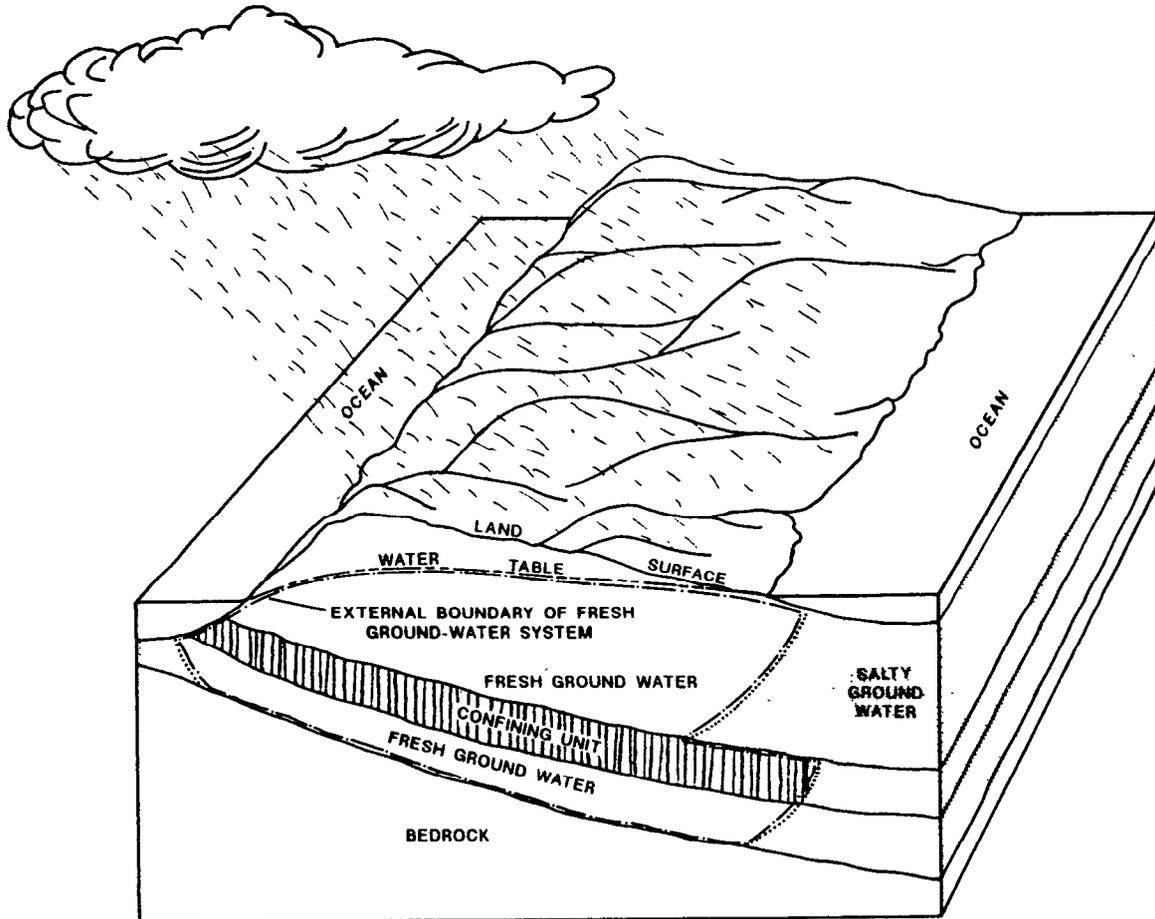


STUDY GUIDE FOR A BEGINNING COURSE IN GROUND-WATER HYDROLOGY: PART I -- COURSE PARTICIPANTS



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SECTION (2)--PRINCIPLES OF GROUND-WATER FLOW AND STORAGE

The keystone of this section and the entire course is Darcy's law, which provides the basis for quantitative analysis of ground-water flow. The final outcome of this section, after establishing the necessary supporting relationships, is a simplified development of the ground-water flow equation.

Darcy's Law

Assignments

*Study Fetter (1988), p. 75-85, 123-131; Freeze and Cherry (1979), Darcy's law--p. 15-18, 34-35, 72-73; physical content of permeability--p. 26-30; Darcy velocity and average linear velocity--p. 69-71; or Todd (1980), p. 64-74.

*Work Exercise (2-1)--Darcy's law.

*Define the following terms, using the glossary in Fetter (1988), an unabridged dictionary, or other available sources--steady state, unsteady state, transient, equilibrium, nonequilibrium.

*Study Note (2-1)--Dimensionality of a ground-water flow field.

The importance of Darcy's law to ground-water hydrology cannot be overstated; it provides the basis for quantitative analysis of ground-water flow. Several important points related to Darcy's law that are covered in Fetter (1988) are emphasized below.

(1) The physical content of hydraulic conductivity. The reason for the statement by some writers that hydraulic conductivity is a coefficient of proportionality in Darcy's experiment is demonstrated in the first part of Exercise (2-1). Theory and experiment indicate that the coefficient of hydraulic conductivity represents the combined properties of both the flowing fluid (ground water) and the porous medium. The physical content of hydraulic conductivity is developed in connection with equations (4-8) and (4-9) in Fetter (1988). The term "intrinsic permeability" designates the parameter that describes only the properties of the porous medium, irrespective of the flowing fluid. Explicit use of fluid properties and intrinsic permeability instead of hydraulic conductivity is required in analyzing density-dependent flows (for example, movement of water with variable density in fresh ground water-salty ground water problems) or flows that involve more than one phase or more than one fluid, as occurs in the unsaturated zone, in petroleum reservoirs, and in many situations that involve contaminated ground water.

(2) The Darcy velocity (or specific discharge) and the average linear velocity. The Darcy velocity (equation (5-24) in Fetter, 1988) is an apparent average velocity that is derived directly from Darcy's law. The average linear velocity (equation (5-25), the Darcy velocity divided by the porosity (n), is a better approximation of the actual average velocity of flow in the openings within the solid earth material. In most practical problems, particularly those involving movement of contaminants, the average linear velocity is applicable.

(3) Dimensionality of flow fields. Flow patterns in real ground-water systems are inherently three dimensional. Often, hydrologists analyze ground-water flow patterns in two or even one dimension. The purpose of Note (2-1) is to introduce the concept of flow system dimensionality. The hydrologist must differentiate between the actual ground-water flow patterns that occur in a real ground-water system and what is assumed about these flow patterns as an approximation in order to simplify their quantitative analysis.

Exercise (2-1)--Darcy's Law

The purpose of this exercise is to develop an increased familiarity with Darcy's experiment and to practice using Darcy's law in some typical problems.

A sketch of a laboratory seepage system is shown in figure 2-1. The "seepage system" may be thought of as a steady flow of water through the square prism of fine sand. The system input is the volume of water flowing through any cross-section of the sand prism per unit of time (Q). The system response is the hydraulic gradient in the sand prism, defined as the difference in head (Δh) between the two piezometers divided by the distance between them (l). The water input may be changed by adjusting the control valve. This water input, which equals the discharge from the system, is measured at the downstream end of the experimental apparatus.

The results of a series of hypothetical experiments on this flow system are listed in table 2-1. For each experiment, Q is changed by adjusting the control valve, and both Q and Δh are measured. The most convenient way to consider these data is to make a graphical plot. We will assume that we already know, based on previous experiments, that Q is proportional to A , the cross-sectional area of the sand prism. In other words, if all other experimental conditions are the same, doubling the prism cross-sectional area A will double Q . In our experiments A equals 1.21 ft^2 .

Complete the entries in table 2-1 and make a plot of $\Delta h/l$ (y-axis) against Q/A (x-axis) on the worksheet provided (fig. 2-2). After you have prepared the graph, answer the following questions:

- (1) How would you describe or characterize the relationship between the two variables $\Delta h/l$ and Q/A ?
- (2) Assuming that we are dealing with a linear relationship, write an equation for this relationship between the two variables. The first step is to recall the basic form of a linear equation in terms of x and y .
- (3) Make a graphical determination of the slope of the "experimental" curve (in this case, straight line).
- (4) Express the relationship in (2) in terms of Q ; that is, $Q = ?$. This is a form of Darcy's law that we see frequently, except that the slope of the "experimental" curve is in the denominator.

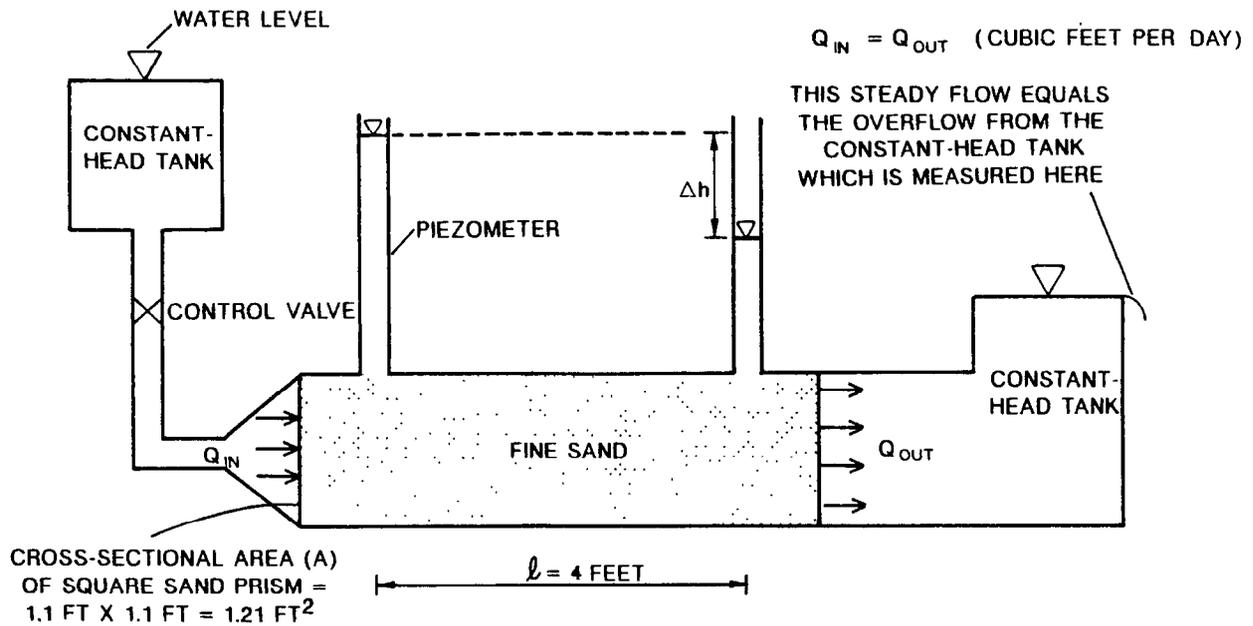


Figure 2-1.--Sketch of laboratory seepage system.

Table 2-1.--Data from hypothetical experiments with the laboratory seepage system

| Test number | Q^1 (cubic feet per day) | Δh (feet) | $\Delta h / l$ | Q/A (feet/day) |
|-------------|-------------------------------|----------------------|----------------|---------------------|
| 1 | 2.2 | 0.11 | | |
| 2 | 3.3 | .17 | | |
| 3 | 4.6 | .23 | | |
| 4 | 5.4 | .26 | | |
| 5 | 6.7 | .34 | | |
| 6 | 7.3 | .38 | | |
| 7 | 7.9 | .40 | | |

¹ Q is steady flow through sand prism, Δh is head difference between two piezometers; l is distance between two piezometers; A is constant cross-sectional area of sand prism (fig. 2-1).

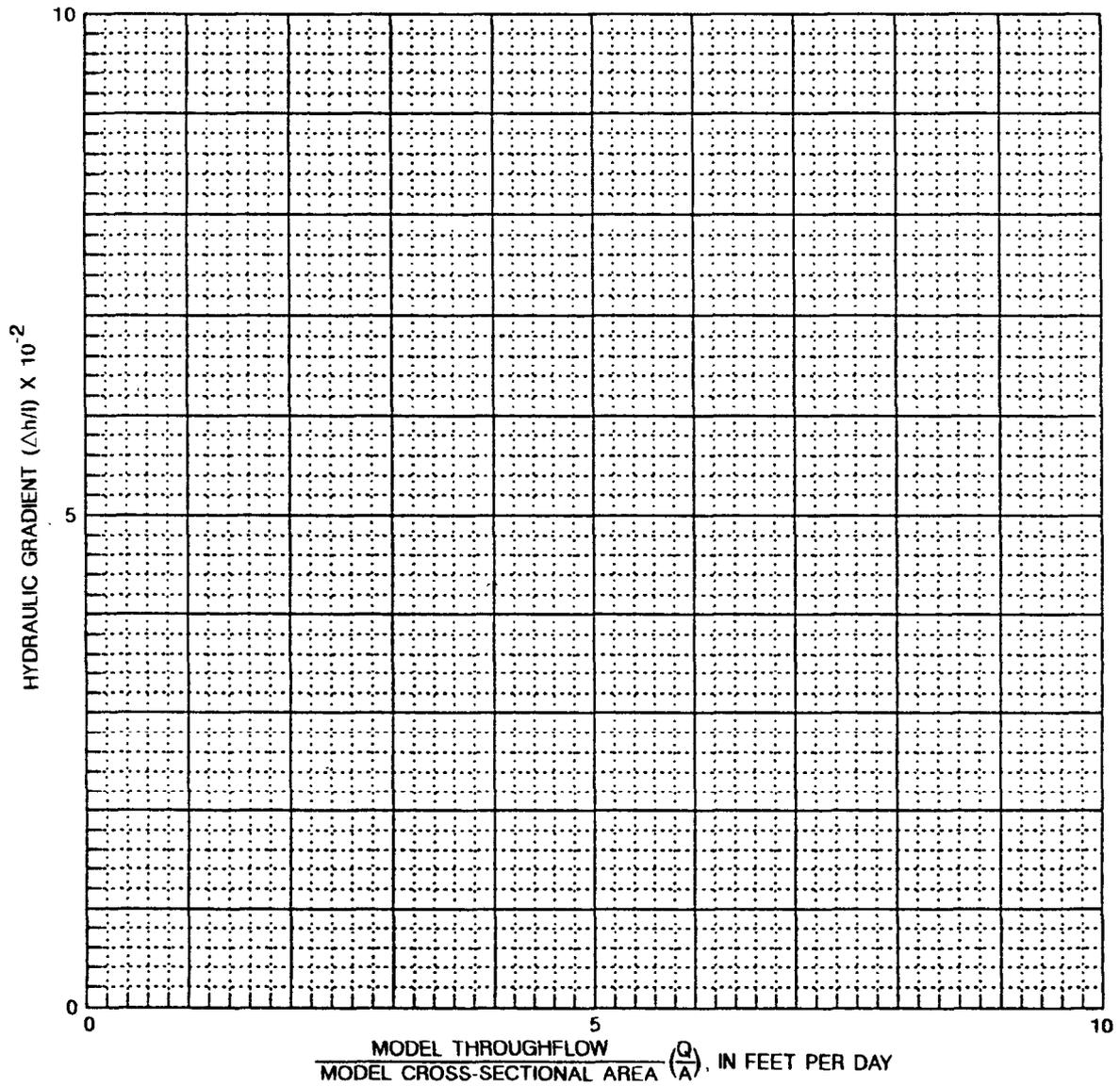


Figure 2-2.--Worksheet for plotting data from hypothetical experiments with the laboratory seepage system.

- (5) If we express Darcy's law in the form $Q = K \cdot \Delta h / l \cdot A$ or $Q = K \cdot i \cdot A$, where K is the hydraulic conductivity of the sand, what is the numerical value of K in this "experiment"?
- (6) On the sketch of the laboratory seepage system describe the conditions of head and flow at the boundaries of the sand prism (upstream end, downstream end, and walls of prism); that is, describe the BOUNDARY CONDITIONS of the sand prism. Also sketch some typical streamlines and equipotential lines within the sand prism.
- (7) Solve problems 6 and 7, p. 159 in Applied Hydrogeology by C. W. Fetter (1988). Make a sketch of the stated problem, and write the appropriate formulas before performing numerical calculations.

Note (2-1).--Dimensionality of a Ground-Water Flow Field

Ground-water velocity at a point in a ground-water flow field is a vector; that is, it possesses both magnitude and direction. In general, the magnitude and direction of the velocity vector is a function of location in the flow field.

Several examples of velocity fields, in which velocity vectors are drawn at selected points in the flow field, are shown in figure 2-3. Also shown in figure 2-3 are x and y Cartesian coordinate axes. These axes are mutually perpendicular and located in the plane of the figure. The third dimension is represented conceptually by the z coordinate axis (not shown) which is oriented perpendicular to the plane of the figure. In figure 2-3(A) all the velocity vectors are parallel to one another, equal in magnitude, and oriented parallel to the x coordinate axis. In figure 2-3(B) the same conditions apply except that the velocity vectors are not equal in magnitude.

In both figures 2-3(A) and 2-3(B) we assume that the illustrated velocity vectors, which are drawn parallel to the x coordinate axis, are replicated exactly in the y and z coordinate directions. In other words, in this special situation, if the velocity vectors are known or defined by an equation at all points in the x coordinate direction, the velocity vectors are also known at any point in the x - y plane, the x - z plane, and the y - z plane. Thus, in this special situation, the velocity distribution in the ground-water flow field is completely described if it is defined only in the x coordinate direction, or, more formally, velocity (v) = $f(x)$. Such a velocity field is termed "one-dimensional."

Examples of two-dimensional velocity fields are shown in figures 2-3(C) and 2-3(D). Again, in these examples we assume that the velocity field is replicated exactly perpendicular to the plane of the figures in the z coordinate direction. In these cases, because velocity varies from point to point in the two-dimensional x - y plane, two coordinates in the plane are required to specify the velocity field, or $v = f(x,y)$. Similarly, the concept of flow-field dimensionality is extended to three dimensions; that is, velocity varies from point to point in three-dimensional x - y - z space, or $v = f(x,y,z)$.

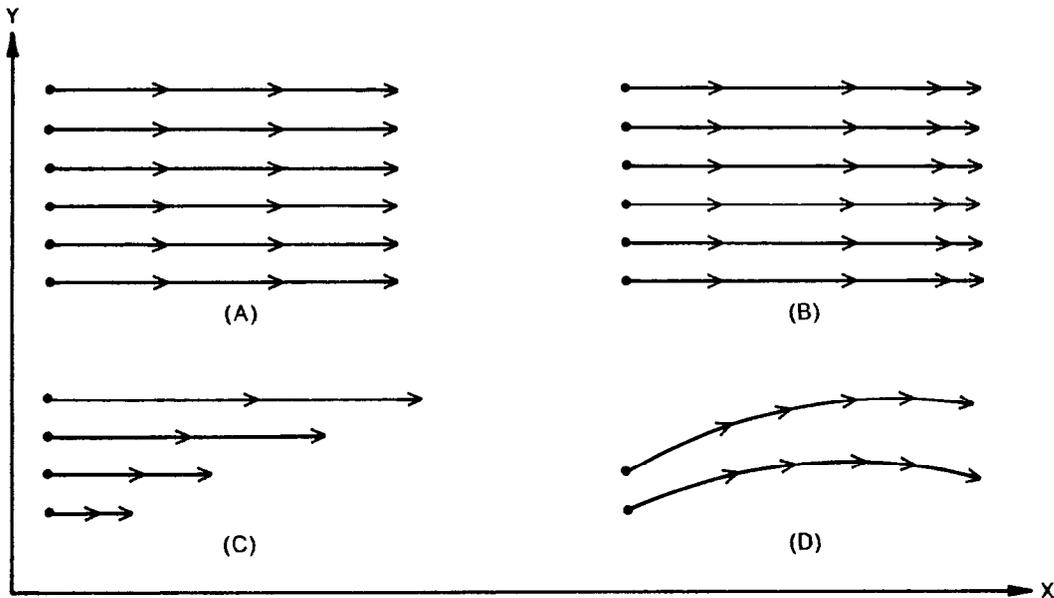


Figure 2-3.--Examples of ground-water flow fields depicted by velocity vectors at selected points: (A) and (B) are one-dimensional flow fields; (C) and (D) are two-dimensional flow fields.

In the Darcy experiment the earth material in the prism in which flow and head differences are measured is assumed to be isotropic and homogeneous. This assumption, together with measured linear head drops in the prism, result in equally spaced head contours (potential lines) and streamlines that are perpendicular to them as shown in figure 2-4. This pattern of potential lines and streamlines represents a flow field in which the velocity vectors are parallel and equal everywhere in the prism as in figure 2-3(A). Therefore, the flow in an ideal Darcy prism is an example of one-dimensional flow.

The Darcy prism in figure 2-4 is not horizontal as is the prism in figure 2-3. If the same hydraulic conditions at the boundaries of the prism are maintained during the experiment, the results of a Darcy experiment are independent of the orientation of the sand prism--that is, the prism can be horizontal, vertical, or tipped at any intermediate angle. This is one of the conclusions drawn by Fetter (1988) in his discussion of figure 5.3 on page 57.

In real ground-water systems, ground-water flow fields are always three-dimensional. However, in order to simplify problem analysis, we often assume as an approximation that the flow field is two-dimensional, or sometimes even one-dimensional. Problem solutions of acceptable accuracy sometimes can be obtained by using such simplifying assumptions regarding the flow field; however, in other situations the results obtained by employing such simplifications may be grossly in error.

Transmissivity

Assignments

*Study Fetter (1988), p. 105, 108-111; Freeze and Cherry (1979), p. 30-34, 59-62; or Todd (1980), p. 69, 78-81.

*Work Exercise (2-2)--Transmissivity and equivalent vertical hydraulic conductivity in a layered sequence.

Transmissivity is a convenient composite variable that applies only to horizontal or nearly horizontal hydrogeologic units. In order to analyze vertical ground-water flow, we must use values of hydraulic conductivity that are appropriate to the vertical direction. Exercise (2-2) provides practice in the use of formulas (4-16), (4-17), (4-22), and (4-23) in Fetter (1988).

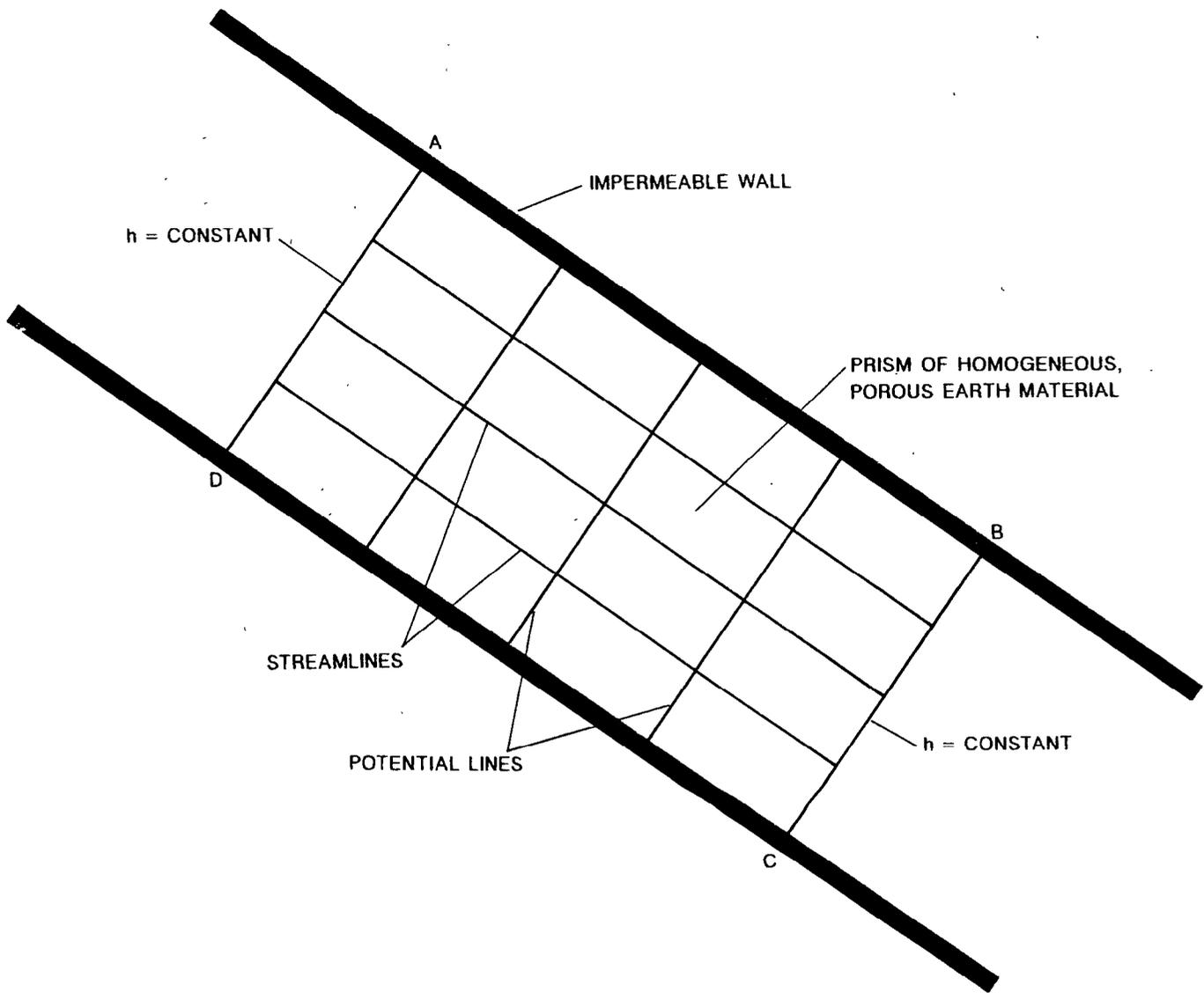


Figure 2-4.--Idealized flow pattern in a Darcy prism, ABCD, composed of homogeneous, porous earth material.

Exercise (2-2)--Transmissivity and Equivalent Vertical Hydraulic Conductivity in a Layered Sequence

The definitions of transmissivity and equivalent vertical hydraulic conductivity and the relevant formulas for their calculation can be found in any standard textbook on ground-water hydrology (see Fetter, 1988, p. 105, 110, 111).

The four horizontal beds described in the table below are assumed to be isotropic and homogeneous (not a generally realistic assumption for a field situation).

- (1) Calculate the equivalent horizontal hydraulic conductivity (K_x) and the transmissivity (T) for the four layers.
- (2) Calculate the equivalent vertical hydraulic conductivity (K_z) for the four layers.
- (3) In these calculations (a) what bed or beds exert the greatest control on the equivalent horizontal hydraulic conductivity and transmissivity, and (b) what bed or beds exert the greatest control on the equivalent vertical hydraulic conductivity?

| Bed number | Bed thickness (feet) | Bed hydraulic conductivity (K) (feet/day) |
|------------|----------------------|---|
| 1 | 25 | 10 |
| 2 | 30 | 100 |
| 3 | 20 | 0.001 |
| 4 | 50 | 50 |

Aquifers, Confining Layers, Unconfined and Confined Flow

Assignment

*Study Fetter (1988), p. 101-105; Freeze and Cherry (1979), p. 47-49; or Todd (1980), p. 25-26, 37-45.

The physical mechanisms by which ground-water storage in saturated aquifers or parts of aquifers is increased or decreased (described in the next section of the outline) are determined by the hydraulic conditions under which the ground water occurs. In nature, ground water in the saturated zone occurs in unconfined aquifers and confined aquifers. The upper bounding surface of an unconfined aquifer is a water table, which is overlain by an unsaturated zone and is subject to atmospheric pressure, whereas confined aquifers are overlain and underlain by confining beds. A confining bed has a relatively

low hydraulic conductivity compared to that of the adjacent aquifer. Ratios of hydraulic conductivity generally are at least 1,000 (aquifer) to 1 (confining bed), and commonly are much larger. In addition, the head at the top of the confined aquifer always is higher than the bottom of the overlying confining bed. This means that the entire thickness of the confined aquifer is fully saturated.

Ground-Water Storage

Assignments

*Study Fetter (1988), p. 73-76, 105-107; Freeze and Cherry (1979), p. 51-62; or Todd (1980), p. 36-37, 45-46.

*Study Note (2-2)--Ground-water storage.

*Work Exercise (2-3)--Specific yield.

Hydraulic parameters for earth materials may be divided into (a) transmitting parameters and (b) storage parameters. We already have encountered the principal transmitting parameters, hydraulic conductivity (K) or intrinsic permeability (k), and transmissivity (T). In this section the principal storage parameters, storage coefficient (S), specific storage (S_s), and specific yield (S_y), are introduced.

The physical mechanisms involved in unconfined storage and confined storage are different. A change in storage in an unconfined aquifer involves a physical dewatering of the earth materials; that is, earth materials that previously were saturated become unsaturated. When a change in storage takes place in a confined aquifer, the earth materials in the confined aquifer remain saturated.

Note (2-2).--Ground-Water Storage, by Gordon D. Bennett¹.

Originally it was thought that a porous medium acted only as a conduit--in other words, that it simply transmitted water according to Darcy's law. Approximately 50 years ago, hydrologists recognized that a porous medium could also act as a storage reservoir--that water could be accumulated in an aquifer, retained for a certain time, and then released.

In problems of steady-state ground-water flow, the inflow to a unit volume of aquifer always balances the outflow. When inflow equals outflow in this manner, no water is accumulating in the system, and storage need not be considered. In general, however, inflow and outflow are not in balance. The

¹ U.S. Geological Survey, Reston, Virginia.

general equation for systems in which storage is a factor is termed the equation of continuity, and may be written as follows:

$$\text{Inflow} - \text{Outflow} = \text{Rate of accumulation.} \quad (1)$$

Steady-state flow refers to the special case where the rate of accumulation is zero.

The general equation indicated above could be applied to a water tank which is being filled at a rate Q_1 , and drained simultaneously at a rate Q_2 . The rate of accumulation in the tank (fig. 2-5) is $Q_1 - Q_2$. Negative accumulation, or depletion, occurs if Q_2 exceeds Q_1 . If, for example, water is flowing in at 5 cubic feet per second, and flowing out at 6 cubic feet per second, the volume of water in the tank will diminish at a rate of 1 cubic foot per second; and if the area of the bottom of the tank is 10 square feet, the water level in the tank will fall at a rate of 0.1 feet per second. In a tank, therefore, the factor that relates the rate of change of water level to the rate of accumulation of fluid is simply the base area of the tank, A . If V is the volume of water in the tank and h is the water level, then

$$V = Ah, \quad (2)$$

and if we add a volume of water ΔV to the tank, the water level rises by an increment Δh such that

$$\Delta V = A\Delta h. \quad (3)$$

Now suppose a length of time Δt is required to add the volume of water ΔV . Dividing the above equation by this time interval Δt gives

$$\frac{\Delta V}{\Delta t} = A \frac{\Delta h}{\Delta t} \quad (4)$$

where $\frac{\Delta V}{\Delta t}$ is the rate at which water is added to the tank which may be expressed, for example, in cubic feet per second; and

$\frac{\Delta h}{\Delta t}$ is the rate at which the water level rises in the tank--expressed, for example, in feet per second.

In the customary notation of differential calculus, we would use $\frac{dV}{dt}$ in place of $\frac{\Delta V}{\Delta t}$ for the rate at which fluid is added to the tank, or taken into storage; and $\frac{dh}{dt}$ in place of $\frac{\Delta h}{\Delta t}$ for the rate at which the fluid level rises in the tank.

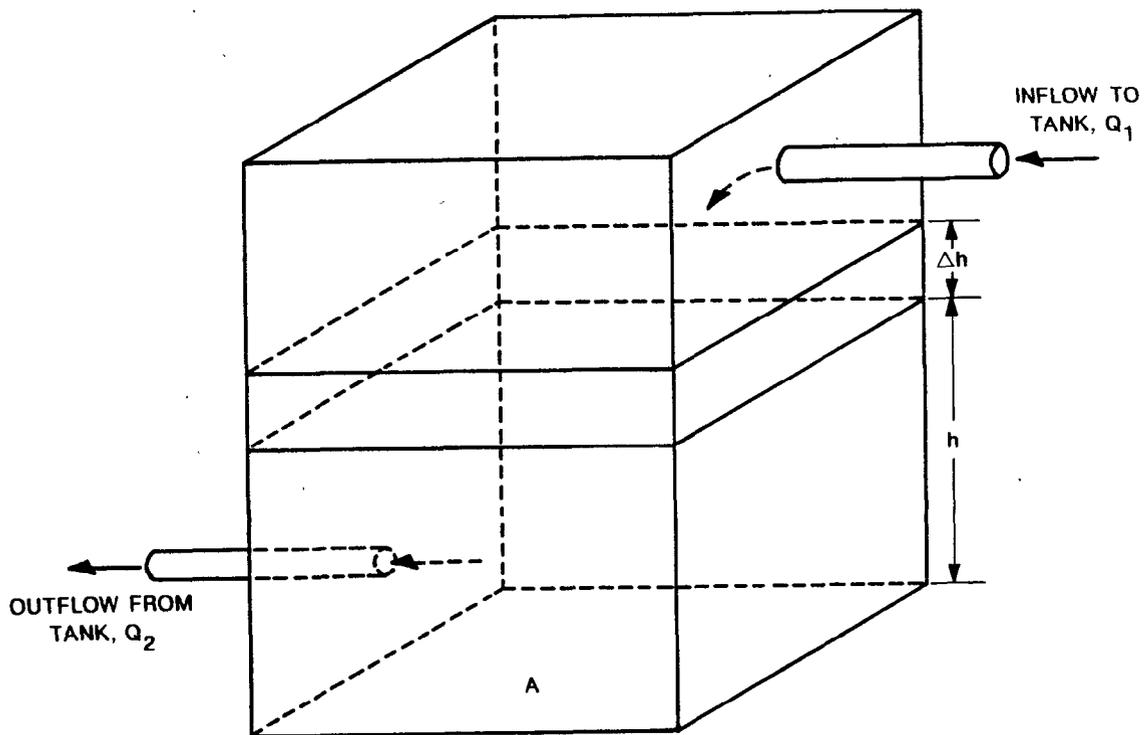


Figure 2-5.--Inflow to and outflow from a tank.

When $\frac{dh}{dt}$ is zero, $\frac{dV}{dt}$ must also be zero, and inflow must equal outflow.

Because h is not changing with time, the system is said to be at equilibrium, or to be in the steady state. When $\frac{dh}{dt}$ differs from zero, inflow and outflow are out of balance; accumulation or depletion is occurring, and the system is described as nonequilibrium.

The mechanism of storage in an unconfined aquifer is essentially similar to that of storage in an open tank. Consider a prism (as shown in fig. 2-6) through an unconfined aquifer which is bounded below by an impervious layer. Let the base area of the prism be A , and the porosity be n . If the prism is saturated to a height h above the base, the volume of water contained in the prism is $(n A h)$; and if the water level is falling at a rate $\frac{dh}{dt}$, the volume of water in the prism is decreasing at a rate

$$\frac{dV}{dt} = \frac{d}{dt} (n A h) = n A \frac{dh}{dt}, \quad (5)$$

assuming that the sand is fully drained as the water level falls.

In general, however, a certain fraction of the water is retained in the pores by capillary forces as the water level falls. When the water level is lowered a distance Δh , therefore, the volume of water removed will not be $nA\Delta h$, but rather $\alpha n A \Delta h$, where α is the percentage of the water, expressed as a fraction, that can be drained by gravity. The fraction that is retained by capillary forces is $(1-\alpha)$. In this case, then,

$$\Delta V = \alpha n A \Delta h \quad (6)$$

is the volume of water removed and

$$\frac{dV}{dt} = \alpha n A \frac{dh}{dt} \quad (7)$$

is the volumetric rate of removal of water from the prism. The quantity αn is called the specific yield¹ or storage coefficient of the aquifer, and is usually denoted as S . Using this notation, the expression for ΔV , the volume of water that must be removed to achieve a drop in water level of Δh , is

$$\Delta V = S A \Delta h. \quad (8)$$

¹ The specific yield of an unconfined aquifer is often denoted by S_y or S.Y. The storage coefficient S and specific yield S_y of an unconfined aquifer are approximately equivalent (see Fetter, 1988, p. 107, eqn. 4-20). The storage coefficient S is used to describe storage properties of both unconfined and confined aquifers.

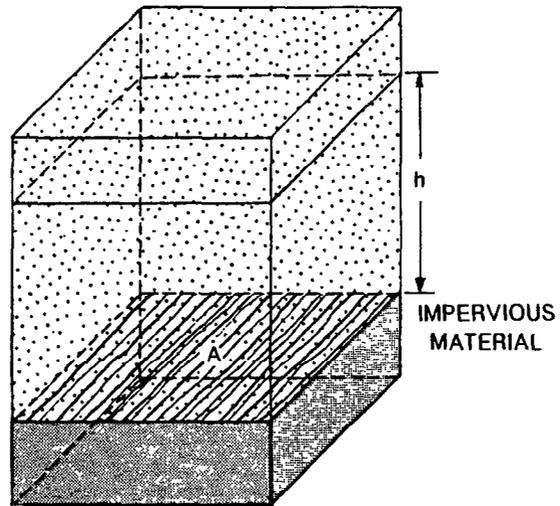


Figure 2-6.--Reference prism in an unconfined aquifer bounded below by an impervious layer.

Therefore,
$$\frac{\Delta V}{\Delta h} = SA \quad (9)$$

or, expressed as a derivative,

$$\frac{dV}{dh} = SA \quad (10)$$

Storage coefficient, for the unconfined case, therefore is given by the equation

$$S = \frac{1}{A} \cdot \frac{dV}{dh} \quad (11)$$

This equation states that storage coefficient is the volume of water released per unit decline in head per unit surface area of aquifer. The expression given previously for the volumetric rate of removal of water from the prism was

$$\frac{dV}{dt} = \alpha n A \frac{dh}{dt} \quad (12)$$

or, in terms of storage coefficient,

$$\frac{dV}{dt} = SA \frac{dh}{dt} \quad (13)$$

The same result can be obtained from the general rules that govern derivatives, as follows:

$$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt} = SA \frac{dh}{dt} \quad (14)$$

In the given prism through the unconfined aquifer, therefore, it is not necessary for inflow to equal outflow. If the inflow to the prism exceeds outflow, water will accumulate in the prism at a rate equal to the difference in flow, and a rise in water level with time will be observed. If outflow exceeds inflow, water is being depleted within the prism, and a fall of water level with time will be observed. Thus, any record of water level versus time in an aquifer is essentially a record of water taken into storage or released from storage in the vicinity of the recording station.

The property of storage is observed in confined aquifers as well as in unconfined aquifers. The mechanism of confined storage depends, at least in part, on compression and expansion of the water itself and of the porous framework of the aquifer; for this reason confined storage sometimes is referred to as "compressive storage." In this discussion we do not attempt to analyze of the mechanisms of confined storage, but concentrate instead on developing a mathematical description of its effects that is suitable for hydrologic calculations. In order to describe the effects of confined storage, we will consider an imaginary experiment.

Consider a vertical prism of unit cross-sectional area, cut from a horizontal confined aquifer and extending the full thickness of the aquifer (fig. 2-7). Assume that water is pumped into the prism through a pipe and that the sides of the prism are sealed so that all of the water that is pumped in must accumulate in storage within the prism. Assume that the prism is saturated to begin with, and that as additional water is pumped in, the pressure within the prism, as measured by the height of water in the measuring piezometer, increases. Assume further that the pressure increase is observed to occur in the following way: a constant incremental change (or increase), Δh , occurs in the water level in the piezometer tube for each constant incremental change (or increase), ΔV , in fluid volume injected into the prism. Thus, if the total injected volume, V , is plotted against the water level in the piezometer, the graph will be a straight line with slope $\frac{dV}{dh} = \frac{\Delta V}{\Delta h}$.

Finally, assume that the following is observed: if the cross-sectional area of the prism is doubled, twice as much water must be injected in order to produce the increase in water level Δh ; if the area is tripled, three times as much water must be injected in order to produce the incremental increase Δh ; and so on. Thus, if the area of the prism is allowed to vary, the quantity $\frac{dV}{dh}$ will not be the same in each case, but the quantity $\frac{dV/dh}{A}$ will be the same, where A is the base area of the prism. This latter quantity, then, is a constant, independent of the area of the prism under consideration, as well as of V and h . It is presumably a function only of the properties of the aquifer material and the thickness of the aquifer. If the aquifer is homogeneous and of uniform thickness, the quantity $\frac{dV/dh}{A}$ will have the same value for any prism through the aquifer, and may be considered to be a constant for the aquifer. It is denoted as the storage coefficient, S --that is, $S = \frac{dV/dh}{A}$.

Figure 2-8 illustrates the relation between V , h , and the cross-sectional area A for the prism of aquifer we have considered. For any given base area, the ratio of the increment of injected water to the resulting head increment is a constant, $\frac{\Delta V}{\Delta h}$. Thus, a plot of V against h will have a constant slope. However, if the base area is varied, the slope of the graph must vary proportionately. Thus, if we are given three prisms for which

$$A_1 = \frac{1}{2} A_2 = \frac{1}{3} A_3 \quad (15)$$

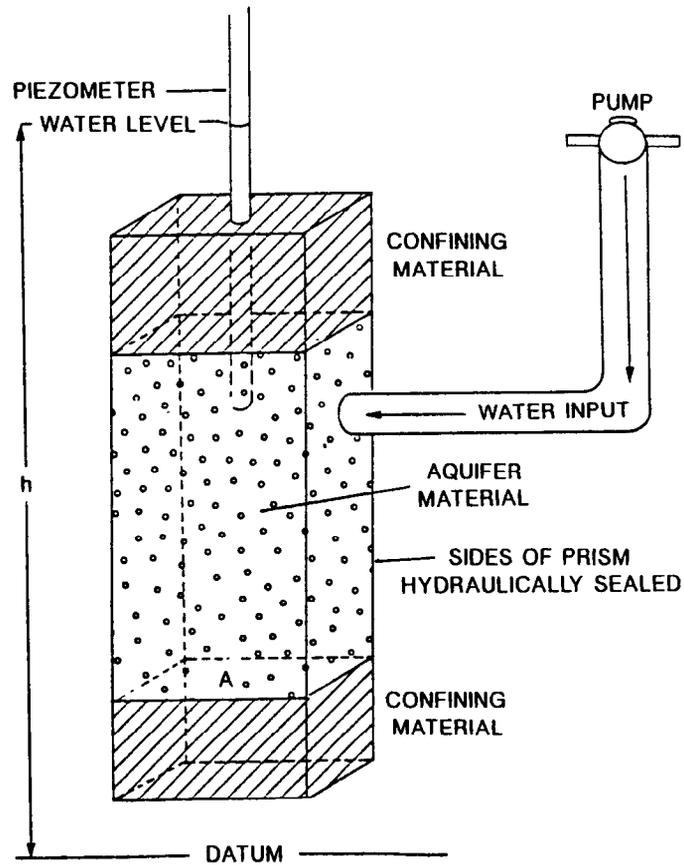


Figure 2-7.--Vertical prism in a confined aquifer bounded above and below by confining material and laterally by walls that are hydraulically sealed. (Modified from Bennett, 1976, p. 55.)

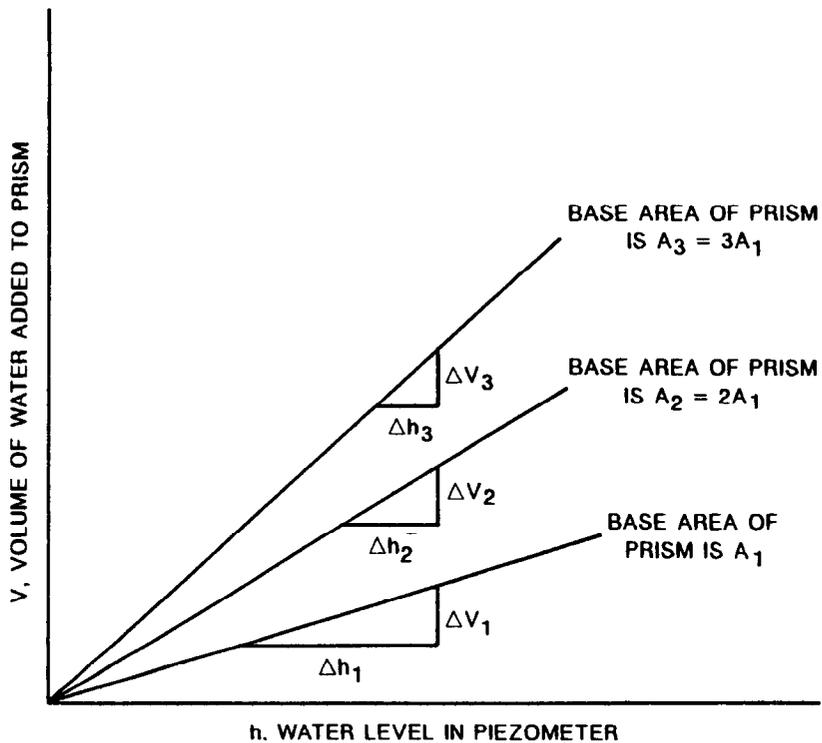


Figure 2-8.--Relation between volume of water injected into a prism of earth material that extends the full thickness of a confined aquifer and head in the prism for different values of prism base area.

we will observe that

$$\frac{\Delta V_1}{\Delta h_1} = \frac{1}{2} \frac{\Delta V_2}{\Delta h_2} = \frac{1}{3} \frac{\Delta V_3}{\Delta h_3} \quad (16)$$

as indicated in figure 4. Therefore,

$$\frac{1}{A_1} \frac{(\Delta V_1)}{\Delta h_1} = \frac{1}{A_2} \frac{(\Delta V_2)}{\Delta h_2} = \frac{1}{A_3} \frac{(\Delta V_3)}{\Delta h_3} = \frac{dV}{dh} = S. \quad (17)$$

Storage coefficient in the confined case is similar to that in the unconfined case in that it is the volume of water taken into storage per unit increase in head per unit surface area of aquifer, or the volume released from storage per unit decline in head per unit surface area of aquifer. Storage coefficient in a confined aquifer is thus a measure of the capacity of the aquifer to absorb water under pressure, and to release water in response to pressure drop, where pressure is measured in feet of water. If head is

observed to be increasing at a rate $\frac{dh}{dt}$ in the prism of aquifer under consideration, water is being taken into storage in the prism at a volumetric rate $\frac{dV}{dt}$ where

$$\frac{dV}{dt} = \frac{dV}{dh} \frac{dh}{dt} = S A \frac{dh}{dt}. \quad (18)$$

Although the definition of confined storage coefficient by the equation

$$S = \frac{1}{A} \frac{dV}{dh} \quad (19)$$

gives no information about the reasons for storage, it describes accurately the effects of storage, and is therefore an adequate definition for the purpose of engineering calculations. As in the case of an unconfined aquifer,

if head in the given prism of confined aquifer is increasing at the rate $\frac{dh}{dt}$,

water is being taken into storage in the prism at a rate $\frac{dV}{dt}$ where

$$\frac{dV}{dt} = \frac{dV}{dh} \frac{dh}{dt} = S A \frac{dh}{dt}. \quad (20)$$

In either the unconfined or the confined case, therefore, the equation

$$\text{Inflow} - \text{Outflow} = \text{Rate of Accumulation, } \frac{dV}{dt} \quad (21)$$

becomes

$$\text{Inflow} - \text{Outflow} = S A \frac{dh}{dt} \quad (22)$$

when applied to a prism of aquifer that has a base area A. If $\frac{dh}{dt} = 0$, inflow to the prism equals outflow.

Exercise (2-9)--Specific Yield

A rectangular prism whose base is a square with sides equal to 1.5 ft and height equal to 6 ft is filled with fine sand whose pores are saturated with water. The porosity (n) of the sand equals 34 percent. The prism is drained by opening a drainage hole in the bottom and 2.43 ft³ of water is collected. Calculate the following quantities:

total volume of prism

volume of sand grains in prism

total volume of water in prism before drainage

volume of water drained by gravity 2.43 ft³

volume of water retained in prism (not drained by gravity)

specific yield

specific retention

- (1) Assuming the value of specific yield determined above, what volume of water, in ft³, is lost from ground-water storage per mi² for an average 1-ft decline in the water table?

Express this volume as a rate for 1 day in ft³/sec.

Express this volume as depth of water in inches over the mi².

- (2) Assuming the value of specific yield determined above, a volume of water added as recharge at the water table that is equal to (a) 1 in. and (b) 4.8 in. per unit area would represent what average change in ground-water levels, expressed in feet?

Ground-Water Flow Equation

Assignments

*Study Fetter (1988), p. 131-136; Freeze and Cherry (1979), p. 63-66, 174-178, 531-533; or Todd (1980), p. 99-101.

*Study Note (2-3)--Ground-water flow equation.

*Define the following terms by referring to any available mathematics text that covers differential equations--independent variable, dependent variable, order, degree, linear, nonlinear.

A differential equation that describes or "governs" ground-water flow under a particular set of physical circumstances may be regarded as a kind of mathematical model. In ground-water flow equations head generally is the dependent variable. If the flow equation is solved, either analytically or numerically, values of head can be calculated as a function of position in space in the ground-water reservoir (coordinates x, y, and z) and time (t). The differential equation provides a general rule that describes how head must vary in the neighborhood of any and all points within the flow domain (ground-water flow system). Numerical algorithms that are amenable to solution by digital computers (for example, the finite-difference approximation of a differential equation) may be developed directly from the differential equation.

The ground-water flow equation developed in Note (2-3) is widely applicable. Note that the steady-state form of this equation represents the mathematical combination of (a) the equation of continuity and (b) Darcy's law.

Note (2-3).--Ground-Water Flow Equation--A Simplified Development,

by Thomas E. Reilly¹

The following development of the ground-water flow equation is simplified in that it (1) employs an intuitive and physical rather than a mathematical approach and (2) implicitly makes the assumption that the flowing fluid (water) is incompressible. This assumption will be discussed further during the course of the presentation.

The continuity principle often is expressed by the "hydrologic equation" as:

$$\text{Inflow (of water)} = \text{Outflow (of water)} \pm \Delta \text{ Storage (of water)} \quad (1)$$

(The symbol Δ (delta) means "change in".) The units of (1), because we are dealing with liquid water, are L^3/T , the same units as discharge Q .

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Let us first consider the steady-state form of (1) in relation to a hypothetical rectangular block of aquifer material (fig. 2-9). The steady-state form of (1)

$$\text{Inflow} = \text{Outflow} \quad (2)$$

may be written

$$\text{Inflow} - \text{Outflow} = 0,$$

or expressed equivalently in different words:

$$\text{Quantity (of water) in (to block)} - \text{Quantity (of water) out (from block)} = 0.$$

Let us define flow into the block as positive and out of the block as negative. Then, with reference to figure 2-9 the previous equation may be written:

$$Q_{x_{\text{left}}} + Q_{y_{\text{front}}} + Q_{z_{\text{top}}} + Q_{x_{\text{right}}} + Q_{y_{\text{back}}} + Q_{z_{\text{bottom}}} = 0 \quad (3)$$

Let: $\Delta Q_x = Q_{x_{\text{left}}} + Q_{x_{\text{right}}}$ = Quantity of flow gained or lost in the x direction

$\Delta Q_y = Q_{y_{\text{front}}} + Q_{y_{\text{back}}}$ = Quantity of flow gained or lost in the y direction; and

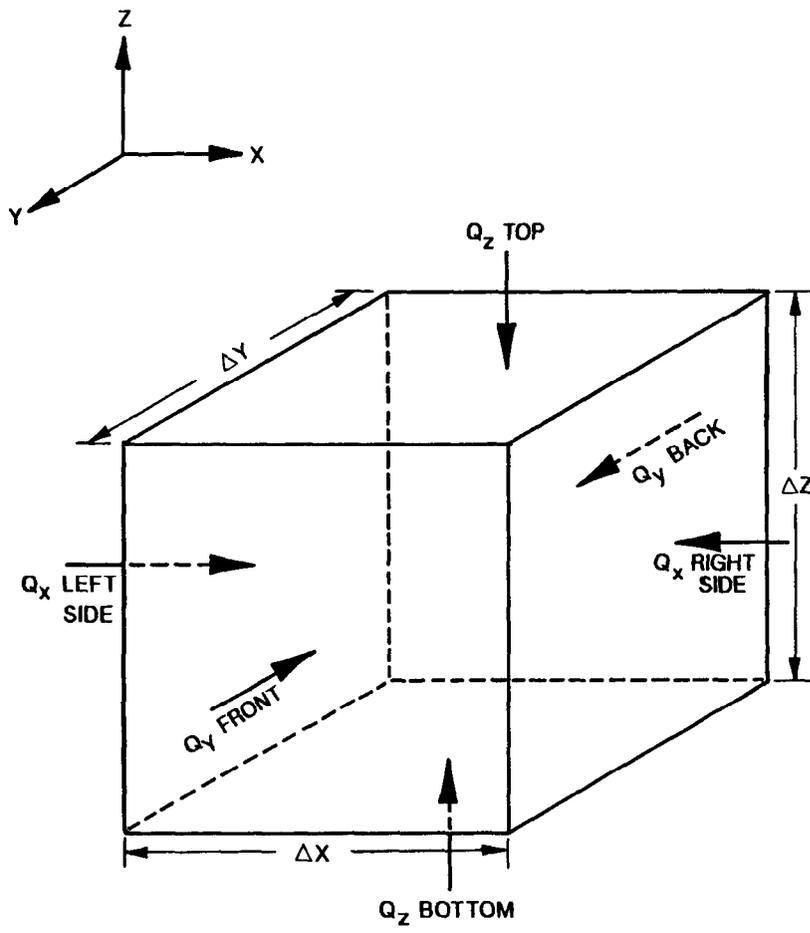
$\Delta Q_z = Q_{z_{\text{top}}} + Q_{z_{\text{bottom}}}$ = Quantity of flow gained or lost in the z direction.

Then:

$$\Delta Q_x + \Delta Q_y + \Delta Q_z = 0. \quad (4)$$

Equation (4), which is expressed in terms of gains or losses in flow in the three coordinate directions relative to the aquifer block rather than absolute flow magnitudes at the faces of the block, is a convenient form of the steady-state continuity equation (2) and (3) for our purposes.

Equation (4) is specific in that we are dealing with the flow of water relative to a block of aquifer material. However, it is too general and, therefore, useless in practical applications unless we have a specific rule for calculating $Q_{y_{\text{front}}}$, $Q_{z_{\text{top}}}$, and so forth in (3) or ΔQ_x , and so forth in (4). The rule that relates (4) specifically to ground-water flow is Darcy's law. Relative to the block of aquifer material in figure 2-9, Darcy's law may be written:



QUANTITY IN - QUANTITY OUT = 0
(CONSERVATION OF MASS)

Figure 2-9.--Steady-state (equilibrium) flow of ground water through a typical block of aquifer material.

$$\begin{aligned}
Q_x &= K_x \Delta y \Delta z \frac{\Delta h}{\Delta x} \\
Q_y &= K_y \Delta x \Delta z \frac{\Delta h}{\Delta y} \\
Q_z &= K_z \Delta x \Delta y \frac{\Delta h}{\Delta z}
\end{aligned} \tag{5}$$

where K_x , K_y , and K_z are values of hydraulic conductivity in the respective coordinate directions, and Δh is the change in head between respective pairs of block centers.² We assume that the hydraulic conductivity remains constant within the aquifer block in each coordinate direction. Expressed more formally, we assume that the aquifer block is anisotropic and homogeneous with respect to hydraulic conductivity. Note that Darcy's law in equation (5) calculates the flow across each of the six block faces in the three coordinate directions x , y , and z and does not calculate the change in flow, ΔQ_x , ΔQ_y , or ΔQ_z in each of the three coordinate directions as expressed in equation 4.

For example, the first equation in (5) $Q_x = K_x \Delta y \Delta z \frac{\Delta h}{\Delta x}$ would be used to calculate $Q_{x_{\text{left}}}$ and $Q_{x_{\text{right}}}$, and ΔQ_x is the algebraic sum of these two across-face flows, as expressed in equation (4).

Substituting equations (5) into (4) and dividing all terms by $\Delta x \Delta y \Delta z$ we obtain:

$$\frac{\Delta(K_x \Delta y \Delta z \frac{\Delta h}{\Delta x})}{\Delta x \Delta y \Delta z} + \frac{\Delta(K_y \Delta x \Delta z \frac{\Delta h}{\Delta y})}{\Delta x \Delta y \Delta z} + \frac{\Delta(K_z \Delta x \Delta y \frac{\Delta h}{\Delta z})}{\Delta x \Delta y \Delta z} = 0$$

or

$$\frac{\Delta}{\Delta x} \left(K_x \frac{\Delta h}{\Delta x} \right) + \frac{\Delta}{\Delta y} \left(K_y \frac{\Delta h}{\Delta y} \right) + \frac{\Delta}{\Delta z} \left(K_z \frac{\Delta h}{\Delta z} \right) = 0. \tag{6}$$

If we take the limit of this equation as the block of aquifer material becomes smaller and smaller (Δx , Δy , and Δz approach zero), we may write (6) in terms of partial derivatives

$$\frac{\partial}{\partial x} \left(K_x \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left(K_y \frac{\partial h}{\partial y} \right) + \frac{\partial}{\partial z} \left(K_z \frac{\partial h}{\partial z} \right) = 0. \tag{7}$$

² Here and in the immediately succeeding discussion, reference is made to the six aquifer blocks and heads at their respective centers that are located adjacent to the six faces of the reference block of the aquifer depicted in figure 2-9.

This equation is related to the widely utilized Laplace equation³ but differs from it in that this equation accounts for differences in hydraulic conductivity in the three coordinate directions (K_x , K_y , and K_z).

Let us now add storage to the basic steady-state flow equations (2), (4), (6), and (7). With reference to equation (4), we can write:

$$\Delta Q_z + \Delta Q_y + \Delta Q_x = \frac{\Delta V}{\Delta t} \quad (8)$$

as a form of equation (1) which includes changes in storage. The right-hand side of (8) symbolizes that the volume of water in our hypothetical block of aquifer material (V) (fig. 2-9) changes (ΔV) during a specified time interval

(Δt). This symbolic representation of a rate of change in storage $\frac{\Delta V}{\Delta t}$ is

general and non-specific in the sense that it could represent types and configurations of flow other than ground-water flow. Our problem now is to

obtain an expression equivalent to $\frac{\Delta V}{\Delta t}$ that relates specifically to ground-water flow--that is, to our (or any) block of aquifer material.

At this point review (if necessary) the previous note on ground-water storage by G. D. Bennett, which provides a physical discussion of the following required relationships:

$$\Delta V = S A \Delta h = S \Delta x \Delta y \Delta h \quad (9)$$

and

$$\frac{\Delta V}{\Delta t} = S \Delta x \Delta y \frac{\Delta h}{\Delta t} \quad (10)$$

³ Laplace's equation, a second-order partial-differential equation that is used in diverse fields of science, may be written as a three-dimensional ground-water flow equation in the form

$$\frac{\partial}{\partial x} \left(\frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial h}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{\partial h}{\partial z} \right) = 0$$

or

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = 0.$$

Because hydraulic conductivity K does not appear in this equation explicitly, it must be constant in all directions. In other words, this equation implies that the flow medium is isotropic and homogeneous with respect to hydraulic conductivity.

where S is the storage coefficient (dimensionless) and Δh is the change in head as measured in a tightly cased well (piezometer) in the block of aquifer material during the time interval Δt .

In reviewing the discussion leading to equations (9) and (10) in the note on storage by Bennett, we recall that the storage coefficient S is defined with reference to a vertical prism of earth material that extends completely through any given aquifer. Thus, the storage coefficient S is a storage parameter that relates to representations of ground-water flow systems in which the entire aquifer thickness is treated as a single unit or layer.⁴ The appropriate storage parameter for more fully three-dimensional flow is the coefficient of specific storage S_s . This coefficient is equivalent to S/b where b is the aquifer thickness. Thus, specific storage represents the volume of water released from or taken into storage per unit volume (instead of "per unit area" as for storage coefficient S) of the porous medium per unit change in head. Specific storage S_s has units of L^{-1} .

We see from the preceding definition that specific storage S_s is a storage parameter that relates to a unit volume of aquifer material; that is, a unit area of aquifer in the xy plane times a unit thickness of aquifer. Thus, in equations (9) and (10), we can express S , relative to the original reference block of aquifer material with dimensions of Δx , Δy , and Δz , as $S_s \Delta z$. Therefore, we can express equation (10) as

$$\frac{\Delta V}{\Delta t} = S_s \Delta z \Delta x \Delta y \frac{\Delta h}{\Delta t} \quad (10a)$$

To add storage to the continuity equation that describes the flow through a block of aquifer material, we substitute Darcy's law and equation (10a) into the transient continuity equation (8), which becomes

$$\Delta \left(K_x \Delta y \Delta z \frac{\Delta h}{\Delta x} \right) + \Delta \left(K_y \Delta x \Delta z \frac{\Delta h}{\Delta y} \right) + \Delta \left(K_z \Delta x \Delta y \frac{\Delta h}{\Delta z} \right) = S_s \Delta z \Delta x \Delta y \frac{\Delta h}{\Delta t} \quad (11)$$

By dividing all terms in (11) by $\Delta x \Delta y \Delta z$, we obtain

$$\frac{\Delta}{\Delta x} \left(K_x \frac{\Delta h}{\Delta x} \right) + \frac{\Delta}{\Delta y} \left(K_y \frac{\Delta h}{\Delta y} \right) + \frac{\Delta}{\Delta z} \left(K_z \frac{\Delta h}{\Delta z} \right) = S_s \frac{\Delta h}{\Delta t} \quad (12)$$

Taking the limit of this equation (Δx , Δy , Δz , and Δt approach zero), we may write equation (12) in terms of partial derivatives:

$$\frac{\partial}{\partial x} \left(K_x \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left(K_y \frac{\partial h}{\partial y} \right) + \frac{\partial}{\partial z} \left(K_z \frac{\partial h}{\partial z} \right) = S_s \frac{\partial h}{\partial t} \quad (13)$$

⁴ As a result of its definition, the storage coefficient S is generally used in flow equations that represent two-dimensional nearly horizontal flows.

Equation (13) is a widely applicable form of a transient, three-dimensional ground-water flow equation. One- and two-dimensional simplifications of this equation can be found in most ground-water texts.

To review, a steady-state form of the ground-water flow equation always involves the combination of two rules or principles:

- (1) a statement of the continuity principle in the form of an equation which is appropriate to the problem we are trying to solve, and
- (2) Darcy's law, which provides a specific rule for calculating the system flux; in our case this flux involves the flow of liquid water through porous earth materials.

If our problem is a transient one, we must add a storage term to the equation, which represents a change in the dependent variable (in our problems h or head) as a function of time. In the preceding development this storage term was represented in equation (10):

$$\frac{\Delta V}{\Delta t} = S \Delta x \Delta y \frac{\Delta h}{\Delta t}$$

In more formal developments of the ground-water flow equation the continuity equation is written in terms of mass flux instead of volume flux. Instead of equation (2) relative to our block of aquifer material

$$\text{Quantity of water in} - \text{Quantity of water out} = 0, \quad (2)$$

we express the continuity equation in more general terms as:

$$\text{Mass of fluid in} - \text{Mass of fluid out} = 0. \quad (14)$$

More specific to our problem, the steady-state continuity equation that involves fluid mass is expressed in terms of ρ (fluid density) and the fluid velocity components v_x , v_y , and v_z as:

$$\frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} = 0. \quad (15)$$

Note that (ρv) has the units of mass flux, $M/L^2 T$.

Writing Darcy's law in the form

$$v_x = K_x \frac{\partial h}{\partial x} \quad (16)$$

for one coordinate direction and substituting (16) in the first term of (15) we obtain

$$\frac{\partial}{\partial x} \left(\rho K_x \frac{\partial h}{\partial x} \right) = 0. \quad (17)$$

Assuming that ρ is constant we can write

$$\frac{\partial}{\partial x} \left(K_x \frac{\partial h}{\partial x} \right) = 0, \quad (18)$$

which is exactly the same as the first term in (7). This last discussion indicates that our simplified approach to developing the flow equation involved the implicit assumption that the fluid density remains constant throughout the flow field. This is equivalent to the assumption that liquid water is incompressible. Physically meaningful solutions to many practical problems in ground-water hydrology employ this assumption.