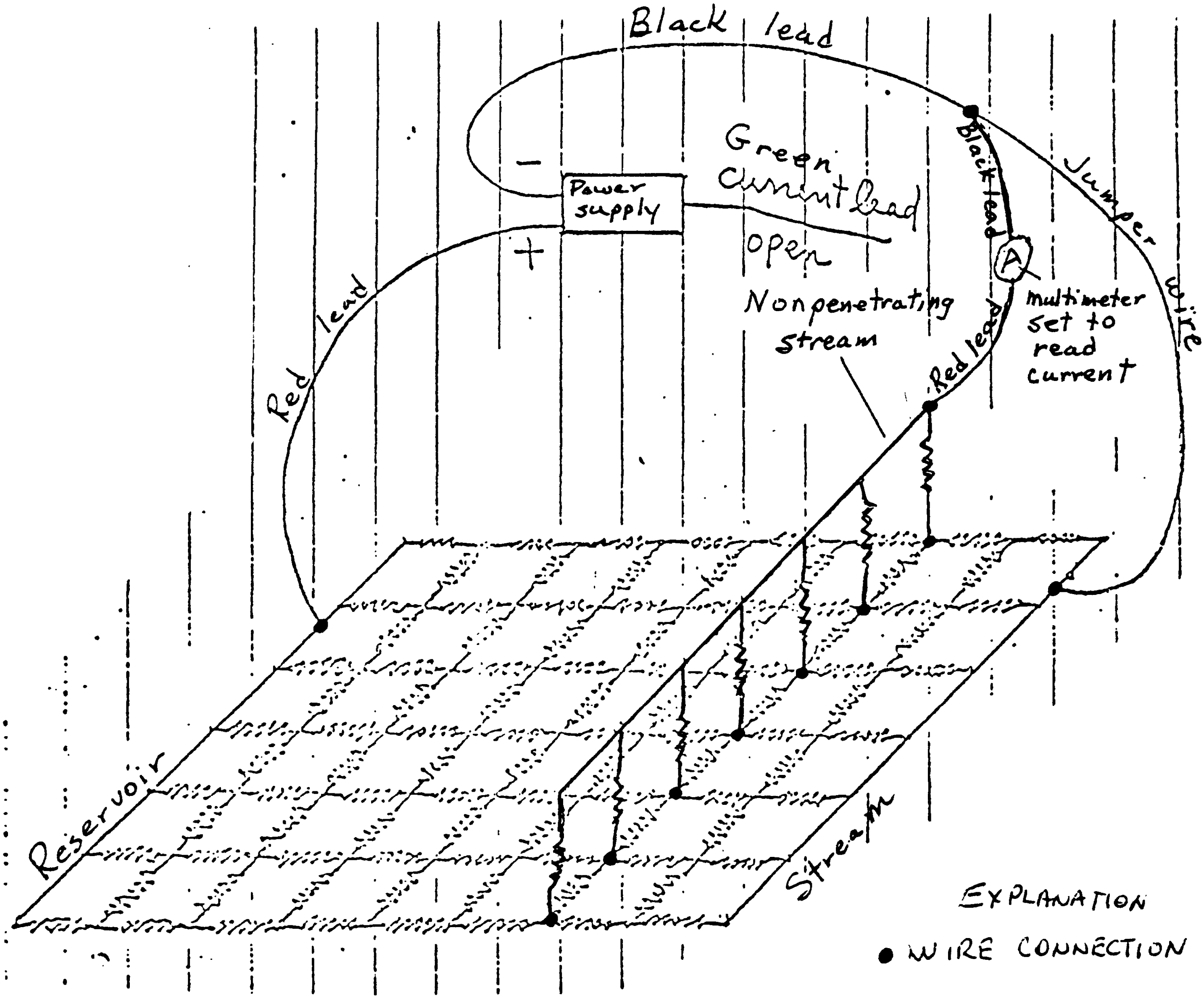


Figure 3.5.18.--Connections for measurement of current leaving the nonpenetrating stream boundary.

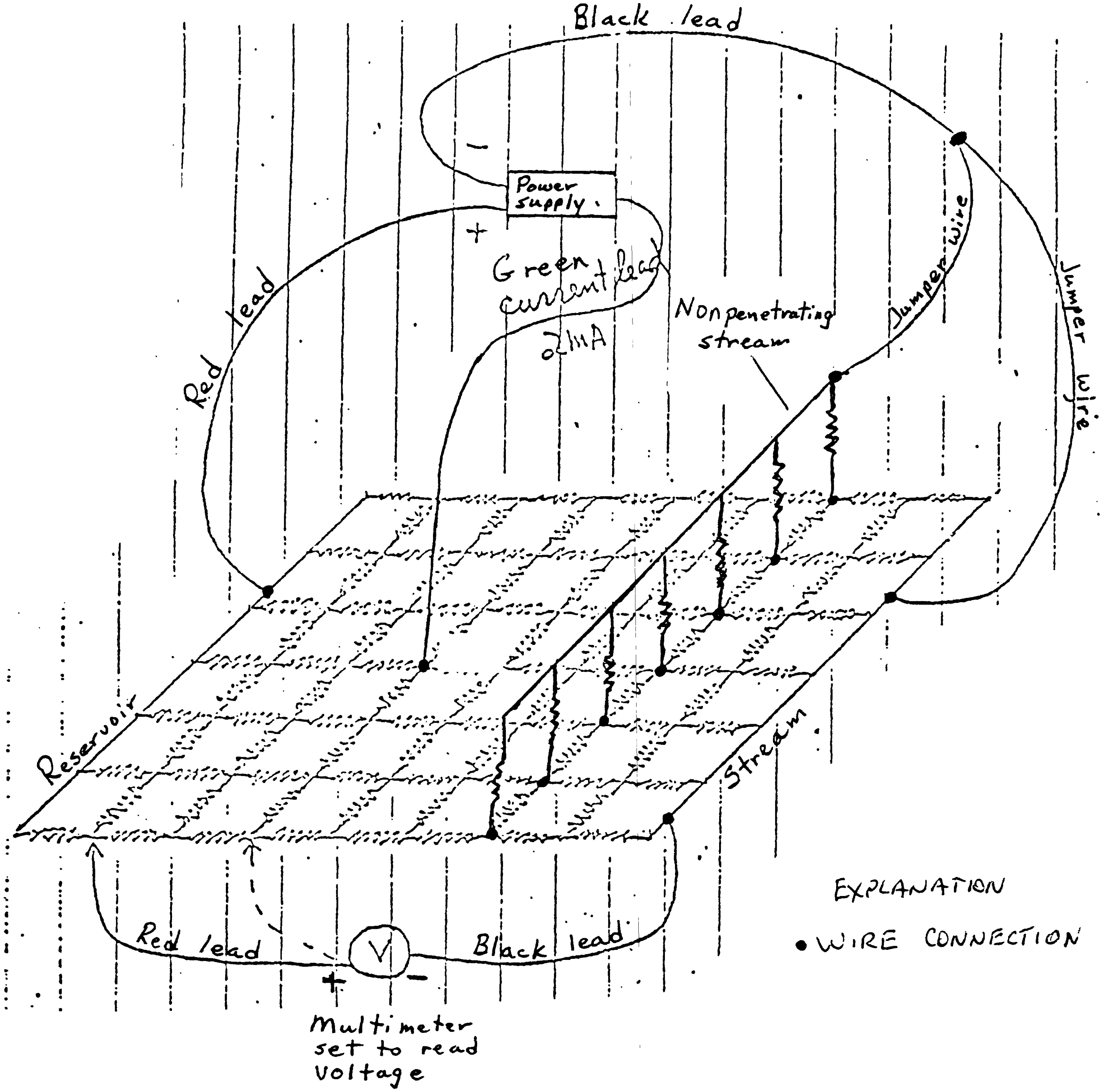
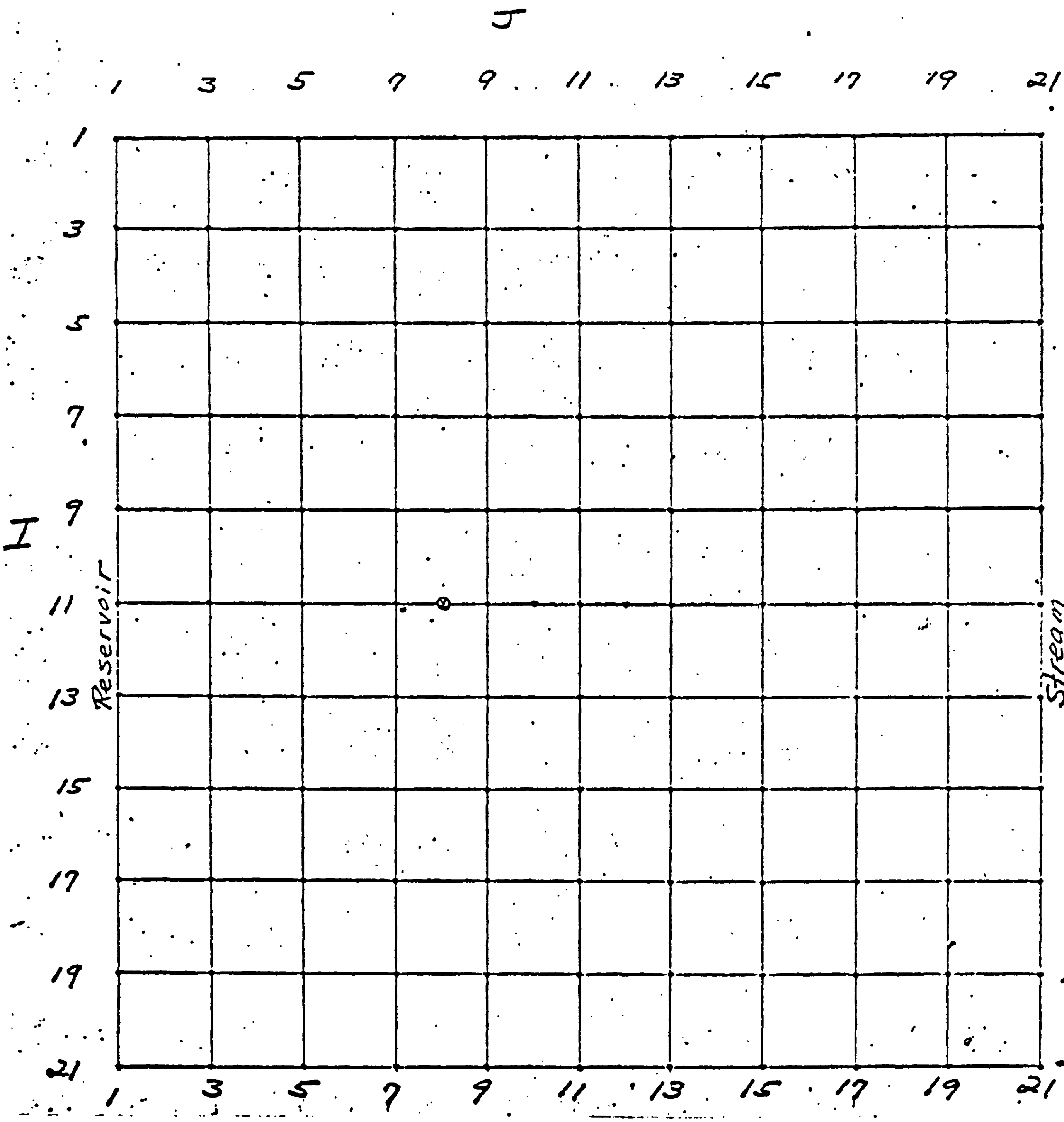


Figure 3.5.19.--Connections for measurement of voltages within the system when the simulation includes a nonpenetrating stream and a discharging well.

Worksheet 3.5.7.--Worksheet for plotting and contouring heads when simulation includes nonpenetrating stream and a discharging well.



2. Measure current and convert to flow for the reservoir boundary, the nonpenetrating streams, and the fully penetrating stream boundary (see figs. 3.5.20, 3.5.21, 3.5.22).

3. Construct a profile of heads along an east-west line from the reservoir through the well to the fully penetrating stream boundary.

4. Does any effect of the pumping extend to the opposite side of the nonpenetrating stream from the well?

5. The discharge of the well must be supplied by (a) increased inflow along the reservoir boundary; (b) decreased outflow (and possibly induced inflow) along the nonpenetrating stream; and (c) decreased outflow (and possibly induced inflow) along the fully penetrating stream boundary. Determine what percentage of the 2 million gallons per day (Mgal/d) of well discharge is accounted for by each of these three changes in the original flow distribution.

6. Does induced inflow actually occur from either of the two streams, or is the effect simply one of reduced seepage into the stream? Explain how you arrived at your answer.

7. Suppose we measure the discharge in the two streams at some point downstream from the aquifer both before adding the well

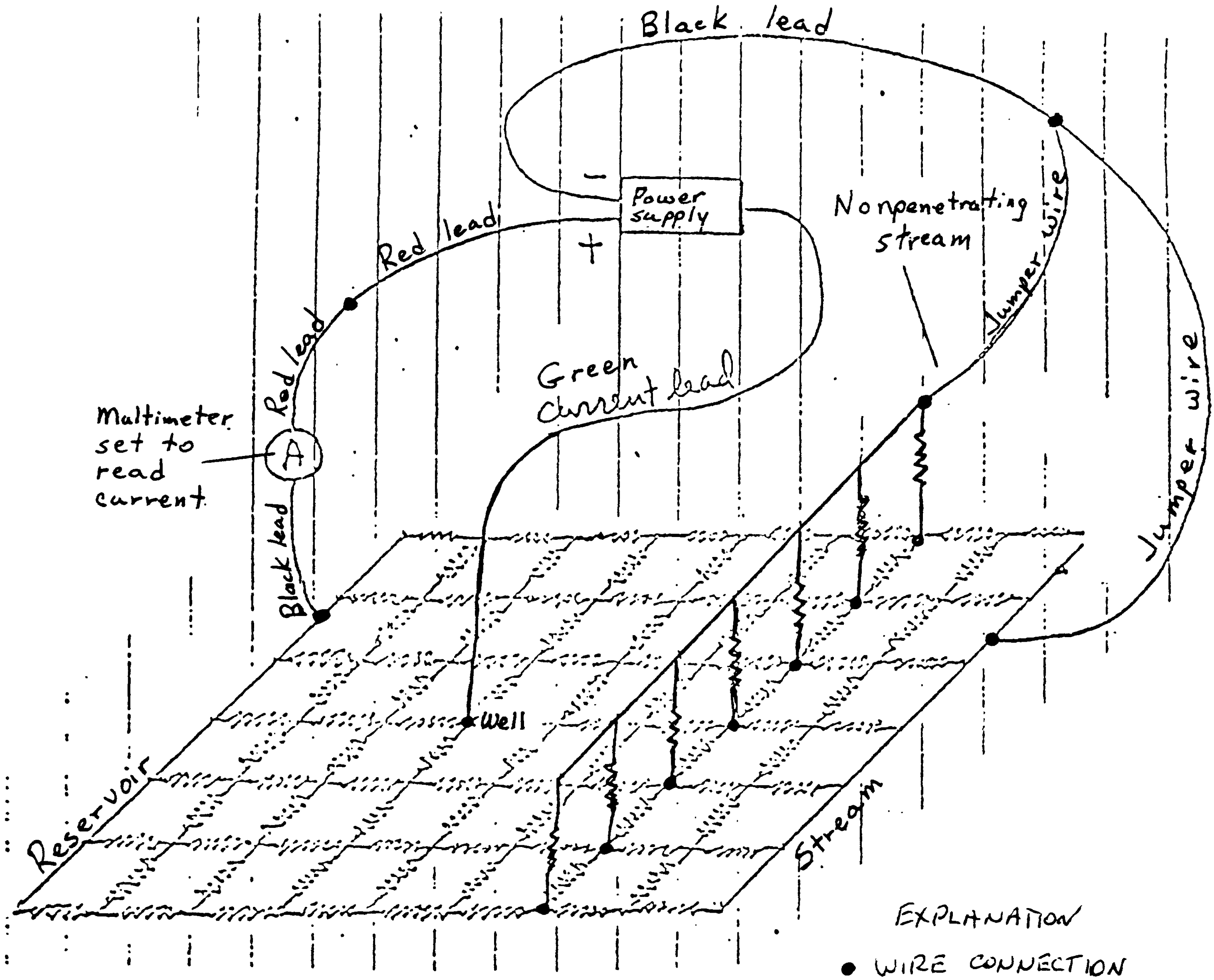


Figure 3.5.20.--Connections for measurement of current into the reservoir boundary when simulation includes a nonpenetrating stream and a discharging well.

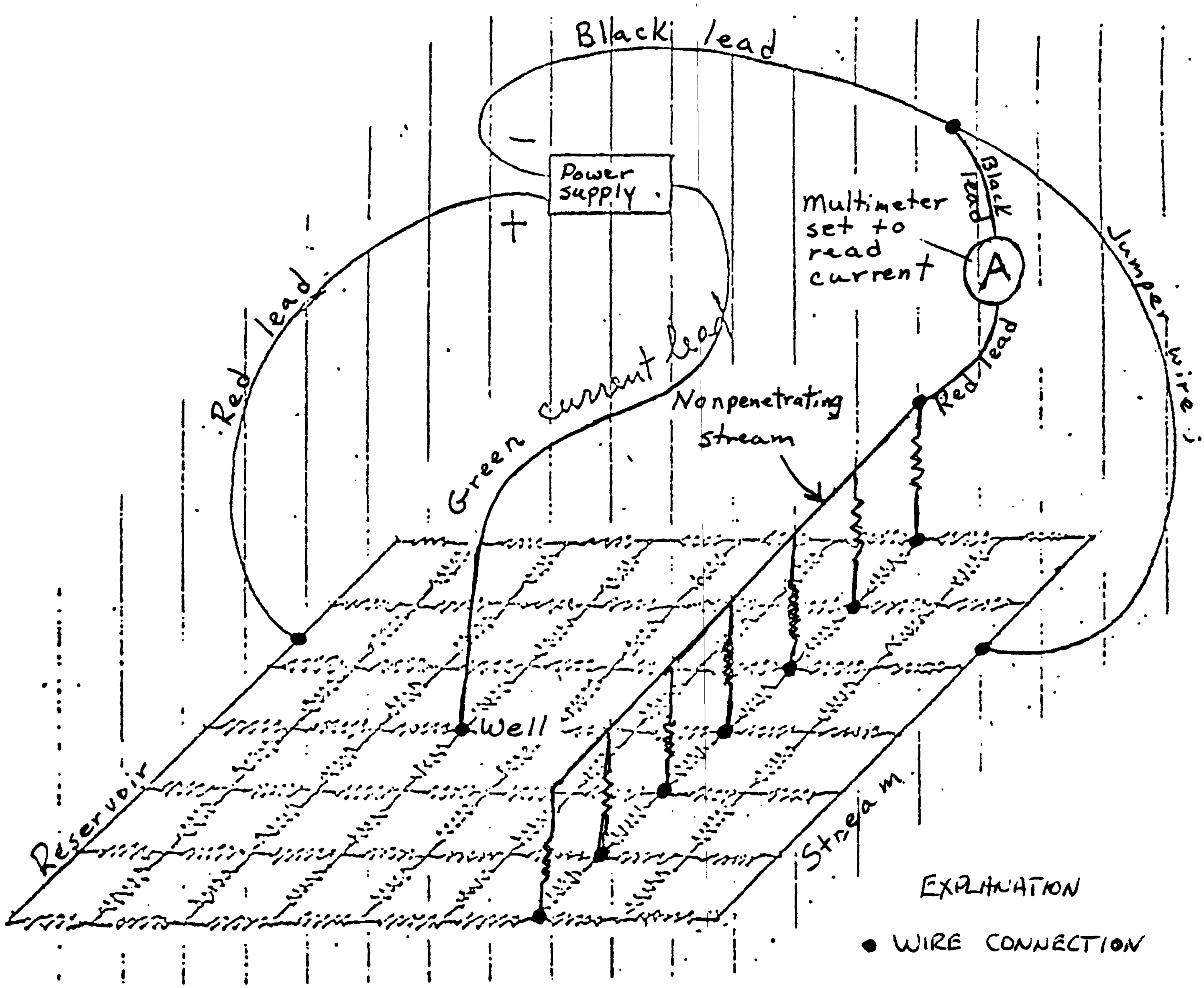


Figure 3.5.21.--Connections for measurement of current leaving through the nonpenetrating stream boundary when simulation includes a discharging well.

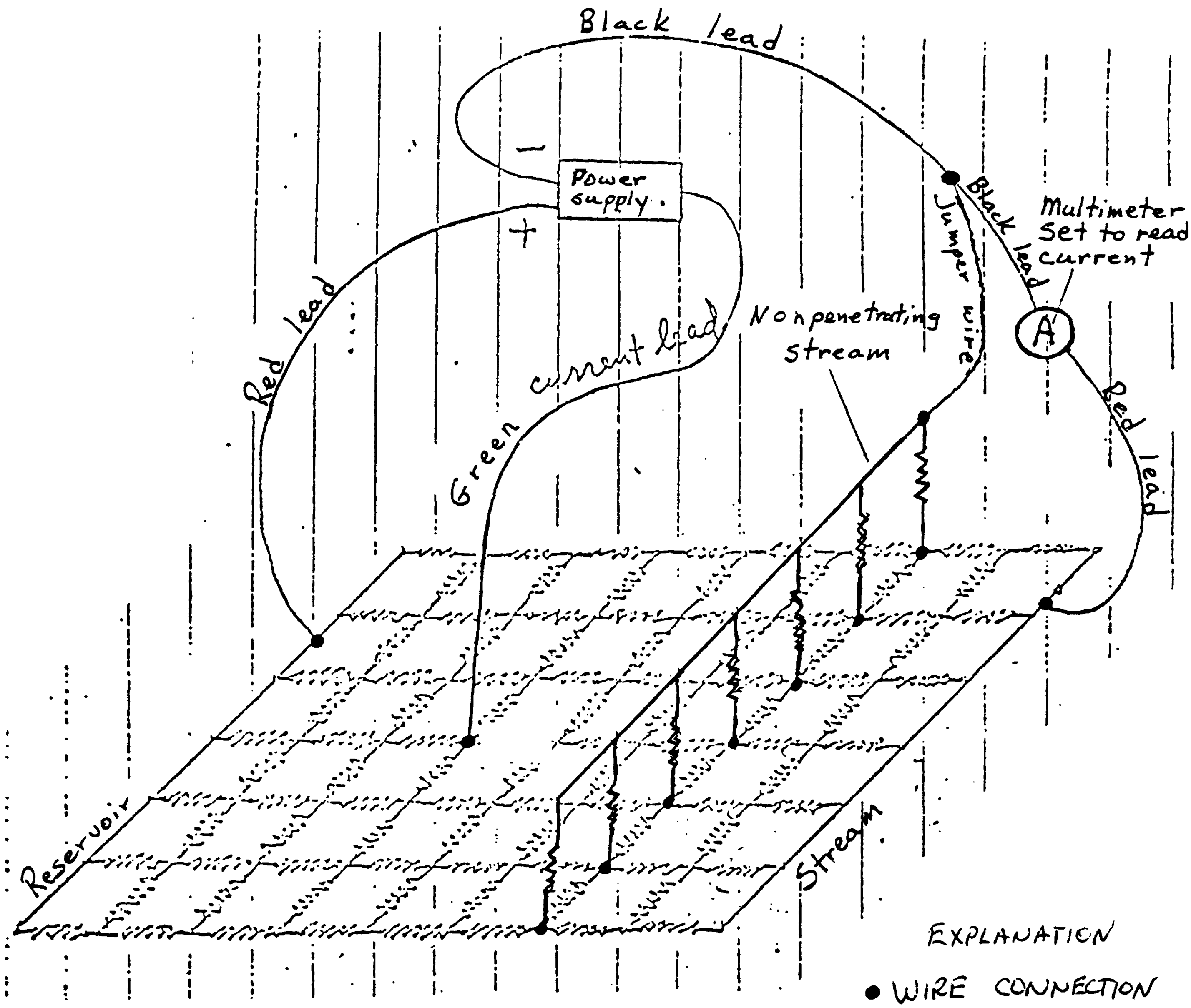


Figure 3.5.22.--Connections for measurement of current leaving the stream boundary when simulation includes a nonpenetrating stream and a discharging well.

and after the equilibrium response to the well had been attained, and note the reduction in streamflow caused by the pumping. Could we tell, from these measurements alone, whether the reduction represented only reduced seepage into the stream or a combination of reduced seepage and direct flow from the stream into the aquifer?

8. Can you give an example of a problem in which it would be important to know whether or not direct flow from the stream into the aquifer was taking place?

9. Can you give an example of a problem in which this information would not be required?

10. With the method that we have used to simulate the nonpenetrating stream, there is theoretically no limit to the current we can move through the resistor; the lower we cause the potential in the network to be, the greater will be the current through the vertical resistors. Is this a valid simulation of field conditions or would there be a practical limit to the actual flow that could be induced from the stream? Can you give an estimate for such a limit? (Assume the top of the aquifer to be at an elevation of 100 feet below datum.)

F.--Confined Aquifer with Nonpenetrating Stream and Discharging Well in Superposition

We now wish to rerun the simulation in part E, but using superposition as in part C. To do this, we connect all of the boundaries--the reservoir, the nonpenetrating stream, and the fully penetrating stream at the eastern boundary--to the ground lead of the power supply, and we again set the well current to 2mA.

1. Measure the current entering or leaving through the three boundaries; convert these to flow values (see fig. 3.5.23).

2. What percentage of the well discharge is represented by each of these boundary flows? Compare these percentages with the values obtained in part E for the percentages of the well discharge accounted for by the changes in the three boundary flows.

3. Measure the drawdowns at representative points in the network according to the setup shown in figure 3.5.24, and plot the data on worksheet 3.5.8.

4. Is it possible to determine, from the results of this exercise (F) alone, whether the effect on the streams consists only of a reduction in outflow or includes induced inflow as well? If not, is it possible to make this determination using the

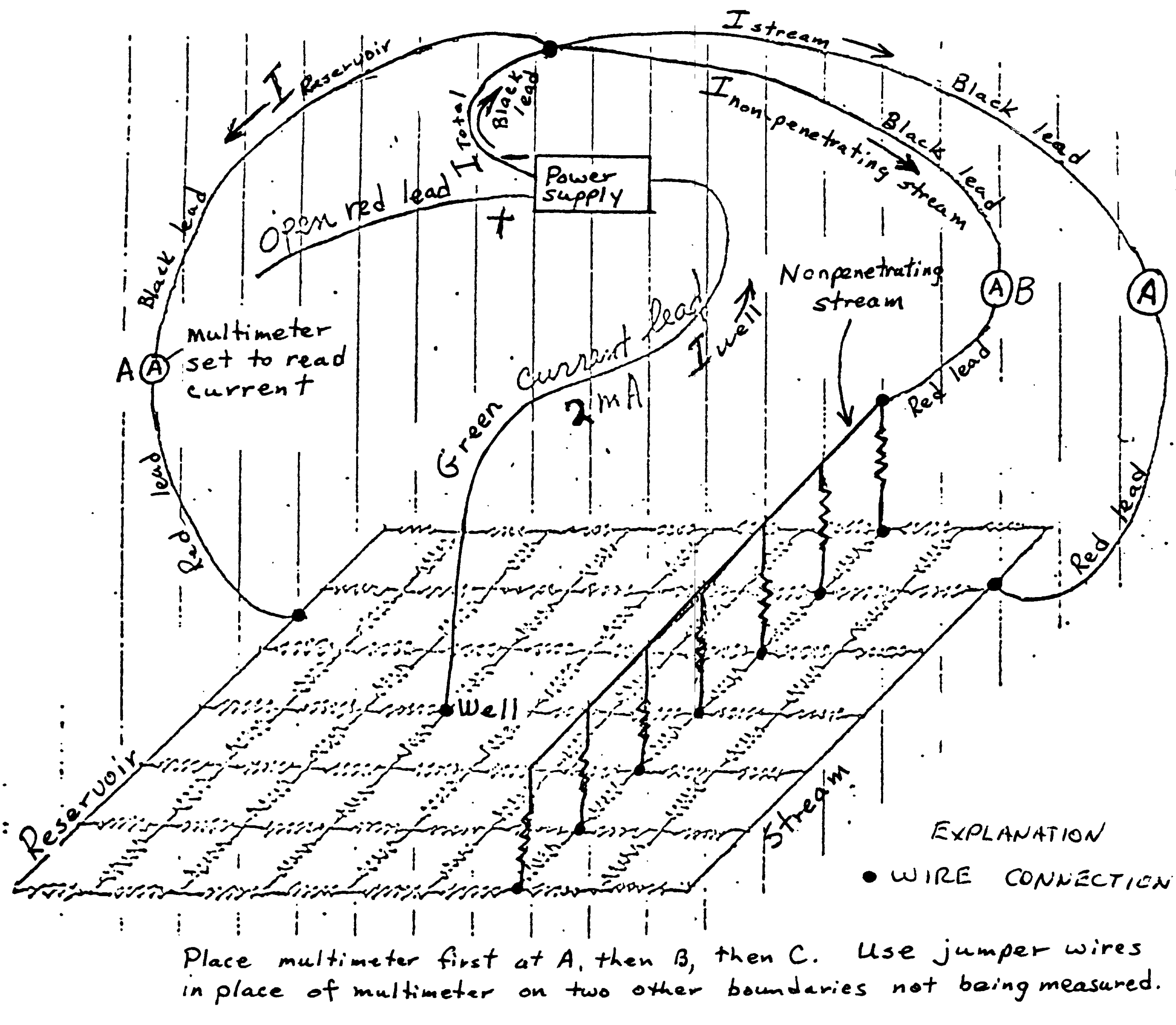


Figure 3.5.23.--Connections for measurement of current for each boundary with zero potentials at the reservoir, nonpenetrating stream, and stream boundaries.

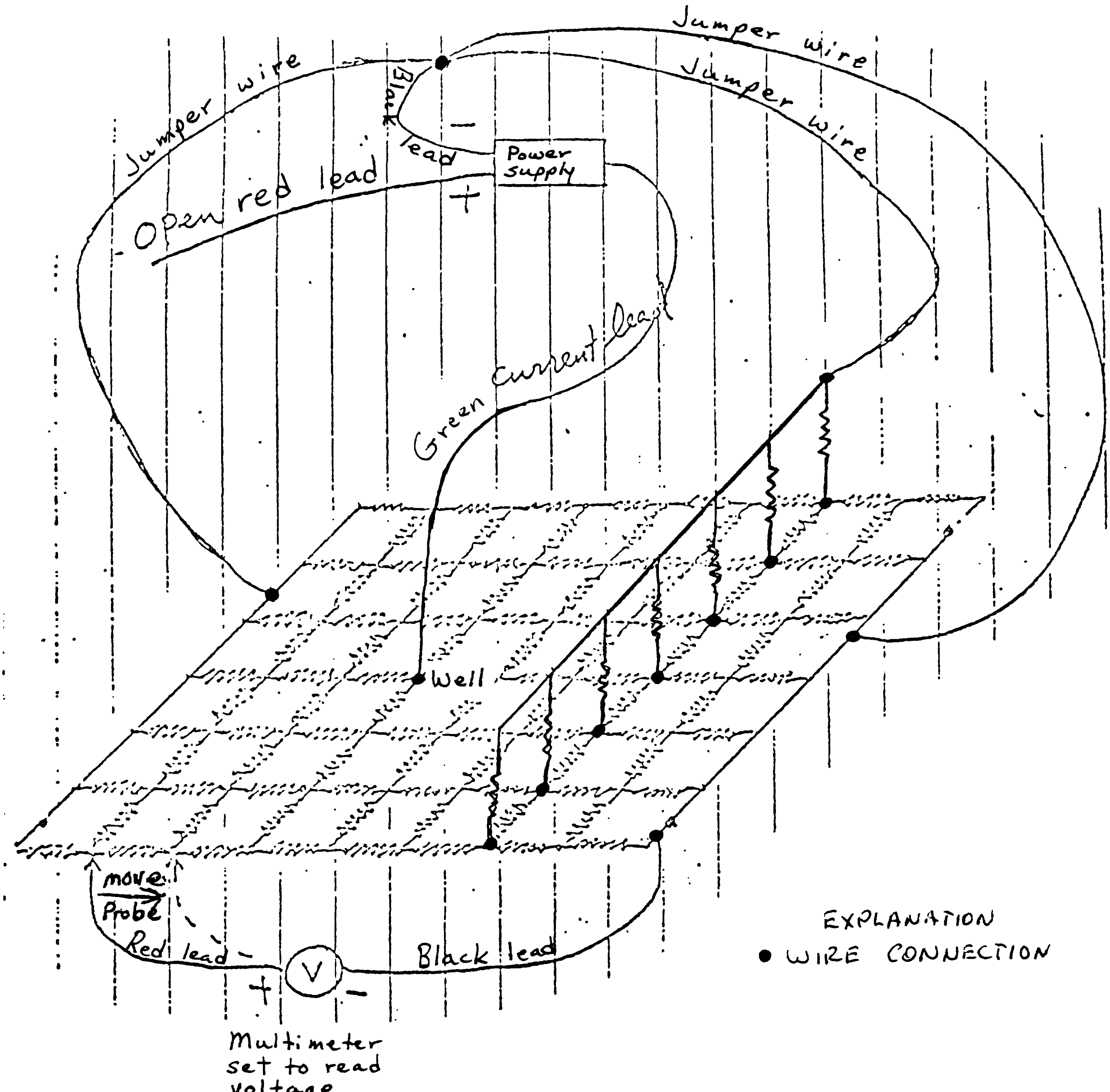
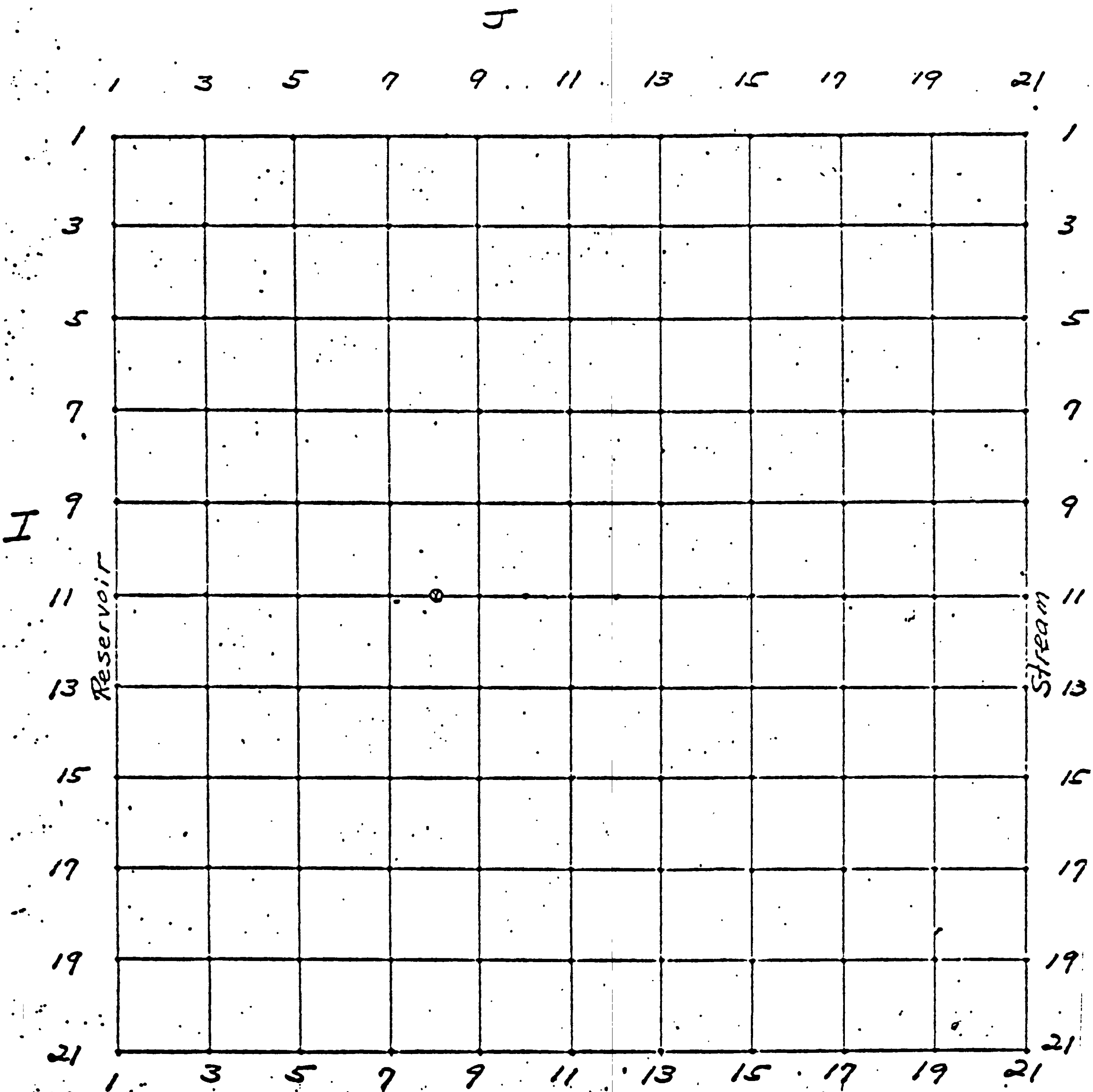


Figure 3.5.24.--Connections to measure voltages throughout the analog system with a discharging well and zero potentials at the reservoir, nonpenetrating stream and stream boundaries.

Worksheet 3.5.8.--Worksheet for plotting and contouring drawdowns in response to a discharging well with zero potentials at the reservoir, nonpenetrating stream and stream boundaries.



results of part F combined with those from part D? How would you go about doing this?

G.--Confined Aquifer as in Part F with Discharging Well 500 Feet Closer to Nonpenetrating Stream

With the boundaries connected to ground as in part F, move the well 500 feet closer to the nonpenetrating stream.

1. Again, measure three boundary currents and convert to flow values.

2. Measure the drawdown at the well node. What does this drawdown actually represent?

3. Repeat this procedure for several other well locations along an east-west axis through the original well position and record the data in table 3.5.1; at each well location, check to make sure that the well current is still 2 milliamperes; include well locations both east and west of the nonpenetrating stream. On a single graph sheet, with distance of the well from the reservoir boundary on the horizontal axis, plot (a) drawdown at the discharging well node; (b) percentage of the well discharge derived from the reservoir; (c) percentage of the well discharge derived from the nonpenetrating stream; and (d) percentage of the well discharge derived from the fully penetrating stream.

Table 3.5.1.--Format for recording boundary flows and drawdowns at the discharging-well node for specified well locations in the simulation experiment in part G

[flows in millions of gallons per day; drawdown in feet]

Nodal location of pumping well	(11,2)	(11,3)	(11,4)	(11,5)	(11,6)	(11,7)	(11,8)	(11,9)	(11,10)	(11,11)
Flow to or from reservoir boundary										
Flow to or from non-penetrating stream boundary										
Flow to or from stream boundary.										
Drawdown at well node										

Nodal location of pumping well	(11,12)	(11,13)	(11,14)	(11,15)	(11,16)	(11,17)	(11,18)	(11,19)	(11,20)
Flow to or from reservoir boundary									
Flow to or from non-penetrating stream boundary									
Flow to or from stream boundary									
Drawdown at well node									

Indicate the position of the nonpenetrating and fully penetrating streams on the horizontal axis.

4. What well location gives minimum drawdown? At what location does the well cause the greatest change in flow in the aquifer from the nonpenetrating stream? At what location does a well cause the least change in flow in the aquifer from the stream?

H.--Confined Aquifer as in Part G with Fully Penetrating Stream in Place of Nonpenetrating Stream

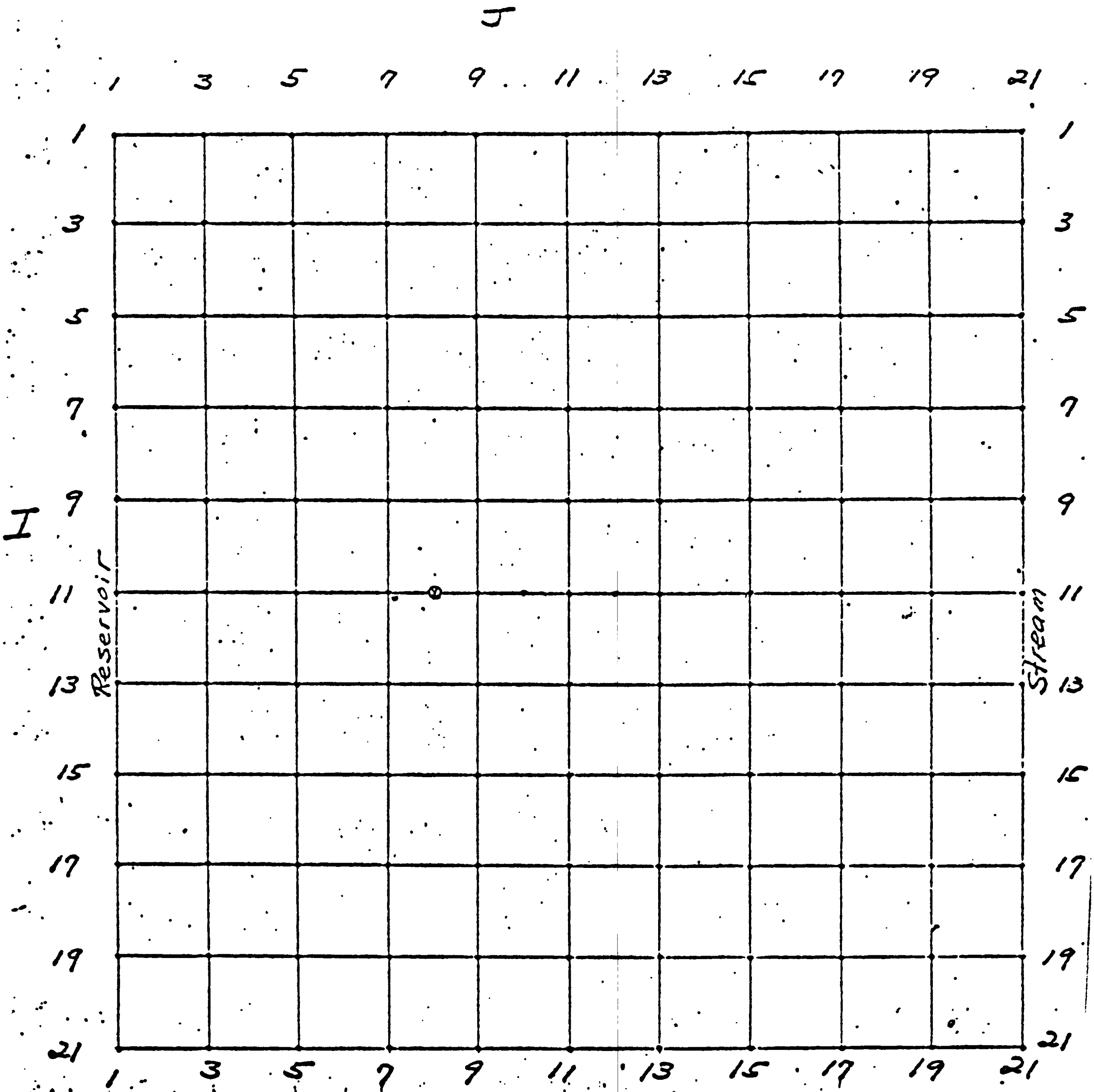
Replace the nonpenetrating stream with a fully penetrating stream by connecting the bus wire with alligator clips in place of the resistor rack. The bus wire and alligator clips can be a little difficult to attach. Be certain that each clip is in solid contact with a resistor wire at each node. Return the well to its original position and rerun the simulation as in part F, with all boundaries connected to ground.

1. Measure all boundary flows.

2. What percentage of the well discharge is now derived from the "interior" stream boundary?

3. Measure voltages in the network, convert to drawdowns, and plot on worksheet 3.5.9.

Worksheet 3.5.9.--Worksheet for plotting and contouring drawdowns in response to a discharging well with an interior fully penetrating stream and the reservoir at zero potential.



4. Suppose we had no information on the vertical hydraulic conductivity of the confining unit and, therefore, could not calculate the vertical resistances to use in simulating a nonpenetrating stream. Would a simulation using a fully penetrating stream have any value? Could it be used to provide an upper limit for the percentage of the well discharge derived from the stream?

I.--Additional Discussion on Simulation of Streams

In these exercises, we have used vertical resistors to represent a nonpenetrating stream of the type shown in figure 3.5.5 of Electrical-Analog Model Design, Part B, and we have seen that a direct connection or bus wire, is used to represent a fully penetrating stream.

1. What could we do to simulate a stream that partially penetrates an aquifer?

2. Could we simulate the hydraulic conductance of a low permeability streambed deposit in such a case?

PROBLEM 6, DIGITAL STREAM-AQUIFER INTERACTION PROBLEM

Simulation of a Fully Penetrating Stream

Figure 3.6.1 shows a cross section and map of a sand aquifer that is overlain by a semipermeable confining unit and underlain by impermeable bedrock; the aquifer is bounded by impermeable bedrock hills on the north, south, and west, and by a fully penetrating stream on the east. The dimensions of the aquifer are 50,000 feet from north to south, 50,000 feet from east to west, and 100 feet in thickness. Recharge through the confining unit occurs at a uniform rate of 0.0002 gallons per day per square foot. The hydraulic conductivity of the sand is 100 gallons per day per square foot and its specific storage is 2.67 x 10^{-7} per foot. The stream surface altitude is 100 feet above sea level, and the altitude of the top of the aquifer is 10 feet above sea level. Before a digital model can be designed, the boundary conditions of the flow system must be designated. On figure 3.6.1 indicate the boundary conditions for the system.

System Under Natural Conditions

Design a digital model for this aquifer using a mesh spacing of 5,000 feet. In this problem set, gallons will be used as one of the primary units of measure. Thus, some quantities such as storage capacity will require conversion of cubic feet (ft^3) to gallons (1 ft^3= 7.48 gals).

(1) Worksheets 3.6.1, 3.6.2, and 3.6.3 show the array of nodes[1] for the model with the i value indicated for each row and j value for each column. On worksheet 3.6.1 indicate the x distance associated with each column (taking x = 0 at the western edge) and the y distance for each row (taking y = 0 along the northern edge). In rows 1, 2, and 11 on worksheet 3.6.1, outline the blocks between nodes for which hydraulic conductance in the x direction must be specified. Calculate the required x-direction hydraulic conductance for a block in each of these rows.

In columns 1, 2, and 11 on worksheet 3.6.2, outline the blocks for which hydraulic conductance in the y direction must be specified. Calculate the required y-direction hydraulic conductance for a block in each of these columns. On worksheet 3.6.3, sketch the blocks for which storage capacity must be specified around each of the following nodes: 1,1; 10,1 (For node 10,1, i=10, j=1); 11,1; 7,3; 7,7; 1,5; 10,5; and 11,5. Calculate the required storage capacity for each of these blocks (the units of ft^2 should be converted to gals/ft).

For the same set of nodes, calculate the recharge to the aquifer through the confining unit within each node block. (Multiply the recharge rate by the planar area associated with the node.)

---

[1] A point-centered finite-difference grid is used in this digital model. The nodes are located at the intersection of the lines in the grid shown in worksheet 3.6.1. Thus, conductances and storage capacities in this problem set are defined as described in notes 2 and 3.

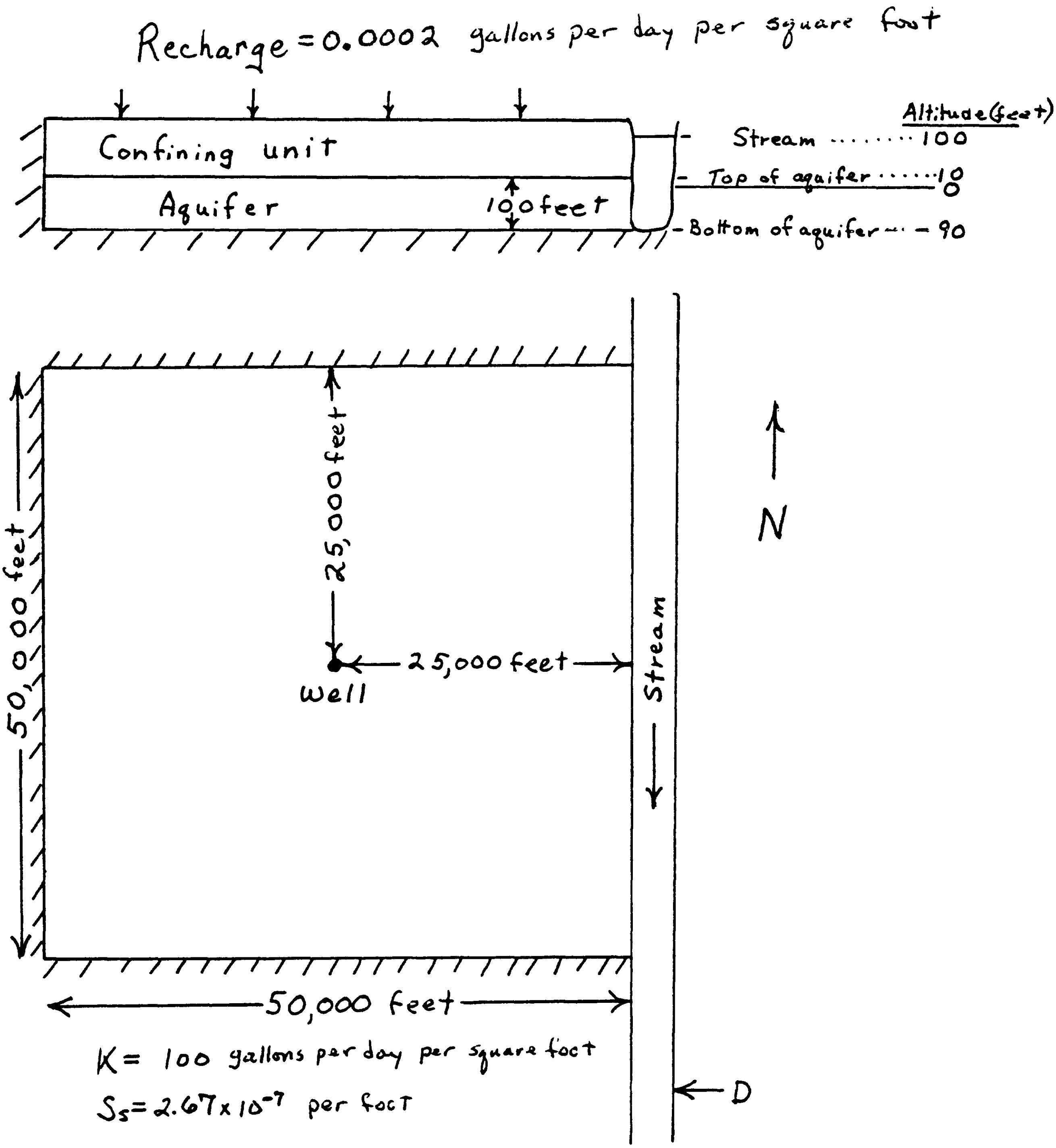
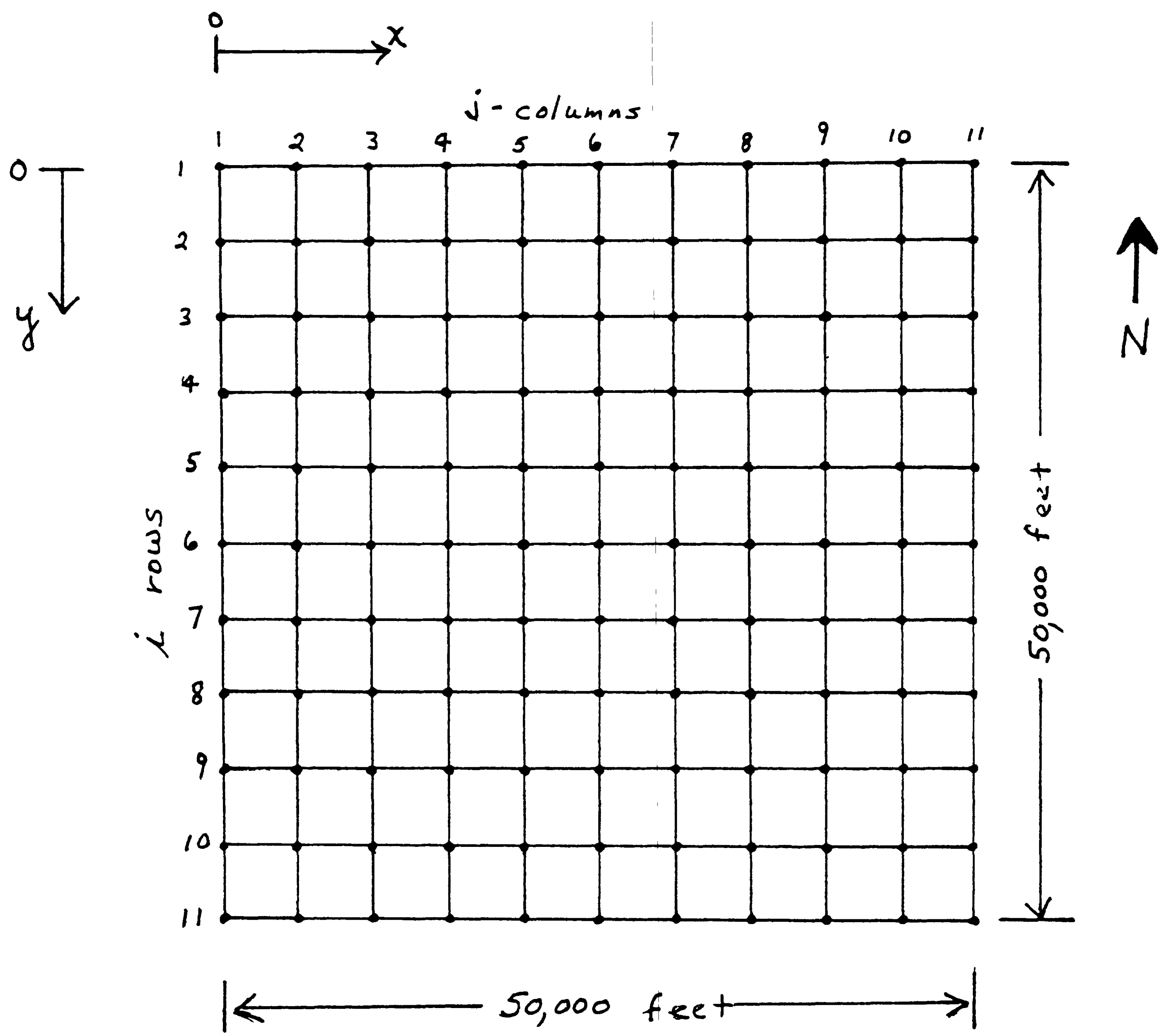
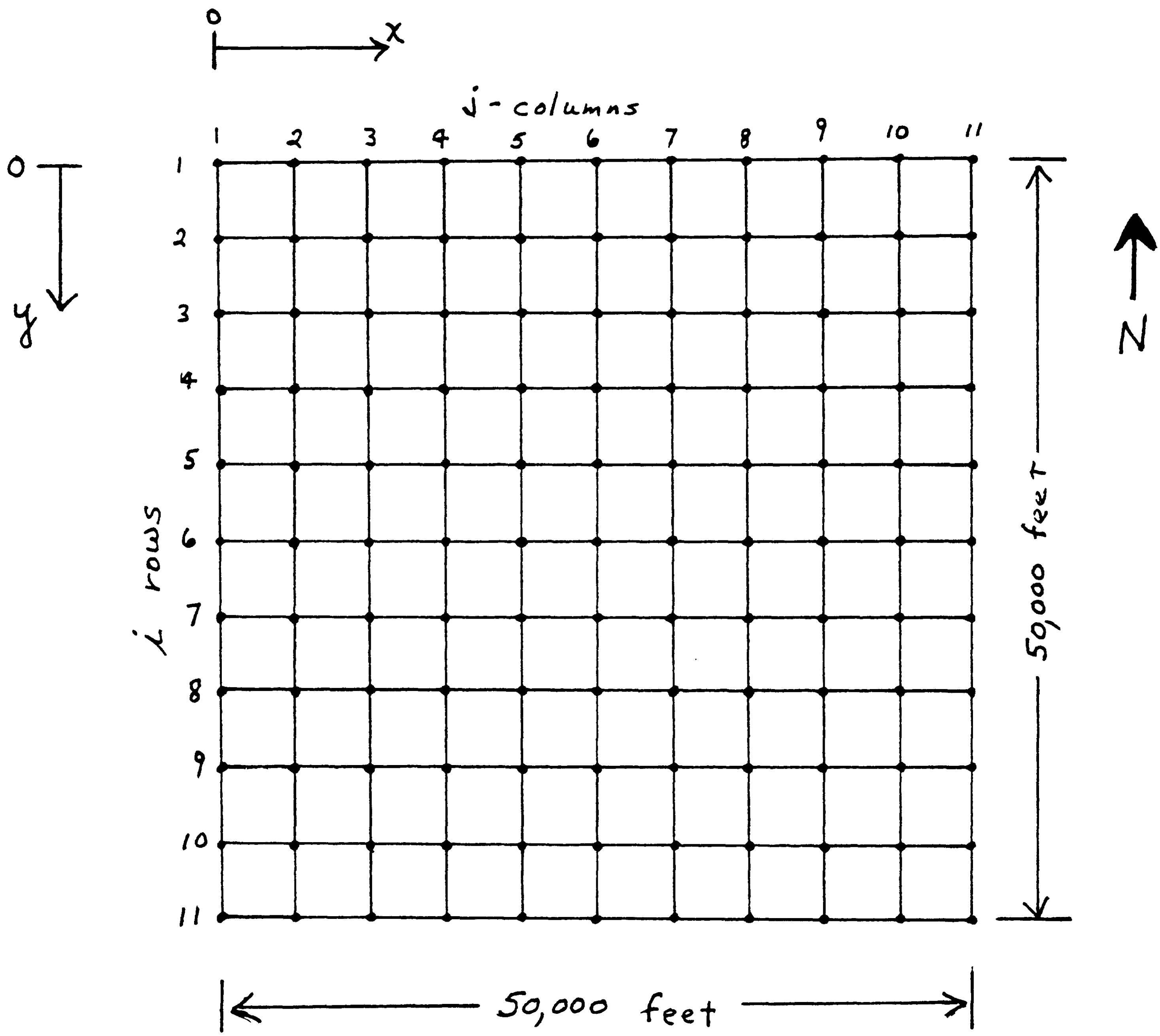


Figure 3.6.1.--Cross section and map of sand aquifer.

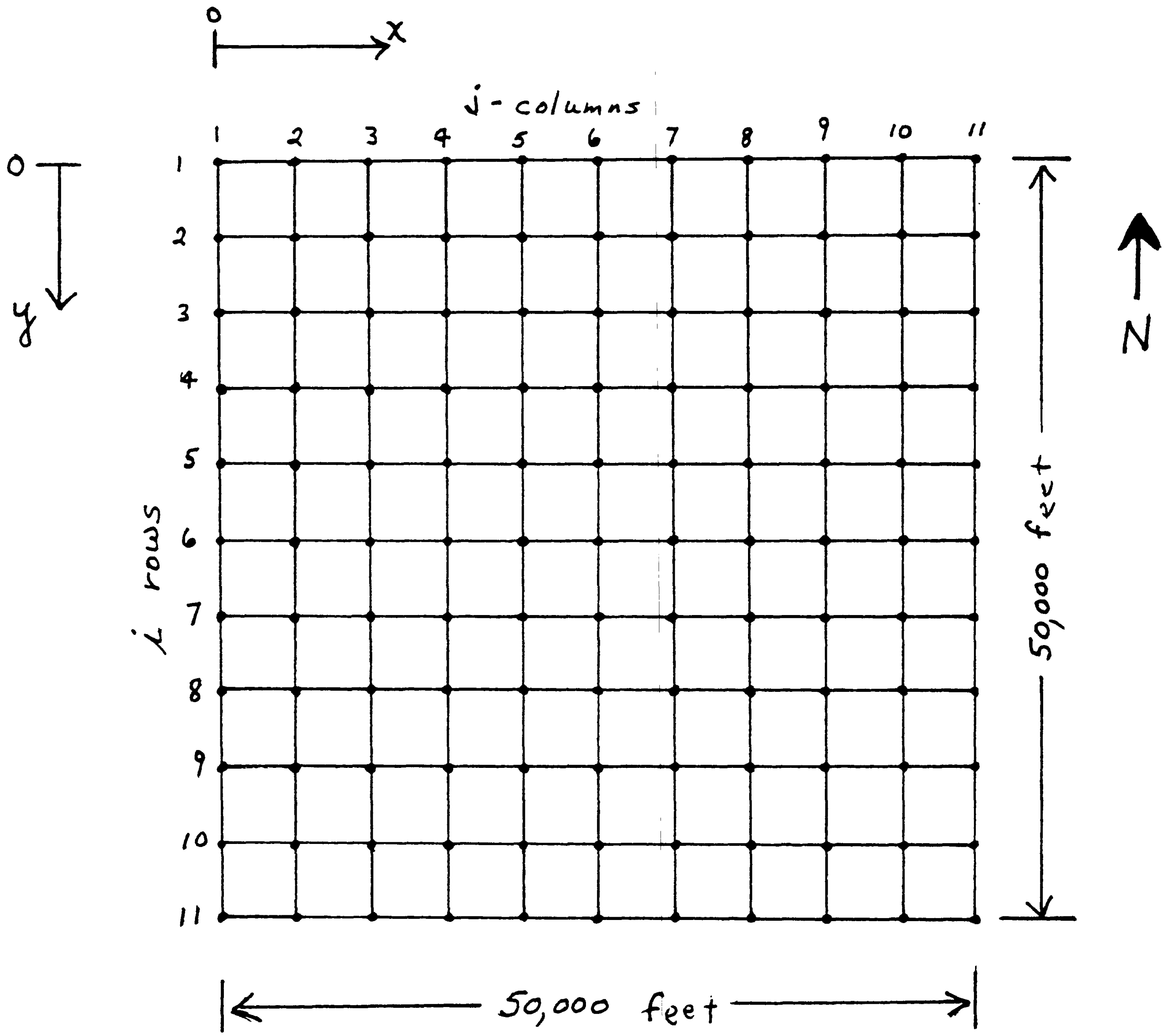
Worksheet 3.6.1.--Array of model nodes for showing x values, y values, and blocks for defining hydraulic conductance in the x direction.



Worksheet 3.6.2.--Array of model nodes for showing blocks for defining hydraulic conductance in the y direction.



Worksheet 3.6.3.--Array of model nodes for showing blocks for defining storage capacity and for calculating recharge.



(2) Write the specific backward-difference formulation of the finite-difference equation that the model must solve at a typical interior node using the general equation (see note 4 equation 2). Allow for storage, flow between nodes, and a net withdrawal rate, Q from the node; assume that recharge will be handled through this net withdrawal term--that is, as a negative withdrawal from each node--so that a term for recharge need not be specified separately in the equation.

The stream is fully penetrating, and we assume that its level is unaffected by what happens in the aquifer. Thus it can be considered a constant-head boundary. The edge of the stream follows column 11 of the model, and we require some method of keeping the head along this column at a constant level throughout the simulation. To accomplish this, we set the initial head in these nodes at the stream level, 100 feet, and assign a very high value of storage capacity,say, 10^{40}, to each node in the column. This is one method of maintaining a node at constant head in a digital model. (Another method is to flag the constant-head nodes and treat them as known values.) (3) Using the definition of storage capacity, S_c (Sectiuon 2.3, Note 3), explain how this causes the head along column 11 to remain constant during the simulation.

Table 3.6.1 shows a steady-state head distribution obtained using a model of the design that you just developed. The result

Table 3.6.1.--Steady-state head distribution in aquifer shown in figure 3.6.1

Rows \ Columns	1	2	3	4	5	6	7	8	9	10	11
1	125.1	124.8	124.1	122.8	121.1	118.8	116.0	112.8	109.0	104.8	100.0
2	125.1	124.8	124.1	122.8	121.1	118.8	116.0	112.8	109.0	104.8	100.0
3	125.1	124.8	124.1	122.8	121.1	118.9	116.0	112.8	109.0	104.8	100.0
4	125.1	124.8	124.1	122.8	121.1	118.8	116.0	112.8	109.0	104.8	100.0
5	125.1	124.8	124.1	122.8	121.1	118.8	116.0	112.8	109.0	104.8	100.0
6	125.1	124.8	124.1	122.8	121.1	118.8	116.0	112.8	109.0	104.8	100.0
7	125.1	124.8	124.1	122.8	121.1	118.8	116.0	112.8	109.0	104.8	100.0
8	125.1	124.8	124.1	122.8	121.1	118.8	116.0	112.8	109.0	104.8	100.0
9	125.1	124.8	124.1	122.8	121.1	118.9	116.0	112.8	109.0	104.8	100.0
10	125.1	124.8	124.1	122.8	121.1	118.8	116.1	112.8	109.0	104.8	100.0
11	125.1	124.8	124.1	122.8	121.1	118.8	116.1	112.8	109.0	104.8	100.0

was obtained by using an iterative technique to solve the set of simultaneous algebraic equations that represent this system.

(4) Using the data in table 3.6.1, plot a profile of heads along row 6, from the western edge of the aquifer to the stream. How would you describe this profile? What is the direction of flow? Why does the hydraulic gradient become steeper as the stream is approached? (5) Using head differences from table 3.6.1 and hydraulic conductance values, calculate the total ground-water flow within the aquifer perpendicular to north-south lines at the following distances from the stream: 50,000 feet, 42,500 feet, and 27,500 feet. Use the table on worksheet 3.6.4 for all calculations. Calculate the recharge to the aquifer to the west of each of these lines for which ground-water flow was calculated; compare the recharge in each case to the ground-water flow value. Calculate the total flow from the aquifer to the stream and compare it to the total recharge to the aquifer. (6) How would you describe this flow system?

Response of System to Pumping

A well is located at a distance of 25,000 feet from the stream and 25,000 feet from the northern edge of the aquifer (fig. 3.6.1). This well is pumped at a rate of 2 million gallons per day. We wish to determine the head distribution after (approximately) 5, 13, and 155 days of pumping, assuming that prior to pumping, the aquifer is in the equilibrium condition given on table 3.6.1. The recharge is assumed to continue

Worksheet 3.6.4. Steady-state flow calculations, no discharging well. [distances and head differences in feet; hydraulic conductance in gallons per day per foot; flows and recharge in gallons per day]

Distance from stream		50,000		42,500		27,500		0	
Row	Hydraulic conductance	Head diff-erence	Flow	Head diff-erence between columns 2 and 3	Flow	Head diff-erence between columns 5 and 6	Flow	Head diff-erence between columns 10 and 11	Flow
1	5,000								
2	10,000								
3	10,000								
4	10,000								
5	10,000								
6	10,000								
7	10,000								
8	10,000								
9	10,000								
10	10,000								
11	5,000								
Total flow									
Area of aquifer to west of line of calculation									
Recharge to aquifer to west of line of calculation									

throughout the period of pumping. Our model is set up in such a way that we can specify only one net input or output at any given node in any time step.

When we operate this model in the transient mode, we can assign the lengths of the time steps in a number of different ways. (7) Suppose we have reason to think that changes in head with time in the aquifer will follow a logarithmic trend. Then for two successive (unequal) time intervals, t_1 to t_2 and t_2 to t_3, the change in head between t_2 and t_3 will be approximately equal to that between t_1 and t_2 if we set $(\log t_3 - \log t_2) = (\log t_2 - \log t_1)$. Would it then be efficient to use a time step of constant duration in our simulation? If not, how should the time step durations be varied during the simulation? Explain your answers.

The head distribution after pumping (approximately) 5, 13, and 155 days was solved for in a transient simulation. Its initial condition was the steady-state solution shown in Table 3.6.1. The simulation was run through 25 time steps. The duration of the first time step was taken as 1 day; the duration of the second was 1.2 times that of the first; the duration of the third was 1.2 times that of the second, and so on. The 19th time step falls after 155 days. The 25th time step falls after 472 days. (8) By the end of the 25 time steps, successive output sheets were showing identical head distributions. Explain the significance of this result.

(9) To simulate the system in terms of absolute heads, what should we use for initial heads in this simulation? (10) Suppose we use an initial time-step length of one day and take the length of each successive time step as 1.2 times that of the preceding step. What time-step number corresponds to a cumulative time of pumping of approximately 5 days? (11) At what node should the well be located? If recharge and discharge occur in the same block, the quantity of recharge to and discharge from that block is accounted for in the variable Q, which represents the net value of recharge vs. discharge. (12) What should be the net value of the input or withdrawal function, Q, for the block surrounding the node representing the well? (13) Should it vary from one time step to the next or should it remain constant?

Tables 3.6.2, 3.6.3, and 3.6.4 show the head distributions after 5, 13, and 155 days of pumping, respectively. (14) Using these head distributions and the hydraulic conductance values, calculate the total net inflow to or outflow from the stream and the change in this total flow from its value prior to pumping for each of these times (for purposes of this problem, use the original ground-water flow to the stream as calculated from the gradient in (5)--i.e., 480,000 gpd--rather than the actual flow of 500,000 gpd). Use the table on worksheet 3.6.5 for all calculations. Then plot this <u>change</u> as a function of time of pumping. (15) Direct flow from the stream is the quantity of water flowing from the stream into the aquifer. Calculate the

Table 3.6.2.--Head distribution after 5 days of pumping in aquifer shown in figure 3.6.1

Rows \ Columns	1	2	3	4	5	6	7	8	9	10	11
1	114.2	113.6	111.8	109.3	106.6	104.3	103.0	102.3	101.8	101.1	100.0
2	113.8	113.1	111.2	108.3	105.2	102.8	101.7	101.4	101.3	100.9	100.0
3	112.7	111.8	109.2	105.4	101.1	97.8	97.6	98.6	99.7	100.2	100.0
4	111.2	110.0	106.4	100.6	93.4	87.7	89.9	94.0	97.2	99.1	100.0
5	109.8	108.3	103.4	94.7	81.6	66.9	78.2	88.3	94.6	98.1	100.0
6	109.3	107.5	101.9	90.7	68.6	17.6	65.2	84.3	93.2	97.6	100.0
7	109.8	108.3	103.5	94.7	81.6	66.9	78.2	88.3	94.6	98.1	100.0
8	111.2	110.0	106.4	100.6	93.4	87.7	89.9	94.0	97.2	99.1	100.0
9	112.7	111.8	109.2	105.4	101.1	97.8	97.6	98.6	99.7	100.2	100.0
10	113.8	113.1	111.2	108.3	105.2	102.8	101.7	101.4	101.3	100.9	100.0
11	114.2	113.5	111.8	109.3	106.6	104.3	103.0	102.3	101.8	101.1	100.0

Table 3.6.3.--Head distribution after 13 days of pumping in aquifer shown in figure 3.6.1.

Rows	Columns 1	2	3	4	5	6	7	8	9	10	11
1	90.3	89.8	88.4	86.7	85.3	85.0	86.4	89.1	92.7	96.4	100.0
2	89.8	89.2	87.7	85.6	83.8	83.4	85.0	88.1	92.1	96.2	100.0
3	88.4	87.7	85.5	82.4	79.3	78.1	80.5	85.1	90.3	95.4	100.0
4	86.6	85.6	82.3	77.2	71.2	67.4	72.5	80.2	87.6	94.2	100.0
5	85.1	83.6	79.1	71.0	59.1	46.4	60.5	74.2	84.8	93.1	100.0
6	84.4	82.8	77.4	66.8	45.9	-3.1	47.4	70.1	83.3	92.6	100.0
7	85.1	83.6	79.1	71.0	59.1	46.4	60.5	74.2	84.8	93.1	100.0
8	86.7	85.6	82.3	77.2	71.2	67.4	72.5	80.2	87.6	94.2	100.0
9	88.4	87.7	85.5	82.4	79.3	78.1	80.5	85.1	90.3	95.4	100.0
10	89.8	89.2	87.7	85.6	83.8	83.4	85.0	88.1	92.1	96.2	100.0
11	90.3	89.8	88.4	86.7	85.3	85.0	86.4	89.1	92.7	96.4	100.0

Table 3.6.4.--Head distribution after 155 days of pumping in aquifer shown in figure 3.6.1.

Rows \ Columns	1	2	3	4	5	6	7	8	9	10	11
1	28.0	28.3	29.2	31.1	34.7	40.7	49.4	60.6	73.2	86.6	100.0
2	27.5	27.7	28.4	30.0	33.2	39.0	48.0	59.5	72.6	86.3	100.0
3	26.2	26.2	26.2	26.7	28.7	33.7	43.6	56.5	70.8	85.5	100.0
4	24.4	24.0	23.0	21.5	20.6	23.1	35.6	51.6	68.1	84.3	100.0
5	22.8	22.1	19.8	15.3	8.4	2.0	23.5	45.6	65.3	83.2	100.0
6	22.1	21.2	18.1	11.1	-4.7	-47.5	10.4	41.5	63.8	82.7	100.0
7	22.8	22.1	19.8	15.3	8.4	2.0	23.5	45.6	65.3	83.2	100.0
8	24.4	24.0	23.0	21.5	20.6	23.1	35.6	51.6	68.1	84.3	100.0
9	26.2	26.2	26.2	26.7	28.7	33.7	43.6	56.5	70.8	85.5	100.0
10	27.5	27.7	28.4	30.0	33.2	39.0	48.0	59.5	72.6	86.3	100.0
11	28.0	28.3	29.2	31.1	34.7	40.7	49.4	60.6	73.2	86.6	100.0

Worksheet 3.6.5--Flow between stream and aquifer, and change in flow from steady-state conditions after 5, 13, and 155 days of pumping. [head difference in feet; flow in gallons per day]

		5 days pumping (table 3.6.2)		13 days pumping (table 3.6.3)		155 days pumping (table 3.6.4)	
Row	Hydraulic conductance C_x	Head diff-erence between columns 10 and 11	Flow	Head diff-erence between columns 10 and 11	Flow	Head diff-erence between columns 10 and 11	Flow
1	5,000						
2	10,000						
3	10,000						
4	10,000						
5	10,000						
6	10,000						
7	10,000						
8	10,000						
9	10,000						
10	10,000						
11	5,000						
Total net flow between stream and aquifer							
Change in flow from steady-state conditions							

Negative sign indicates flow from stream into aquifer.

rate at which water is being supplied by direct flow from the stream to the well at each time. Diversion of flow that would otherwise reach the stream from the aquifer is defined as the decrease in ground-water flow from the aquifer to the stream. Calculate the rate at which water is being supplied by diversion of the original flow toward the stream at the end of each pumping period. How much is being supplied by withdrawal from aquifer storage at each of these times of pumping? (16) On the same graph sheet on which the change in flow to the stream is plotted, plot the rate of withdrawal of water from storage as a function of time of pumping. (17) What will the rate of withdrawal of water from storage be when a new equilibrium is attained? (18) What will be the total change in the flow of the stream at a point D (fig. 3.6.1) downstream from the aquifer, when a new equilibrium is attained?

(19) Construct new plots of head vs. distance from the stream, along an axis through the well (row 6), for each of the above times of pumping (tables 3.6.2, 3.6.3, and 3.6.4). Use the same graph sheet as in question 4 for the plot of head vs. distance of the steady-state head distribution.

(20) Estimate the total drawdown and actual water level (head) within the pumping well after 155 days of pumping, assuming the well radius to be 0.5 foot and neglecting well entrance losses (see note 7).

(21) What would be the effect on the plots of change in flow to the stream vs. time, and on the head profiles, if storage coefficient were increased?

<u>Simulation of a Fully Penetrating Stream by Means of Superposition</u>

Consider the same aquifer described previously with the discharging well again as described. Now, however, we wish to model the system in terms of drawdown utilizing the principle of superposition; that is, we wish to model only the disturbance created by the well, leaving out any representation of the original steady state. (22) What should we use for the initial head distribution, prior to pumping, in this case? (23) What should we use as the net input into each node representing the recharge? (24) What should the net withdrawal be from the node block containing the well?

Table 3.6.5 shows the head distribution after 155 days of pumping using this mode of simulation. (25) Using the data in Table 3.6.5, construct a plot of head vs. distance from the stream for this time along an axis running through the well (row 6). Compare these head values with the <u>differences</u> between "heads after 155 days of pumping" and "heads prior to pumping" on the profiles constructed in question 4 and question 19. (26) Calculate the total flow from the stream to the aquifer after 155 days of pumping. Compare this with the total <u>change in flow to the stream</u> determined for the same time of pumping in question 14.

Table 3.6.5.--Drawdown distribution showing effects of only the pumping well after 155 days in aquifer shown in figure 3.6.1.

Columns

Rows	1	2	3	4	5	6	7	8	9	10	11
1	-96.9	-96.4	-94.8	-91.7	-86.3	-78.1	-66.5	-52.2	-35.8	-18.2	-0.0
2	-97.4	-97.0	-95.6	-92.8	-87.8	-79.7	-68.0	-53.2	-36.4	-18.5	-0.0
3	-98.8	-98.6	-97.8	-96.0	-92.3	-85.0	-72.4	-56.2	-38.2	-19.3	-0.0
4	-100.6	-100.7	-101.0	-101.2	-100.4	-95.7	-80.4	-61.2	-40.9	-20.4	-0.0
5	-102.2	-102.7	-104.2	-107.4	-112.6	-116.7	-92.5	-67.1	-43.7	-21.5	-0.0
6	-102.9	-103.5	-105.9	-111.6	-125.7	-166.2	-105.6	-71.3	-45.2	-22.1	-0.0
7	-102.2	-102.7	-104.2	-107.4	-112.6	-116.7	-92.5	-67.1	-43.7	-21.5	-0.0
8	-100.6	-100.7	-101.0	-101.2	-100.4	-95.7	-80.4	-61.2	-40.9	-20.4	-0.0
9	-98.8	-98.6	-97.8	-96.0	-92.3	-85.0	-72.4	-56.2	-38.2	-19.3	-0.0
10	-97.4	-97.0	-95.6	-92.8	-87.8	-79.7	-68.0	-53.2	-36.4	-18.5	-0.0
11	-96.9	-96.5	-94.8	-91.7	-86.3	-78.1	-66.6	-52.2	-35.8	-18.2	-0.0

(27) On the basis of these results, are there any problems for which useful answers could be obtained without knowing either the original recharge rate or the stream elevation? Are there any problems for which this approach would not be adequate? (28) If we make a model computation in this way, obtaining drawdowns and <u>flow</u> changes, can we then calculate actual <u>head values</u> and <u>values of flow</u> after a specified time of pumping? If so, explain how this can be done and what additional information is required to do it.

Simulation of a Non-Penetrating Stream

Consider the aquifer shown in figure 3.6.2, which has the same dimensions and properties as that described in the previous exercise, but is bounded on four sides by impermeable barriers---that is, the eastern boundary is no longer a fully penetrating stream, but rather an impervious valley wall. In this case, however, a stream crosses the area from north to south at a distance of 20,000 feet from the western edge of the aquifer. This stream does not penetrate the aquifer but rather is separated from it by an 80-foot thickness of a confining unit consisting of semipermeable material as shown in the cross section. The vertical hydraulic conductivity of the semipermeable material in the confining unit is 10 gpd/ft^2, the width of the stream is 80 ft, the altitude of the stream surface is 110 ft above sea level, and the altitude of the top of the aquifer is 10 ft above sea level. It is assumed that flow QZ(i,j) to or from the river through the confining unit is entirely vertical and occurs only within the

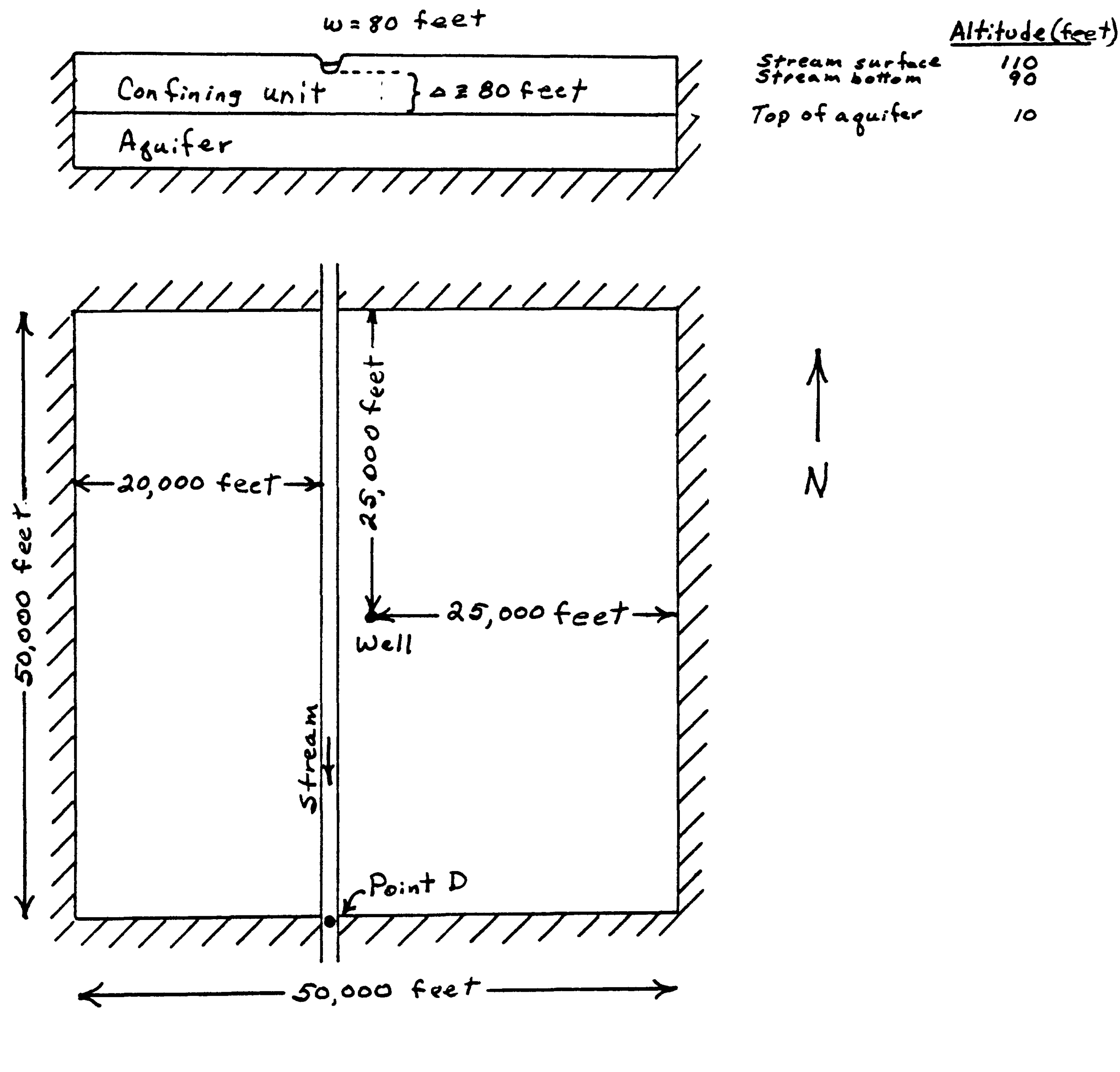


Figure 3.6.2.--Cross section and map of sand aquifer used in simulation of a nonpenetrating stream.

80-foot strip beneath the river itself. The top of this vertical-flow interval is the streambed; the head at this point is simply the stream elevation. The bottom of this vertical-flow interval is the top of the aquifer; the head at this point is the water level in the aquifer directly below the stream unless this water level has fallen below the top of the aquifer. If this has happened, the upper part of the aquifer is dewatered, and we assume that a saturated column exists beneath the stream through the confining unit and that the head at the base of this column is the altitude of the top of the aquifer.

System Under Natural Conditions

Consider a model node located along the stream. (29) Sketch the block of semiconfining material "above" this node through which flow between the stream and the aquifer occurs according to the assumptions outlined above. Show all dimensions. What is the hydraulic conductance of this block of semiconfining material in the vertical direction? (30) Write two equations which describe, according to the above assumptions, the flow between the stream and the aquifer through this block of confining unit (that is, write one equation for the condition where the head is above the top of the aquifer and one for the condition where the upper part of the aquifer is dewatered). (31) Outline the steps to determine which of the equations should be used for the calculation of flow between the stream and aquifer in the model. (32) Sketch a graph of flow between the stream and the aquifer, as calculated by these the stream. Plot flow on the ordinate and head on the abscissa; take the flow as positive when it is directed from the stream to the aquifer and include both positive and negative flow ranges on your sketch. Indicate the value of head at which the graph crosses the zero-flow axis and at which the change from one equation to the other takes place.

(33) Describe the changes that would have to be made in the model which you designed in the section "Simulation of a fully penetrating stream" in order to represent the aquifer shown in figure 3.6.2. Indicate the nodes at which any changes must be made, the new input terms which must be specified at nodes along the stream, and the new form of the finite-difference equation which the model must solve at nodes along the stream for both specified conditions (head above the top of the aquifer or head below the top of the aquifer). Under these conditions, if the aquifer becomes unconfined the horizontal hydraulic conductances and storage capacities would change. For the sake of this problem, assume that these changes are negligible and do not have to be considered.

Table 3.6.6 shows a steady-state head distribution for the aquifer of figure 3.6.2, obtained using the model with the modifications developed in the preceding question. (34) Using the data on table 3.6.6, plot a profile of heads across the aquifer in an west-east direction at a distance of 25,000 feet from the

Table 3.6.6.--Steady-state head distribution for the aquifer shown in figure 3.6.2 (vertical hydraulic conductivity of confining unit between the aquifer and the stream is 10 gallons per day per square foot).

Columns

Rows	1	2	3	4	5	6	7	8	9	10	11
1	115.0	114.7	114.0	112.7	111.0	113.7	116.0	117.7	119.0	119.7	119.9
2	115.0	114.7	114.0	112.7	111.0	113.7	116.0	117.7	119.0	119.7	119.9
3	115.0	114.7	114.0	112.7	111.0	113.7	116.0	117.7	118.9	119.7	119.9
4	115.0	114.7	114.0	112.7	111.0	113.7	116.0	117.7	119.0	119.7	119.9
5	115.0	114.7	114.0	112.7	111.0	113.7	116.0	117.7	119.0	119.7	119.9
6	115.0	114.7	114.0	112.7	111.0	113.7	116.0	117.7	119.0	119.7	119.9
7	115.0	114.7	114.0	112.7	111.0	113.7	116.0	117.7	119.0	119.7	119.9
8	115.0	114.7	114.0	112.7	111.0	113.7	116.0	117.7	119.0	119.7	120.0
9	115.0	114.7	114.0	112.7	111.0	113.7	116.0	117.7	119.0	119.7	120.0
10	115.0	114.7	114.0	112.7	111.0	113.7	116.0	117.7	119.0	119.7	120.0
11	115.0	114.7	114.0	112.7	111.0	113.7	116.0	117.7	119.0	119.7	120.0

northern edge of the aquifer. Indicate the water level of the stream on this plot. (35) Compare each half of this plot with the western half of the head profile obtained in question 4. Explain the similarities or differences that you note. (36) Calculate the total discharge from the aquifer to the stream and compare this to the total recharge to the aquifer.

Response of System to Pumping

A well located 25,000 feet from the northern edge of the aquifer and 5,000 feet east of the stream is pumped at a rate of 2 million gallons per day. We wish to simulate both this pumping rate and the recharge (0.0002 gallons per day per square foot) considered previously. (37) At which node of the model should the pumping be represented? (38) What should be the net withdrawal from this node? (39) What should be used as the initial head distribution?

Table 3.6.7 shows the head distribution in the aquifer after 155 days of pumping. (40) Using the data in table 3.6.7, plot a profile of heads in the same direction as in question (34). Is there any effect of pumping on the opposite side of the stream from the well? Could this occur if the stream were fully penetrating <u>and</u> its level not affected by the pumping? (41) Calculate the flow between the aquifer and each reach (node) of the stream. (42) What is the <u>total net flow</u> and the <u>change</u> in flow between the aquifer and the stream? (43) What is the change in stream flow from its value prior to pumping, at a point D

Table 3.6.7.--Head distribution after pumping 155 days in aquifer shown in figure 3.6.2 (vertical hydraulic conductivity of confining unit between the aquifer and the stream is 10 gallons per day per square foot).

Rows \ Columns	1	2	3	4	5	6	7	8	9	10	11
1	111.6	111.4	111.0	110.3	109.4	105.1	101.8	99.8	98.7	98.2	98.1
2	111.5	111.3	110.8	110.1	109.2	104.4	100.9	99.0	98.1	97.8	97.7
3	111.2	111.0	110.4	109.6	108.7	101.7	98.0	96.7	96.5	96.7	96.7
4	110.8	110.6	109.8	108.7	107.6	95.2	92.3	92.9	94.1	95.0	95.3
5	110.5	110.2	109.1	107.3	104.9	78.8	82.5	87.9	91.5	93.5	94.1
6	110.3	109.9	108.6	105.9	99.8	32.1	70.4	84.2	90.2	92.8	93.6
7	110.5	110.2	109.1	107.3	104.9	78.8	82.5	87.9	91.5	93.5	94.1
8	110.8	110.6	109.8	108.7	107.6	95.2	92.3	92.9	94.1	95.0	95.3
9	111.2	111.0	110.4	109.6	108.7	101.7	98.0	96.7	96.5	96.7	96.7
10	111.5	111.3	110.8	110.1	109.2	104.4	100.9	99.0	98.1	97.8	97.7
11	111.6	111.4	111.0	110.3	109.4	105.1	101.8	99.8	98.7	98.2	98.1

(fig 3.6.2) downstream from the aquifer? (44) Does this indicate that a new equilibrium has been reached? Explain your answer.

Effect of Stream-Bed Conductance

Consider the same problem of a non-penetrating stream with pumpage with the vertical hydraulic conductivity of the material between the aquifer and the stream now changed to 1,000 gpd/ft^2. (45) What is the vertical hydraulic conductance of the block of confining unit which you sketched in question 29 using this value of conductivity? Table 3.6.8 shows the head distribution after 155 days of pumping using this conductivity. Plot a head profile in the same direction as was done in questions (34) and (40). (46) Does the effect of pumping extend to the opposite side of the stream in this case? (47) Would it be permissible to consider the stream a fully penetrating constant-head boundary in simulating this problem? Explain your answer.

Simulation of a Non-Penetrating Stream by Means of Superposition

Suppose we were to redo the simulation of the non-penetrating stream in terms of drawdown rather than head using the principles of superposition (see the section in the Course Guide on "Principle of superposition and its application in ground-wter modeling"). (48) What would we use as the initial head distribution? (49) What would we use as the net withdrawal from the node at which the well is simulated? (50) What would we use as

Table 3.6.8.--Head distribution after pumping 155 days in aquifer shown in figure 3.6.2 (vertical hydraulic conductivity of confining unit between aquifer and stream is 1,000 gallons per day per square foot).

Columns

Rows	1	2	3	4	5	6	7	8	9	10	11
1	114.0	113.7	113.0	111.7	110.0	106.6	103.9	102.2	101.4	101.0	100.9
2	114.0	113.7	113.0	111.7	110.0	106.0	103.2	101.6	100.9	100.6	100.6
3	114.0	113.7	113.0	111.7	110.0	103.8	100.6	99.5	99.4	99.6	99.7
4	114.0	113.7	113.0	111.7	110.0	98.2	95.4	96.0	97.2	98.1	98.4
5	114.0	113.7	113.0	111.7	110.0	83.2	86.3	91.4	94.8	96.7	97.2
6	114.0	113.7	112.9	111.7	109.9	37.9	74.7	87.9	93.5	96.0	96.7
7	114.0	113.7	113.0	111.7	110.0	83.2	86.3	91.4	94.8	96.7	97.2
8	114.0	113.7	113.0	111.7	110.0	98.2	95.4	96.0	97.2	98.1	98.4
9	114.0	113.7	113.0	111.7	110.0	103.8	100.6	99.5	99.4	99.6	99.6
10	114.0	113.7	113.0	111.7	110.0	106.0	103.2	101.6	100.9	100.6	100.6
11	114.0	113.7	113.0	111.7	110.0	106.6	103.9	102.2	101.4	101.0	100.9

the net input, representing recharge, at the remaining nodes? Table 3.6.9 shows the results of a simulation carried out in this fashion, in terms of drawdown, for 155 days of pumping. (51) Compare the drawdown values obtained in this run to the differences between the absolute heads prior to pumping and after 155 days of pumping obtained previously. Calculate the total seepage from the stream and compare it to the total change in seepage caused by the pumping in question 42. (52) Again, having obtained drawdowns and flow changes, how would you calculate actual head values and values of flow after a specified time of pumping?

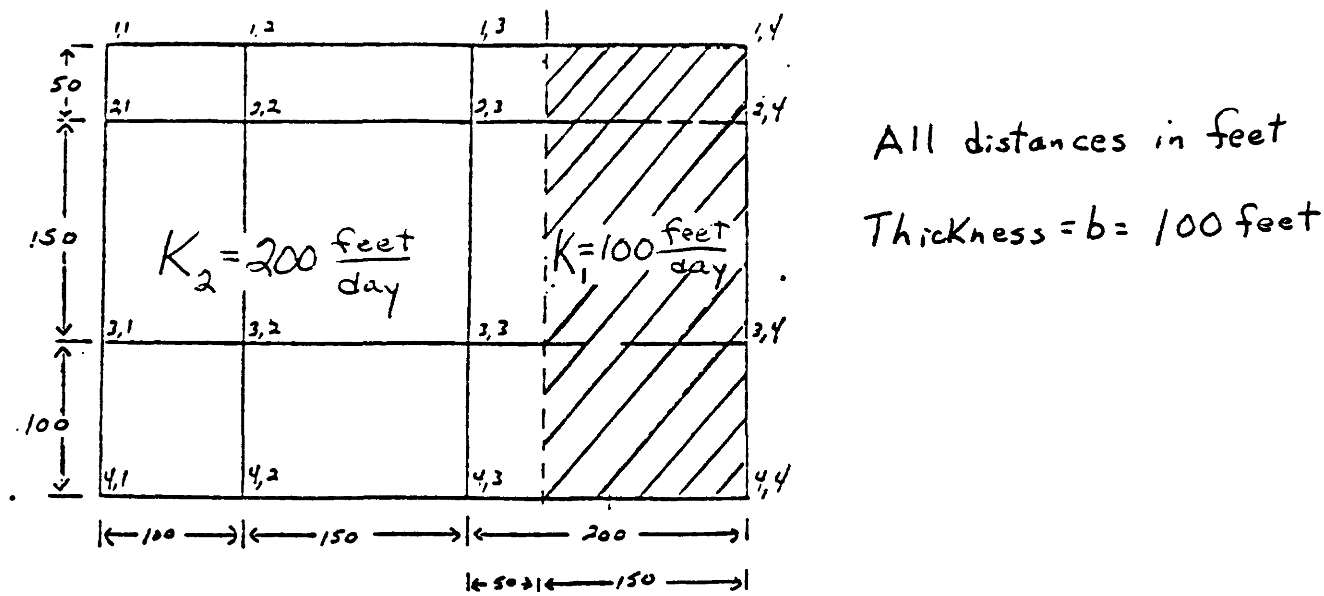
Table 3.6.9.--Drawdown distribution showing effects of only the pumping well after 155 days of pumping in the aquifer shown in figure 3.6.2 (hydraulic conductivity of confining unit between the aquifer and the stream is 10 gallons per day per square foot).

Columns

Rows	1	2	3	4	5	6	7	8	9	10	11
1	-3.4	-3.3	-3.0	-2.5	-1.6	-8.6	-14.2	-18.0	-20.3	-21.5	-21.9
2	-3.5	-3.4	-3.2	-2.6	-1.8	-9.4	-15.1	-18.7	-20.9	-21.9	-22.3
3	-3.8	-3.7	-3.6	-3.1	-2.3	-12.0	-18.0	-21.0	-22.5	-23.1	-23.3
4	-4.2	-4.2	-4.2	-4.1	-3.4	-18.5	-23.7	-24.9	-24.9	-24.7	-24.6
5	-4.5	-4.6	-4.9	-5.5	-6.1	-35.0	-33.5	-29.8	-27.4	-26.2	-25.9
6	-4.7	-4.8	-5.4	-6.9	-11.2	-81.7	-45.6	-33.5	-28.8	-26.9	-26.4
7	-4.5	-4.6	-4.9	-5.5	-6.1	-35.0	-33.5	-29.8	-27.4	-26.2	-25.9
8	-4.2	-4.2	-4.2	-4.1	-3.4	-18.5	-23.7	-24.9	-24.9	-24.7	-24.6
9	-3.8	-3.7	-3.6	-3.1	-2.3	-12.0	-17.9	-21.0	-22.5	-23.1	-23.3
10	-3.5	-3.4	-3.2	-2.6	-1.8	-9.4	-15.1	-18.7	-20.8	-21.9	-22.2
11	-3.4	-3.3	-3.0	-2.5	-1.6	-8.6	-14.2	-18.0	-20.3	-21.5	-21.9

ANSWERS TO PROBLEM 1, CALCULATION OF LUMPED HYDRAULIC CONDUCTANCES AND STORAGE CAPACITIES IN RECTANGULAR GRIDS.

Worksheet 4.1.1--Answers to worksheet 3.1.1, calculation of hydraulic conductances.



HYDRAULIC CONDUCTANCES FOR THE BRANCHES LISTED BELOW:

BRANCH	FORMULA USED (series, parallel, or, single)		$C\left[\frac{\text{feet squared}}{\text{Day}}\right]$
1,1-1,2	$\frac{KA}{L} = \frac{200(100)(25)}{100}$	SINGLE	5,000
1,2-2,2	$\frac{KA}{L} = \frac{200(100)(75+50)}{50}$	SINGLE	50,000
1,3-1,4	$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{\frac{200(25)100}{50}} + \frac{1}{\frac{100(25)100}{150}}$	SERIES	1,428.6
2,3-3,3	$C_{eq} = C_1 + C_2 = \frac{200(100)(75+50)}{150} + \frac{100(100)50}{150}$	PARALLEL	20,000
3,1-4,1	$\frac{KA}{L} = \frac{200(100)50}{100}$	SINGLE	10,000
3,2-3,3	$\frac{KA}{L} = \frac{200(100)(75+50)}{150}$	SINGLE	16,666.7

Single

$$C = \frac{KA}{L}$$

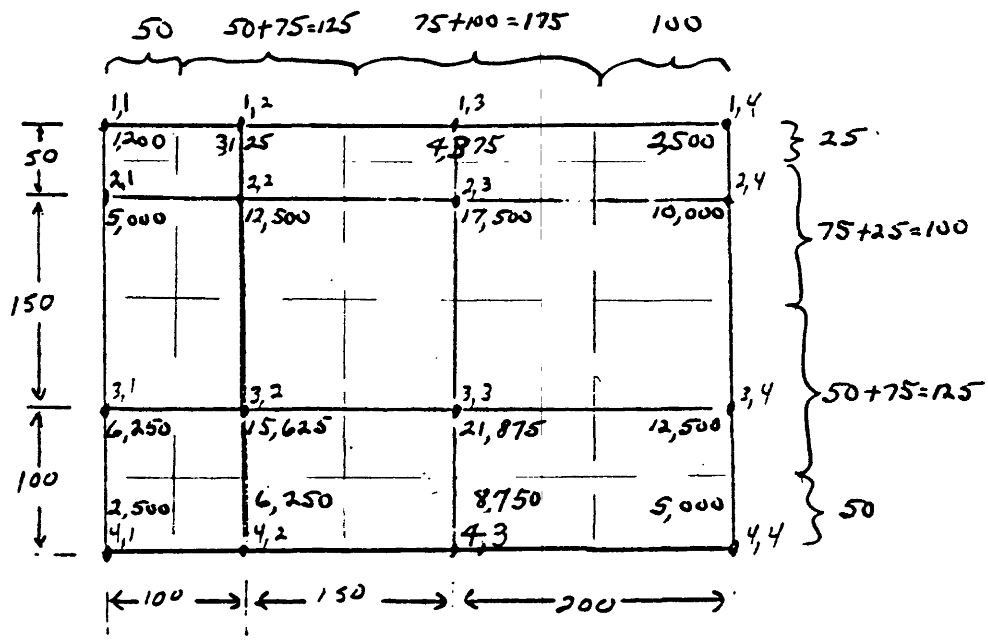
Series

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$

Parallel

$$C_{eq} = C_1 + C_2$$

Worksheet 4.1.2--Answers for worksheet 3.1.2, calculation of storage capabilities.



Thickness(b)=100 feet

Distances in feet

$S_s = 1.\times 10^{-5}$ per foot

STORAGE CAPACITIES FOR EACH NODE

Node	Storage Area	S_c	Node	Storage Area	S_c
1,1	1,250	1.25	3,1	6,250	6.25
1,2	3,125	3.125	3,2	15,625	15.625
1,3	4,375	4.375	3,3	21,875	21.875
1,4	2,500	2.50	3,4	12,500	12.50
2,1	5,000	5.00	4,1	2,500	2.50
2,2	12,500	12.5	4,2	6,250	6.25
2,3	17,500	17.5	4,3	8,750	8.75
2,4	10,000	10.0	4,4	5,000	5.0

ANSWERS TO PROBLEM 2, NUMERICAL ANALYSIS, STEADY STATE

Exercise: [values of head are in feet]

Step 1: Subtract equation 8c from equation 8b.

$$h_{2,3} = 1/4(h_{3,3} + h_{2,2} + 2)$$

$$-\{h_{3,2} = 1/4(h_{3,3} + h_{2,2} + 2)\}$$

$$h_{2,3} - h_{3,2} = 0$$

This gives us that $h_{2,3} = h_{3,2}$.

Step 2: Because $h_{2,3} = h_{3,2}$, substitute $h_{2,3}$ for $h_{3,2}$ in equations 8a and 8d. This then gives us the following set of three equations and three unknowns:

$$h_{2,2} = 1/4(2h_{2,3}) \quad (9a)$$

$$h_{3,3} = 1/4(2h_{2,3} + 9) \quad (9b)$$

$$h_{2,3} = 1/4(h_{3,3} + h_{2,2} + 2) \quad (9c)$$

Fill in the rest of equation (9b) for $h_{3,3}$. The three unknown values are now $h_{2,2}$, $h_{3,3}$, and $h_{2,3}$.

Step 3: Substitute the right hand side of equations 9a and 9b into equation 9c for $h_{2,2}$ and $h_{3,3}$.

$$h_{2,3} = 1/4(1/4(2h_{2,3} + 9) + 1/4(2h_{2,3}) + 2)$$

$$h_{2,3} = 1/4h_{2,3} + \frac{17}{16}$$

This gives us an equation all in terms of $h_{2,3}$.

Step 4: Solve this equation for $h_{2,3}$.

$$h_{2,3} = \frac{h_{2,3}}{4} + \frac{17}{16}$$

$$\frac{3}{4} h_{2,3} = \frac{17}{16}$$

$$h_{2,3} = 1.417$$

Step 5: Substitute the value of $h_{2,3}$ into equation 9a and solve for $h_{2,2}$.

$$h_{2,2} = 1/4[2(1.417)]$$

$$h_{2,2} = .708$$

Step 6: Substitute the value of $h_{2,3}$ into equation 9b and solve for $h_{3,3}$.

$$h_{3,3} = 1/4[2(1.417) +9)$$

$$h_{3,3} = 1/4(11.834)$$

$$h_{3,3} = 2.958$$

Setp 7: The values of the four unknown heads are:

$$h_{2,3} = 1.417$$

$$h_{3,2} = 1.417$$

$$h_{2,2} = .708$$

$$h_{3,3} = 2.958$$

Worksheet 4.2.1.--Answers for worksheet 3.2.1--Iterative method, steady state

[values of head are in feet]

$$h_{i,j}^{m+1} = (1/4)\,(h_{i-1,j}^{m+1} + h_{i+1,j}^{m} + h_{i,j+1}^{m} + h_{i,j-1}^{m})$$

→ j

↓ i

0.	0.	0.	0.
0.	$h_{2,2}^{0} = 0.$ $h_{2,2}^{1} = 0$ $h_{2,2}^{2} = .250$ .594 .680 .702 .707	$h_{2,3}^{0} = 0.$ $h_{2,3}^{1} = .500$ 1.188 1.360 1.403 1.414 1.416	2.
0.	$h_{3,2}^{0} = 0.$ $h_{3,2}^{1} = .500$ 1.188 1.360 1.403 1.414 1.416	$h_{3,3}^{0} = 0.$ $h_{3,3}^{1} = 2.500$ 2.844 2.930 2.951 2.957 2.958	4.
0.	2.	5.	6.

ANSWERS TO PROBLEM 3, NUMERICAL ANALYSIS, TRANSIENT STATE

[values of head are in feet]

<u>Solution of the Nonequilibrium Problem</u>

Forward-Difference Equation Using Time-Step Duration = 1 Day

$$h_{i,j,n+1} = (1)\ [h_{i-1,j,n} + h_{i+1,j,n} + h_{i,j-1,n} + h_{i,j+1,n} - 3h_{i,j,n}]$$

Backward-Difference Equation Using Time-Step Duration = 1 Day

$$h^{m+1}_{i,j,n} = \left(\frac{1}{5}\right)\left[h^{m+1}_{i-1,j,n} + h^{m}_{i+1,j,n} + h^{m+1}_{i,j-1,n} + h^{m}_{i,j+1,n} + (1)\ h_{i,j,n-1}\right]$$

Solution Using Method 1-- Forward Difference and Time-Step Duration = 1 Day

Values of $h_{i,j,n+1}$ at the end of 1 day are:

$h_{2,2,n+1}$ = 0.71;
$h_{2,3,n+1}$ = -0.58;
$h_{3,2,n+1}$ = 1.42;
$h_{3,3,n+1}$ = -1.04

The negative values for water levels shown above are not reasonable. Recall the discussion in Note 4, page 2.4-5 and 2.4-6 where it is stated, "Thus, the method (forward-difference method) appears at first glance to be straight-forward and simple. However, if the time increment is taken too large, any error that appears for any reason at any time step is guaranteed to grow in successive time steps until the finite-difference solution is eventually dominated by error and bears no relation at all to the analytical, or exact, solution of the differential equations of flow. <u>"This condition is termed numerical instability; because of it, forward-difference techniques are generally not used."</u>

Bennett (1976, p. 137) cautions that unless the ratio $\Delta t/a^2$ in the forward-difference equation is kept sufficiently small, errors that grow in magnitude with each step of the calculation may appear in the final result. Unless the ratio is sufficiently small, this error will increase in magnitude at each succeeding time step of calculation until eventually the error completely dominates the solution. The term "error" as used here, refers to any differences between the computed head at a node i,j and time $n\Delta t$, and the actual value of head--that is, the value that would be given by the exact solution to the differential equation at that point and time (Bennett, 1976, p. 137-138). Neither the discussion in Note 4 nor that in Bennett (1976) define how small the ratio $\Delta t/a^2$ would have to be to make the forward-difference calculation stable, thus preventing the error from growing in magnitude at each succeeding time step. Because we are applying the forward-difference method to the flow equation, this stability criterion also involves the hydrologic parameters. Mathematically, it can be proven for a 2-dimensional flow equation that if $(\Delta t T/a^2 S)<1/4$ the forward-difference calculation will be stable (Remson, and others, 1971, p. 71-77; Wang and Anderson, 1983, p. 70; Rushton and Redshaw, 1979, p. 166-168).

In the example in problem 3,

$a=1,000$ ft,

$T=2X10^5$ ft^2/day, and

$S=0.2$

then, if Δt=1 day, $(\Delta tT/a^2S)=(1)\ (2X10^5)/(0.2)(1000)^2=1$, which violates the stability creterion. Therefore the results obtained by Δt=1 day is not near the true solution. To achieve the stability creterion, Δt must be smaller than $(1000)^2(0.2)/4(2X10^5)=1/4$ day. The following are head-calculations for Δt=1 day by using Δt=1/4 day and Δt=1/8 day. The following are the calculated heads using Δt=1/4 day.

• .709	• 1.417
.709	.917
.584	.667
.459	.605
.428	.542
• 1.417	• 2.958
1.417	1.959
1.167	1.834
1.105	1.709
1.042	1.678

The following are the calculated heads using Δt-1/8 day.

• .709	• 1.417
.709	1.167
.678	.979
.631	.846
.582	.753
.538	.686
.501	.638
.472	.603
.449	.577
• 1.417	• 2.958
1.417	2.458
1.354	2.177
1.284	2.005
1.222	1.894
1.171	1.819
1.130	1.769
1.099	1.731
1.075	1.706

As shown, there are no negative water levels.

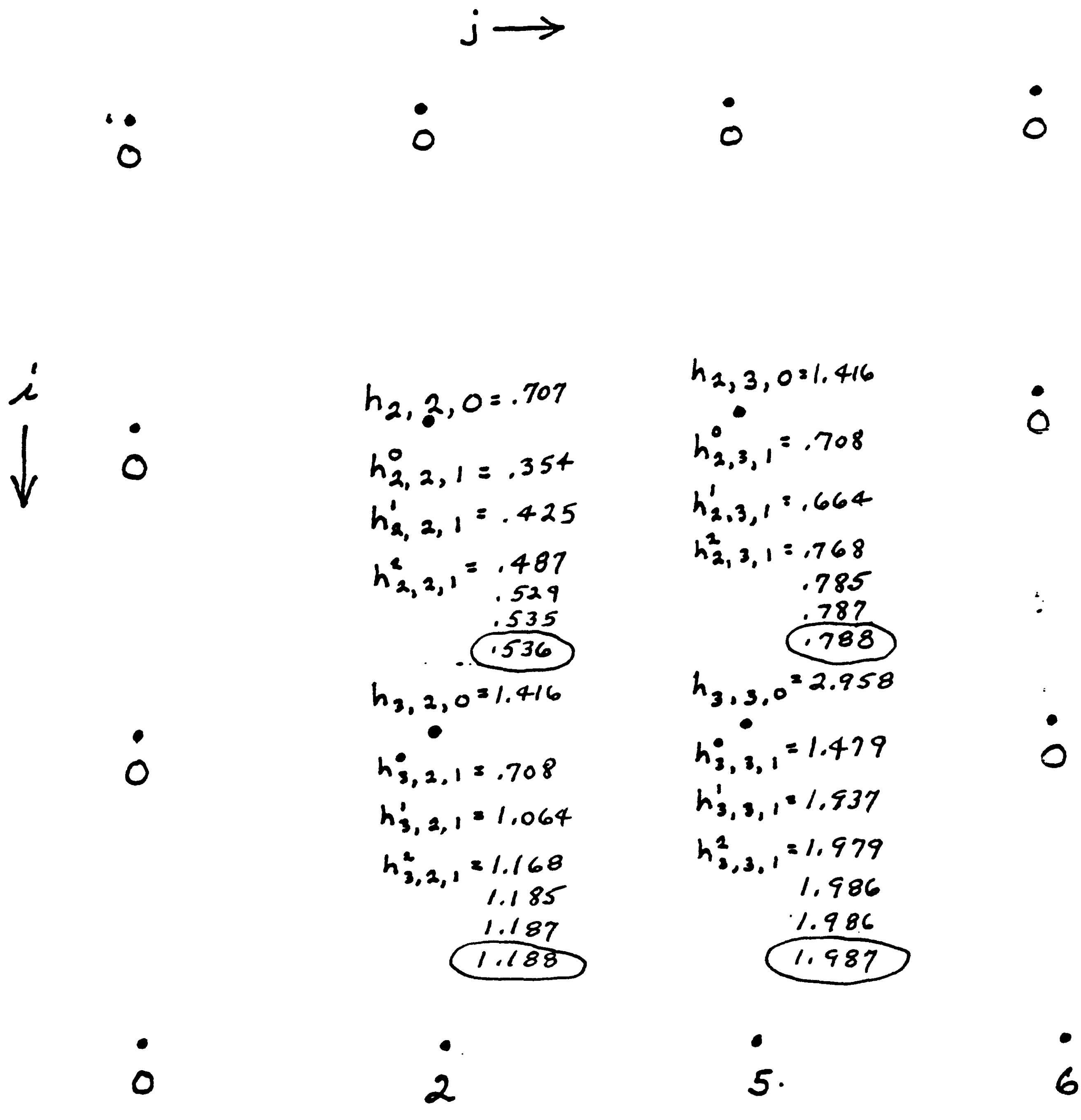
Solution Using Method 2--Backward-Difference and Time-Step Duration = 1 Day (See Worksheet 4.3.1)

Solution Using Method 3--Backward-Difference and Time-Step Duration = 1/2 Day

$$h^{m+1}_{i,j,n} = \left(\frac{1}{6}\right) [h^{m+1}_{i-1,j,n} + h^{m}_{i+1,j,n} + h^{m+1}_{i,j-1,n} + h^{m}_{i,j+1,n} + (2)h_{i,j,n-1}]$$

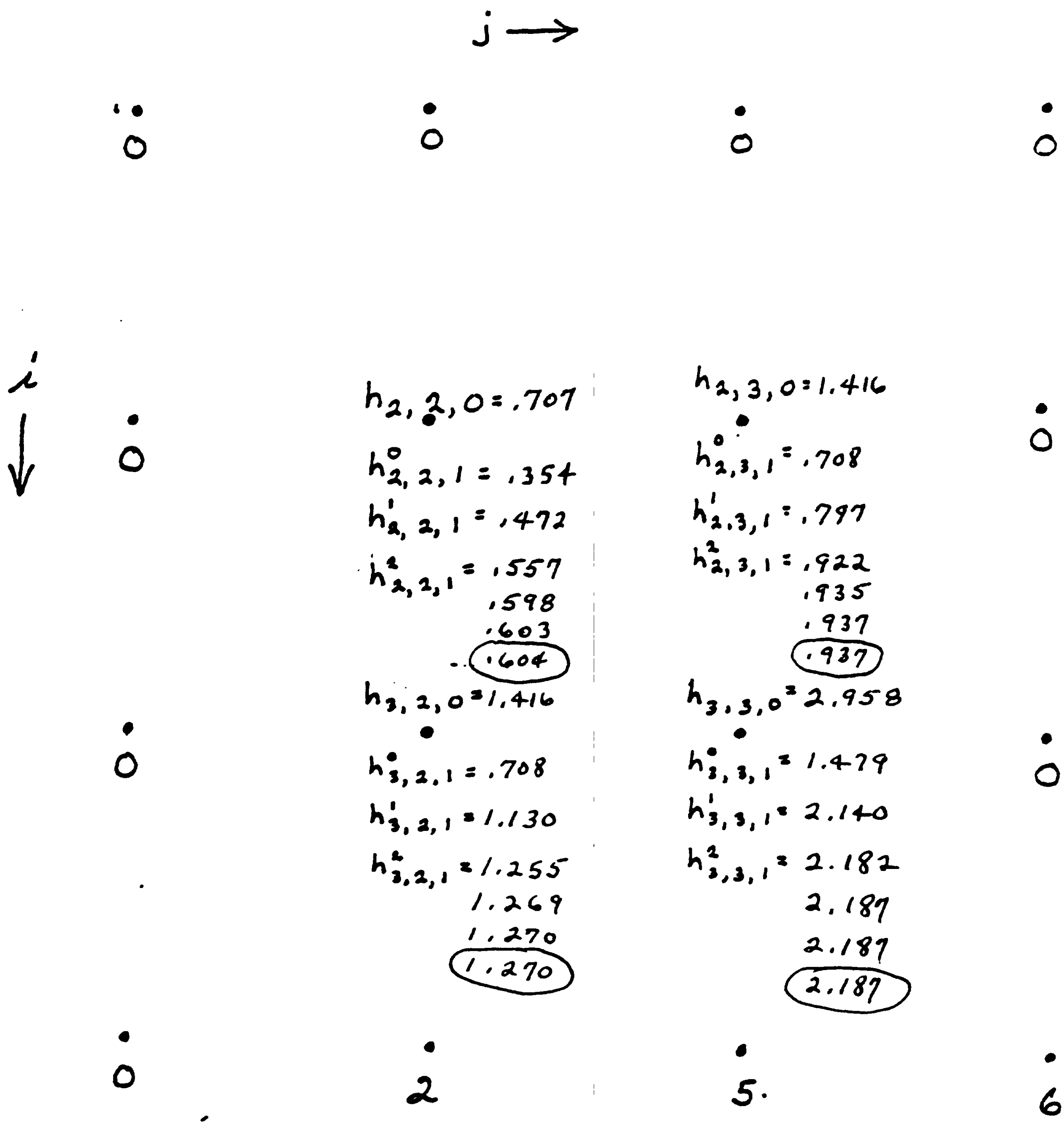
For first half-day, see worksheet 4.3.2.
For second half-day, see worksheet 4.3.3.

Worksheet 4.3.1--Answers for worksheet 3.3.1, transient state, backward difference, time-step duration = 1 day



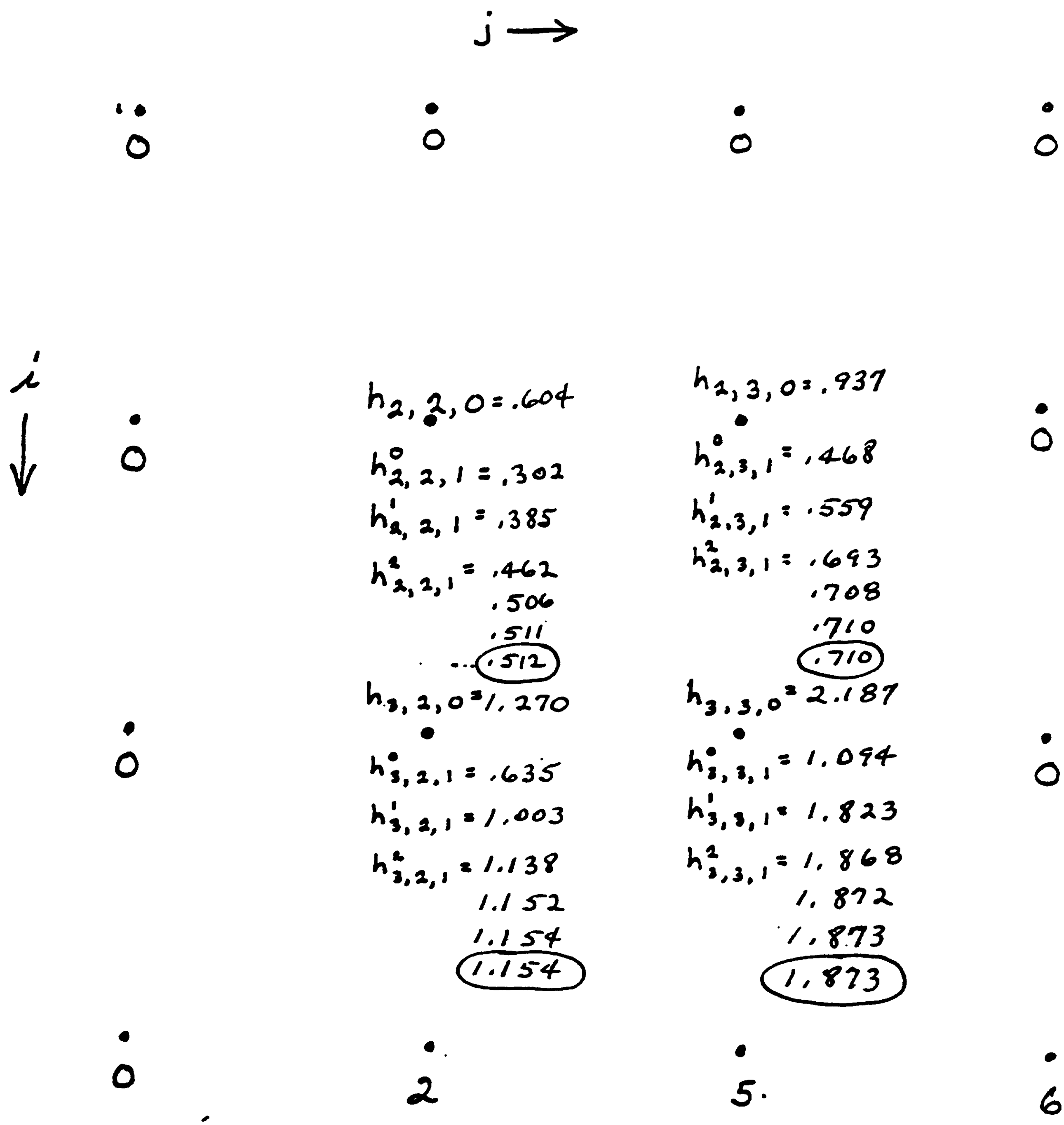
The values of $h_{i,j,n}$ are circled.

Worksheet 4.3.2.--Answers for worksheet 3.3.2, transient state, backward difference, time step duration = 1/2 day (first time step; t = 1/2 day)



The values of $h_{i,j,n}$ are circled.

Worksheet 4.3.3.--Answers for Worksheet 3.3.3, transient state, backward difference, time-step duration = 1/2 day (second time step; t = 1 day).



The values of $h_{i,j,n}$ are circled.

Answers Related to Worksheet 3.3.4 and Figure 3.3.1

The simulated head at the end of one day is slightly different for each of the four curves. The reason for this is that as the time step gets smaller the error in the finite-difference approximation of the time derivative gets smaller. Figure 3.3.1 shows that as the time step is decreased the simulated head at the end of one day is changing less as the error in the time derivative becomes smaller. If a good approximation of the head in the aquifer were required at day one or information on the response of the aquifer at shorter times were required, then the selection of the time step would have to be thought out and perhaps sensitivity analysis undertaken to insure that the simulation is an accurate representation of the real system.

ANSWERS TO PROBLEM 4, IMPERMEABLE WALL PROBLEM

Ground-Water Flow System

In this problem set we assume that flow is two dimensional, that is, the flow pattern is replicated exactly in planes parallel to the plane of the figures illustrating the impermeable wall ground-water system (section 3.4, fig. 3.4.1). It is convenient to consider the plane of the figures in this problem set as the x-z plane. The external geometry of the flow system is defined by the external boundary of the fine sand in figure 3.4.1. Because the flow medium is isotropic and homogeneous, no layering or internal geometry exists in this sytem, and no differences in hydraulic conductivity occur in different parts of the system.

Some course participants designate the upper right-hand horizontal boundary, the discharge boundary, as a constant-flux boundary. In principle, this boundary could be designated as a constant-flux boundary if the flow through the system is known. Generally, however, this flow is not known, and one reason for performing a quantitative analysis of the system is to determine this flow. Furthermore, if some depth of standing water, however small, were present above this boundary, most hydrologists would conceptualize this boundary as a constant-head boundary because, if the system were stressed (not an issue in this problem set), the response of the system with a constant-head boundary would differ markedly from its response with a constant-flux boundary.

Water enters the system along the upper left-hand, constant-head boundary and discharges from the system along the upper right-hand, constant-head boundary. Two bounding streamlines connect the inflow and outflow boundaries. The purpose of requesting participants to sketch several internal streamlines and potential lines is to emphasize that, given the external geometry and boundary conditions, we can conceptualize the approximate flow pattern within the ground-water system without detailed data or analysis.

The foregoing comments relate to the simplified ground-water system depicted in figure 3.4.1 of problem set 4. However, a comparison of the system depicted in figure 3.4.1 with similar real ground-water systems indicates that the position and possibly the type of boundaries on ST and VU are arbitrary. In nature, the flow system may extend, perhaps for a considerable distance, beyond these two boundaries as depicted in figure 3.4.1. The purpose of simulation in this type of problem is to achieve realistic heads and flows, compared with the real system, in the neighborhood of the structure. A logical approach to simulation of this problem type is to perform a "sensitivity analysis" on the position of boundaries ST and VU, that is, execute a series of simulations in which the distance of these two vertical boundaries from the impermeable wall increases continuously until two successive simulations exhibit negligible differences in heads and flows near the wall.

The question of assigning boundary conditions to these two vertical boundaries still remains. Possibilities include (a) constant head, (b) constant flux, and (c) streamline. Our concept of the flow pattern in this system envisions streamlines starting at the upper left-hand constant head boundary and flowing beneath the wall. Because head is dissipated along streamlines, a vertical constant-head boundary close to the wall is not appropriate. Wherever we establish the position of the vertical boundary ST, we may be neglecting a small quantity of lateral ground-water inflow. Thus, a lateral constant-flux boundary along ST is physically reasonable. However, a realistic estimate of this flux would require a simulation in which this lateral boundary was positioned considerably further from the impermeable wall than the proposed constant-flux boundary. For this reason, the simplest and usual approach in this type of problem is to treat these lateral boundaries as streamlines. Their position is such that heads and flows near the impermeable wall are insensitive to a further increase in the distance of these boundaries from the wall.

The previous considerations did not play a role in positioning the lateral boundaries ST and VU in this problem set. These boundaries would be placed further from the wall in an actual quantitative analysis of this ground-water system.

Information requested on heads in connection with figure 3.4.1.

Location	Elevation head (z), in feet	Pressure head (p/r), in feet	Hydraulic head (h), in feet
Upper left-hand constant-head boundary	50	5	55
Upper right-hand constant-head boundary	25	0	25

(head at upper left-hand boundary) - (head at upper right-hand boundary) = 55 - 25 = 30 feet.

In this system, water flows from a higher-head, constant-head boundary (h equals 55 feet at upper left-hand boundary) to a lower-head, constant-head boundary (h equals 25 feet at upper right-hand boundary).

Flow Net

The three calculated values of total flow through the impermeable wall ground-water system vary slightly (see answers to table 3.4.1, p. 4.4-8). The difference between the maximum and minimum calculated values is about 1% of the total flow. This variation in calculated flow may be attributed to (a) round-off error and (b) discretization error. In theory, if the head values at nodes are expressed with three or four significant figures after the decimal point instead of two, and if the discretization interval between nodes is reduced to less than 5 feet, then the variation in calculated values of total flow along different traverses through the ground-water system would decrease.

According to Darcy's law, head is dissipated along flowlines. In this system 30 feet of head is dissipated between the two ends of all flowlines. The lengths of flowlines in this system vary continuously from a maximum for the outer bounding streamline to a minimum for the streamline along the sides and bottom of the impermeable wall. Thus, the average distance between intersection points of potential lines and flowlines in the flow net on worksheet 3.4.1 decreases toward the impermeable wall.

The actual distance between intersection points of potential lines on any flowline in the flow net varies widely. Head dissipation is concentrated near the bottom of the wall where the potential lines are closely spaced. If the impermeable wall were deeper, the potential lines would be spaced even more closely in this region, and the opposite is true if the wall were less deep. Sometimes, it is simpler to think in terms of "resistance to flow" instead of "relative ease of flow" in a ground-water system. The greatest "resistance to flow" occurs beneath the impermeable wall because the minimum cross-sectional area of flow is located there.

The spacing flowlines that bound flow tubes containing equal proportions of the total flow is related to the pattern of head dissipation. The widths of the five flow tubes along the upper left-hand constant-head boundary decrease slightly, but continuously from left to right toward the impermeable wall. All the heads in the row of heads immediately below the upper left-hand constant head boundary must be equal in order for the width of flow tubes along this boundary to be equal. In fact, heads in this row immediately below the constant head boundary decrease from left to right toward the impermeable wall. On the right, or discharge, side of the impermeable wall, the widths of flow tubes along the upper right-hand constant-head boundary decrease markedly from right to left toward the impermeable wall, corresponding to a sharply increasing vertical gradient from right to left along this boundary. In general, in flow nets for systems with isotropic and homogeneous media, the spacing between potential lines and flowlines varies in a continuous and orderly manner.

Worksheet 4.4.1--Answers for worksheet 3.4.1, construction of a flow net.

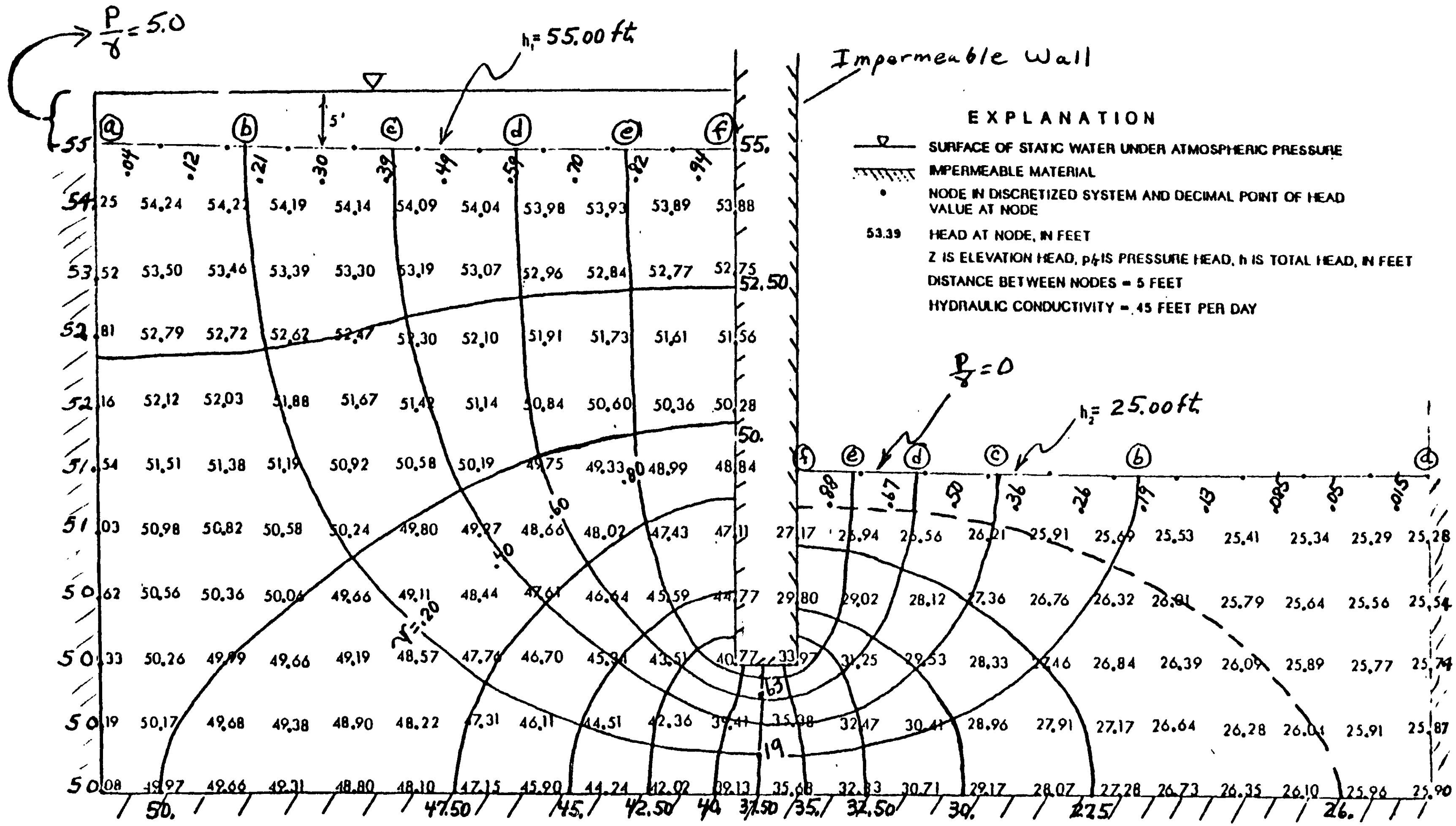


Table 4.4.1--Answers, table 3.4.1, format for calculations of stream functions in impermeable-wall problem [ft^2/d, feet squared per day; ft, feet; ft^3/d, cubic feet per day]

Format for calculation of stream functions in impermeable wall problem.

	Block number	$C_{block} = KA/L$ (ft^2/d)	h_1 (ft)	h_2 (ft)	Δh_{block} (ft)	$q_{block} = C\Delta h$ (ft^3/d)	$\sum q_{block}$ (ft^3/d)	$\Psi = \sum q_{block} / Q_{total}$	Plotting position for stream functions
Inflow boundary; Traverse DE	1	22.5	55.00	54.25	.75	16.88	16.88	.04	a
	2	45.0		54.24	.76	34.2	51.08	.12	b
	3			54.22	.78	35.1	86.18	.21	c
	4			54.19	.81	36.45	122.63	.30	d
	5			54.14	.86	38.7	161.33	.39	e
	6			54.09	.91	40.95	202.28	.49	f
	7			54.04	.96	43.2	245.48	.59	g
	8			53.98	1.02	45.9	291.38	.70	h
	9			53.93	1.07	48.15	339.53	.82	i
	10	45.0		53.89	1.11	49.95	389.48	.94	j
	11	22.5	55.00	53.88	1.12	25.2	414.68	1.00	K
						414.68			
Traverse FG	1	22.5	39.13	35.68	3.45	77.63	77.63	.19	
	2	45.0	39.41	35.38	4.03	181.35	258.98	.63	
	3	22.5	40.77	33.97	6.80	153.0	411.98	1.00	
						411.98			
Below wall									

Table 4.4.1.--Answers, table 3.4.1, format for calculations of stream functions in impermeable-wall problem (cont.)

Format for calculation of stream functions in impermeable wall problem.

Block number	$C_{block} = KA/L$ (ft^2/d)	h_1 (ft)	h_2 (ft)	Δh_{block} (ft)	$q_{block} = C\Delta h$ (ft^3/d)	$\sum q_{block}$ (ft^3/d)	$\Psi = \frac{\sum q_{block}}{Q_{total}}$
Outflow boundary 1	22.5	25.28	25.00	.28	6.3	6.3	.015
2	45.0	25.29		.29	13.05	19.35	.047
3		25.34		.34	15.30	34.65	.085
4		25.41		.41	18.45	53.10	.13
5		25.53		.53	23.85	76.95	.19
6 (Traverse H1)		25.69		.69	31.05	108.00	.26
7		25.91		.91	40.95	148.95	.36
8		26.21		1.21	54.45	203.40	.50
9		26.56		1.56	70.20	273.60	.67
10	45.0	26.94		1.94	87.30	360.90	.88
11	22.5	27.17	25.00	2.17	48.83	409.73	1.00
					409.73		

Water-Pressure

Our expectation, which is borne out by the pattern of equal-water-pressure contours on worksheet 3.4.2, is that pressure will vary smoothly and continuously in an isotropic and homogeneous flow medium (worksheet 4.4.2). However, further inspection of the contours on worksheet 3.4.2 and additional calculations indicate that the variation of water pressure with depth along a vertical section (1) is not equal to a hydrostatic pressure distribution at any location and (2) is not exactly a linear relation at any location. (A linear relation between pressure and depth would necessitate equally spaced pressure contours on a vertical section.) Furthermore, the rates of change of pressure with depth vary dramatically near the bottom of the impermeable wall. Finally, a comparison of the pressure contours on worksheet 3.4.2 and the streamlines and potential lines on worksheet 3.4.1 indicates three completely different patterns of lines, in keeping with their differing physical definitions.

For discussions of answers to the calculation of uplift force on the impermeable wall, refer to figure 3.4.4. Integration of the uplift pressure along the bottom of the impermeable wall is simple in this case because we assumed a linear change in pressure beneath the wall between the two edges.

Uplift force acting at base of 1-foot wall section = average pressure times the area

$$= \left(1708\ \frac{\text{pounds}}{\text{square foot}}\right)(5\ \text{square feet})$$

= 8,540 pounds

(2) Downward force at base of 1-foot wall section = weight of 1-foot wall section

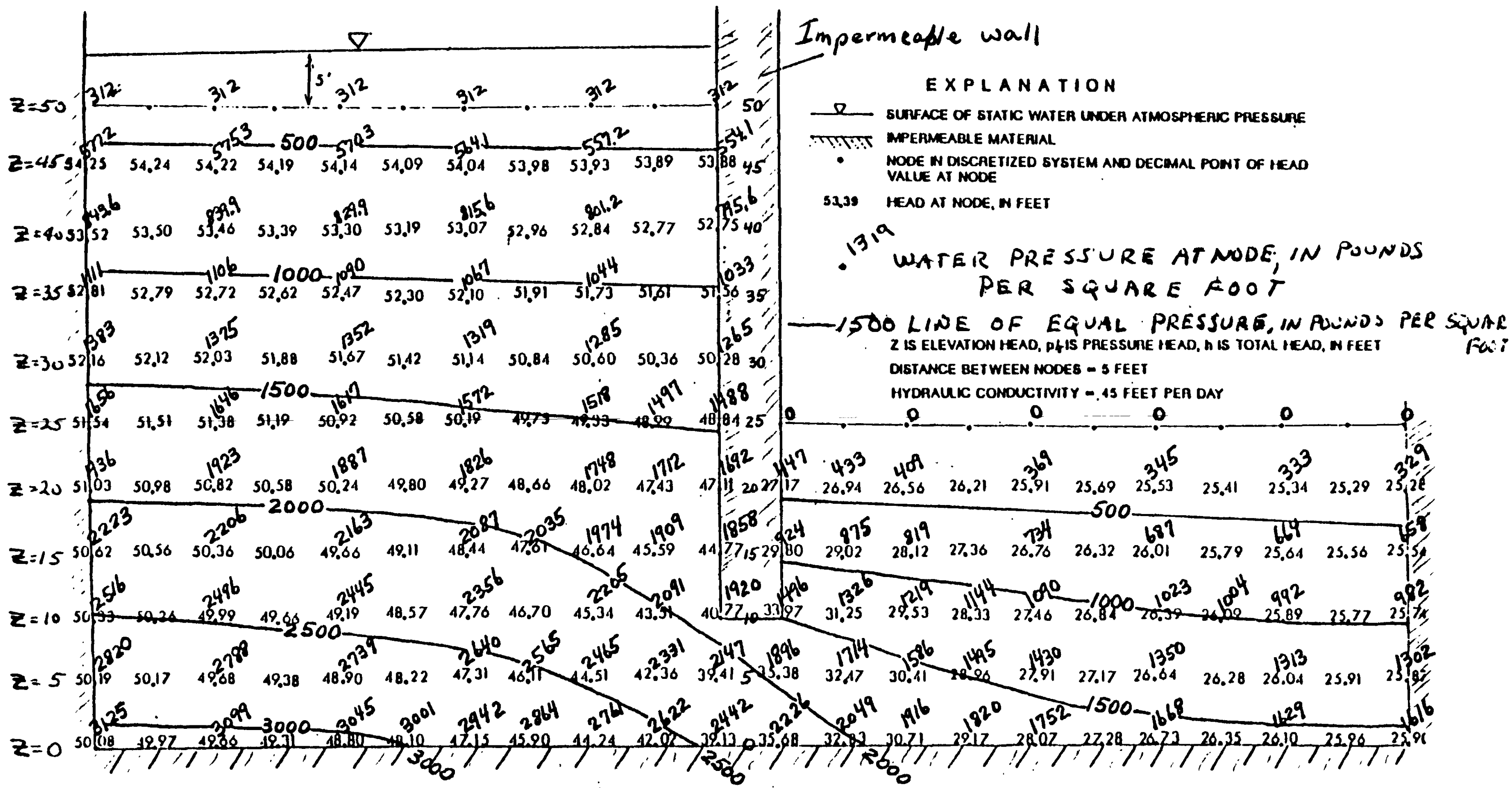
$$= \begin{pmatrix}\text{weight per}\\ \text{unit volume}\end{pmatrix}\begin{pmatrix}\text{volume of 1-foot}\\ \text{section}\end{pmatrix}$$

$$= \left(165\ \frac{\text{pounds}}{\text{cubic foot}}\right)(50\ \text{feet})\ (5\ \text{feet})$$

= 41,250 pounds

Comparison of results in (1) and (2) shows that the downward force (weight) acting at the base of the structure is about 5 times greater than the uplift force.

Worksheet 4.4.2--Answers for worksheet 3.4.2, plotting and contouring point values of water pressure

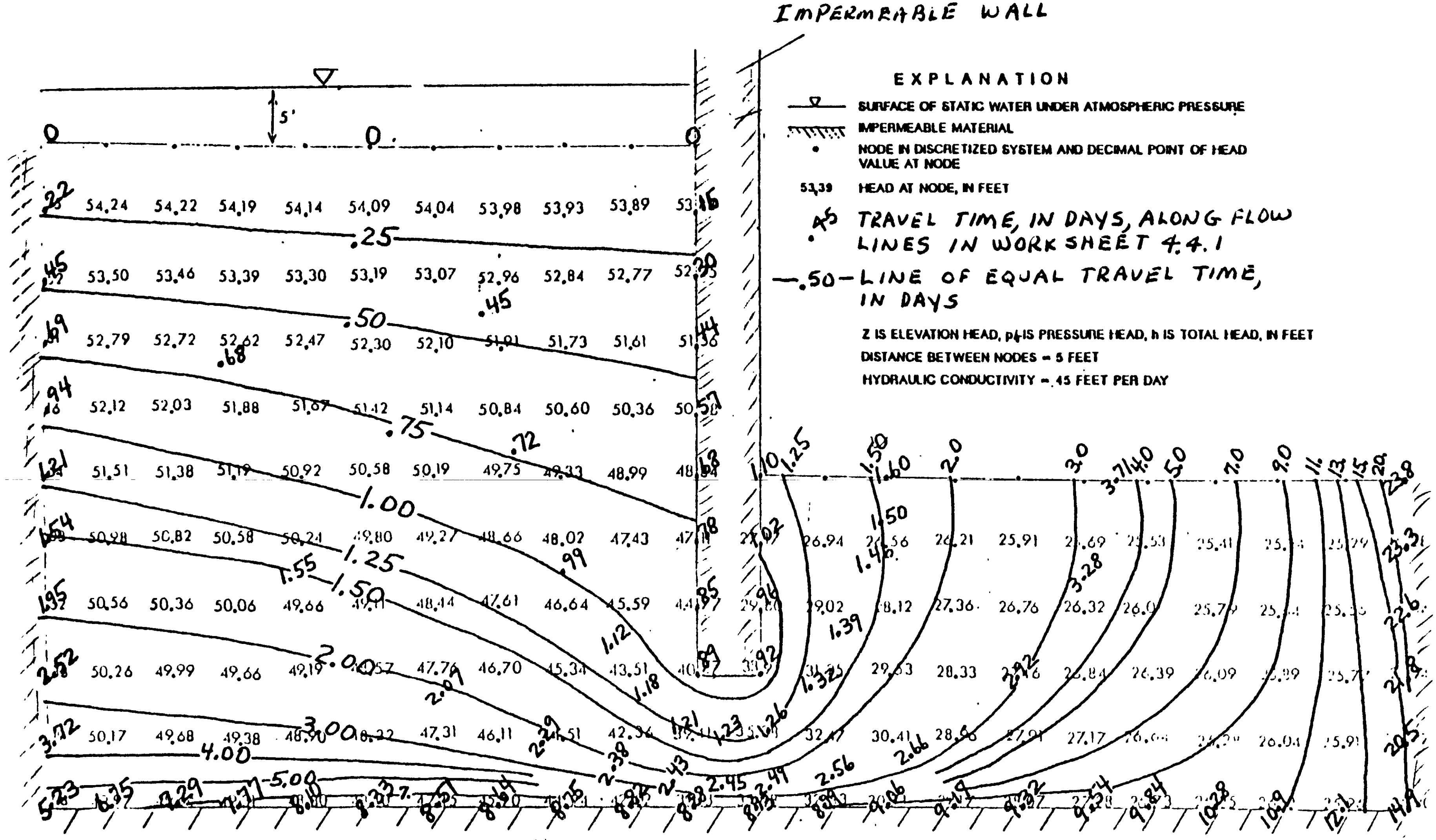


Time of Travel

With reference to the plotted time-of-travel values and related equal-time contours on worksheet 3.4.3, the total time of travel from the inflow boundary to the discharge boundary along the longest bounding streamline is about twenty time greater than the total travel times along the shortest bounding streamline around the impermeable wall. The time of travel for increments of the longest streamline vary widely. The longest travel times per unit length of streamline occur in the lower left-hand corner, lower right-hand corner, and right-hand vertical boundary. This observation is predictable from the low head gradients in these regions. The shortest times of travel per unit length of streamline occur beneath the impermeable wall, where head gradients are largest in this system.

Hydrologists generally are not accustomed to calculating the positions of time-of-travel contours and visualizing their general pattern in ground-water systems. The pattern of these contours does not bear a simple relationship to the more familiar potential lines and streamlines in flow nets; compare, for example, the line patterns in worksheet 3.4.3 and worksheet 3.4.1. Because of our present-day concern with contamination problems and the advent of particle-tracking algorithms in association with digital flow models, we can expect ever-increasing applications of equal-time-of-travel contours and "surfaces" in ground-water studies.

Worksheet 4.4.3--Answers for worksheet 3.4.3, plotting and contouring calculated point values of time of travel



Contours of equal time of travel from recharge boundary, in days (Note different contour intervals)

Table 4.4.2--Answers, table 3.4.2--Calculations of time of travel along selected flow lines in impermeable wall problem

Long streamline - a $t = \frac{.167}{\Delta h}$

Detailed time of travel

h	Δh	t (days)	Σt (days)	h	Δh	t (days)	Σt (days)
55.00				35.68			
54.25	0.75	0.223	0.223	32.83	2.85	0.059	8.985
53.52	.73	.229	.452	30.71	2.12	.079	9.064
52.81	.71	.235	.687	29.17	1.54	.108	9.172
52.16	.65	.257	.944	28.07	1.10	.152	9.324
51.54	.62	.269	1.213	27.28	.79	.211	9.535
51.03	.51	.327	1.54	26.73	.55	.304	9.839
50.62	.41	.407	1.947	26.35	.38	.439	10.278
50.33	.29	.576	2.523	26.10	.25	.668	10.946
50.19	.14	1.193	3.716	25.96	.14	1.193	12.139
50.08	.11	1.518	5.234	25.90	.06	2.783	14.922
49.97	.11	1.518	6.752	25.87	.03	5.567	20.489
49.66	.31	.539	7.291	25.74	.13	1.285	21.774
49.31	.35	.477	7.768	25.54	.20	.835	22.609
48.80	.51	.327	8.095	25.28	.26	.642	23.251
48.10	.70	.239	8.334	25.00	.28	.596	23.847
47.15	.95	.176	8.51		30.00		
45.90	1.25	.134	8.644				
44.24	1.66	.101	8.745				
42.02	2.22	.075	8.82				
39.13	2.89	.058	8.878				
35.68	3.45	.048	8.926				

Table 4.4.2--Answers, table 3.4.2, calculations of time of travel along selected flow lines in impermeable wall problem (cont.)

Short streamline - f
Detailed time of travel

$t = \frac{0.167}{\Delta h}$

h	Δh	t (days)	Σt (days)
55.00			
53.88	1.12	0.149	0.149
52.75	1.13	.148	.297
51.56	1.19	.140	.437
50.28	1.28	.130	.567
48.84	1.44	.116	.683
47.11	1.73	.097	.780
44.77	2.34	.071	.851
40.77	4.00	.042	.893
33.97	6.80	.025	.918
29.80	4.17	.040	.958
27.17	2.63	.063	1.021
25.00	2.17	.077	1.098
	30.00		

Table 4.4.2--Answers, table 3.4.2, calculations of time of travel along selected flow lines in impermeable-wall problem (cont.)

h (feet)	L (feet)	Δh (feet)	t (days) = $\frac{6.67 \times 10^{-3} L^2}{\Delta h}$	Σt (days)
		Flowline b, length 112 feet		
55	—	—	—	—
52.50	16	2.5	0.68	0.68
50	18		.86	1.54
47.50	14		.52	2.06
45	8		.17	2.23
42.50	6		.10	2.33
40	5		.07	2.40
37.50	3		.02	2.42
35	4		.04	2.46
32.50	5		.07	2.53
30	7		.13	2.66
27.50	9	2.5	.22	2.88
26	10	1.5	.44	3.32
25	8	1.0	.43	3.75
		Flowline d, length 76 feet		
55	—	—	—	—
52.50	12	2.5	0.38	0.38
50	13		.45	.83
47.50	10		.27	1.10
45	7		.13	1.23
42.50	4		.04	1.27
40	5		.07	1.34
37.50	2		.01	1.35
35	2		.01	1.36
32.50	5		.07	1.43
30	4		.04	1.47
27.50	6	2.5	.10	1.57
26	4	1.5	.07	1.64
25	3	1.0	.06	1.70

ANSWERS TO PROBELM 5, ELECTRICAL ANALOG PROBLEM

Electrical Analog Model Design

A.--Confined aquifer bounded by impermeable bedrock Hills and fully penetrating stream and reservoir

A-1. $C_e = 1/1{,}000 = 10^{-3}$ Siemen for 1,000 ohm resistor.

$C_e = 1/2{,}000 = 5 \times 10^{-4}$ Siemen for 2,000 ohm resistor.

A-2. (See figure 4.5.1.)

A-3. $C_h = Q/\Delta h = KA/L$

For internal blocks

$$C_h = 100 \left[\frac{(500)(100)}{500}\right] \frac{(gal)(ft)(ft)}{(day)(ft^2)(ft)} = 10^4 \frac{gal}{(day)(ft)}$$

For impermeable boundary blocks

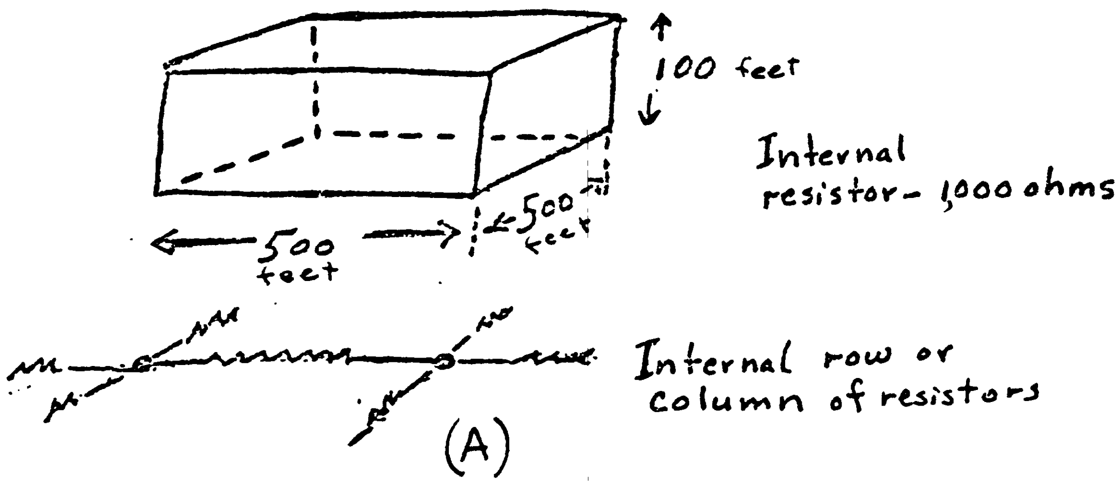
$$C_h = 100 \left[\frac{(250)(100)}{500}\right] = \left(5 \ X \ 10^3\right) \frac{gal}{(day)(ft)}$$

A-4. For internal blocks

$$K_c = \frac{C_h}{C_e} = \frac{10^4}{10^{-3}} = 10^7 \frac{gal}{(day)(ft)(Siemen)}$$

For impermeable boundary blocks

$$K_c = \frac{C_h}{C_e} = \frac{\left(5 \ X \ 10^3\right)}{\left(5 \ X \ 10^{-4}\right)} = 10^7 \frac{gal}{(day)(ft)(Siemen)}$$



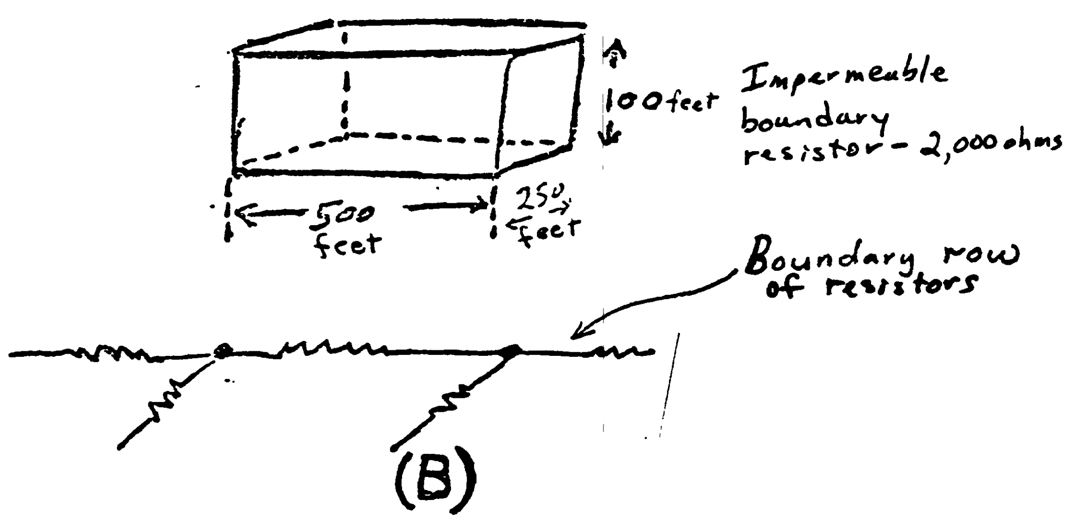


Figure 4.5.1.--Answers to question A-2 sketch of blocks of aquifer represented by (A) a single internal resistor of the network and (B) a single resistor along one of the impermeable boundaries.

A-5. $K_v = \frac{\Delta h}{\Delta v}$

$= \frac{200 \text{ ft}}{2 \text{ volt}}$

$= 100\text{ft/volt}$

A-6. $K_i = \frac{Q}{I} \frac{\text{gal}}{(\text{day})\ (\text{amp})} = \frac{C_h \Delta h}{C_e \Delta v} = K_c K_v = 10^9 \frac{\text{gal}}{(\text{day})\ (\text{amp})}$

A-7. $K_i = \frac{Q}{I} \frac{C_e \Delta h}{C_e \Delta v} = K_c K_v$

If any two of K_i, K_C, and K_V are known, the third can be calculated; two are sufficient to describe the model.

A-8. See worksheet 4.5.1 for answers to worksheet 3.5.1.

$K_V = 100$ ft/volt

$\Delta h = 100$ ft

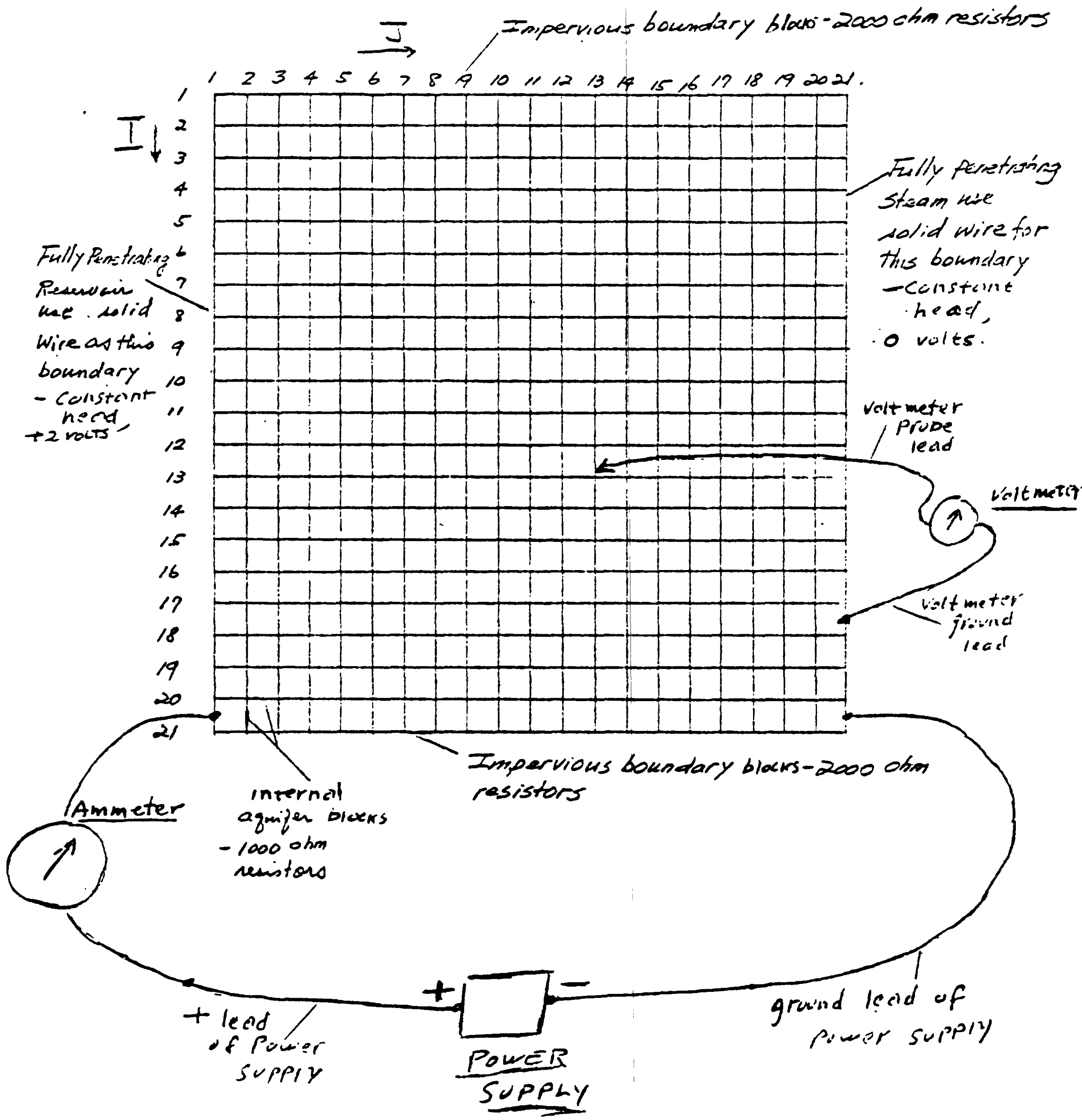
$\Delta v = \frac{200}{100} = 2$ volts

Voltage to head conversion: multiply the measured voltage by K_v, 100 ft/volt.

Current to flow conversion: multiply the measured current by K_i, 10^9 gal/day/amp.

The flow represents inflow to the aquifer from the reservoir. The flow is equal to the discharge to the stream; because we have

A-8. Worksheet 4.5.1-- Answers for worksheet 3.5.1, electric analog model grid for showing boundaries and method of connecting power supply, ammeter, and voltmeter.



steady-state conditions inflow to the aquifer must equal outflow from it.

A-9. From (6), $K_i = 10^9 \frac{\text{gal}}{(\text{day})\ (\text{amp})} = \frac{Q}{I}$

$$I = \frac{Q}{K_i} = \frac{2 \times 10^6}{10^9} = 2 \times 10^{-3} \text{ amp}$$

or $I = 2$ mA.

B.--Partially Penetrating Stream

B-1. C_z = vertical hydraulic conductance of the confining bed

$$C_z = \frac{Q}{\Delta h} = \frac{K'A}{b} = \frac{K'aw}{b}$$

$$= \frac{(0.98)\ (40)\ (500)}{10} \frac{(\text{gal})\ (\text{ft})\ (\text{ft})}{(\text{day})\ (\text{ft}^2)\ (\text{ft})}$$

$$= 1{,}960 \text{ gal/day ft}$$

2. $K_c = 10^7 \frac{\text{gal}}{(\text{day})(\text{ft})(\text{Siemen})}$ (previously calculated)

$$K_c = \frac{C_z}{C_e}$$

$$C_e = \frac{C_z}{K_c} = \frac{1{,}960}{10^7} = 1.96 \times 10^{-4} \text{ Siemen}$$

$$R = \frac{1}{C_e} = \frac{10^4}{1.96} = 5{,}100 \text{ ohms}$$

B-3. The simulation of the nonpenetrating stream is an extension of the aquifer simulation. Voltages and currents in the simulation of the nonpenerating stream must mean the same thing, in terms of heads and flows, as in the simulation of the aquifer and its original boundaries; therefore, K_v and K_i must be the same so that according to the relation $K_c = K_i/K_v$, K_c must also be the same.

B-4. At the upper end of each vertical resistance (i.e., at the end not connected to the network). In our case, because we will assume that the water-surface elevation is the same along the entire stream channel, this can be accomplished by connecting the upper ends of the vertical resistances to a bus wire and holding the voltage on the bus wire at a value corresponding to the stream elevation.

Electrical Analog Experimental Procedures

A.--Confined Aquifer Bounded by Impermeable Bedrock Hills and Fully Penetrating Stream and Reservoir

A-1. See Worksheet 4.5.2 for answers to worksheet 3.5.2

A-2. Total discharge into the stream is calculated using Darcy's Law, Q = KIA. The hydraulic conductivity, K, is 100(gal/d)/ft^2; the gradient, I, is 200 ft/10,000 ft or 0.02; and the area is the length of the stream between the impermeable bedrock hills multiplied by the thickness of the aquifer, 10,000 (100) or 10^6 ft. Thus, Q = (100) (0.02) (10^6) = 2 Mgal/d.

A-3. Total ground-water discharge to stream by current measurement.

I_s = 1.98 mA

$Q = K_i I$

$K_i = 10^9$ gal/d/ampere = 1 Mgal/mA

Q = 1.98 Mgal

Compare this result to that calculated in A-2.

B.--Confined Aquifer with a Discharging Well

B-1. Pumping well is represented by the node (11, 8)

Worksheet 4.5.2--Answers for worksheet 3.5.2, initial head distribution

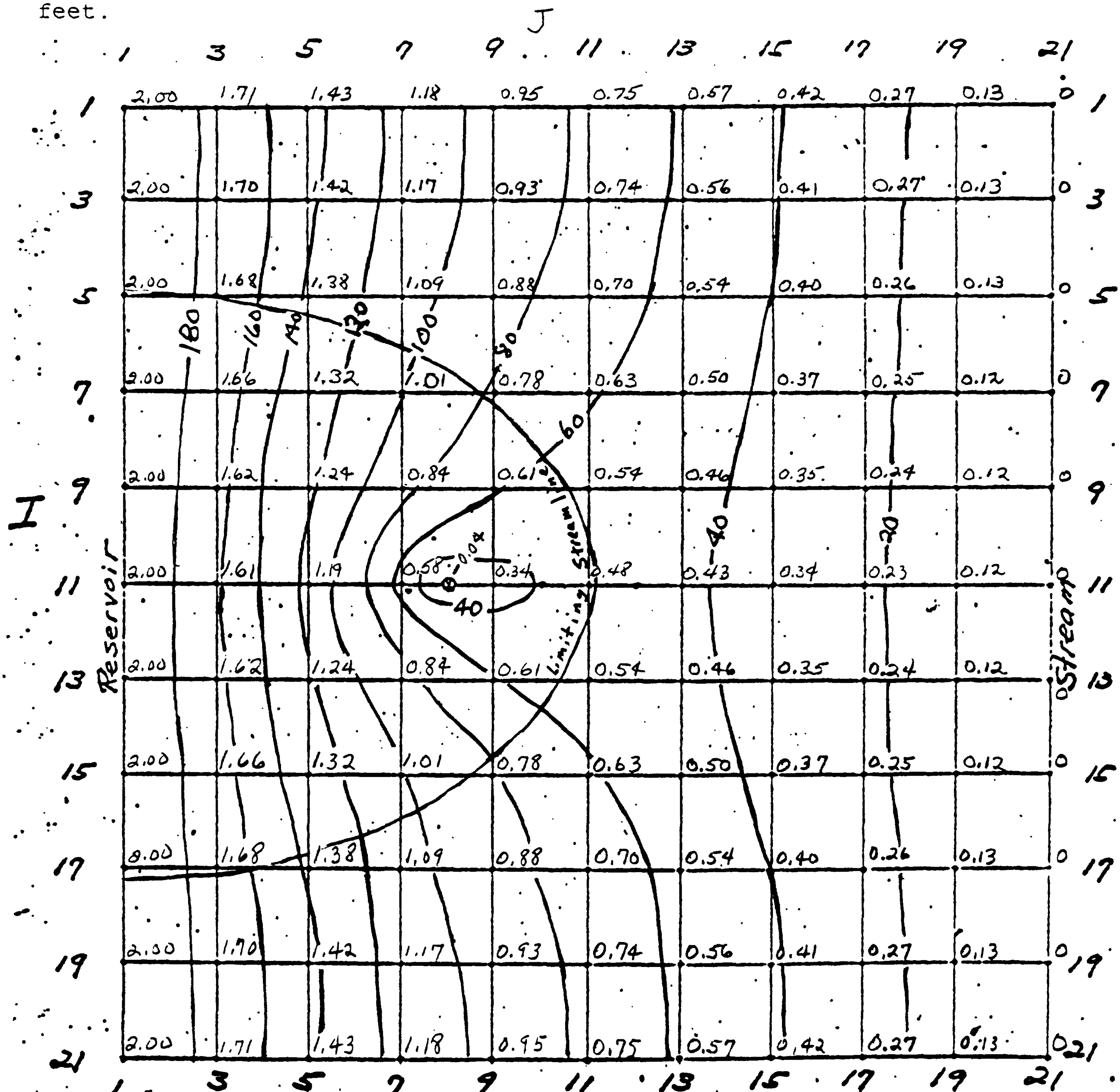
J

I — Reservoir (left edge, J = 1); Stream (right edge, J = 21)

I \ J	1	3	5	7	9	11	13	15	17	19	21
1	2.00	1.80	1.60	1.40	1.20	1.00	0.80	0.60	0.40	0.20	0
3	2.00	1.80	1.60	1.40	1.20	1.00	0.80	0.60	0.40	0.20	0
5	2.00	1.80	1.60	1.40	1.20	1.00	0.80	0.60	0.40	0.20	0
7	2.00	1.80	1.60	1.40	1.20	1.00	0.80	0.60	0.40	0.20	0
9	2.00	1.80	1.60	1.40	1.20	1.00	0.80	0.60	0.40	0.20	0
11	2.00	1.80	1.60	1.40	1.20	1.00	0.80	0.60	0.40	0.20	0
				1.30	1.10	0.90					
13	2.00	1.80	1.60	1.40	1.20	1.00	0.80	0.60	0.40	0.20	
15	2.00	1.80	1.60	1.40	1.20	1.00	0.80	0.60	0.40	0.20	0
17	2.00	1.80	1.60	1.40	1.20	1.00	0.80	0.60	0.40	0.20	0
19	2.00	1.80	1.60	1.40	1.20	1.00	0.80	0.60	0.40	0.20	0
21	2.00	1.80	1.60	1.40	1.20	1.00	0.80	0.60	0.40	0.20	

Worksheet 4.5.3--Answers for worksheet 3.5.3, head distribution that reflects the influence of a discharging well.

Multiply recorded values of voltage by 10^2 to obtain drawdown in feet.



The influence of the well (measureable drawdowns) extends throughout the aquifer; flow is diverted to the well within the limiting streamlines.

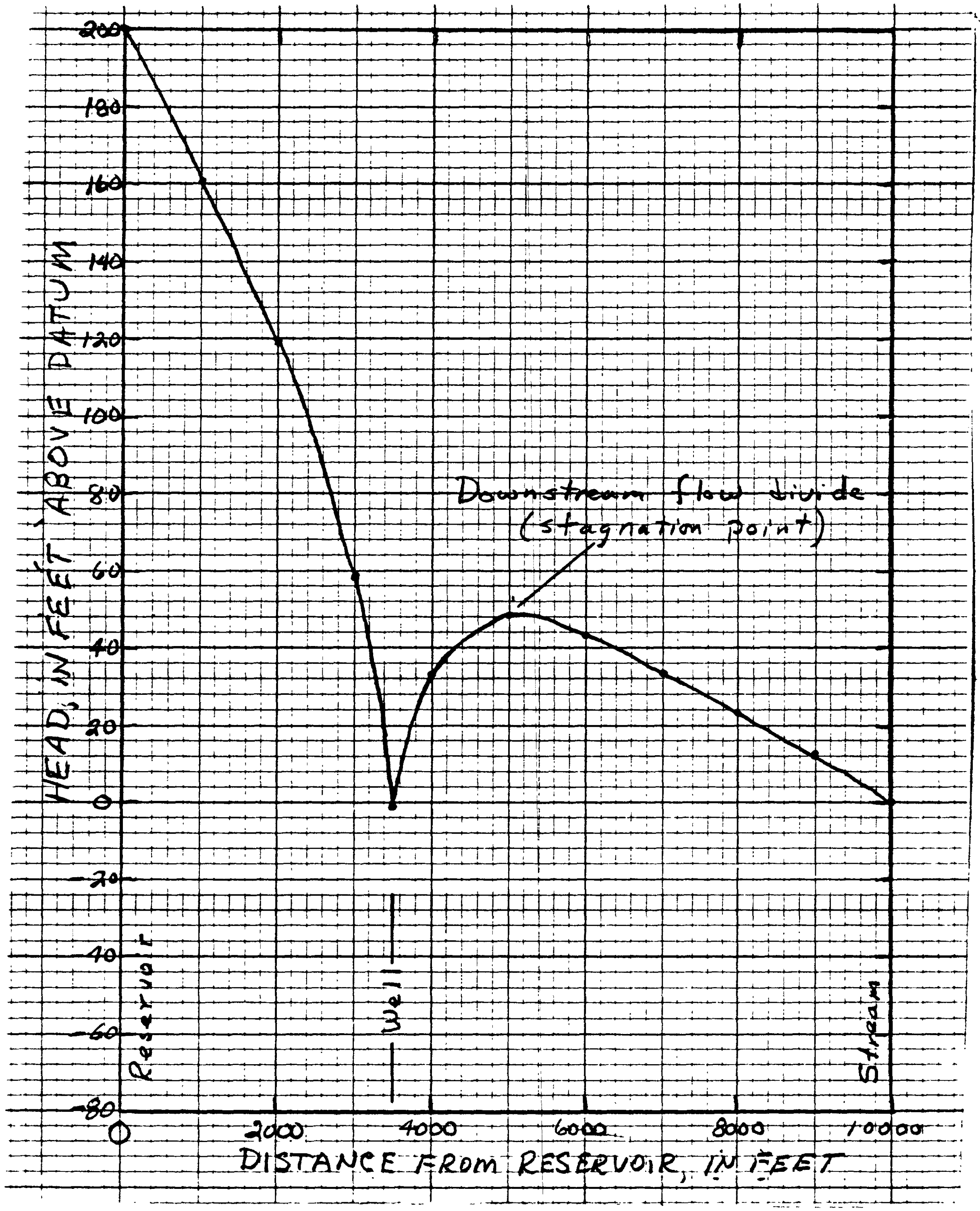


Figure 4.5.2--Answer to question B-3, east-west profile of head along a line extending from the reservoir through the well to the stream.

B-4. Flow from reservoir, 3.31 Mgal/d

Flow into stream, 1.30 Mgal/d

B-5. Total flow in: 3.31 Mgal/d at resevoir

Total flow out: 1.30 Mgal/d at stream
+ 2.00 Mgal/d from well
3.30 Mgal/d

Except for experimental error, total inflow = total outflow.

B-6. When a new equilibrium has been achieved, the well discharge is balanced by increases in inflow to the aquifer plus decreases in natural outflow.

	New Inflow	Previous Inflow	Change
Increase in inflow:	3.31	2.0	= 1.31
	Previous Outflow	New Outflow	Change
Decrease in outflow:	2.0	1.30	= 0.70

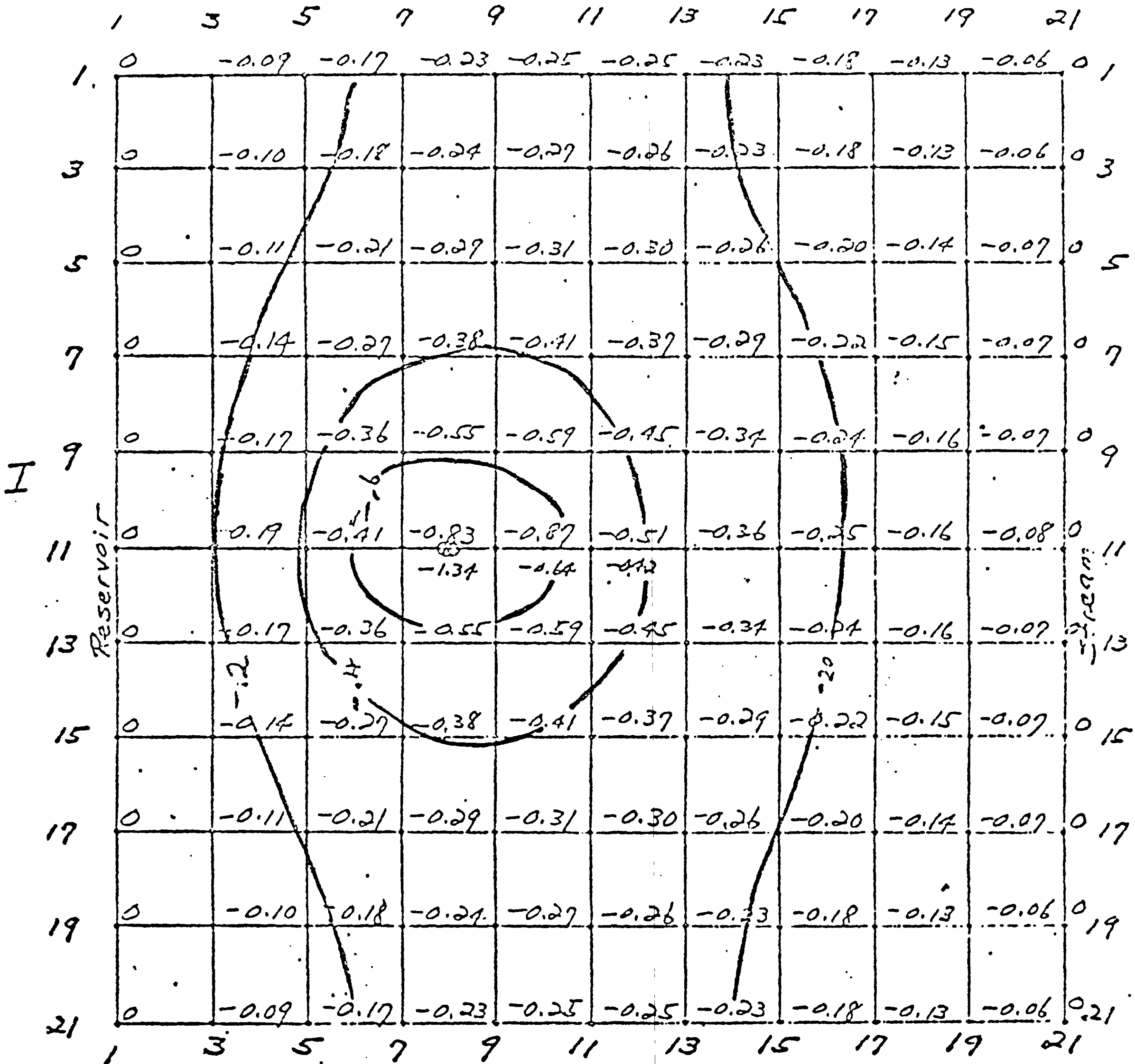
The sum of these, 1.31 + 0.70 = 2.01 Mgal/d, is equal to the well discharge within experimental error.

B-7. If the storage coefficient of the aquifer is known, the drawdown distribution can be used to calculate the total volume of water taken from storage during the nonequilibrium period. No other information is provided on the nonequilibrium period by this analysis.

C.--Confined Aquifer with Discharging Well in Superposition

C-1

Worksheet 4.5.4--Answers for worksheet 3.5.4, drawdown distribution when well is discharging and potentials at reservoir and stream boundaries equal zero.



C-2, Measured inflow from reservoir = 1.29 Mgal/d
Measured inflow from stream = .71 Mgal/d

Multiply recorded values of voltage by 10^2 to obtain drawdown in feet.

Worksheet 4.5.5--Answers for worksheet 3.5.5, algebraic sum of drawdowns measured in C-1 and heads measured in A-1.

J

I \ J	1	3	5	7	9	11	13	15	17	19	21
1	2 00	1 71	1 43	1 17	95	75	57	42	27	14	0
3	2.00	1 70	1 42	1 16	93	74	57	42	27	14	0
5	2.00	1 69	1 39	1 11	89	70	54	.40	26	13	0
7	2.00	1 66	1 33	1 02	79	63	0.51	38	25	13	0
9	2.00	1 63.	1 24	0 85	61	55	46	36	21	13	0
11	2.00.	1 61	1 19	0 57	33	49	44	35	24	12	0
				−4	46	48					
13	200	1 63	1 24	0 85	61	55	46	36	24	13	0
15	200	1 66	1 33	1 02	79	63	51	38	25	13	0
17	200	1 69	1 39	1 11	89	70	54	40	26	13	0
19	2.00	1 70	1 42	1 16	93	74	57	42	27	14	0
21	200	1 71	1 43	1 17	95	75	57	42	27	14	0

Reservoir (along J = 1)

C-4. The heads recorded in question A-1 (answers to worksheet 3.5.2, p. 4.5-8) show the effect of the reservoir and stream boundaries alone; the values recorded in question C-1 (answers to worksheet 3.5.4, p. 4.5-13) show the effect of the pumping well alone. The algebraic sum, as recorded in question C-3 (answers to worksheet 3.5.5, p. 4.5.-14) gives heads that are equal to those recorded in question B-2 (answers to worksheet 3.5.3, p. 4.5-10) in which both the boundaries and the well were represented.

C-5 and C-6. The measured inflow along the reservoir in Part C actually represents the increase in flow from the reservoir to the aquifer in response to the pumpage. The measured inflow along the stream represents the decrease in outflow from the aquifer to the stream as a result of the pumpage. Thus each flow actually represents the algebraic change in flow induced by the pumpage.

D.--Nonpenetrating Stream

Worksheet 4.5.6--Answers for worksheet 3.5.6, head distribution when simulation includes a nonpenetrating stream (no discharging well).

Reservoir (J = 1) — Stream (J = 21)

I \ J	1	3	5	7	9	11	13	15	17	19	21
1	2.0	1.76	1.52	1.28	1.04	0.80	0.56	0.32	0.21	0.10	0
3	2.0	1.76	1.52	1.28	1.04	0.80	0.56	0.32	0.21	0.10	0
5	2.0	1.76	1.52	1.28	1.04	0.80	0.56	0.32	0.21	0.10	0
7	2.0	1.76	1.52	1.28	1.04	0.80	0.56	0.32	0.21	0.10	0
9	2.0	1.76	1.52	1.28	1.04	0.80	0.56	0.32	0.21	0.10	0
11	2.0	1.76	1.52	1.28	1.04	0.80	0.56	0.32	0.21	0.10	0
				1.16	0.92	0.68					
13	2.0	1.76	1.52	1.28	1.04	0.80	0.56	0.32	0.21	0.10	
15	2.0	1.76	1.52	1.28	1.04	0.80	0.56	0.32	0.21	0.10	0
17	2.0	1.76	1.52	1.28	1.04	0.80	0.56	0.32	0.21	0.10	0
19	2.0	1.76	1.52	1.28	1.04	0.80	0.56	0.32	0.21	0.10	0
21	2.0	1.76	1.52	1.28	1.04	0.80	0.56	0.32	0.21	0.10	0

(Multiply values by 10^2 to get heads)

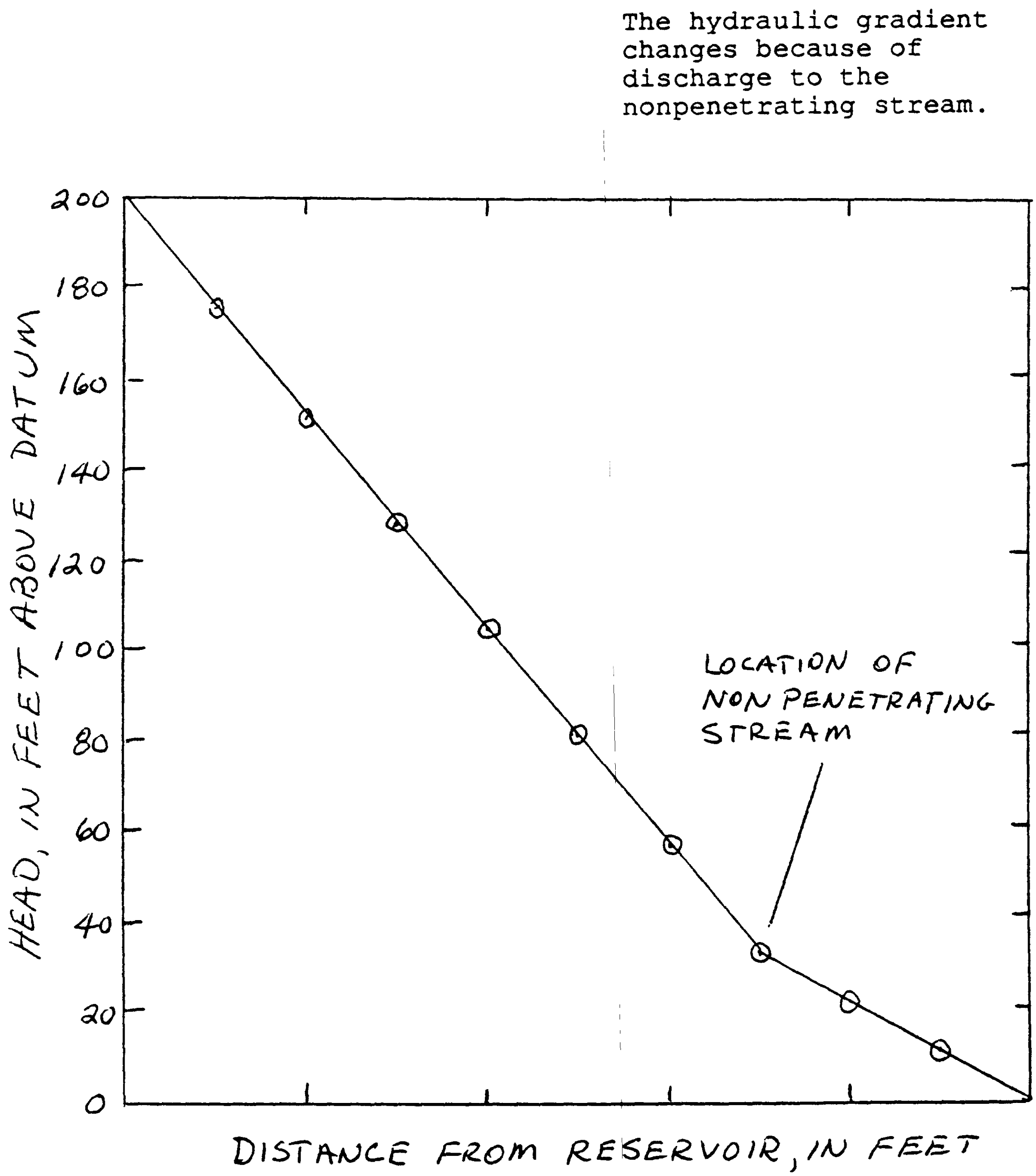


Figure 4.5.3--Answer to question D-2, east-west head profile when simulation includes a nonpenetrating stream.

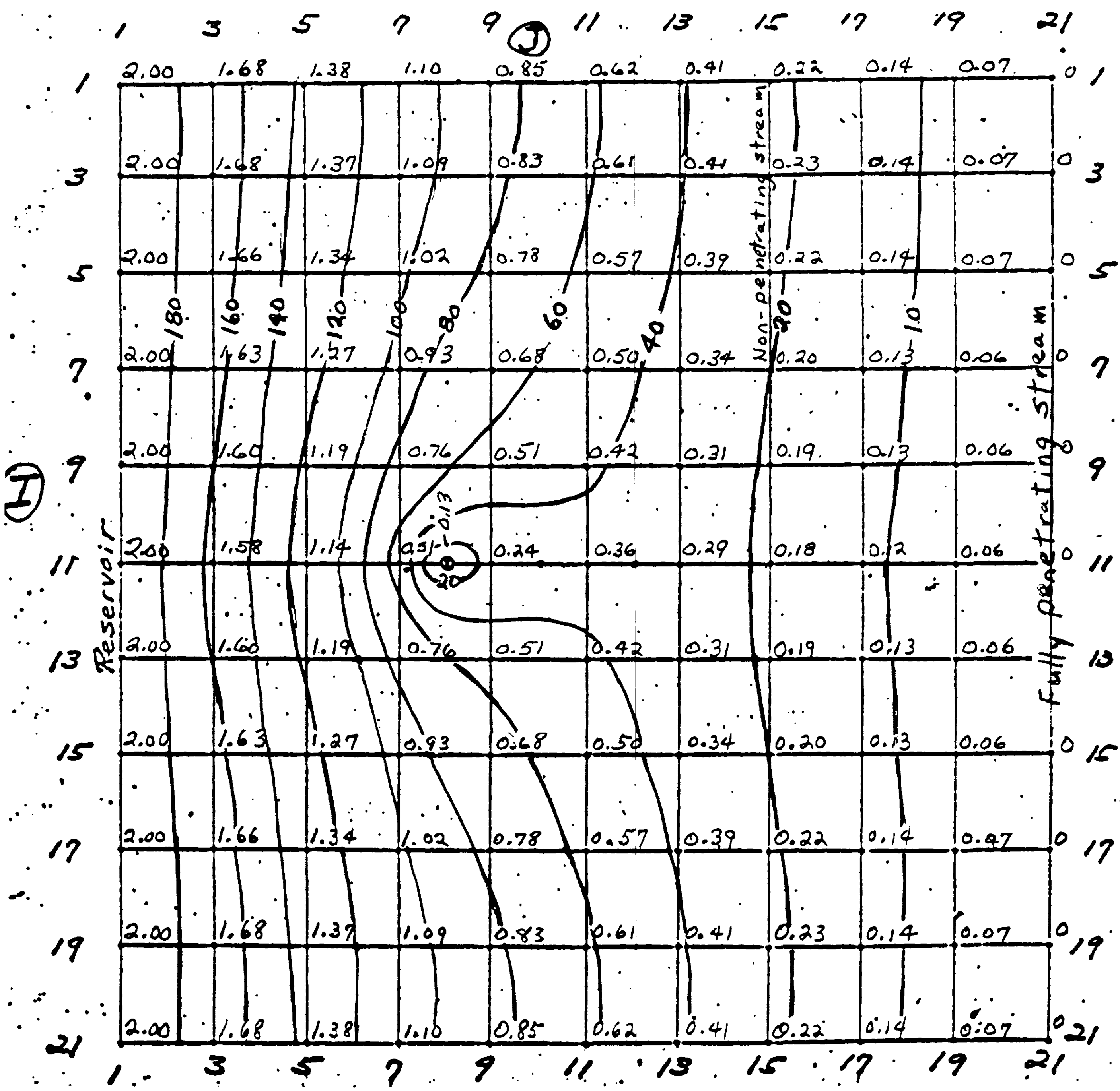
D-3. Flow from reservoir	2.36 Mgal/d
Flow into nonpenetrating stream	1.28 Mgal/d
Flow into fully penetrating stream	1.08 Mgal/d

Total inflow (reservoir) = total outflow (2 streams)

E. Confined Aquifer with Nonpenetrating Stream and Discharging Well

E-1

Worksheet 4.5.7--Answers for worksheet 3.5.7, head distribution when simulation includes a nonpenetrating stream and a discharging well.



E-2, Inflow from reservoir = 3.56 Mgal/d
Outflow to nonpenetrating stream = .84 Mgal/d
Outflow to fully penetrating stream = .68 Mgal/d

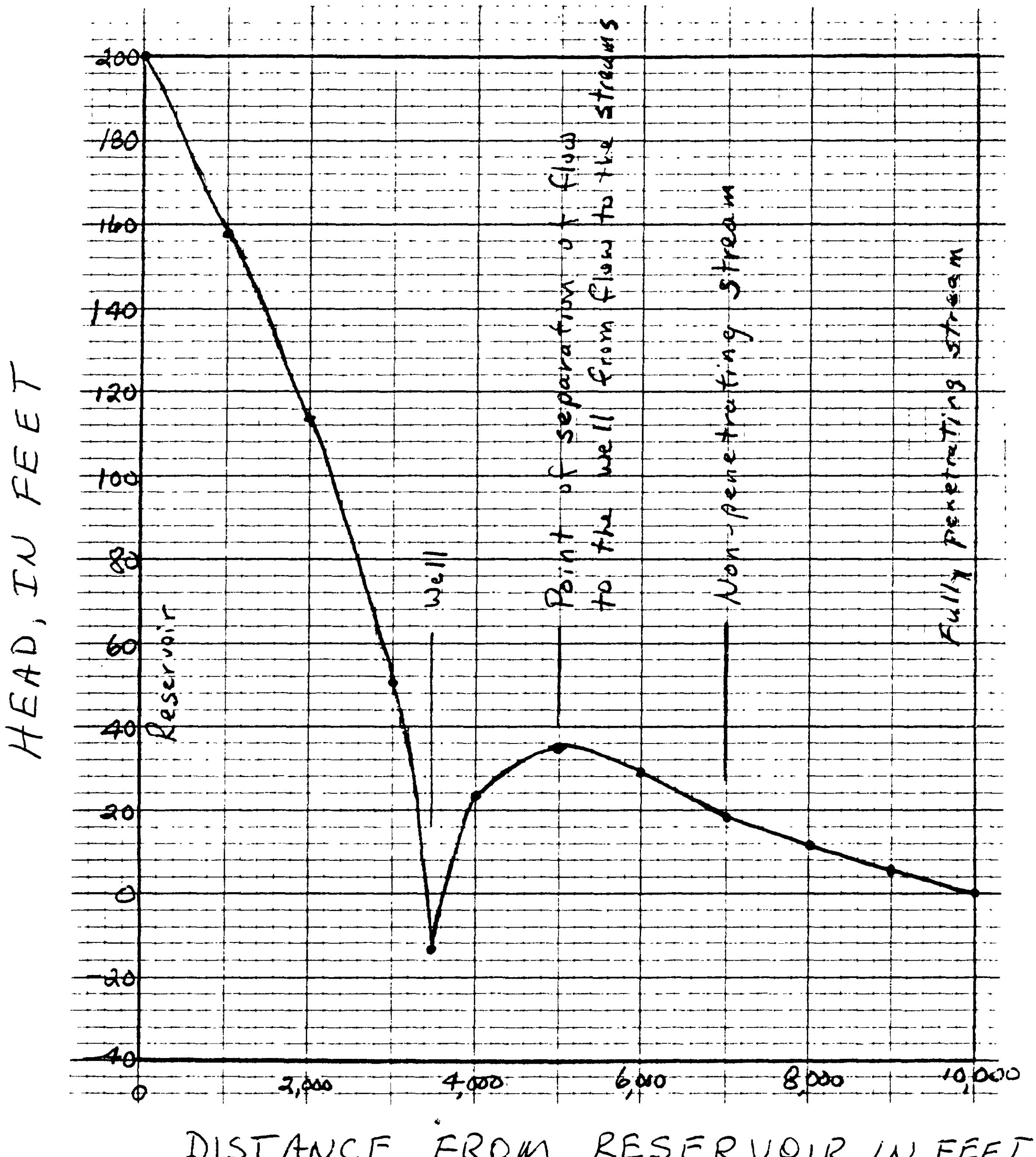


Figure 4.5.4--Answers to question E-3, east-west head profile through pumping well when simulation includes a nonpenetrating stream.

E-4. Yes, there is an effect on the opposite side of the nonpenetrating stream.

E-5. Increase in flow from reservoir, 3.56 - 2.36 = 1.20 Mgal/d

Decrease in flow to nonpenetrating stream, 1.28 - 0.84 = 0.44 Mgal/d

Decrease in flow to fully penetrating stream, 1.08 - 0.68 = 0.40 Mgal/d

Percent from reservoir $= \frac{1.20}{2.0}(100) = 60\%$

Percent from nonpenetrating stream $= \frac{0.44}{2.0}(100) = 22\%$

Percent from fully penetrating stream $= \frac{0.40}{2.0}(100) = 20\%$

2% error

E-6. Only reduced inflow; we know this because potentials in the aquifer are above zero everywhere beneath the nonpenetrating stream and everywhere adjacent to the penetrating stream; thus flow everywhere is from the aquifer into these streams.

E-7. The gaging results alone would not indicate this.

E-8 and E-9. If we were concerned about the effects of seepage from the stream on the quality of water in the aquifer--e.g., if polluted stream water might be infiltrating,--we would have to know whether or not seepage from the stream was occurring. On the other hand, if only hydraulic effects--such as drawdowns, or the reduction in streamflow downstream from the aquifer--were of interest we would not have to distinguish between reduced outflow to the stream and direct inflow from it.

E-10. In a field situation, if the head beneath the stream fell below the top of the aquifer, an unsaturated zone would develop at the top of the aquifer. Presumably, the confining material would remain saturated. The head at the base of the confining material--i.e., just at the top of the aquifer--would equal the elevation since pressure in an unsaturated zone is essentially zero. Further decrease in head in the aquifer cannot change this. Thus once an unsaturated zone develops at the top of the aquifer, the head difference across the confining zone would remain constant, i.e.:

The difference between the elevation of base of the confining unit, and stream head is 100 ft and the discharge per unit area of streambed associated with this head difference would also remain constant, i.e.:

$$\frac{Q}{A} = k' \frac{h_s - h_e}{b'} = 0.98 \left(\frac{100\text{ft}}{10\text{ft}}\right) = 9.8 \frac{\text{gal/d}}{\text{ft}^2}$$

Because w = 40 ft, this is equivalent to about 390 gal/d per foot of channel, assuming vertical flow, beneath the stream only.

F.--Confined Aquifer with Nonpenetrating Stream and Discharging Well in Superposition

F-1. Flow from reservoir	1.15 Mgal/d
Flow from nonpenetrating stream	0.45 Mgal/d
Flow from fully penetrating stream	0.40 Mgal/d
Total inflow	2.0 Mgal/d

F-2. Reservoir percent $\frac{1.15}{2}(100) = 57.5\%$

Nonpenetrating stream percent $\frac{0.45}{2}(100) = 22.5\%$

Fully penetrating stream percent $\frac{0.40}{2}(100) = 20\%$

Percentages are equal to those obtained in E, within limit of experimental error.

F-3. See answers to worksheet 3.5.8 on next page.

F-4. No; this determination cannot be made on the basis of F alone. It can be made by adding, algebraically, the heads measured in D to those measured in F (superposition) and noting the net direction of the head differences along the streams.

Worksheet 4.5.8--Answers for worksheet 3.5.8, drawdown in response to a discharging well and zero potentials at the reservoir, nonpenetrating stream and stream boundaries.

J

I \ J	1	3	5	7	9	11	13	15	17	19	21
1	0	-0.07	-0.14	-0.18	-0.19	-0.18	-0.14	-0.08	-0.06	-0.03	0
3	0	-0.08	-0.15	-0.19	-0.21	-0.19	-0.15	-0.09	-0.06	-0.04	0
5	0	-0.10	-0.18	-0.25	-0.26	-0.23	-0.17	-0.11	-0.07	-0.04	0
7	0	-0.13	-0.25	-0.34	-0.36	-0.30	-0.21	-0.12	-0.08	-0.04	0
9	0	-0.16	-0.33	-0.51	-0.53	-0.38	-0.25	-0.14	-0.09	-0.04	0
11	0	-0.17	-0.39	-0.78	-0.81	-0.44	-0.27	-0.15	-0.09	-0.05	0
				-1.29	-0.58	-0.35					
13	0	-0.16	-0.33	-0.51	-0.53	-0.38	-0.25	-0.14	-0.09	-0.04	
15	0	-0.13	-0.25	-0.34	-0.36	-0.30	-0.21	-0.12	-0.08	-0.04	
17	0	-0.10	-0.18	-0.25	-0.26	-0.23	-0.17	-0.11	-0.07	-0.04	
19	0	-0.08	-0.15	-0.19	-0.21	-0.19	-0.15	-0.09	-0.06	-0.04	0
21	0	-0.07	-0.14	-0.18	-0.19	-0.18	-0.14	-0.08	-0.06	-0.03	0

Reservoir (left boundary, J = 1); NON-PENETRATING STREAM (between J = 15 and J = 17, rows 1–11); Fully Penetrating Stream (right boundary, J = 21)

Multiply voltages by 100 to obtain drawdown in feet

G.--Confined Aquifer as in Part F with Discharging Well 500 Feet Closer to Nonpenetrating Stream

G-1.	Flow from reservoir	1.03 Mgal/d
	Flow from nonpenetrating stream	.52 Mgal/d
	Flow from fully penetrating stream	0.45 Mgal/d

G-2. Drawdown at well node is -131 ft. This actually represents only drawdown or head change due to the pumping because we are using superposition.

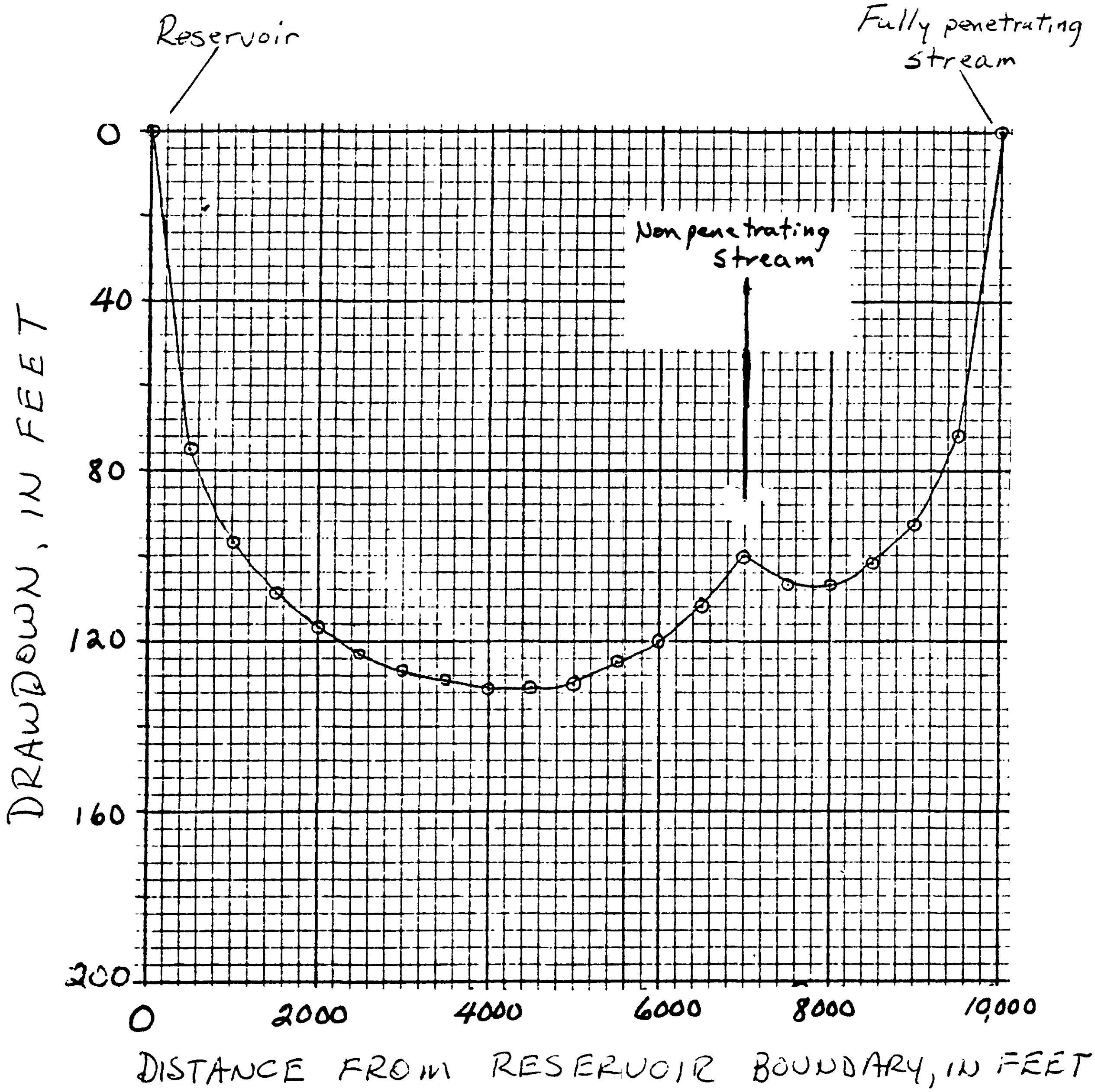
G-3. See answers to table 3.5.1 on next page and profiles in figures 4.5.5, 4.5.6, 4.5.7, 4.5.8.

G-4. A discharging well at the reservoir or fully penetrating stream boundary gives the minimum drawdown. The greatest flow from the nonpenetrating stream occurs when the discharging well is located at node (11,15). The least amount of flow from the nonpenetrating stream occurs when the discharging well is located at node (11,2) near the reservoir.

Table 4.5.1--Answers for table 3.5.1, flow from reservoir, nonpenetrating stream, and fully penetrating stream boundaries and drawdown at the well node as the discharging well is moved along an east-west profile through the original well position
[ft = feet; Mgal/d = million of gallons per day]

Nodal location of pumping well	(11,2)	(11,3)	(11,4)	(11,5)	(11,6)	(11,7)	(11,8)	(11,9)	(11,10)	(11,11)
Reservoir (Mgal/d)	-1.86	-1.74	-1.62	-1.50	-1.39	-1.27	-1.15	-1.03	-0.91	-0.79
Nonpenetrating stream (Mgal/d)	-0.06	-0.13	-0.19	-0.26	-0.32	-0.39	-0.45	-0.52	-0.58	-0.64
Fully penetrating stream (Mgal/d)	-0.08	-0.13	-0.19	-0.24	-0.29	-0.34	-0.40	-0.45	-0.51	-0.57
Drawdown X 10^2 (ft)	-0.74	-0.97	-1.09	-1.17	-1.23	-1.27	-1.29	-1.31	-1.31	-1.30

Nodal location of pumping well	(11,12)	(11,13)	(11,14)	(11,15)	(11,16)	(11,17)	(11,18)	(11,19)	(11,20)
Reservoir (Mgal/d)	-0.67	-0.55	-0.44	-0.33	-0.27	-0.22	-0.16	-0.11	-0.05
Nonpenetrating stream (Mgal/d)	-0.71	-0.77	-0.83	-0.89	-0.75	-0.60	-0.45	-0.30	-0.15
Fully penetrating str. (Mgal/d)	-0.62	-0.68	-0.73	-0.78	-0.98	-1.18	-1.39	-1.59	-1.80
Drawdown X 10^2 (ft)	-1.25	-1.20	-1.12	-1.00	-1.07	-1.07	-1.02	-0.93	-0.72



Minimum drawdown is at nodes 2 and 20 (excluding boundaries).

Figure 4.5.5--Answer to question G-3(a), drawdown in discharging well as the well is moved along an east-west profile through the original well location.

G-3(b)

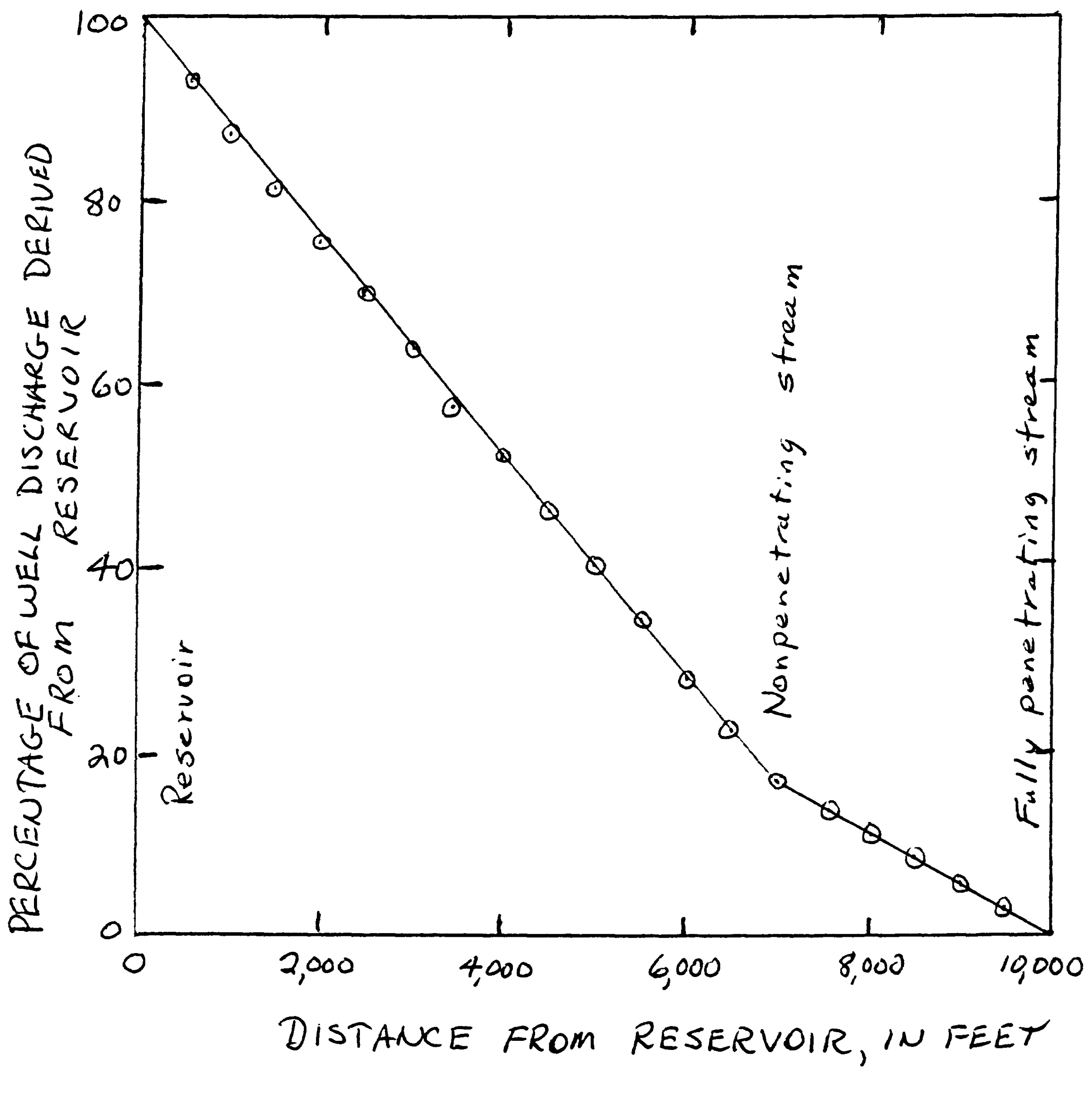


Figure 4.5.6--Answer to question G-3(b), percentage of the well discharge derived from the reservoir as the well location is moved along an east-west profile through the originial well location.

G-3(c)

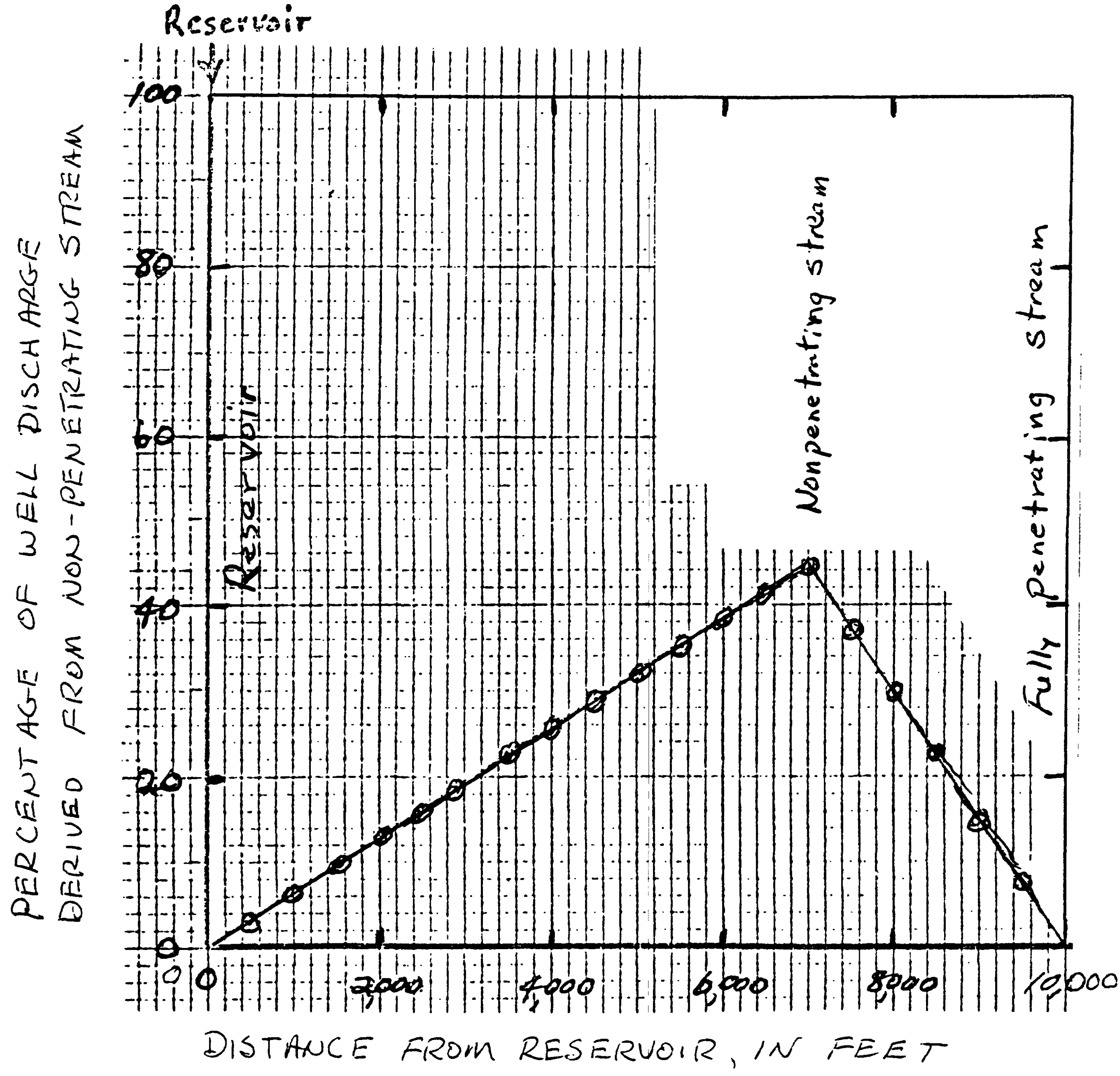


Figure 4.5.7--Answer to question G-3(c), percentage of the well discharge derived from the nonpenetrating stream as the well location is moved along an east-west profile through the original well location.

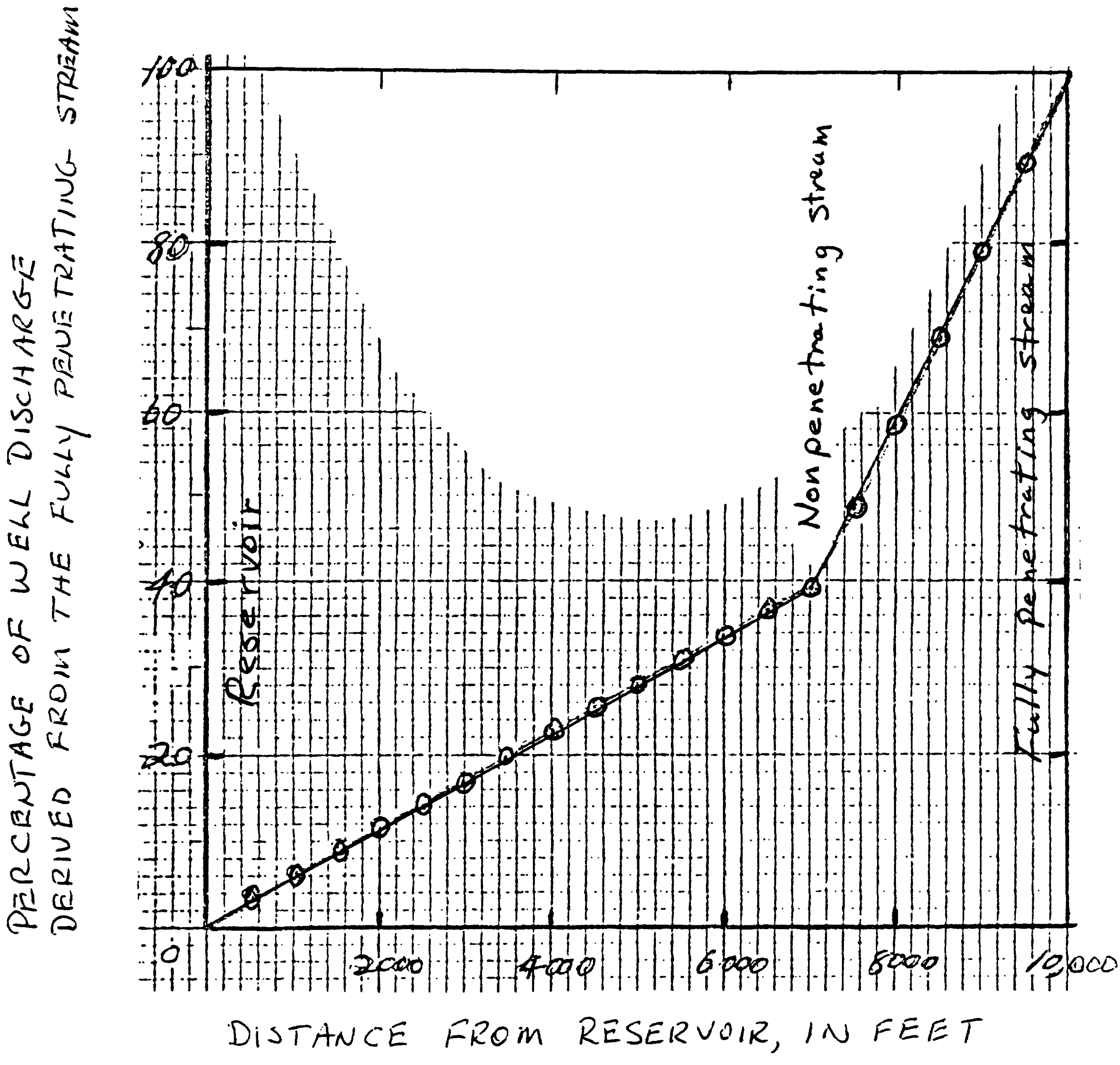


Figure 4.5.8--Answer to question G3(d), percentage of the well discharge derived from the fully penetrating stream as well location is moved along an east-west profile through the original well location.

H.--Confined Aquifer as in Part G with Fully Penetrating Stream in Place of Nonpenetrating Stream

H-1. Flow from reservoir	99 Mgal/d
Flow from the new (interior) fully penetrating stream	<u>96 Mgal/d</u>
Total inflow	195 Mgal/d
	error → 5 Mgal/d

Flow through the original fully penetrating stream boundary should theoretically be zero; if any inflow is measured at this boundary, it is probably caused by a less-than-perfect connection of the new fully penetrating stream to the network.

H-2. Approximately 50 percent.

H-3. See answers to worksheet 3.5.9.

H-4. The simulation would have value in that it would indicate the maximum possible contribution of the stream to the well's discharge; that is, it would give us an upper limit for this contribution.

I.--Additional Discussion on Simulation of Streams

I-1. (1) The head value given by our network voltage represents an average over the vertical thickness of the aquifer, or, approximately, the value which we would expect to measure at or near the vertical midpoint (mid-depth) of the aquifer. If the partially penetrating stream extends into the aquifer for, say, 10 percent of the aquifer thickness, we could use a vertical resistor to represent the 40 percent of the aquifer thickness between the stream bottom and the aquifer midpoint. The permeability which we would use in calculating the required resistance value would then be the vertical permeability of the aquifer material itself.

(2) A low permeability stream layer would constitute a conductance in series with the vertical segment of the aquifer itself described in (1). To account for such a layer, we would calculate the equivalent conductance of the two elements in series, and choose a vertical resistance corresponding to this equivalent conductance.

Worksheet 4.5.9--Answers for worksheet 3.5.9, drawdown distribution in response to a discharging well with an interior fully penetrating stream and the reservoir at zero potential.

J

I \ J	1	3	5	7	9	11	13	15	17	19	21
1	0	-0.06	-0.11	-0.14	-0.14	-0.11	-0.06	0	0	0	0
3	0	-0.06	-0.12	-0.15	-0.15	-0.12	-0.06	0	0	0	0
5	0	-0.08	-0.15	-0.19	-0.19	-0.15	-0.08	0	0	0	0
7	0	-0.11	-0.21	-0.29	-0.29	-0.21	-0.11	0	0	0	0
9	0	-0.14	-0.29	-0.46	-0.46	-0.29	-0.14	0	0	0	0
11	0	-0.16	-0.35	-0.73	-0.73	-0.35	-0.16	0	0	0	0
				-1.23	-0.50	-0.25					
13	0	-0.14	-0.29	-0.46	-0.46	-0.29	-0.14	0	0	0	0
15	0	-0.11	-0.21	-0.29	-0.29	-0.21	-0.11	0	0	0	0
17	0	-0.08	-0.15	-0.19	-0.19	-0.15	-0.08	0	0	0	0
19	0	-0.06	-0.12	-0.15	-0.15	-0.12	-0.06	0	0	0	0
21	0	-0.06	-0.11	-0.14	-0.14	-0.11	-0.06	0	0	0	0

I

Reservoir (J = 1)

FULLY PENETRATING STREAM (J = 15)

Stream (J = 21)

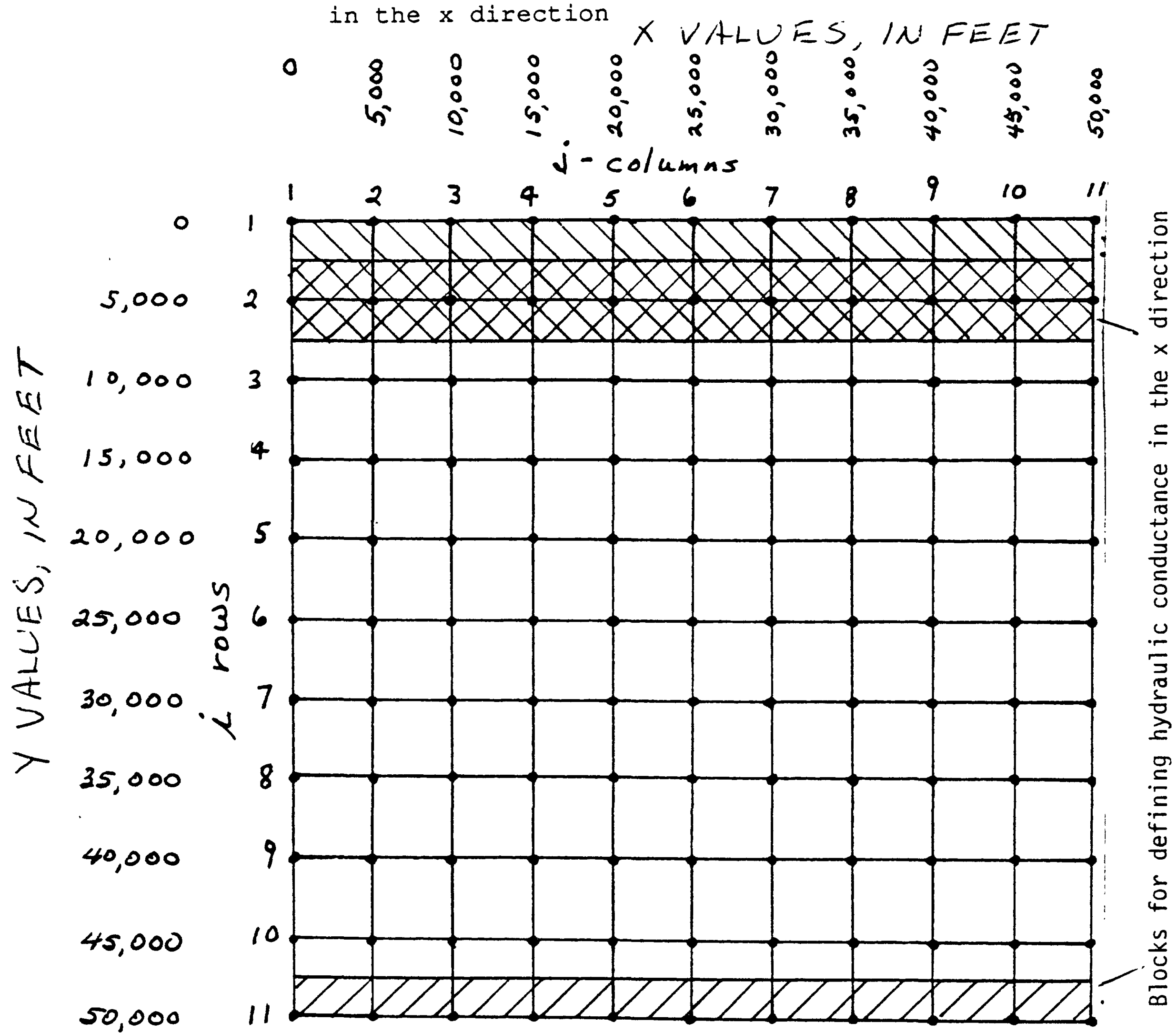
ANSWERS TO PROBLEM 6, DIGITAL STREAM-AQUIFER PROBLEM

Simulation of a Fully Penetrating Stream

System Under Natural Conditions

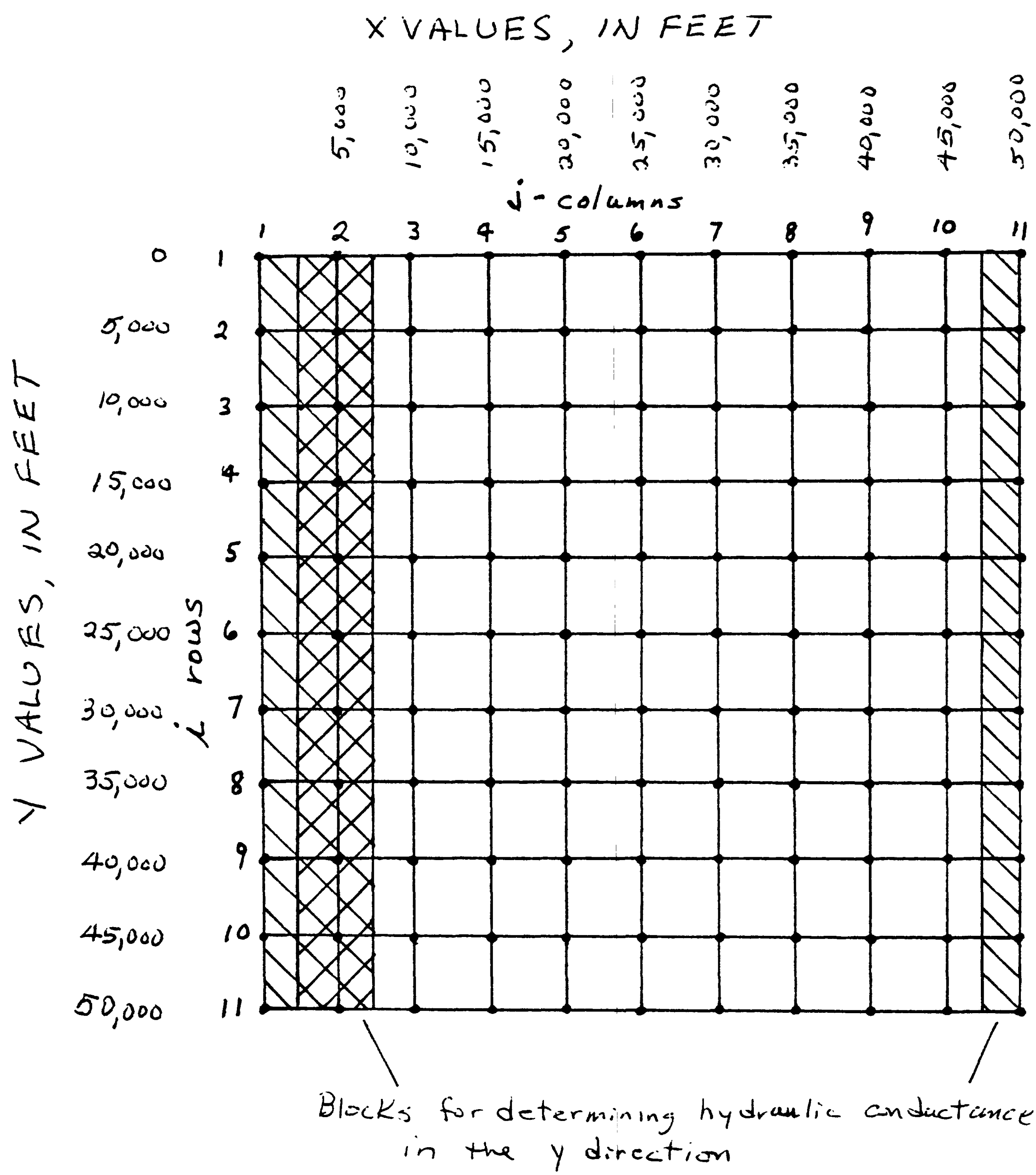
1.

Worksheet 4.6.1--Answer for worksheet 3.6.1, x values, y values, and blocks for defining hydraulic conductance in the x direction



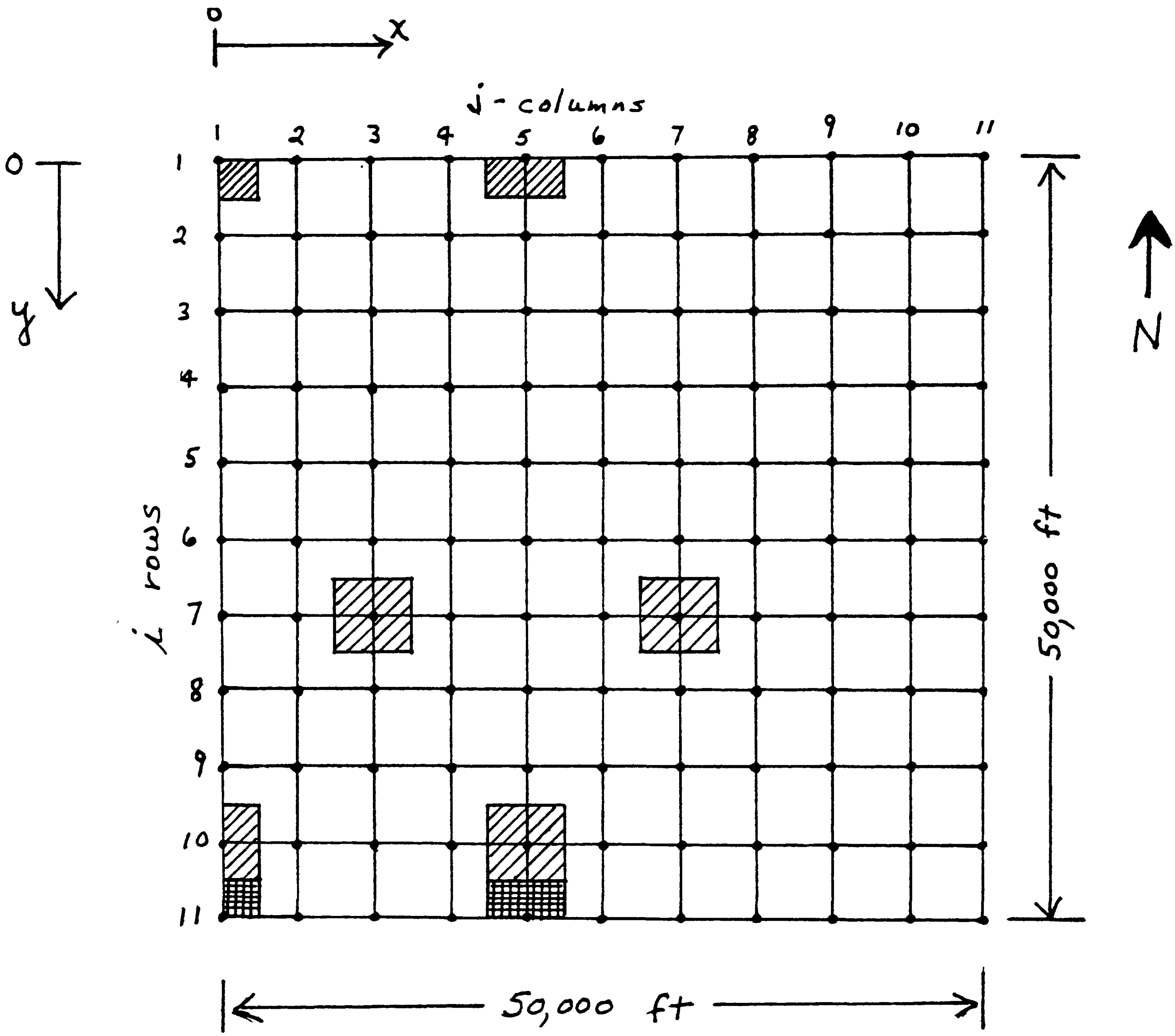
1. (cont.)

Worksheet 4.6.2--Answers for worksheet 3.6.2, blocks for defining hydraulic conductance in the y direction.



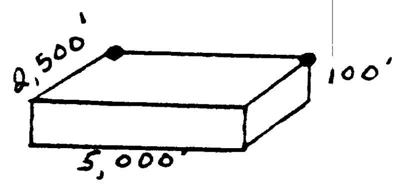
1. (cont.)

Worksheet 4.6.3--Answers for worksheet 3.6.3, blocks for defining storage capacity and for calculating recharge.



1. (cont.) <u>Hydraulic conductance between nodes in x direction</u>

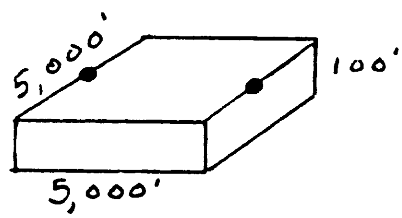
Rows 1 and 11:



$$C_x = \frac{K_x A}{L}$$

$$C_x = \frac{(100\ (gal/d)/ft^2)\ (100\ ft)\ (2,500\ ft)}{5,000\ ft} = 5,000 (gal/d)/ft$$

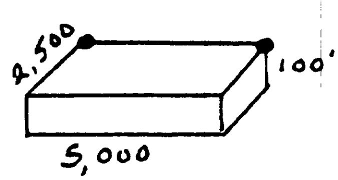
Row 2 (and all other interior rows):



$$C_x = \frac{(100\ (gal/d)/ft^2)\ (100\ ft)\ (5,000\ ft)}{5,000} = 10,000\ (gal/d)/ft$$

<u>Hydraulic conductance between nodes in y direction</u>

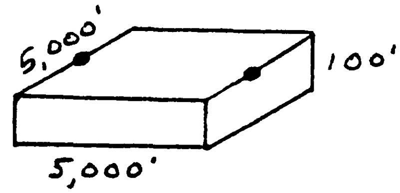
Columns 1 and 11:



$$C_y = \frac{K_y A}{L}$$

$$C_y = \frac{(100\ (gal/d)/ft^2)\ (100\ ft)\ (2,500\ ft)}{5,000\ ft} = 5,000 (gal/d)/ft$$

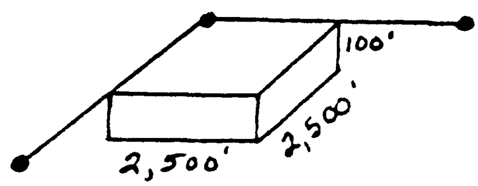
Column 2 (and all other interior columns):



$$C_y = \frac{(100\ (gal/d)/ft^2)\ (100\ ft)\ (5{,}000\ ft)}{5{,}000\ ft} = 10{,}000 (gal/d)/ft$$

<u>Storage capacity (S_c) and recharge</u>

Corner nodes (1,1 and 11,1) =



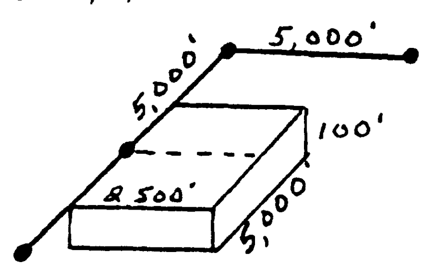
$S_c = S_s bA$

$$S_c = (7.48 (gal/d)/ft^3) (2.67 \times 10^{-7}) \frac{ft^3/ft^3}{ft} (100ft)$$

$$(2{,}500ft)(2{,}500ft) = 1{,}250 gal/ft$$

$$R = (0.0002\ (gal/d)/ft^2)\ (2{,}500\ ft)\ (2{,}500\ ft) = 1{,}250 gal/d$$

Side nodes (10,1; 1,5; and 11,5):

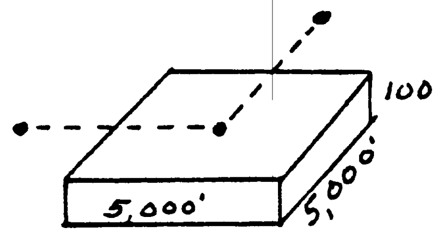


$$S_c = (7.48\ gal/ft^3)\ (2.67 \times 10^{-7})\ \frac{ft^3/ft^3}{ft}\ (100\ ft)$$

$$(2{,}500\ ft)\ (5{,}000\ ft) = 2{,}500\ gal/ft$$

$$R = 0.0002\ (gal/d)/ft)\ (2{,}500\ ft)\ (5{,}000\ ft) = 2{,}500\ gal/d$$

Interior nodes (7,3; 7,7; and 10,5):



$$S_c = (7.48\ \text{gal/ft}^3)\ (2.67 \times 10^{-7})\frac{\text{ft}^3/\text{ft}^3}{\text{ft}}(100\ \text{ft})$$

$$(5{,}000\ \text{ft})\ (5{,}000\ \text{ft}) = 5{,}000\ \text{gals/ft}$$

$$R = (0.0002\ (\text{gal/d})/\text{ft}^2)\ (5{,}000\ \text{ft})\ (5{,}000\ \text{ft}) = 5{,}000\ \text{gal/d}$$

2. $$-C_x(h_{i,j,n} - h_{i,j-1,n}) + C_x(h_{i,j+1,n} - h_{i,j,n})$$

$$-C_y(h_{i,j,n} - h_{i-1,j,n}) + C_y(h_{i+1,j,n} - h_{i,j,n})$$

$$-Q_{i,j,n} = S_c\left[\frac{h_{i,j,n} - h_{i,j,n-1}}{t_n - t_{n-1}}\right]$$

3. $$S_c = \frac{\Delta v}{\Delta h}$$

$$\Delta h = \frac{\Delta v}{S_c}$$

If $S_c = 10^{40}$, Δh will be very small no matter how big ΔV becomes within practical limits. Since Δh stays approximately zero, h must remain constant.

4. The head profile is a parabola (see graph, figure 4.6.1). The direction of flow is toward the stream. The gradient becomes steeper as we approach the stream because the discharge carried by the aquifer increases in this direction, as a result of progressive accretion to the flow through recharge.

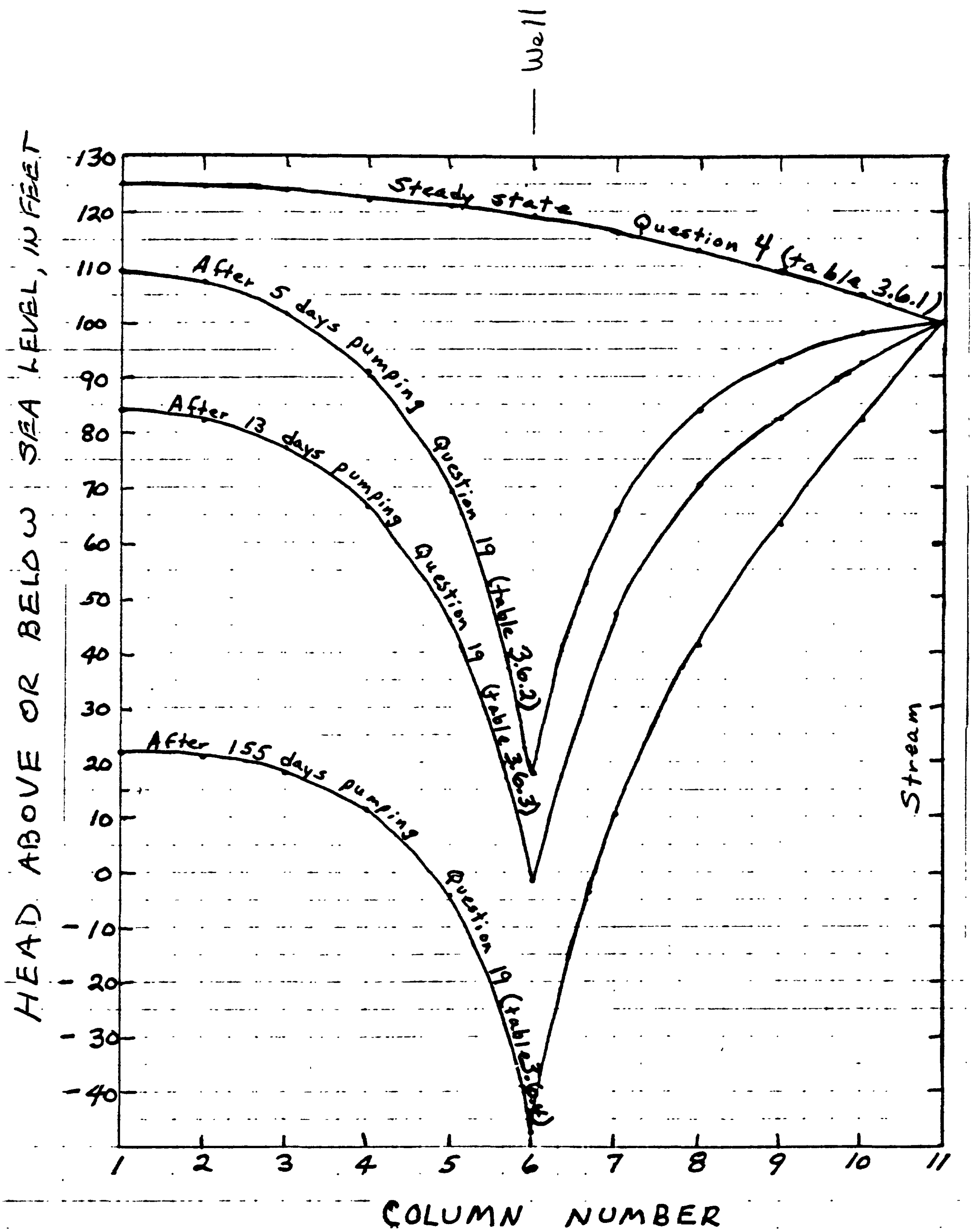
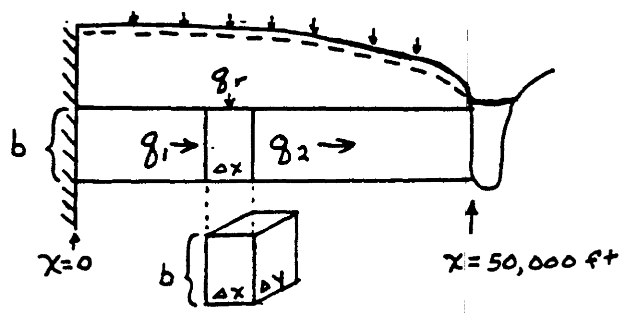


Figure 4.6.1--Answers to questions 4 and 19--Head profiles along row 6 of aquifer in figure 3.6.1 for steady-state conditions and after 5, 13, and 155 days of pumping.

The parabolic form of the head profile observed here can be predicted by developing the differential equation of flow for this problem and obtaining the solution for the boundary conditions in effect. This is done on the following pages.

We imagine a prism extending through the aquifer as shown in the figure



The height of the prism is the aquifer thickness, b; the width in the direction of flow is the distance Δx, while the other horizontal dimension, normal to the flow, is Δy.

The inflow through the left face of the prism, q_1 , is by Darcy's law the product of hydraulic conductivity, flow area and head gradient, i.e.,

$$q_1 = -Kb\Delta y \left(\frac{dh}{dx}\right)_1$$

where $\left(\frac{dh}{dx}\right)_1$ refers to the head gradient at the left face of the prism. The flow through the right face is similarly given by

$$q_2 = -Kb\Delta y \left(\frac{dh}{dx}\right)_2$$

where $\left(\frac{dh}{dx}\right)_2$ is the head gradient at the right face.

The vertical recharge over the surface of the prism, q_r, is given by

$$q_r = w\Delta x\Delta y$$

where w is the recharge rate per unit area.

Because this is an equilibrium condition, the equation of continuity states that inflow minus outflow must be zero, i.e.,

$$q_r + q_1 - q_2 = 0.$$

Substituting the expressions for q_r, q_1 , and q_2 gives

$$w\Delta x\Delta y - Kb\Delta y \left(\frac{dh}{dx}\right)_1 + Kb\Delta y \left(\frac{dh}{dx}\right)_2 = 0$$

or

$$w\Delta x\Delta y + Kb\Delta y \left\{\left(\frac{dh}{dx}\right)_2 - \left(\frac{dh}{dx}\right)_1\right\} = 0.$$

The difference in the head gradient $\left(\frac{dh}{dx}\right)_2 - \left(\frac{dh}{dx}\right)_1$ can be viewed as a product of a finite-difference approximation to the second derivative and the spacing, Δx, between the two faces as follows:

$$\left(\frac{dh}{dx}\right)_2 - \left(\frac{dh}{dx}\right)_1 = \frac{d\left(\frac{dh}{dx}\right)}{dx}\Delta x = \frac{d^2h}{dx^2}\Delta x.$$

Substitution into the previous equation gives

$$w\Delta x\Delta y + Kb\frac{d^2h}{dx^2}\Delta x\Delta y = 0.$$

Dividing by $\Delta x\Delta y$ and rearranging, we obtain as the differential equation of flow

$$\frac{d^2h}{dx^2} = -\frac{w}{Kb}.$$

Direct integration of this equation yields the general solution

$$h = -\frac{w}{2Kb}x^2 + C_1x + C_2,$$

where C_1 and C_2 are constants that have to be determined from the boundary conditions. To prove that this is, in fact, a solution, we need only differentiate it twice; that is,

$$\frac{dh}{dx} = -\frac{w}{Kb}x + C_1, \qquad \text{(a)}$$

and

$$\frac{d^2h}{dx^2} = -\frac{w}{Kb}.$$

The second differentiation takes us back to the original differential equation, indicating that our equation for head versus x is, in fact, a solution to this differential equation. If we look at equation (a) above, we can see that C_1 must be the value of the head gradient, $\frac{dh}{dx}$ at x = 0; that is, at the western boundary of the aquifer. At this point, as indicated by the head profile which you plotted, the gradient is zero. There is no flow, because there is no area of aquifer to the west over which recharge can accumulate; and zero flow, by Darcy's law, requires a zero gradient. Thus we have established that the constant C_1 of our solution must be zero, and the solution therefore takes the form

$$h = -\frac{w}{2Kb}x^2 + C_2.$$

This tells us that the constant C_2 must actually be the head at x = 0 (i.e., at the western boundary of the aquifer). If we designate this boundary head h_0, the solution can be written

$$h = h_0 - \frac{w}{2Kb}x^2$$

or

$$h_0 - h = \frac{w}{2Kb}x^2.$$

These equations indicate that a plot of h vs x must have a parabolic form as, in fact, your head profile should indicate. The term $\frac{w}{2Kb}$ is given for this problem by

$$\frac{w}{2Kb} = \frac{2 \times 10^{-4} \text{ gpd/ft}^2}{(2)\ (100 \text{ gpd/ft}^2)\ (100 \text{ ft})} = 10^{-8} \text{ per foot}$$

so that we have

$$h_0 - h = 10^{-8}x^2$$

at x = 50,000 ft, we have

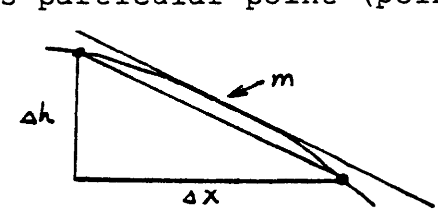
$$h_0 - h = (10^{-8})\ (50{,}000)^2 = 25 \text{ feet.}$$

This agrees with the head difference across the aquifer as given in the digital model solution (table 3.6.1). You may verify that the solution is satisfied at other points across the aquifer.

5. See completed worksheet 4.6.4

The discrepancies between calculated ground-water flows and calculated recharges are due to truncation errors and roundoff errors which affect both the heads on table 3.6.1 and our calculations of flow as discussed in the following paragraphs.

In calculating ground-water flows, we used the average head gradient, $\frac{\Delta h}{\Delta x}$ (where Δh is the head difference between successive columns and Δx is the node spacing, 5,000 ft) to approximate the hydraulic gradient at a particular point between the columns. In effect, we assumed that the average head gradient between the nodes was equal to the slope of a tangent to the head parabola at this particular point (point m) as shown in the sketch:



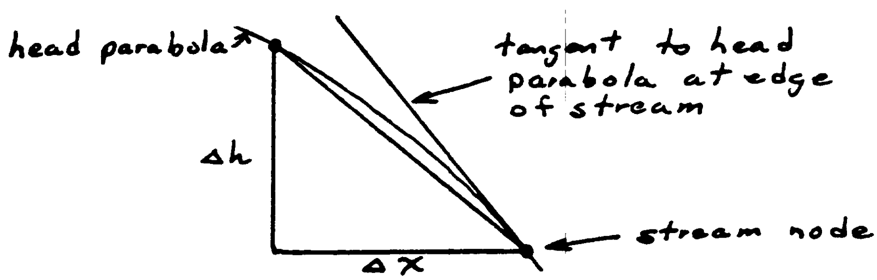
5.

Worksheet 4.6.4--Answers for worksheet 3.6.4, steady-state flow calculations, no pumpage. [distances and head differences in feet; hydraulic conductances in gallons per day per foot; flows and recharge in gallons per day]

Distance from Stream		50,000		42,500		27,500		0	
Row	Hydraulic conductance	Head difference	Flow	Head diff. between columns 2 and 3	Flow	Head diff. between columns 5 and 6	Flow	Head diff. between columns 10 and 11	Flow
1	5,000	0	0	0.7	3,500	2.3	11,500	4.8	24,000
2	10,000	0	0	0.7	7,000	2.3	23,000	4.8	48,000
3	10,000	0	0	0.7	7,000	2.3	23,000	4.8	48,000
4	10,000	0	0	0.7	7,000	2.3	23,000	4.8	48,000
5	10,000	0	0	0.7	7,000	2.3	23,000	4.8	48,000
6	10,000	0	0	0.7	7,000	2.3	23,000	4.8	48,000
7	10,000	0	0	0.7	7,000	2.3	23,000	4.8	48,000
8	10,000	0	0	0.7	7,000	2.3	23,000	4.8	48,000
9	10,000	0	0	0.7	7,000	2.3	23,000	4.8	48,000
10	10,000	0	0	0.7	7,000	2.3	23,000	4.8	48,000
11	5,000	0	0	0.7	3,500	2.3	11,500	4.8	24,000
Total flow			0		70,000		230,000		480,000
Area of aquifer to west of line of calculation			0	3.75×10^8 ft		1.125×10^9		2.5×10^9	
Recharge to aquifer to west of line of calculation			0	75,000		225,000		500,000	

For the calculations at 27,500 and 42,500 feet from the stream, we assumed that the point m, for which we were approximating the hydraulic gradient, was actually the midpoint of the interval between the two nodes. For the calculation at 27,500 feet from the stream, this was apparently not a bad assumption because our calculation of total ground-water flow past this point was relatively close to the total recharge applied to the aquifer to the west of this point. For the calculation at 42,500 feet, the error was similar in magnitude but represented a much larger fraction of the total ground-water flow at that point.

For the calculation of total ground-water flow into the stream, we assumed that the average gradient $\frac{\Delta h}{\Delta x}$ represented the slope of a tangent to the head parabola at the edge of the stream, that is, at one edge of the interval Δx as shown in the diagram below,



and we compared our ground-water flow calculation to the total recharge to the aquifer including that applied in Column 11. Obviously, the approximation here was not as good. Had we taken the point of calculation as the midpoint of the interval Δx and compared the flow just to the recharge west of that point (i.e., excluding the recharge on Column 11), our calculated flow would have come much closer to matching the recharge figure.

The errors discussed above fall into the category of truncation error. These are inherent in the representation of a derivative by a finite-difference approximation. The heads shown on table 3.6.1 were themselves calculated using a finite-difference approximation to the differential equation of flow; thus they contain truncation errors themselves. That is, they differ from the heads which would be given by the analytical solution to the differential equation at the same points. This further contributes to the differences between our calculated ground-water flows and calculated recharges.

In addition to these truncation errors, our results are affected by roundoff errors. These are errors which arise because we cannot handle the finite-difference calculations exactly--rather, we must carry our arithmetic to some number of significant figures, and then give up. Like the truncation errors, roundoff errors are present both in the calculated heads on table 3.6.1 and in our own calculations of ground-water flow. The heads on table 3.6.1, in particular, contain significant roundoff errors because they were printed out only to the nearest tenth of a foot (although calculated to greater accuracy in the program). Particularly for the heads at 40,000 and 45,000 feet from the stream (used in the flow calculation for 42,500 feet from the stream) the error involved in roundoff to the nearest tenth was a significant fraction of the total head difference involved in the calculation.

6. This is a steady-state flow system in which the total inflow from vertical recharge is equal to the total outflow to the stream. Recharge causes a uniform accretion to the ground-water system along the path of flow leading to a progressive steepening of the hydraulic gradient along the path of flow.

Response of System to Pumping

7. No; the time-step length should progressively increase. With a uniform step length, we would be calculating very small head changes as time increased, and thus wasting a lot of computer time.

8. The system had reached equilibrium; inflow and outflow were equal.

9. The steady-state head distribution of table 3.6.1 is taken as the initial head distribution.

10. At the end of the fourth time step, approximately 5.37 days of pumping have elapsed.

11. The well should be located at node 6,6 (that is, at the node for which i = 6 and j = 6).

12. 2,000,000 gpd withdrawal minus 5,000 gal/d recharge = 1,995,000 gal/d net withdrawal.

13. The withdrawal rate should remain the same through successive time steps.

14. See table on worksheet 4.6.5, and graph on figure 4.6.2.

15-17. After 5 days of pumping--The direct flow from the stream to the aquifer takes place through rows 4, 5, 6, 7, and 8. The total direct flow from the stream to the aquifer through these rows, as calculated in the answer to question 14, is 80,000 gal/d.

The original flow toward the stream (from question 5, worksheet 4.6.4) was calculated as: 48,000 gal/d through each of the rows from 2 through 10; 24,000 gpd through row 1; and 24,000 gpd through row 11. We consider first rows 1, 2, 3, 9, 10, and 11. For these rows, the direction of flow was still from the aquifer to the stream after 5 days pumping. The diversion of the original flow, in each row, is simply the difference between the prepumping (steady-state) flow to the stream (from question 5) and the flow after 5 days (from question 14). The results are tabulated below:

Row	Original flow to stream (gpd) (from question 5)		Flow to stream after 5 days pumping (gpd) (from question 14)		Diversion of original flow (gal/d)
1	24,000	-	5,500	=	18,500
2	48,000	-	9,000	=	39,000
3	48,000	-	2,000	=	46,000
9	48,000	-	2,000	=	46,000
10	48,000	-	9,000	=	39,000
11	24,000	-	5,500	=	18,500
			Total		207,000

Now consider rows 4, 5, 6, 7, and 8 in which the pumpage has produced reversal of the flow after 5 days. Before pumping, 48,000 gal/d was entering the stream through each of these rows. In each row, this seepage toward the stream must be reduced to zero before seepage from the stream can begin. Thus for these five rows, the diversion of original flow is 5 x 48,000, or 240,000 gal/d. The total diversion of original flow after 5 days pumping, from all 11 nodes, is thus 207,000 + 240,000, or 447,000 gal/d.

The total contribution of the stream to the well after 5 days pumping is thus 527,000 gal/d--80,000 gal/d in direct flow plus 447,000 gal/d in diversion of original flow toward the stream. The balance, or 1,473,000 gal/d, is the rate of withdrawal from ground-water storage after 5 days of pumping.

Worksheet 4.6.5--Answers for worksheet 3.6.5, flow between stream and aquifer, and change in flow from steady-state conditions after 5, 13, and 155 days of pumping [head differences in feet; hydraulic conductances in gallons per day per foot; flows in gallons per day]

		5 days pumping (table 3.6.2)		13 days pumping (table 3.6.3)		155 days pumping (table 3.6.4)	
Row	Hydraulic conductance C_x	Head difference between columns 10 and 11	Flow	Head difference between columns 10 and 11	Flow	Head difference between columns 10 and 11	Flow
1	5,000	1.1	5,500	-3.6	-18,000	-13.4	-67,000
2	10,000	0.9	9,000	-3.8	-38,000	-13.7	-137,000
3	10,000	0.2	2,000	-4.6	-46,000	-14.5	-145,000
4	10,000	-0.9	-9,000	-5.8	-58,000	-15.7	-157,000
5	10,000	-1.9	-19,000	-6.9	-69,000	-16.8	-168,000
6	10,000	-2.4	-24,000	-7.4	-74,000	-17.3	-173,000
7	10,000	-1.9	-19,000	-6.9	-69,000	-16.8	-168,000
8	10,000	-0.9	-9,000	-5.8	-58,000	-15.7	-157,000
9	10,000	0.2	2,000	-4.6	-46,000	-14.5	-145,000
10	10,000	0.9	9,000	-3.8	-38,000	-13.7	-137,000
11	5,000	1.1	5,500	-3.6	-18,000	-13.4	-67,000
Total net flow between stream and aquifer			-47,000		-532,000		-1,521,000
Change in flow from steady-state conditions			527,000		1,012,000		2,001,000

Negative sign indicates flow from stream into aquifer

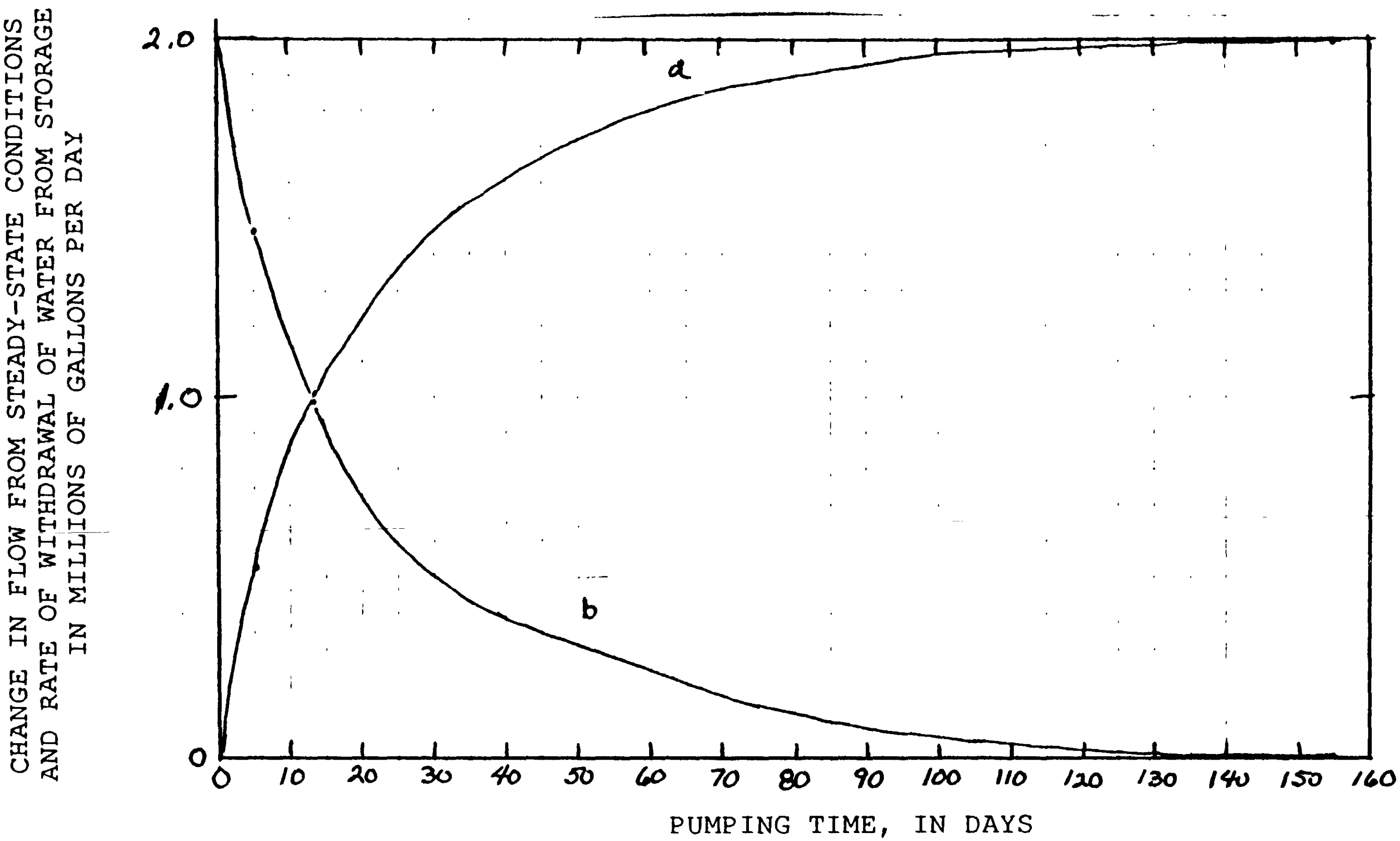


Figure 4.6.2--Answers to questions 14 and 16.
14--Change in flow between the stream and aquifer from steady-state conditions versus time of pumping (curve a), and
16--Rate of withdrawal of water from storage versus time of pumping (curve b).

After 13 days of pumping--Direct flow from the stream to the aquifer occurs through all rows and (from question 14) totals 532,000 gal/d. Again, before any seepage from the stream to the aquifer can occur in a given row, the original seepage toward the stream through that row must be totally diverted. Thus all of the original ground-water flow to the stream, which we calculated in question 5 as 480,000 gal/d, has been diverted to the well after 13 days pumping. The total contribution of the stream to the well is therefore 532,000 + 480,000, or 1,012,000 gal/d, after 13 days pumping. The rate of withdrawal from storage after this interval of pumping is 2,000,000 - 1,012,000 = 988,000 gal/d.

After 155 days of pumping--The direct flow from the stream to the aquifer (from the calculation of question 14) is 1,521,000 gal/d. Again the entire original flow to the stream, which was calculated in question 5 as 480,000 gal/d, has been diverted to the well. The total contribution of the stream to the well is, therefore, approximately 2 million gallons per day. This is equal to the pumpage and indicates that no water is being supplied by withdrawal from storage after 155 days of pumping. This, in turn, means that the system has reached a new equilibrium.

In terms of the principles outlined by Theis (1940), water pumped from a well must be accounted for by withdrawal of water from storage, decrease in natural outflow, or increase in recharge. In this problem, the decrease in original flow to the stream represents the decrease in natural outflow while the direct seepage from the stream to the aquifer represents an increase in recharge. Note that at the new equilibrium there are two types of recharge in the system--areal infiltration of precipitation, which remains constant throughout the pumping period, and direct flow from the stream, which increases until the new equilibrium is attained. In the final equilibrium, all of the infiltrating precipitation actually flows to the well; however, this infiltration of precipitation was never considered explicitly in our calculation because its contribution to the pumpage is fully accounted for as the decrease in natural discharge to the stream.

If we were looking at the system in a broader context--that is, if we were considering the entire stream basin, and all ground-water bodies in connection with the stream, rather than the isolated aquifer of our problem--we might view the direct flow from the stream to our local aquifer as a reduction in natural outflow from the entire stream basin rather than as an increase in recharge. For example, let's suppose the flow of the stream, upstream from our aquifer, to be 4 Mgal/d; and let's suppose that all of this flow is derived from the discharge of other aquifers to the stream. The original flow from the aquifer of our problem to the stream is about 0.5 Mgal/d so that, prior to pumping, the total streamflow at the down stream end of our problem area is 4.5 Mgal/d. The effect of the pumping is to reduce the net ground-water flow into the stream by 2 Mgal/d. In this case, considering the stream basin as a whole, we might consider the entire 2 Mgal/d reduction in natural ground-water discharge to the stream.

(Upstream from the aquifer of our problem, the ground-water flow to the stream would still be 4 Mgal/d while in our aquifer it is minus 1.5 Mgal/d. Thus the net flow from the ground-water system to the stream is 2.5 Mgal/d, which represents a decrease of 2 Mgal/d from the original value.

This illustrates the point that when considering the total ground-water contribution to stream flow, reduction in discharge to a stream and increase in seepage from a stream are basically the same thing. Each represents a change in the flow between the ground-water system and the stream; the only real difference is in direction, and thus sign.

18. No matter how the system is viewed, the reduction in streamflow at point D at the new equilibrium is equal to the pumpage, 2 Mgal/d.

19. (See graph of plots of head versus distance, question 4.)

20. From the note "Well drawdown correction at a pumping node," the additional drawdown, s, in a real well at node 6,6 is:

$$S = \frac{2.3Q}{2\pi T} \log \frac{r_a}{r_w}$$

$$r_a = \frac{a}{4.81} = \frac{5,000}{4.81} = 1,039$$

$$s = \frac{2.3Q}{2\pi T} \log \frac{1,039}{0.5} = \frac{(2.3)\ (2 \times 10^6)}{6.28 \times 10^4} \quad (3.32)$$

$$= 243 \text{ ft}$$

From table 3.6.1 and table 3.6.4, total drawdown from digital model = 118.8 - (-47.5) = 166.3 ft (at node 6,6). Estimated total drawdown in well = 166.3 + 243 = 409.3 ft. (Obviously, this drawdown could not be attained in the field for the system we have described without violating some of our assumptions.)

21. At any given time prior to equilibrium, both drawdown and change in flow to the stream would be reduced. Drawdowns and reductions in streamflow at equilibrium would be the same as with the lower storage coefficient; however, the time required to reach the new equilibrium will be longer if the storage coefficient is increased. Furthermore, when the new equilibrium is attained, the total volume of water obtained from storage with the larger storage coefficient is greater than with the lower storage coefficient. The ratios of the two volumes from storage must equal the ratios of the storage coefficients because the volume of the cone of depression is the same in both cases.

Simulation of a Fully Penetrating Stream by Means of Superposition

22. Zero everywhere.

23. Zero everywhere.

24. Two mgd.

25. (See figure 4.6.3)

26. The flow from the stream to the aquifer after 155 days of pumping is 2 Mgal/d which is equal to the algebraic change in flow to the stream calculated for the same time of pumping in question 14.

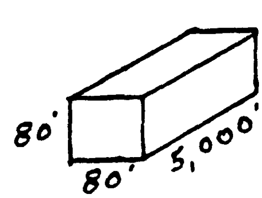
27. If only hydraulic changes are required, we can calculate head differences and flow changes caused by the pumping without knowing the original condition. However, if the resultant water levels and flow values in the aquifer are required (as in a problem involving contaminant movement from a stream to a well), this method of calculation would not be sufficient.

28. Yes, by algebraic addition of drawdowns to the steady-state head distribution prior to pumping and algebraic addition of flow changes to the original flow values--that is, by superposition. The additional information required is the original distribution of head and flow.

Simulation of a Non-penetrating Stream

System Under Natural Conditions

29.



$$C_z = \frac{KA}{L} = \frac{(10)\ (80)\ (5{,}000)}{80} = 50{,}000\ \text{(gal/d)/ft}$$ (25,000 (gal/d)ft for each of the two end blocks--rows 1 and 11)

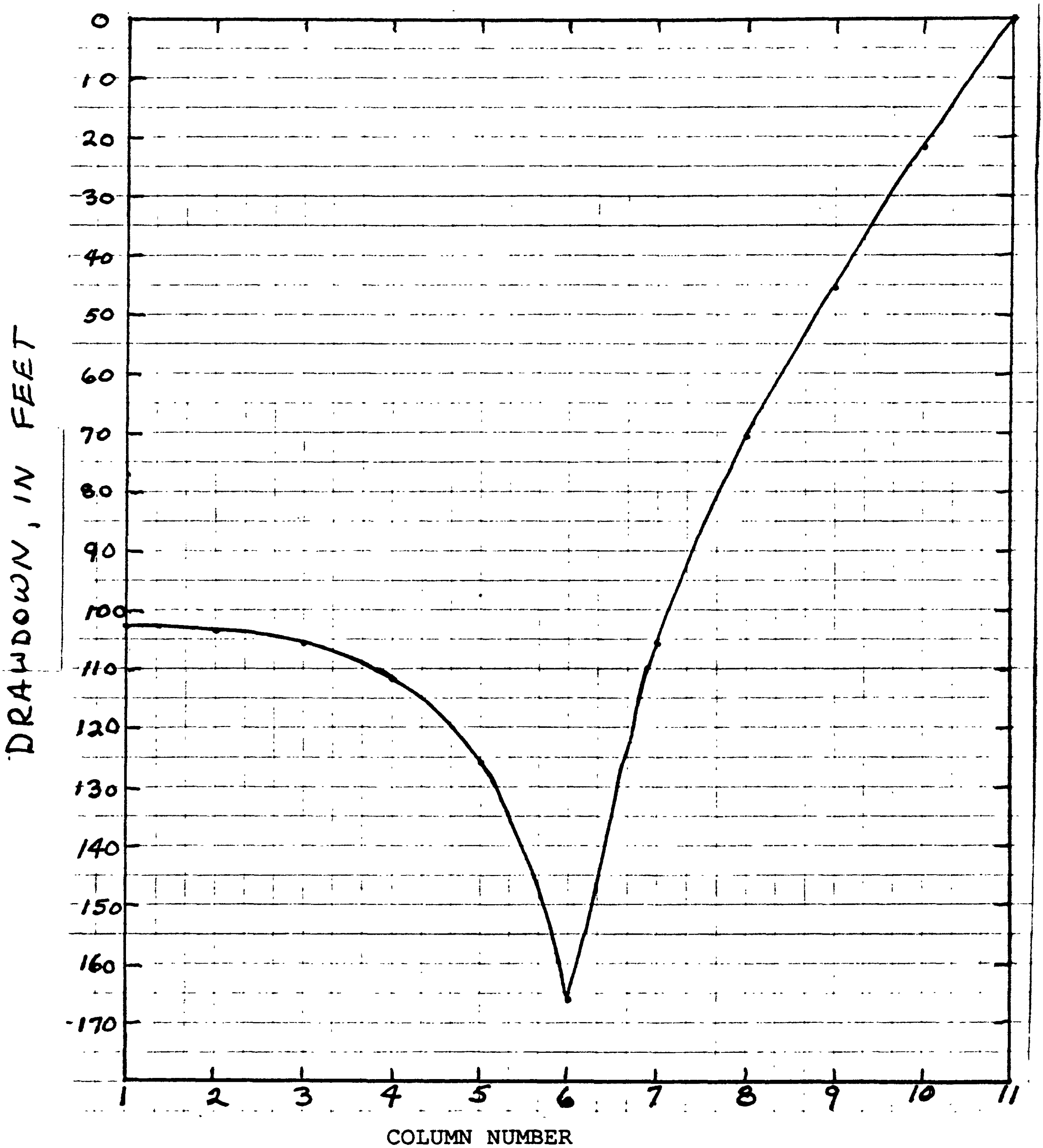


Figure 4.6.3--Answer to question 25--Head profile along row 6 showing the effects of only the pumping well after 155 days of pumping.

30. $QZ_{i,j} = C_z\ (110 - h_{i,j})$ --For $h_{i,j}$ above top of aquifer

$QZ_{i,j} = C_z\ (100)$ --For $h_{i,j}$ below top of aquifer that is, for $h_{i,j} < 10.0$ feet,

31. $QZ_{i,j} = C_z\ (110.0 - h_{i,j})$ --If $h_{i,j} > 10$

$QZ_{i,j} = C_z\ (100)$ --If $h_{i,j} < 10$

32. (See figure 4.6.4)

33. The statements given in (31) would have to be implemented at each node along column 5 (that is, at each node along the stream) in this case, because we know that the stream head is always 100 feet and the elevation of the top of the aquifer is always 10 feet, the only new input information that we would need would be the value of C_z, 50,000 (gal/d)/ft. In an actual problem, we would have to specify the stream elevation, the aquifer top elevation, and the value of C_z at each node along the stream.

The finite-difference equation for nodes along the stream would take the form

$$-C_x(h_{i,j,n} - h_{i,j-1,n}) + C_x(h_{i,j+1,n} - h_{i,j,n})$$
$$-C_y(h_{i,j,n} - h_{i-1,j,n}) + C_y(h_{i+1,j,n} - h_{i,j,n})$$
$$-Q_{i,j,n} + QZ_{i,j,n} = S_c \frac{h_{i,j,n} - h_{i,j,n-1}}{t_n - t_{n-1}}$$

where

$$QZ_{i,j,n} = C_z(110 - h_{i,j,n}), \text{ when } h_{i,j,n} \geq 10$$

$$QZ_{i,j,n} = C_z\ (100), \text{ when } h_{i,j,n} < 10.$$

34. (See figure 4.6.5)

35. Both the differences and the similarities between the head profile of this problem and that for the fully penetrating stream problem can be explained through the analytical solution to the flow equation which we obtained previously.

Following the answer to question 4, a development of the equation for unidirectional flow with uniform recharge was given. In the final form of this equation

$$h_o - h = \frac{w}{2Kb}\ x^2$$

h_0 is the head at x = 0, which is the beginning of the flow path or the point where the cumulative ground-water flow is zero; w is the recharge rate per unit area; K is hydraulic conductivity; and b is aquifer thickness. We can apply this equation to the western part (columns 1-5) of the nonpenetrating stream problem taking x = 0 along the western edge of the aquifer and letting x increase to

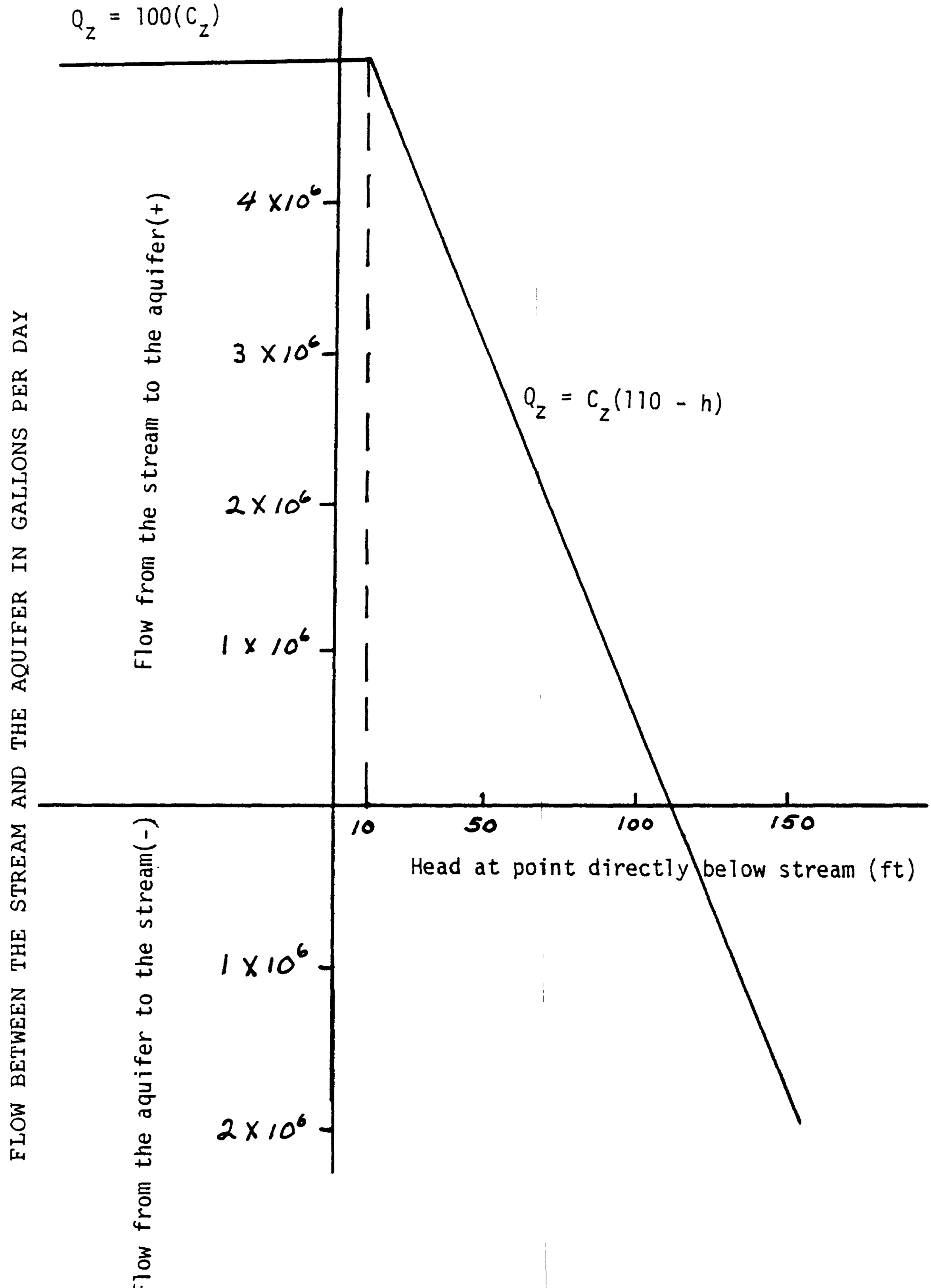


Figure 4.6.4--Answer to question 32, flow (QZ) between the stream and aquifer.

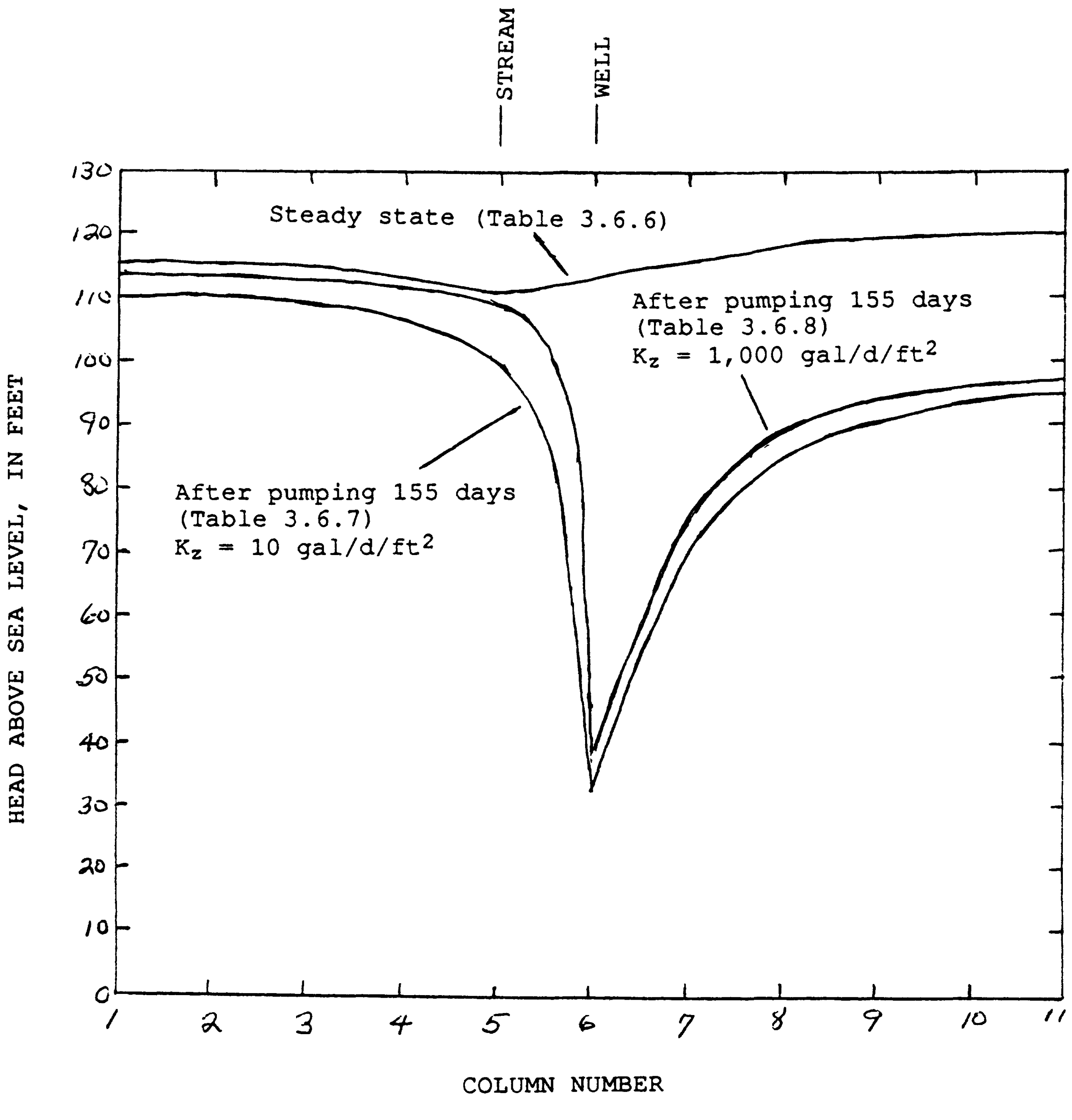


Figure 4.6.5--Answers to questions 34, 40, and 45--Head profiles along row 6 for the aquifer shown in figure 3.6.2 for steady state and after after pumping 155 days with vertical hydraulic conductivity of material between the aquifer and the stream equal to 10 gpd/ft^2 and 1,000 gpd/ft^2.

the east; and we can apply it to the eastern part (columns 6-11) taking x = 0 along the eastern edge of the aquifer and letting x increase to the west. The heads in column 5 on table 3.6.6 represent those in the aquifer directly beneath the river. If we calculate the head difference, $h_0 - h_1$, between each edge of the aquifer and a point directly beneath the river we obtain, for the western part,

$$h_0 - h = \frac{w}{2Kb}x^2 = \frac{(2 \times 10^{-4})}{2(100)(100)} (2 \times 10^4)^2$$

$$= 10^{-8} (4 \times 10^8) = 4 \text{ ft}$$

and for the eastern part

$$h_0 - h = \frac{2 \times 10^{-4}}{(2)(100)(100)} (3 \times 10^4)^2$$

$$= 10^{-8} (9 \times 10^8) = 9 \text{ ft.}$$

These results agree closely with the head differences between the nodes along the stream (column 5) and the boundaries on the west (column 1) and east (column 11) (table 3.6.6). In the nonpenetrating stream problem, the elevation of water in the stream controls the water level in the aquifer immediately below the stream, in the sense that the head in the aquifer must exceed that in the stream by exactly the amount needed to move the cumulative ground-water flow (i.e., the total recharge) through the vertical conductance separating the stream from the aquifer. In our case, this vertical head difference is exactly 1 foot; the stream-surface elevation is 110 feet and the elevation of water level in the aquifer, immediately below, is 111 feet. Thus, 111 feet is the boundary head for this problem--the head at x = 0. Head differences, as described by the equation above, are added to 111 feet to give absolute values of head.

For the nonpenetrating stream problem, each segment of the head profile is a parabola, as required by our solution; thus each is similar to the initial portion of the profile for a fully penetrating stream (although directions are reversed for the eastern half of the nonpenetrating stream profile). However, the actual magnitudes of head buildup differ because our boundary head, 111 feet, differs from that in the fully penetrating stream problem and because distances from the boundary, x, are now different.

36. The total flow into the stream is 500,000 gal/d which is equal to the total recharge on the aquifer. Note that our calculation of flow into the stream is made across the vertical conductance between the stream and the aquifer, and is thus external to our two-dimensional flow simulation. For this reason, the error that we encountered in (4), relating to the point within the aquifer at which our calculation of flow applied, does not affect the total flow calculation here.

Response of System to Pumping

37-38. The well should be at node (6,6); the withdrawal rate is 2,000,000 gal/d - 5,000 gal/d = 1,995,000 gal/d.

39. The initial head distribution is the steady-state distribution on table 3.6.6.

40. An effect does occur on the opposite side of the stream; this could not happen if the stream were fully penetrating and its level unaffected by the pumping.

41.

Row	$\Delta h (C_Z)$		Flow
1	0.6 (25,000)	=	15,000 gal/d, stream to aquifer
2	0.8 (50,000)	=	40,000 gal/d, stream to aquifer
3	1.3 (50,000)	=	65,000 gal/d, stream to aquifer
4	2.4 (50,000)	=	120,000 gal/d, stream to aquifer
5	5.1 (50,000)	=	255,000 gal/d, stream to aquifer
6	10.2 (50,000)	=	510,000 gal/d, stream to aquifer
7	5.1 (50,000)	=	255,000 gal/d, stream to aquifer
8	2.4 (50,000)	=	120,000 gal/d, stream to aquifer
9	1.3 (50,000)	=	65,000 gal/d, stream to aquifer
10	0.8 (50,000)	=	40,000 gal/d, stream to aquifer
11	0.6 (25,000)	=	15,000 gal/d, stream to aquifer
			1,500,000 gal/d, stream to aquifer

42. The total net flow is 1.5 Mgal/d from the stream to the aquifer. Prior to pumping, the flow was 0.5 Mgal/d from the aquifer to the stream. The total change in flow between the stream and the aquifer is thus 2 Mgal/d.

43. At point D, the change in streamflow from its value before pumping will be 2 Mgal/d.

44. Two Mgal/d represents the sum of a decrease in natural discharge (0.5 Mgal/d) and an increase in recharge (1.5 Mgal/d). Because it equals the pumpage, there is nothing being taken from storage. Thus water levels are no longer declining, and a new equilibrium has been reached.

Effect of Stream-Bed Conductance

45. (5×10^6) gpd/ft.

46. There is a slight effect; heads directly beneath the stream (column 5) have been lowered by the pumping. This has changed the "boundary condition" for the flow field to the west of the stream so that heads throughout the area west of the stream are lower. However, there is no clearly defined "cone of depression" west of the stream. In fact, heads in this area exhibit a parabolic decline with increasing x, as in the prepumping condition.

47. As an approximation, we could treat the stream as fully penetrating. The results would show no drawdown at all west of the stream (or along the stream) but might be a close enough approximation for some purposes. In general, the higher the permeability of the stream-aquifer connection, the more acceptable the fully penetrating stream assumption becomes; and in any case, it can always be used to provide an upper limit for the stream-

aquifer interaction--that is, to indicate the maximum possible flow between stream and aquifer.

Simulation of a Non-penetrating Stream by Means of Superposition

48. Zero head everywhere.

49. Two Mgal/d.

50. Zero everywhere.

51. (See figure 4.6.6.) Total seepage from stream = 2 mgd (same as change in seepage from (42)).

52. Superposition--add the head changes and flow changes algebraically to the corresponding original values of head and flow.

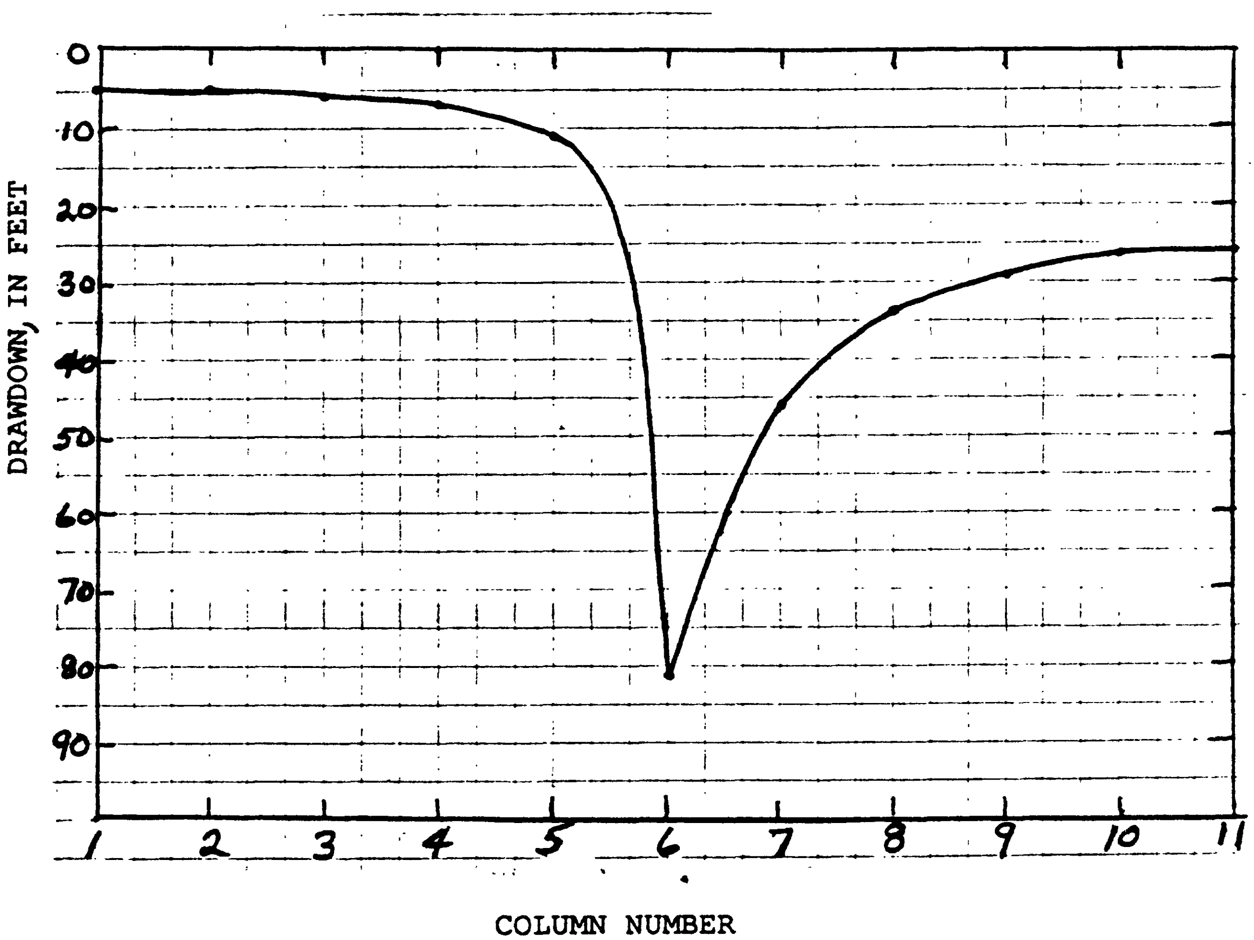


Figure 4.6.6--Answer to question 51--head profile along row 6 of aquifer in figure 3.6.2 showing the effects of only the pumping well after 155 days of pumping (table 3.6.9)