U.S. DEPARTMENT OF THE INTERIOR

U.S. GEOLOGICAL SURVEY

FRACTALS AND THE PARETO DISTRIBUTION APPLIED TO PETROLEUM ACCUMULATION-SIZE DISTRIBUTIONS

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Open-File Report 91-18

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1993

ABSTRACT

During the 1980's the Pareto distribution began to displace the lognormal distribution as a model for oil and gas accumulation-size distributions. Adoption of the Pareto distribution was independent of the study of fractals; however, the Pareto distribution is the probability distribution uniquely characteristic of fractals. The difference between the lognormal and Pareto distributions can be extremely significant in that the estimates of resources are higher for the Pareto than for the lognormal distribution. The Pareto distribution is a fractal probability distribution that has a power-law form with two important scaling (self-similar) properties: scaling under lower truncation and asymptotic scaling under addition. The objective of this paper is to establish the Pareto distribution as a model for petroleum accumulation-size distributions in the context of fractals. We present insights about petroleum accumulation-size distributions gained from fractal properties of the Pareto distribution.

INTRODUCTION

This paper is about probability modeling of petroleum accumulation-size distributions. In the 1980's fractals became widely known, and the Pareto distribution began to displace the lognormal distribution as a model for petroleum accumulation-size distributions. Adoption of the Pareto distribution was independent of, and seemingly unrelated to, the study of fractals; however, the Pareto distribution is the probability distribution characteristic of fractals (Mandelbrot, 1982). The objective of this paper is to establish the Pareto distribution as a model for oil and gas field-size distributions in the context of fractals. The Pareto distribution is related to fractals because it has a power-law form with two important scaling (self-similar) properties: scaling under lower truncation and asymptotic scaling under addition. The fractal approach is used to further substantiate the application of the Pareto distribution as a model for oil and gas accumulation-size distributions. In this paper, we examine the two fractal properties of the Pareto distribution with respect to petroleum accumulation sizes. This provides insight into the economic truncation observed in oil and gas accumulation-size distributions. The fractal approach also yields a new quantitative description of petroleum accumulation-size distributions.

Three decades ago, the Pareto and lognormal distributions were first introduced separately and virtually simultaneously as probability models for petroleum field-size distributions. Kaufman (1962) proposed the lognormal distribution, while Mandelbrot (1962) proposed the Pareto distribution. Mandelbrot advocated the Pareto distribution more than a decade before his development of geometric fractals.

FRACTALS

A fractal is an object made of parts similar to the whole in some way; either exactly the same except for scale or statistically the same. The concept underlying fractals is self-similarity or scaling, that is, invariance against changes in scale or size (scale invariance).

Fractal relationships are of the form:

$$\text{number} = \text{prefactor} \times (\text{quantity})^{\text{exponent}}$$

where the exponent is a fraction. In many natural systems one encounters different power-law expressions of this form. Taking logarithms produces the linear relationship

$$\log(\text{number}) = \log(\text{prefactor}) + [\text{exponent} \times \log(\text{quantity})].$$

Solving for the exponent gives

$$\text{exponent} = \log(\text{number})/\log(\text{quantity}) - \log(\text{prefactor})/\log(\text{quantity}).$$

There are two major types of fractals: geometric fractals and probabilistic fractals. Geometric fractals are geometric shapes that have fractional dimension. Probabilistic fractals are random variables, or equivalently probability distributions, that are self-similar (scale invariant). As will be shown below, the probability distribution characteristic of fractals is the hyperbolic law, which is defined to be either the Pareto law (continuous case) or the Zipf law (discrete case). The overview of fractal power laws set out above is summarized in a Venn diagram (figure 1). This paper is primarily concerned with the Pareto distribution; however, the geometric aspect of fractals is introduced next to establish a concept of self-similarity.

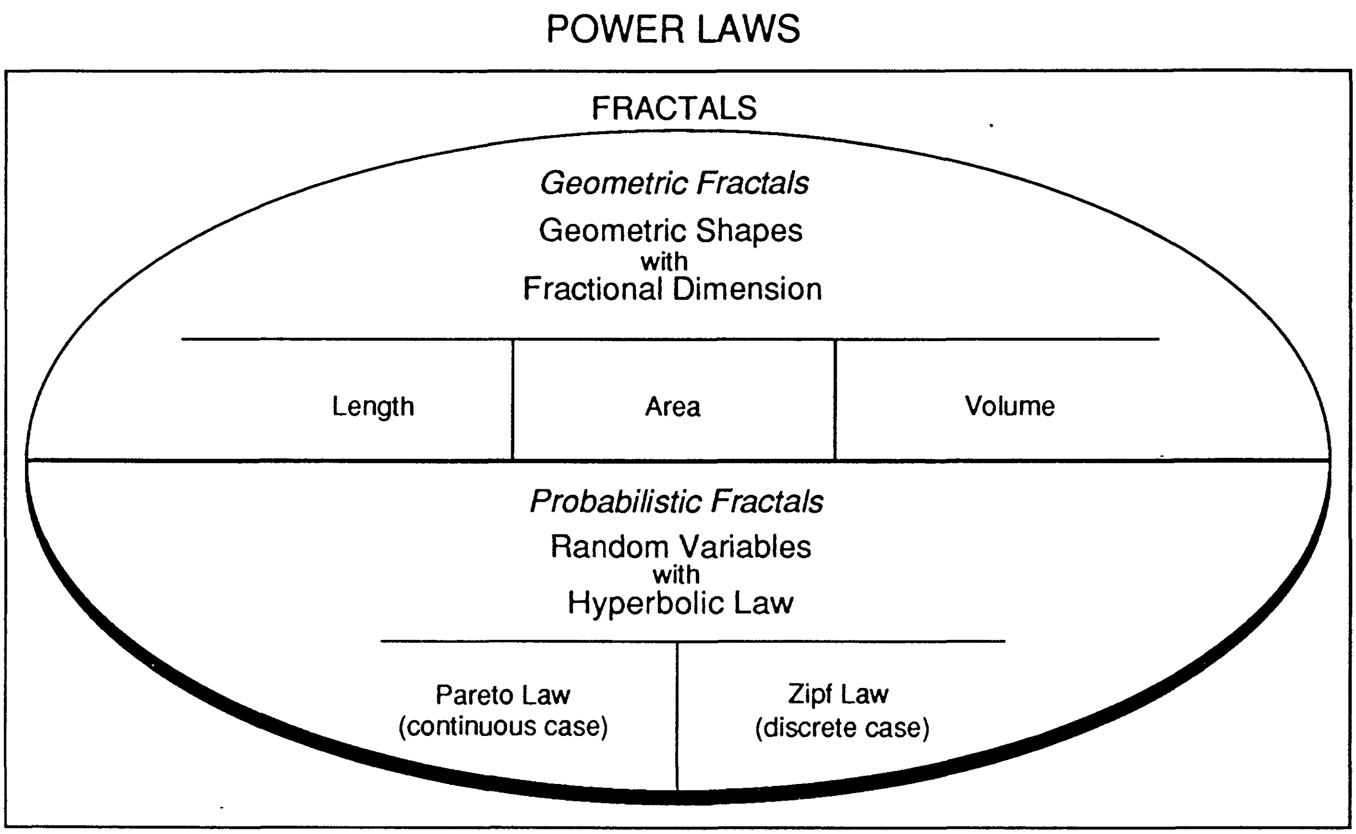

Figure 1.--Venn diagram of fractal power laws. Showing relationships: fractals are a subset of all power laws; fractals include geometric shapes that have fractional dimension and random variables that are hyperbolic. Geometric shapes comprise length, area, and volume. Hyperbolic distributions comprise Pareto and Zipf laws.

Geometric Fractals

A geometric fractal is a geometric shape (pattern) that has two fundamental properties:

1. Self-similar (scale invariant), either exactly or statistically.
2. Fractional dimension.

A coastline is an often cited example of a statistically fractal pattern (Mandelbrot, 1982; Feder, 1988). Determination of the size scaling of a coastline requires measurement of its length. If the coastline is fractal, its length will increase as a power-law function of the size of square boxes or straight-line segments needed to cover it. The power law is of the form

$$N(\delta) = A\delta^{-D} \qquad \delta > 0$$

where

$N(\delta)$ = number of "boxes" ($\delta \times \delta$ squares) or straight-line segments needed to cover the coastline as a function of δ,

δ = side-length of square box or line segment,

D = fractal dimension (fraction > 0),

A = constant of proportionality (prefactor parameter).

Taking logarithms of the power-law equation, we have

$$\log N(\delta) = \log A - D \log \delta$$

A plot of $\log N(\delta)$ versus $\log \delta$ produces a straight line with negative slope ($-D$). The fractal dimension of coastlines has been found to range from 1.1 to 1.3 (Mandelbrot, 1982). The dimension, D, estimated by counting the number of boxes needed to cover the pattern as a function of the box size, is called the <u>box counting dimension</u>.

The fractal approach to geometric patterns introduces concepts and principles for the probabilistic side of fractals.

Probabilistic Fractals

A random variable that has a hyperbolic distribution, i.e., a hyperbolic random variable, is a probabilistic fractal. Thus, a hyperbolic distribution is a fractal distribution. A hyperbolic distribution is either a Pareto distribution (continuous case) or a Zipf distribution (discrete case). The Pareto distribution is a continuous probability distribution that is defined over a continuous sample space, e.g., petroleum accumulation sizes. The Zipf distribution is a discrete probability distribution that is defined over a countable sample space, e.g., word lengths.

Pareto's Law (Pareto, 1897) was developed to describe the distribution of personal annual income, random variable X, over a population and can be stated as a power law in the following form:

$$N(x) = Ax^{-a} \qquad x > 0$$

where

$N(x)$ = number of persons having annual income $\geq x$ (cumulative frequency),

x = annual incomes (values of random variable X),

a = shape parameter called Pareto's constant or exponent (real number > 0),

A = constant of proportionality (prefactor parameter).

Taking logarithms, we have the linear equation

$$\log N(x) = \log A - a \log x$$

The Pareto complementary cumulative distribution function of X is

$$P(X \geq x) = \left(\frac{x}{k}\right)^{-a} \qquad a > 0,\ x \geq k > 0$$

where k is a scale parameter.

The Pareto probability density function of X is

$$f(x) = \frac{ak^a}{x^{a+1}} \qquad a > 0,\ x \geq k > 0$$

Note that f(k) = a/k. Two comparative shapes of the Pareto density function f(x) are shown in figure 2 for shape parameter a = 1 and a = 2.

The expected value or mean of X is

$$E(X) = \frac{ak}{a-1} \qquad \text{if } a > 1$$

The variance of X is

$$V(X) = \frac{ak^2}{(a-1)^2(a-2)} \qquad \text{if } a > 2$$

For $a \leq 1$, E(X) does not exist, and for $a \leq 2$, V(X) does not exist.

Four disparate nongeological examples of data sets that when plotted on axes with logarithmic units, the hyperbolic distributions fit to the data sets plot as straight lines:

(a) The populations within plant genera plotted against the number of genera with at least such populations (Willis, 1922).

(b) The number of journal articles published plotted against the number of scientists publishing at least that many (Lotka, 1926).

(c) Word length in the English language plotted against the usage of words of at least that word length (Zipf, 1949).

(d) The frequencies contained in a cardiac pulse plotted against the occurrence of at least those frequencies (Goldberger et al., 1985).

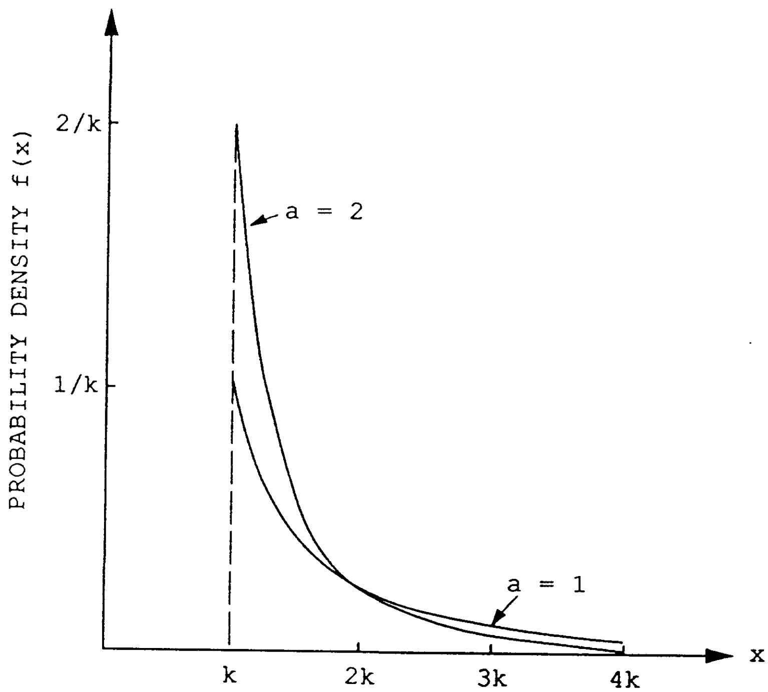

Figure 2.--Two comparative shapes of the Pareto probability density function f(x) for shape parameter a = 1 and a = 2.

Scaling (Self-similar) Probability Distributions

Mandelbrot (1982) defined scaling of a random variable or probability distribution as follows: A random variable X is scaling (self-similar) under the transformation T(X) if the probability distributions of X and T(X) are identical except for scale. Transformation is used in a broad sense. For example, the summation of n independent, identically distributed random variables is viewed as a transform of the common X, and the corresponding variables are called scaling under addition. Scaling under two types of transformations for the Pareto distribution are important: (1) scaling under lower truncation, and (2) asymptotic scaling under addition.

1. Scaling under lower truncation.

Mandelbrot (1966) found that the Pareto distribution possesses the fractal property of scaling under lower truncation. This can be shown by conditioning the Pareto complementary cumulative probability distribution with truncation at k′ where $0 < k < k' \leq x$. We obtain

$$P(X \geq x | X \geq k') = \frac{(x/k)^{-a}}{(k'/k)^{-a}} = (x/k')^{-a} \qquad x \geq k'$$

which is itself a Pareto distribution with scale parameter k′. Thus, the original Pareto distribution is self-similar (under lower truncation).

An important additional property of the conditioned or truncated variable is that it is scale-free in the sense that its distribution does not depend upon the original scale parameter k (Mandelbrot, 1966). Conversely, this property characterizes the Pareto distribution (and the Zipf distribution in the discrete case). A proof can be found in Mandelbrot (1966, 1982). The proof begins with the necessary condition for being scale-free under truncation which, when satisfied, implies the Pareto distribution:

$$P(X \geq x | X \geq k') = P(X \geq cx | X \geq ck')$$

where c is any positive number. The hyperbolic distribution is the only distribution such that the rescaled truncated variable has a distribution independent of c.

An application of the above properties of the Pareto distribution would be to obtain the original Pareto distribution from knowledge of the shape parameter of a lower truncated Pareto distribution and an estimate of the original scale parameter. This will be illustrated in an example below when we estimate the distribution and parameters of the parent (original) population of oil and gas fields in a play from an economically truncated field-size distribution.

The effect of truncating at k′ a Pareto distribution with original scale parameter k can be graphically demonstrated by comparing the original Pareto and the lower truncated Pareto with respect to their probability density functions. Plotted in figure 3 are two probability density functions with shape parameter a = 1.

i. Original Pareto with scale parameter k

$$f(x) = k/x^2 \qquad x \geq k$$

ii. Lower truncated Pareto with scale parameter k′ = 2k

$$f(x) = k'/x^2 = 2k/x^2 \qquad x \geq 2k$$

In figure 3 the original Pareto and the lower truncated Pareto probability density functions have the same total area of unity under their respective curves. The lower truncated Pareto curve has twice the height of the original Pareto curve where $x \geq 2k$.

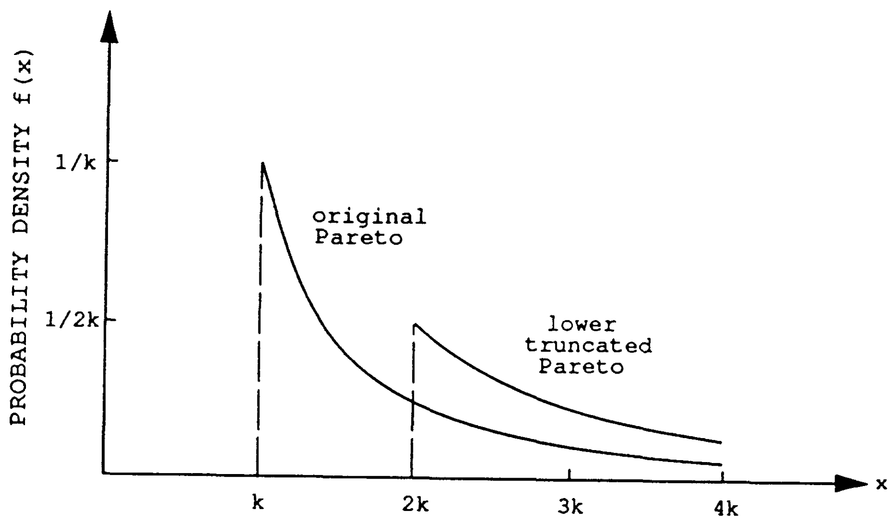

Figure 3.--Scaling under lower truncation of the Pareto probability density function (a = 1) with scale parameter k becoming k′ = 2k.

2. Asymptotic scaling under addition.

Asymptotic scaling was also defined by Mandelbrot (1982). For many transformations, invariance requires an asymptotically hyperbolic distribution. This means that there must exist an exponent a>0 such that

$$\lim_{x\to\infty} P(X < x)x^{a} \quad \text{and} \quad \lim_{x\to\infty} P(X > x)x^{a}$$

are defined and finite, and one of the limits is positive. Note that here the random variable X exceeding a large value x (right tail) is approximately hyperbolically distributed. Hyperbolic distributions invariant under addition are only asymptotically hyperbolic. The hyperbolic distribution plays a key role in nonstandard central limits. Note that asymptotic scale invariance under addition concerns the right side (tail) of the distribution.

Mandelbrot (1960) gives an alternative formulation of the asymptotically hyperbolic distribution. The asymptotic form of the Pareto distribution is

$$P(X \geq x) \sim \left(\frac{x}{k}\right)^{-a} \qquad \text{as } x \to \infty$$

This implies that if log $P(X \geq x)$ is plotted against log x, the resulting curve should be asymptotic to a straight line with slope equal to -a as x approaches infinity. It has been shown (Lévy, 1925) that there is a class of distributions which follows the asymptotic form of the Pareto distribution and is characterized by $0 < a < 2$. For economists concerned with the right tails of distributions, the Pareto distribution is probably more useful than the lognormal distribution, which generally gives a poor fit in the tails (Aitchison and Brown, 1957).

Mandelbrot (1960, 1963, 1964, 1967) used the Pareto distribution and its asymptotic form to explain many empirical phenomena. The Mandelbrot papers address a class of distributions termed "stable Paretian." Johnson and Kotz (1970) discuss various classes of the Pareto distribution.

West and Shlesinger (1990) discuss scale invariance of the hyperbolic distribution (they use the equivalent terms "inverse power-law distribution" and "1/f distribution") in the context of noise in natural phenomena. Although not stated, they deal with asymptotic scale invariance under addition of the hyperbolic distribution. They point out that the procession from the normal distribution to the lognormal distribution, and on to the Pareto distribution (a 1/f distribution) is one of increasing complexity in a system containing many independent random components. They state that as lognormal systems become even more complex, their distributions become broader and take on more of the qualities associated with a power-law distribution. This means that increasingly complex lognormal phenomena take on more of the fractal, or scale-invariant, characteristics of systems governed by power laws.

Petroleum (oil or gas) accumulation size is determined by the interaction of multiple and complex multiplicative geologic processes. The size of an oil or gas accumulation is a function of many geologic variables that can be treated as random variables in a set of petroleum engineering reservoir equations (Garb and Smith, 1987). Thus, petroleum accumulation size is a random variable that is a transformation of the product of many geologic random variables that are themselves functions of other multiplicative geologic processes.

It is possible that an argument similar to the one by West and Shlesinger can be used to show that petroleum accumulation-size distributions have the heavier tail of the Pareto distribution. That is, petroleum accumulation size possesses the fractal property of asymptotic scaling under addition. Further work needs to be done to show this, and a more rigorous approach needs to be used than the one by West and Shlesinger. This is an area for future work.

PETROLEUM ACCUMULATION-SIZE DISTRIBUTIONS

Mandelbrot's Approach

Mandelbrot (1962) was the first to point out that the size distributions of oil, gas, and mineral resources can be modeled with the Pareto distribution. He fit the Pareto distribution to the cumulative frequency of oil field areas (figure 4) and the cumulative frequency of their total ultimate recovery (figure 5), i.e., the sums of the total production and of the currently estimated reserves. He also fit the Pareto to total capacity valuations estimated for several gold, uranium, and pyrite mines in South Africa, for flood levels of rivers, and for glacial varve thicknesses.

The data in figure 4 are cumulative Paretian, i.e., they are doubly logarithmic plots of the numbers of oil fields, in the entire U.S. and in Texas alone, having an area greater than a given number of acres. The data are from The Oil and Gas Journal (1958). The smaller fields, those producing between 1000 and 3000 bbl. per day, are included in the data for all states except Texas. The data points are fit by a straight line with negative slope and Pareto's exponent of 1.3. Mandelbrot's data (1958) were less representative than the data we work with today (see figure 6); the range of

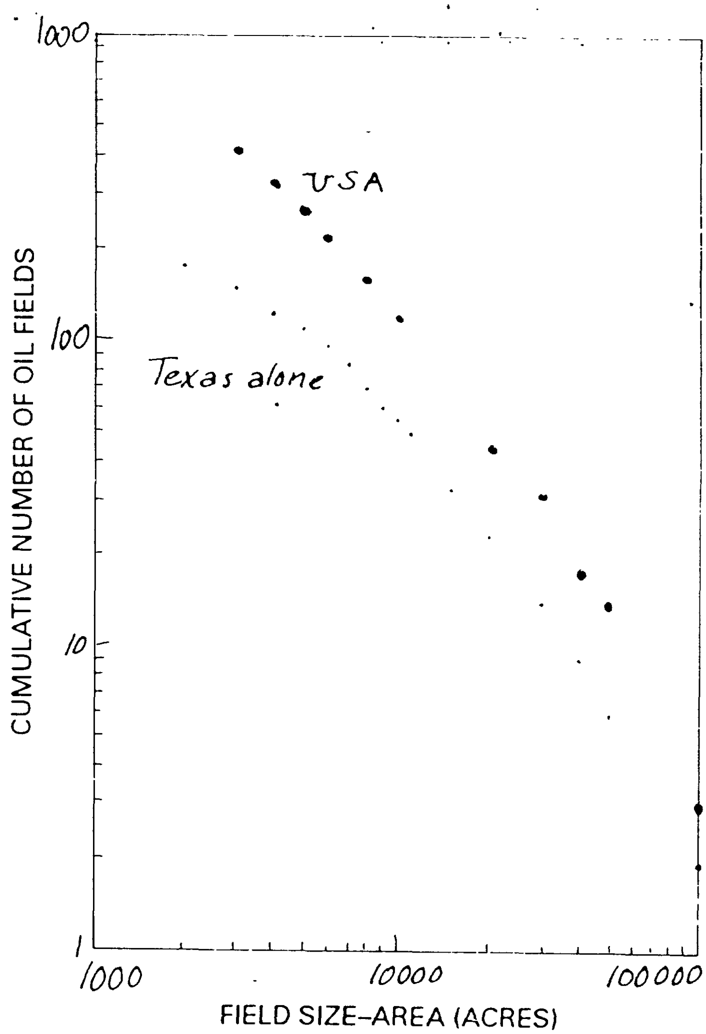

Figure 4.--Reproduction of Mandelbrot's 1962 Paretian plot of oil field sizes based on area. Cumulative frequency plot; axes labels added by authors.

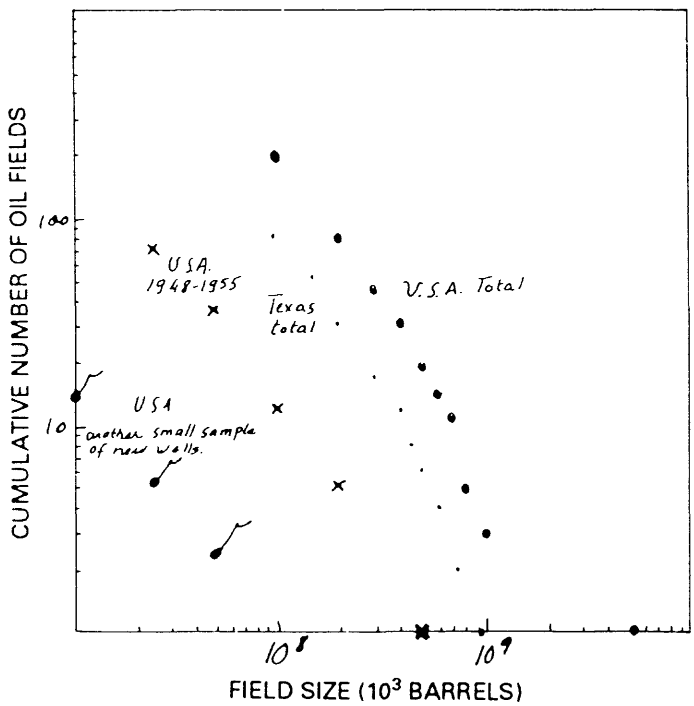

Figure 5.--Reproduction of Mandelbrot's 1962 Paretian plot of oil field sizes based on estimated ultimate recovery. Cumulative frequency plot; axes labels added by authors.

exponents for updated data for plays, basins, and continent-size areas is between 0.81 and 1.08 (Barton and Scholz, this volume).

Mandelbrot (1962) plotted cumulative Paretian graphs for the estimated ultimate recovery (figure 5 USA 1948-1955), i.e., the sum of the cumulated production and of the estimated reserves. The data are from McKie (1960) and The Oil and Gas Journal (1958). He noted that the Paretian exponent may be slightly larger than for USA on figure 4; but, he continued, if one takes into account the enormous size of the largest U.S. field, figures 4 and 5 may exhibit the same exponent. The nearly parallel slopes 1.3 and 1.4 suggest that the size distribution of oil deposits is Pareto-scaling overall and for small subsets of the data. A cumulative Paretian plot was also made by Mandelbrot for the reserves of a sample of gas fields discovered during 1947-1951.

Mandelbrot (1962) also comments on the use of the lognormal distribution in the literature of geostatistics. He states that his use of the Pareto distribution contradicted what had been the prevailing view that the size distributions of natural resources follow the lognormal law. He points out that the fit of the lognormal distribution is poor, both graphically and from the viewpoint of predicted values, and he does not believe that the lognormal law can be rendered appropriate by changes such as those proposed in Krige (1960).

Goldberg's Approach

Truncated Pareto distributions, with a lower limit (k) as well as an upper limit (t), have been found by Goldberg (1967) to fit the size distribution of oil fields. An upper truncated Pareto distribution has density function

$$f(x) = \frac{ak^a}{x^{a+1}} [1 - (t/k)^{-a}]^{-1} \qquad k \leq x \leq t$$

and complementary cumulative distribution function

$$P(X \geq x) = [(x/k)^{-a} - (t/k)^{-a}] [1 - (t/k)^{-a}]^{-1} \qquad k \leq x \leq t$$

The lower limit, k, corresponds to the minimum field size, and the upper limit, t, corresponds to the maximum field size. The population moments of the upper truncated Pareto distribution always exist.

Kaufman's Approach

Despite the work of Mandelbrot and Goldberg, the prevailing view has been that petroleum field sizes are lognormally distributed. The difference between the lognormal and Pareto distributions can be extremely significant, in that the estimates of resources are higher for the Pareto than for the lognormal distribution. Kaufman (1962 and Kaufman et al., 1975) has been the main proponent for the application of the lognormal distribution as a model for the parent population of oil and gas field sizes. Kaufman's approach is largely theoretical (probability theory) in deducing attributes from specified fundamental assumptions. A review of arguments for selecting the lognormal law is presented in Crovelli (1984).

Recent Approaches

Recently, the lognormal distribution has been questioned because empirical analyses of field-size data suggest the use of a J-shaped distribution (a Pareto-type distribution) as a model of the sizes of oil and gas fields. The lognormal shape of the observed size-frequency distribution of oil and gas fields is considered to be an artifact of the discovery process, a consequence of an economic truncation or filtering process (Drew et al., 1982; Schuenemeyer and Drew, 1983; Attanasi and Drew, 1985). Houghton (1988) used a truncated shifted Pareto distribution to model size distribution of oil and gas fields. The latest assessment of U.S. petroleum resources by the U.S. Geological Survey (Mast et al., 1989) applied a modified Houghton model. Drew (1990) gives an interesting presentation on the lognormal debate. He reviews how and why the lognormal distribution came to be so widely accepted as a model for the parent population of oil and gas fields. Drew uses the empirical approach of applied statistics based upon the Arps-Roberts (1958) model. He concludes that the underlying parent population of oil and gas fields in any region is distributed as a log-geometric distribution. This distribution belongs to the family of distributions that is commonly known as "J-shaped," or Pareto.

EXAMPLE

The choice of the probability distribution deemed appropriate to fit the size distribution of oil accumulations has evolved over the past 30 years as exploration and development have provided additional data. This development is graphically illustrated in figure 6 (from Drew, 1990).

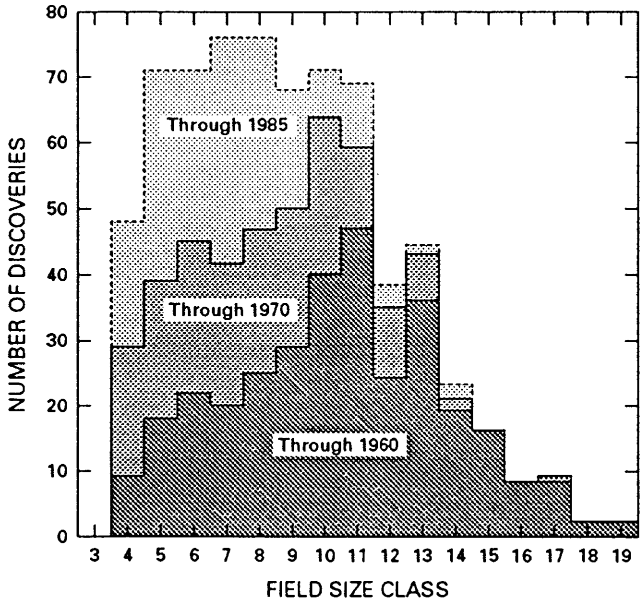

Figure 6.--Observed field-size distribution for the Frio Strandplain play, onshore Texas. The distribution has three segments representing the cumulative number of fields discovered through the specified year (from Drew, 1990).

Based on the data through 1960, a normal distribution would be a reasonable fit to the observed field-size distribution for the Frio Strandplain play, onshore Texas. Through 1970, a lognormal distribution could be considered a good model for the observed distribution. We propose that the data through 1985 are appropriately fit by a fractal (power law) or Pareto distribution since we are actually interested in modeling the ultimate or parent distribution and not simply the observed distribution.

A fractal distribution is scale invariant and exactly or approximately obeys a power-law (size cumulative-frequency) relation, with a fractional exponent. Thus, if the size distribution of petroleum accumulations is fractal, the number of accumulations of volume $V \geq v$, $N(v)$, would obey a relation of the form

$$N(v) = Av^{-a} \qquad v > 0$$

where the exponent, a, is the shape parameter of the Pareto distribution.

Our choice of the fractal distribution is based on our observation (also noted by Drew, 1990) that there is a shift and increase of the size frequency of discoveries to the left with time (figure 6). There is almost no change in the shape of the distribution for field-size classes greater than eleven, and future growth in the larger size classes is not expected to be significant. For size classes less than or equal to eleven, the distribution is strongly affected by economic factors that limit the discovery and development of field-size classes smaller than twelve. So we fit only that part of the distribution greater than size class eleven.

The data through 1985 (figure 6) for field-size classes (in million barrels of oil equivalent) greater than eleven are plotted on the log-log plot in figure 7 and are fit by a straight line whose negative slope, -a, is -0.83, i.e., shape parameter, a, is 0.83. A correlation coefficient of -0.97 indicates a good fit where -1.0 is a perfect negative fit. We propose that this is an appropriate though approximate fit to the data and that extrapolation of the line to smaller field sizes may be a valid basis for prediction of the ultimate undiscovered petroleum in field-size classes less than or equal to eleven.

CONCLUSIONS

1. The Pareto probability distribution is a fractal distribution that yields a reasonably good fit to the data for oil and gas accumulation-size distributions.
2. The scale invariance under lower truncation of the Pareto distribution permits characterization of economic truncation on the left side of the petroleum accumulation-size distribution. This fractal property enables the parent population to be determined from an economically truncated field-size distribution.
3. The asymptotic scale invariance under addition of the Pareto distribution might be representative of the complexity of the geologic multiplicative processes determining the oil and gas accumulation sizes.
4. The Pareto cumulative distribution is most usefully described by a straight line with negative slope on a log-log plot.
5. The absolute value of the slope of the straight line on a log-log plot is an estimate of the shape parameter or Pareto's exponent and provides a useful quantitative measure for comparing petroleum plays.

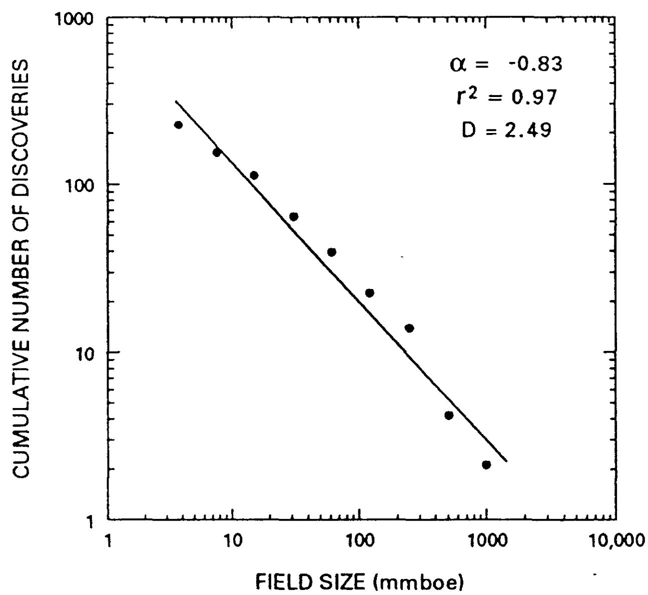

Figure 7.--Cumulative frequency plot of petroleum field sizes for the Frio Strandplain play, onshore Texas.

6. The upper truncated Pareto distribution is also used as a model for petroleum accumulation-size distributions since the largest accumulation is usually discovered early in the exploration process. This distribution has the important property that its moments always exist.

SELECTED NOMENCLATURE (in order of occurrence in text)

$N(\delta)$	= number of "boxes" (δ x δ squares) or straight-line segments needed to cover the coastline as a function of δ
δ	= side-length of square box or line segment (real number > 0)
D	= fractal dimension (fraction > 0)
A	= constant of proportionality (prefactor parameter)
X	= random variable (e.g., personal annual income over a population)
x	= values of random variable X (e.g., annual incomes; real numbers > 0)
N(x)	= number of persons having income $\geq$ x (cumulative frequency)
a	= shape parameter called Pareto's constant or exponent (real number > 0)
$P(X \geq x)$	= probability that random variable X takes on a value at least x (complementary cumulative probability distribution function of X)
k	= scale parameter of the Pareto distribution
f(x)	= Pareto probability density function of X
f(k)	= Pareto probability density function evaluated at k
E(X)	= expected value or mean of X
V(X)	= variance of X

$T(X)$	= random variable that is a transformation of the random variable X
$P(X \geq x \mid X \geq k')$	= probability that random variable X takes on a value at least x, given that X is at least k'
k'	= rescaled truncated value ($0 < k < k' \leq x$)
$P(X \geq cx \mid X \geq ck')$	= probability that random variable X takes on a value at least cx, given that X is at least ck'
c	= any positive number ($c > 0$)
$\lim_{x \to \infty} P(X < x)x^a$	= limit of $P(X < x)x^a$ as $x \to \infty$
$\lim_{x \to \infty} P(X > x)x^a$	= limit of $P(X > x)x^a$ as $x \to \infty$
$P(X \geq x) \sim \left(\frac{x}{k}\right)^{-a}$ as $x \to \infty$	= $P(X \geq x)$ is asymptotically equal to $(x/k)^{-a}$ as $x \to \infty$
V	= volume or size of petroleum accumulation in play (a random variable)
v	= numerical volumes or sizes of petroleum accumulations (values of V)
N(v)	= number of petroleum accumulations having volume or size $\geq$ v (cumulative frequency)

REFERENCES

Aitchison, J., and Brown, J.A.C., 1957, The Lognormal Distribution: London, Cambridge University Press, 176 p.

Arps, J.J., and Roberts, T.G., 1958, Economics of drilling for Cretaceous oil on east flank of Denver-Julesburg Basin: American Association of Petroleum Geologists Bulletin, v. 42, n. 11, p. 2549-2566.

Attanasi, E.D., and Drew, L.J., 1985, Lognormal field size distributions as a consequence of economic truncation: Mathematical Geology, v. 17, p. 335-351.

Crovelli, R.A., 1984, Procedures for petroleum resource assessment used by the U.S. Geological Survey--statistical and probabilistic methodology, in Masters, C.D., ed., Petroleum resource assessment: Ottawa, International Union of Geological Sciences, pub. no. 17, p. 24-38.

Drew, L.J., Schuenemeyer, J.H., and Bawiec, W.J., 1982, Estimation of the future rates of oil and gas discoveries in the Gulf of Mexico: U.S. Geological Survey Professional Paper 1252.

Drew, L.J., 1990, Oil and gas forecasting--Reflections of a petroleum geologist: New York, Oxford University Press, International Association for Mathematical Geology, Studies in Mathematical Geology No. 2, 252 p.

Feder, J., 1988, Fractals: New York and London, Plenum Press, 238 p.

Garb, F.A., and Smith, G.L., 1987, Estimation of oil and gas reserves, Chapter 40, in Bradley, H.B., editor-in-chief, Petroleum Engineering Handbook: Richardson, Texas, U.S.A., Society of Petroleum Engineers, p. 40-1 to 40-38.

Goldberg, G., 1967, The Pareto law and the pyramid distribution, Publication 505: Houston, Texas, Shell Development, 15 p.

Goldberger, A.L., Bhargava, V., West, B.J., and Mandell, A.J., 1985, On a mechanism of cardiac electrical stability: the fractal hypothesis: Biophysics Journal, v. 48, p. 525-528.

Houghton, J.C., 1988, Use of the truncated shifted Pareto distribution in assessing size distribution of oil and gas fields: Mathematical Geology, v. 20, no. 8, p. 907-938.

Johnson, N.L., and Kotz, S., 1970, Continuous univariate distributions-1: New York, John Wiley & Sons, Inc., 300 p.

Kaufman, G.M., 1962 [1963], Statistical decision and related techniques in oil and gas exploration: Ph.D. thesis, Harvard University, Cambridge, Massachusetts. [Reprinted, 1963, New York, Prentice Hall, 307 p.]

Kaufman, G.M., Balcer, Y., and Kruyt, D., 1975, A probabilistic model of oil and gas discovery. In Studies in Geology, No. 1-Methods of estimating the volume of undiscovered oil and gas resources: American Association of Petroleum Geologists, Tulsa, Oklahoma, p. 113-142.

Krige, D.G., 1960, On the departure of ore value distributions from the lognormal model in South African Gold Mines: Journal of South African Institute Mining and Metallurgy, November 1960, p. 231-244.

Lévy, P., 1925, Calcul des Probabilites: Paris, Gauthier-Villars, Chapter 6.

Lotka, A.J., 1926, The frequency distribution of scientific productivity: Journal Washington Academy of Science, v. 16, 317 p.

Mandelbrot, B.B., 1960, The Pareto-Lévy law and the distribution of income: International Economic Review, v. 1, p. 79-106.

Mandelbrot, B.B., 1962, Statistics of natural resources and the law of Pareto: International Business Machines Research Note NC-146, June 29, 1962, 31 p. Reprinted in this volume.

Mandelbrot, B.B., 1963, The variation of certain speculative prices: Journal of Business, University of Chicago, v. 36, p. 394-419.

Mandelbrot, B.B., 1964, Random walks, fire damage amount, and other Paretian risk phenomena: Operations Research, v. 12, p. 582-585.

Mandelbrot, B.B., 1966, Forecasts of future prices, unbiased markets, and 'martingale' models: Journal of Business, University of Chicago, v. 39, p. 242-255. Important errata in a subsequent issue of the same Journal.

Mandelbrot, B.B., 1967, The variation of some other speculative prices: Journal of Business, University of Chicago, v. 40, p. 393-413.

Mandelbrot, B.B., 1982, The fractal geometry of nature: San Francisco, California, W.H. Freeman and Co., 460 p.

Mast, R.F., Dolton, G.L., Crovelli, R.A., Root, D.H., Attanasi, E.D., Martin, P.E., Cooke, L.W., Carpenter, G.B., Pecora, W.C., and Rose, M.B., 1989, Estimates of undiscovered conventional oil and gas resources in the United States--a part of the Nation's energy endowment: U.S. Geological Survey and Minerals Management Service, 44 p.

McKie, J.W., 1960, Market structure and uncertainty in Oil and Gas exploration: Quarterly Journal of Economics, v. 74, p. 543-571.

The Oil and Gas Journal, 1958, Journal survey of proved reserves: where the Nation's oil is stored, January 27, 1958, p. 163-168.

Pareto, V., 1897, Cours d'Economie Politique: Lausanne and Paris, Rouge and Cie.

Schuenemeyer, J.H., and Drew, L.J., 1983, A procedure to estimate the parent population of the size of oil and gas fields as revealed by a study of economic truncation: Mathematical Geology, v. 15, p. 145-162.

West, B.J., and Shlesinger, M., 1990, The noise in natural phenomena: American Scientist, v. 78, p. 40-45.

Willis, J.C., 1922, Age and Area: a study in geographical distribution and origin of species: London, Cambridge University Press.

Zipf, G.K., 1949, Human behavior and the principle of least effort: Cambridge, Massachusetts, Addison-Wesley.