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Glossary of Terms for Chaos, Fractals, and Dynamics

by

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INTRODUCTION

This glossary of terms for chaos, fractals, and dynamics, based on terms in Devaney (1990), is a reference for scientists whose time is limited, but who would like to be exposed to the main ideas. However, the glossary can be used as a reference entirely independent of the Devaney book by anyone interested in this field of study. The purpose of this report is to bring together, in two convenient formats, definitions of important terms found in the subjects of chaos, fractals and dynamical systems. The terms describe many of the definitions, notations, concepts, principles and facts of these subjects. The report consists of two separate independent formats of terms. The first format is a listing of terms in order of occurrence by chapter for Devaney (1990). This format presents a very concise outline of organized ideas for chaos, fractals, and dynamics. The second format is a standard listing of terms in alphabetical order for Devaney (1990). This format allows for an easy reference of terms.

Terms: An Outline for Chaos, Fractals, and Dynamics

(Listed in order of occurrence by chapter for Devaney, 1990)

Chapter 0 - A Mathematical Tour

dynamical systems, 1. The branch of mathematics that studies processes in motion.

iteration, 2. Repeating the same mathematical operation using the output of the previous operation as the input for the next.

chaos, 2. Unpredictability of the system or extremely complicated results.

Julia set, 3. The set of numbers that yield chaotic or unpredictable behavior of quadratic functions. Julia sets are always fractals.

fractals, 3. Sets which, when magnified over and over again, always resemble the original image.

Mandelbrot set, 4. A dictionary, or picture book, of all possible quadratic Julia sets.

Chapter 1 - Iteration

iterate, 7. Repeat a process over and over again.

function, 7. An operation or process that converts certain numbers (the inputs) into other, possibly different numbers (the outputs).

iteration: function, 8. The process of evaluating a function repeatedly.

iteration using a scientific calculator, 8. Selecting an initial x as seed or input and then striking a particular function key repeatedly.

square root function, 9. Iteration of $\sqrt{x}$ eventually yields the number 1, which then remains unchanged or fixed under subsequent iterations.

functional notation, 10. For the square root function $S(x) = \sqrt{x}$, S means the operation of computing the nonnegative square root of the nonnegative number x. The squaring function $T(x) = x^2$, the cosine function $C(x) = \cos x$, and the exponential function $E(x) = \exp x$.

functional iteration notation, 11. $F^j(x)$ denotes the jth iteration of the function F. For example, $F^2(x) = F(F(x))$.

orbit of a point, 13. The list of successive iterates of a point or number.

orbit analysis, 15. The process of understanding all of the orbits of a given dynamical system.

basic question in dynamical systems, 16. Can we predict the fate of all orbits under iteration?

fixed point, 17. A point x_0 for F such that $F(x_0) = x_0$.

periodic orbit or cycle, 17. An orbit that eventually returns to where it began; i.e., the orbit of x_0 is periodic if there is a positive integer N such that $F^N(x_0) = x_0$.

periodic point, 17. The point x_0 of a periodic orbit is called a periodic point of period N.

prime period, 17. The least period N of a cycle.

eventually fixed or eventually periodic points, 18. Points whose orbit is not fixed or periodic but for which some later point on the orbit is fixed or periodic.

ecological application, 20. P_n denotes the percentage of some limiting population that is alive at generation n, and

$$P_{n+1} = cP_n(1-P_n)$$

where c is an ecological constant.

logistic function, 21. The quadratic function of the form $F(x) = cx(1-x)$ where c is a constant.

Chapter 2 - Iteration Using the Computer

program ITERATE1, 23. A BASIC program to iterate the function $2x(1-x)$ 25 times on a given initial input x_0. Note that this function is one of the logistic functions with $c = 2$.

parameter, 28. An arbitrary constant. For each different parameter value, we get a new function to iterate, e.g., the parameter c of the logistic function.

family of dynamical systems, 28. A family of functions with different parameter values, e.g., the logistic family.

bifurcation theory, 31. The study of where dynamical behavior changes as a parameter is varied.

computer orbit graphics, 32. Plot the points of an orbit of a dynamical system in succession on the computer screen.

changing coordinates, 33. Changing the x-coordinates of an orbit from the real line (where the dynamics are occurring) to computer screen coordinates.

program ITERATE2, 34. A BASIC program to display the first 200 points on the orbit of x_0 under $F(x) = 4x(1-x)$ where $0 < x_0 < 1$.

asymptotic behavior of an orbit, 36. The eventual behavior of an orbit, i.e., what happens to very high iterations.

program ITERATE3, 37. A BASIC program that accepts as input an initial value of x_0 together with the endpoints of an interval $1 \leq x_0 \leq r$ in which the orbit is plotted.

Chapter 3 - Graphical Analysis

graphical analysis, 39. A graphical technique to perform orbit analysis geometrically using only the graph of the function and to produce the orbit along the diagonal $F(x) = x$.

attracting fixed point, 45. The fixed point p is called attracting if there is an interval $a < x < b$ containing p in which all points have orbits that tend to p.

basin of attraction, 46. All points whose orbits tend to a given attracting fixed point.

repelling fixed point, 48. The fixed point p is called repelling if there is an interval $a < x < b$ containing p in which all points (except p) have orbits that leave the interval.

"seen", 50. Attracting fixed points can be "seen" using the computer, whereas repelling fixed points cannot.

principal themes, 50. The notions of stability and instability.

stable orbit, 50. An orbit that, if you change the initial input slightly, the resulting orbit behaves similarly.

unstable orbit, 50. An orbit whose nearby orbits have vastly different behaviors. Note that an attracting fixed or periodic point is always stable, whereas a repelling point is never stable (nearby initial conditions tend far away).

attracting and repelling periodic points, 51. Like fixed points, periodic orbits may also be either attracting or repelling.

neutral point, 55. Fixed point that is neither attracting or repelling.

Chapter 4 - The Quadratic Family

quadratic family, 57. The family of quadratic functions

$$Q_c(x) = x^2 + c, \text{ where } c \text{ is a parameter.}$$

nonlinear function, 57. A function not of the form ax + b.

quadratic function: non-escaping orbits, 59. All the "interesting" dynamics of Q_c are confined to the interval $-p \leq x \leq p$ when $c \leq 1/4$ where

$$p = \frac{1 + \sqrt{1-4c}}{2}$$

bifurcation, 59. A change, a splitting apart or a division in two.

saddle-node, or tangent, bifurcation, 59. A fixed point first appears and then suddenly splits into two fixed points as c varies.

quadratic function: attracting fixed point, 62. For $-3/4 \leq c \leq 1/4$, all orbits appear to be attracted to an attracting fixed point.

quadratic function: period 2 cycle, 63. For $-5/4 \leq c < -3/4$, most orbits appear to be attracted to a period 2 cycle.

quadratic function: period 4 and higher cycles, 63. As c decreases below -5/4, most orbits are attracted to a period 4 cycle, then a period 8 cycle, then a period 16 cycle, and so forth.

period-doubling bifurcation, 67. A period N cycle spawns a new attracting cycle of period 2N, which then doubles and gives an orbit of period 4N, and so forth.

chaotic quadratic function, 68. For c = -2 in Q_c, it appears that a "typical" orbit fills up the interval $-2 \leq x \leq 2$.

orbit diagram, 70. A diagram that plots the orbit of a particular point versus the parameter value c.

program ORBITDGM, 70. A BASIC program which produces the orbit diagram. The first 200 points on the orbit of 0 are calculated for each c, but only the last 150 points are plotted to see the eventual, or asymptotic, behavior of the orbit.

Chapter 5 - Iteration in the Complex Plane

complex number, 75. A number of the form z = x + iy, where x and y are real numbers and $i = \sqrt{-1}$.

complex number geometry, 76. Complex numbers may be plotted in the plane in the natural way, with $x + iy$ plotted at the point with coordinates (x,y).

modulus $|x + iy|$, 76. The distance from $x + iy$ to the origin, i.e.,

$$|x + iy| = \sqrt{x^2 + y^2}$$

complex number algebra, 76. Addition and multiplication of complex numbers obey all of the usual rules of algebra, including the commutative, associative, and distributive laws.

triangle inequality, 77. A geometric interpretation of addition of complex numbers gives $|z + w| \leq |z| + |w|$.

program ITERATE4, 78. A BASIC program to iterate the complex squaring function $T(z) = z^2$ or

$$T(x + iy) = x^2 - y^2 + i(2xy)$$

changing complex coordinates, 80. Changing the xy-coordinates of an orbit from the complex plane to computer screen coordinates.

orbit analysis of complex squaring function, 81. Orbits of points under T fall into three categories:

a. Points with modulus less than 1 tend to 0, a fixed point.

b. Points with modulus larger than 1 escape to infinity.

c. Points with modulus exactly 1 remain forever on the circle of radius 1.

Julia set of T, 81. Points with modulus exactly 1, i.e., the circle of radius 1 in the complex plane.

Julia set is invariant under T, 82. If a point starts on the Julia set of T, complex squaring function, its orbit necessarily remains there forever.

Julia set: computing error, 82. Small round-off errors made in computing the orbit throws us off the Julia set, and then the orbit tends to 0 or to infinity.

Julia set of a polynomial, 83. The boundary of the set of points that escape to infinity. A point in the Julia set has an orbit that does not escape to infinity, but arbitrarily nearby there are points whose orbits do escape. The Julia set is precisely the set of unstable orbits of a complex dynamical system.

program ITERATE5, 83. A BASIC program to iterate the quadratic polynomial of the form $Q_c(z) = z^2 + c$, where both z and c are complex numbers.

Chapter 6 - The Julia Set: Basin Boundaries

Julia set: algorithm, 85. An algorithm to compute the computer graphics of the Julia set of $Q_c(z) = z^2 + c$, where c is a complex parameter, that works well when Q_c has an attracting periodic orbit.

Julia set: escaping orbits, 86. An orbit tends to infinity for quadratic functions of the form $z^2 + c$ with $|c| \leq 2$ if any point on the orbit of z_o lies outside the circle of radius 2, i.e., has modulus that exceeds 2.

program JULIA1, 87. A BASIC program for the function $Q_c(z) = z^2 + c$ when $|c| \leq 2$ to display in black the set of points within the square $|x|, |y| \leq 2$ whose orbit has not escaped beyond the circle of radius 2 centered at the origin before the 20th iteration of Q_c.

filled-in Julia set, 90. The set of points whose orbit does not escape to infinity under iteration of Q_c, i.e., the entire black region. The Julia set is the boundary, or edge, of the black region.

self-similarity under magnification, 93. A basic property of fractals. The further we delve into the filled-in Julia set, the more decorations we see, but the pictures generated in smaller windows bear a strong resemblance to our original picture.

Chapter 7 - The Julia Set: Other Algorithms

Julia set: other algorithms, 97. Alternative methods for displaying the Julia set of a polynomial function.

backward iteration method, 97. An algorithm that produces an image of the Julia set itself rather than the filled-in Julia set and works well when the first method fails, notably when the Julia set is "fractal dust."

polar representation of a complex number, 97. A complex number $z = x + iy$ with Cartesian coordinates (x,y) has polar coordinates (r, θ), where r is the modulus and θ is the polar angle.

polar transformations of a complex number, 98. Given the polar coordinates (r, θ) of a complex number z, then the Cartesian coordinates are $x = r \cos \theta$ and $y = r \sin \theta$. Conversely, given the Cartesian coordinates (x, y) of a complex number z, then the polar coordinates are

$$r = \sqrt{x^2 + y^2} \text{ and } \theta = \arctan (y/x) \text{ if } x > 0$$
$$= \arctan (y/x) + \pi \text{ if } x < 0$$

square roots of a complex number, 100. Given the polar coordinates (r, θ) of a complex number z, then the two square roots of z are given by

$$w_1 = \sqrt{r} \cos (\theta/2) + i \sqrt{r} \sin (\theta/2)$$

$$w_2 = -\sqrt{r} \cos (\theta/2) - i \sqrt{r} \sin (\theta/2)$$

backward orbits of T, 101. The Julia set of T, complex squaring function, can be found by computing orbits backward, using the square root since the two square roots of z_0 lie nearer the circle of radius 1 than does z_0. Each time we select one of the two possible square roots, the point we have selected is closer to the Julia set.

program JULIA2, 102. A BASIC program to produce a rough picture of the Julia set of $Q_c(z) = z^2 + c$, using the backward iteration method.

fractal dust, 107. A certain type of Julia set of Q_c that, under magnification over and over, consists of a "cloud" of points, each of which lies in a separate piece of the Julia set. Fractal dust is a Cantor set which is totally disconnected.

totally disconnected set, 107. A subset of the plane such that, given any two points in the set, there is always a closed curve that does not intersect the set and that surrounds one of the points but not the other.

connected set, 108. A subset of the plane such that it is impossible to find a closed curve that is disjoint from the set and that separates the set into two disjoint pieces.

fractal dust: theorem, 110. Whenever the orbit of 0 escapes to infinity, the Julia set of Q_c is fractal dust, i.e., is totally disconnected. On the other hand, if the orbit of 0 does not escape, the Julia set and the filled-in Julia set are connected sets. Either the Julia set of Q_c consists of one piece or else it consists of infinitely many.

critical point, 110. The point 0 is the only point at which the derivative of Q_c vanishes.

critical orbit, 110. The orbit of 0 determines virtually all of the dynamics of Q_c.

Julia set: bifurcation, 111. For $c > 1/4$, the Julia set of Q_c is totally disconnected, but when $-2 \leq c \leq 1/4$, the Julia set is connected.

boundary-scanning method, 111. An algorithm that produces an image of the Julia set (not the filled-in Julia set) and works very well when Q_c has an attracting cycle, but it takes a long time to run on the computer and is based on the very definition of the Julia set.

program JULIA3, 112. A BASIC program to produce a very sharp image of the Julia set of Q_c via the boundary-scanning method.

Chapter 8 - The Mandelbrot Set

Mandelbrot set: description, 113. A compilation of all of the Julia sets of the quadratic function $Q_c(z) = z^2 + c$ that explains how all of these different structures and shapes are related.

critical orbit: fact, 114. Given that Q_c has an attracting periodic orbit, then the critical orbit is attracted to this orbit.

Mandelbrot set: construction, 115. We need simply to understand the orbit of 0 under Q_c for each different c-value. To be precise, the Mandelbrot set, M, is the set of c-values for which the critical orbit of Q_c does not tend to infinity. The Mandelbrot set is a picture in the c-plane, not in the z-plane, where the Julia sets live. (Mandelbrot set: fix $z_0 = 0$ and vary c; Julia set: fix c and vary z_0).

Mandelbrot set: algorithm, 116. The algorithm to compute the Mandelbrot set states: simply compute the orbit of 0

$$0, c, c^2 + c, (c^2 + c)^2 + c, \ldots$$

and check whether any point on this orbit has modulus larger than 2. Once this occurs, we are guaranteed that the critical orbit escapes and c does not belong to the Mandelbrot set.

program MANDELBROT1, 116. A BASIC program to draw the Mandelbrot set. This program is similar to JULIA1; the difference being to treat each grid point as a c-value.

Mandelbrot set: symmetry, 117. Unlike the Julia sets of Q_c, the Mandelbrot set is not symmetric about the origin, but it is symmetric about the x-axis.

complex conjugate, 118. If $z = x + iy$ is a complex number, its complex conjugate is a new complex number given by $\bar{z} = x - iy$.

Mandelbrot set: image, 119. Unlike the situation for Julia sets, where we get a different picture for each different c-value, here there is only one image. In M, each pixel that is colored black corresponds to a quadratic function whose Julia set is connected. The Mandelbrot set tells us which c-values correspond to Julia sets with the same basic shape and which quadratic functions have more or less the same dynamics.

program MANDELBROT2, 121. A BASIC program that allows the user to select a square or window from the output of MANDELBROT1.

Mandelbrot set: baby, 121. All the complexity that occurs in the "main body" of M occurs as well within smaller copies inside certain magnified windows. So there are baby Mandelbrot sets within these small copies of Mandelbrot set, and so on.

Mandelbrot set: connected, 123. The Mandelbrot set is a connected set despite appearances to the contrary.

program MANDELBROT3, 124. A BASIC program to generate only the boundary of the Mandelbrot set by the boundary-scanning method that was used to display Julia sets.

program MANDELBROT4, 124. A BASIC program to understand the relationship between the Julia sets for Q_c and the corresponding c-values in M. The program builds on MANDELBROT2 and allows the user to select a particular c-value from the image displayed by MANDELBROT2. Then the user is given the choice of selecting one of the previous programs, ITERATE5 or JULIA2, as applied to Q_c for the chosen c-value. The screen is then cleared and the appropriate orbit or Julia set displayed.

Mandelbrot set: bulb periods, 126. We can assign to each bulb in M a number that corresponds to the period of the attracting cycle for each Q_c with c in that bulb.

Mandelbrot set: orbit diagram, 126. There is a direct relationship between the orbit diagram and the horizontal slice through the middle of M since this horizontal line represents what happens to the critical orbit for real values of c.

Chapter 9 - Geometric Iteration: Fractals

geometric iteration, 129. Previously, iterated functions gave rise to complicated dynamics. Now iterate certain geometric constructions. These iterations will yield the complicated geometric objects known as fractals.

fractal geometry, 130. A new branch of mathematics to model many of the intricate patterns found in nature.

fractal: definition, 130. A fractal is a geometric shape that has two special properties:

1. The object is self-similar.
2. The object has fractional dimension.

Sierpinski triangle, 130. The Sierpinski triangle is generated by an infinite succession of "removals." Beginning with an equilateral triangle, remove the middle upside-down triangle. Iterate this procedure: from each remaining equilateral triangle, remove the middle triangle, leaving three smaller triangles behind.

Sierpinski triangle: self-similarity, 132. The portion of the triangle contained within a triangle at level n, when magnified by a factor of 2^n, is exactly the same as the whole triangle. That is, small portions of the triangle, when magnified, are similar to the whole triangle.

Cantor set or Cantor middle-thirds set, 133. The Cantor set, the simplest of all fractals, is generated by an infinite succession of "removals." To construct this set, begin with the interval $0 \leq x \leq 1$ and remove the middle third $1/3 < x < 2/3$. Iterate this procedure: from each remaining interval remove the middle third, leaving two smaller intervals behind.

Cantor set: totally disconnected, 134. Between any two points in the set, there must be points that do not belong to the set.

Cantor set: self-similar, 134. Magnification shows that there are just as many points in the left-hand interval after the first interval is removed as there are in the whole set. That is, this small portion of the Cantor set is exactly the same as the whole set.

Cantor set: endpoints, 136. Each endpoint of an interval corresponds to a sequence that consists of a finite number of lefts or rights, followed by an infinite string that is either all lefts or all rights.

Cantor set: fact, 137. Most points in the Cantor set are not endpoints! There is a point in the Cantor set that corresponds to any sequence of lefts and rights.

Koch snowflake, 137. The Koch snowflake is generated by an infinite succession of "additions." Beginning with the boundary of an equilateral triangle with side of length 1, replace each middle third with two pieces of equal length, giving a star-shaped region. Iterating this process, the ultimate result is a curve that is infinitely wiggly--there are no straight lines in it whatsoever.

Koch snowflake: self-similar, 138. At each stage of construction, magnification by a factor of 3 yields the previous image.

Koch snowflake: fact, 138. This fractal has an amazing geometric property: It has finite area but its perimeter is infinite!

computing fractals, 139. Certain fractals such as the Cantor set or the Sierpinski triangle are easy to draw using a computer and an iterative procedure similar to the backward iteration method. These fractals are obtained as the orbit of a single point under the iteration in random order of a collection of functions.

program FRACTAL, 142. A BASIC program to compute a random orbit of any point (M,N). No matter which M and N are input, the same figure results.

iterated function system, 143. A collection of functions iterated in random order. These kinds of dynamical systems are quite important in such applications as image and data compression.

attractor, 143. A fractal that results from an iterated function system because it attracts all of the random orbits.

fractional dimension, 145. The dimension D of a geometric object is given by the formula

$$D = \frac{\log\text{ (number of pieces)}}{\log\text{ (magnification)}}$$

fractional dimension: Sierpinski triangle, 146. The dimension of the Sierpinski triangle is $D = \log 3/\log 2 = 1.585$.

fractional dimension: Koch snowflake, 146. The dimension of the Koch snowflake is $D = \log 4/\log 3 = 1.262$.

fractional dimension: Cantor set, 146. The dimension of the Cantor set is $D = \log 2/\log 3 = 0.6039$.

fractals and dynamics, 147. Fractals arise naturally in dynamical systems. For example, the Cantor middle-thirds set arises naturally in a dynamical system by considering a tent function

$$T(x) = \begin{cases} 3x & x \leq 1/2 \\ 3-3x & x \geq 1/2 \end{cases}$$

It is precisely the Cantor middle-thirds set whose points have orbits that do not tend to $-\infty$. Thus, all the interesting dynamics for T take place on a fractal, the Cantor middle-thirds set.

Chapter 10 - Chaos

chaos: example, 151. Very simple dynamical processes can behave in a very complicated, almost random fashion. For example, the dynamics of the logistic function $F_c(x) = cx(1-x)$, where, for many values of c, the successive points on the orbit of x under F_c seem to hop around the line unpredictably.

chaos: squaring function, 152. The prototype of a chaotic function is the squaring function, $T(z) = z^2$, where z is complex. It is the points on the circle of radius 1, the Julia set, whose orbits are chaotic. Many, many of these points in polar form in the interval $0 \leq \theta \leq 2\pi$ have orbits that are periodic under a doubling function, but the computer fails to find them! This happens because all of these periodic points are repelling and hence unstable.

sensitive dependence on initial conditions, 154. The essential ingredient of a chaotic system. This means that, no matter how close two orbits start out, after just a few iterations they will be very far apart. The orbits separate from each other exponentially. A small error in initial conditions makes a big difference in the results. This is sensitive dependence on initial conditions and is the typical behavior in a chaotic system.

chaos: Julia sets, 156. Points on the Julia set of the squaring function T have orbits that depend very sensitively on initial conditions. Arbitrarily close to any point on the circle is another point whose orbit eventually hits any point whatsoever in the plane, with only one exception. Amazingly, this property is true for all Julia sets. Points in the Julia set have orbits that are the most unstable of all orbits.

Chapter 11 - Julia Sets of Other Functions

Julia sets of higher-degree polynomials, 159. Although every polynomial has a Julia set, the easiest polynomials with which to work are those that have the special form $P_c(z) = z^n + c$ with $n = 3, 4, 5, \ldots$. The exponent n is called the degree of this polynomial. Like the quadratic functions, these polynomials have a single critical point at 0, since 0 is the only point whose image is c. To compute the Julia set of P_c, you may use any of the three methods discussed in Chapters 6 and 7, but some modifications are necessary. Most polynomials have more than one critical point (critical points are simply the points at which the derivative vanishes).

Julia sets of higher-degree polynomials: properties, 160. The polynomials P_c have basically the same properties as the quadratic functions. For example,

a. If the critical orbit escapes, the Julia set is fractal dust.

b. If P_c has an attracting cycle, then the critical orbit is attracted to it.

c. If P_c has an attracting cycle, then the Julia set of P_c is connected.

Mandelbrot sets for higher-degree polynomials, 163. Since the polynomials P_c have a single critical orbit, there is an analogue of the Mandelbrot set for each integer n. As for the quadratic functions, this set is the collection of c-values for which the critical orbit does not escape. This is precisely the set of c-values for which the Julia set is connected. This set is called the <u>degree</u> n <u>bifurcation</u> <u>set</u>.

Euler's formula, 165. Euler's formula relates the exponential function to the trigonometric functions sin x and cos x. Euler's formula is

$$e^{ix} = \cos x + i \sin x$$

complex exponential function, 166. Euler's formula allows us to define the complex exponential function. By Euler's formula, if $z = x + iy$, we have

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y)$$

One major difference between the exponential function and polynomials is the fact that points that are far from the origin in the complex plane are no longer taken even further away by the function.

Julia sets of transcendental functions, 167. Functions such as the exponential, the sine, and the cosine are called transcendental functions. For transcendental functions, we concentrate on the set of points whose orbits escape. For technical reasons, this set contains the Julia set. This may appear to contradict the definition given for polynomials (the Julia set was the boundary of the set of escaping points for polynomials), but actually it does not. For transcendental functions such as the sine or cosine, any point whose orbit escapes is actually on the boundary of the set of escaping orbits.

program JULIAEXP, 168. A BASIC program to compute the Julia set of $E_c(z) = ce^z$, where c is a complex parameter. The algorithm produces a picture whose white points lie in the Julia set. The Julia sets for exponential functions look quite different from those of polynomials.

hyperbolic functions, 170. The hyperbolic cosine and hyperbolic sine functions are defined by

$$\cosh y = \frac{e^y + e^{-y}}{2} \text{ and } \sinh y = \frac{e^y - e^{-y}}{2}$$

program JULIASIN, 171. A BASIC program which computes the Julia set of sin z. This program displays the Julia set in white, not black.

program JULIACOS, 171. A BASIC program which computes the Julia set of c cos z. This program displays the Julia set in white, not black.

exploding Julia sets, 172. The Julia sets of polynomials occasionally underwent dramatic changes when the function experienced a saddle-node or period-doubling bifurcation. The same is true for transcendental functions. Often, these changes are quite spectacular for the complex sine, cosine, or exponential. These bifurcations are called explosions.

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chaos, 2. Unpredictability of the system or extremely complicated results.

chaos: example, 151. Very simple dynamical processes can behave in a very complicated, almost random fashion. For example, the dynamics of the logistic function $F_c(x) = cx(1-x)$, where, for many values of c, the successive points on the orbit of x under F_c seem to hop around the line unpredictably.

chaos: Julia sets, 156. Points on the Julia set of the squaring function T have orbits that depend very sensitively on initial conditions. Arbitrarily close to any point on the circle is another point whose orbit eventually hits any point whatsoever in the plane, with only one exception. Amazingly, this property is true for all Julia sets. Points in the Julia set have orbits that are the most unstable of all orbits.

chaos: squaring function, 152. The prototype of a chaotic function is the squaring function, $T(z) = z^2$, where z is complex. It is the points on the circle of radius 1, the Julia set, whose orbits are chaotic. Many, many of these points in polar form in the interval $0 \leq \theta \leq 2\pi$ have orbits that are periodic under a doubling function, but the computer fails to find them! This happens because all of these periodic points are repelling and hence unstable.

chaotic quadratic function, 68. For $c = -2$ in Q_c, it appears that a "typical" orbit fills up the interval $-2 \le x \le 2$.

changing complex coordinates, 80. Changing the xy-coordinates of an orbit from the complex plane to computer screen coordinates.

changing coordinates, 33. Changing the x-coordinates of an orbit from the real line (where the dynamics are occurring) to computer screen coordinates.

complex conjugate, 118. If $z = x + iy$ is a complex number, its complex conjugate is a new complex number given by $\bar{z} = x - iy$.

complex exponential function, 166. Euler's formula allows us to define the complex exponential function. By Euler's formula, if $z = x + iy$, we have

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y)$$

One major difference between the exponential function and polynomials is the fact that points that are far from the origin in the complex plane are no longer taken even further away by the function.

complex number, 75. A number of the form $z = x + iy$, where x and y are real numbers and $i = \sqrt{-1}$.

complex number algebra, 76. Addition and multiplication of complex numbers obey all of the usual rules of algebra, including the commutative, associative, and distributive laws.

complex number geometry, 76. Complex numbers may be plotted in the plane in the natural way, with $x + iy$ plotted at the point with coordinates (x,y).

computer orbit graphics, 32. Plot the points of an orbit of a dynamical system in succession on the computer screen.

computing fractals, 139. Certain fractals such as the Cantor set or the Sierpinski triangle are easy to draw using a computer and an iterative procedure similar to the backward iteration method. These fractals are obtained as the orbit of a single point under the iteration in random order of a collection of functions.

connected set, 108. A subset of the plane such that it is impossible to find a closed curve that is disjoint from the set and that separates the set into two disjoint pieces.

critical orbit, 110. The orbit of 0 determines virtually all of the dynamics of Q_c.

critical orbit: fact, 114. Given that Q_c has an attracting periodic orbit, then the critical orbit is attracted to this orbit.

critical point, 110. The point 0 is the only point at which the derivative of Q_c vanishes.

-D-

dynamical systems, 1. The branch of mathematics that studies processes in motion.

-E-

ecological application, 20. P_n denotes the percentage of some limiting population that is alive at generation n, and

$$P_{n+1} = cP_n(1-P_n)$$

where c is an ecological constant.

Euler's formula, 165. Euler's formula relates the exponential function to the trigonometric functions sin x and cos x. Euler's formula is

$$e^{ix} = \cos x + i \sin x$$

eventually fixed or eventually periodic points, 18. Points whose orbit is not fixed or periodic but for which some later point on the orbit is fixed or periodic.

exploding Julia sets, 172. The Julia sets of polynomials occasionally underwent dramatic changes when the function experienced a saddle-node or period-doubling bifurcation. The same is true for transcendental functions. Often, these changes are quite spectacular for the complex sine, cosine, or exponential. These bifurcations are called <u>explosions</u>.

-F-

family of dynamical systems, 28. A family of functions with different parameter values, e.g., the logistic family.

filled-in Julia set, 90. The set of points whose orbit does not escape to infinity under iteration of Q_c, i.e., the entire black region. The Julia set is the boundary, or edge, of the black region.

fixed point, 17. A point x_0 for F such that $F(x_0) = x_0$.

fractal: definition, 130. A fractal is a geometric shape that has two special properties:

1. The object is self-similar.
2. The object has fractional dimension.

fractal dust, 107. A certain type of Julia set of Q_c that, under magnification over and over, consists of a "cloud" of points, each of which lies in a separate piece of the Julia set. Fractal dust is a Cantor set which is totally disconnected.

fractal dust: theorem, 110. Whenever the orbit of 0 escapes to infinity, the Julia set of Q_c is fractal dust, i.e., is totally disconnected. On the other hand, if the orbit of 0 does not escape, the Julia set and the filled-in Julia set are connected sets. Either the Julia set of Q_c consists of one piece or else it consists of infinitely many.

fractal geometry, 130. A new branch of mathematics to model many of the intricate patterns found in nature.

fractals, 3. Sets which, when magnified over and over again, always resemble the original image.

fractals and dynamics, 147. Fractals arise naturally in dynamical systems. For example, the Cantor middle-thirds set arises naturally in a dynamical system by considering a <u>tent</u> function

$$T(x) = \begin{cases} 3x & x \leq 1/2 \\ 3-3x & x \geq 1/2 \end{cases}$$

It is precisely the Cantor middle-thirds set whose points have orbits that do not tend to $-\infty$. Thus, all the interesting dynamics for T take place on a fractal, the Cantor middle-thirds set.

fractional dimension, 145. The dimension D of a geometric object is given by the formula

$$D = \frac{\log\text{ (number of pieces)}}{\log\text{ (magnification)}}$$

fractional dimension: Cantor set, 146. The dimension of the Cantor set is $D = \log 2/\log 3 = 0.6039$.

fractional dimension: Koch snowflake, 146. The dimension of the Koch snowflake is $D = \log 4/\log 3 = 1.262$.

fractional dimension: Sierpinski triangle, 146. The dimension of the Sierpinski triangle is $D = \log 3/\log 2 = 1.585$.

function, 7. An operation or process that converts certain numbers (the inputs) into other, possibly different numbers (the outputs).

functional iteration notation, 11. $F^j(x)$ denotes the jth iteration of the function F. For example, $F^2(x) = F(F(x))$.

functional notation, 10. For the square root function $S(x) = \sqrt{x}$, S means the operation of computing the nonnegative square root of the nonnegative number x. The squaring function $T(x) = x^2$, the cosine function $C(x) = \cos x$, and the exponential function $E(x) = \exp x$.

-G-

geometric iteration, 129. Previously, iterated functions gave rise to complicated dynamics. Now iterate certain geometric constructions. These iterations will yield the complicated geometric objects known as fractals.

graphical analysis, 39. A graphical technique to perform orbit analysis geometrically using only the graph of the function and to produce the orbit along the diagonal $F(x) = x$.

-H-

hyperbolic functions, 170. The hyperbolic cosine and hyperbolic sine functions are defined by

$$\cosh y = \frac{e^y + e^{-y}}{2} \text{ and } \sinh y = \frac{e^y - e^{-y}}{2}$$

-I-

iterate, 7. Repeat a process over and over again.

iterated function system, 143. A collection of functions iterated in random order. These kinds of dynamical systems are quite important in such applications as image and data compression.

iteration, 2. Repeating the same mathematical operation using the output of the previous operation as the input for the next.

iteration: function, 8. The process of evaluating a function repeatedly.

iteration using a scientific calculator, 8. Selecting an initial x as seed or input and then striking a particular function key repeatedly.

-J-

Julia set, 3. The set of numbers that yield chaotic or unpredictable behavior of quadratic functions. Julia sets are always fractals.

Julia set: algorithm, 85. An algorithm to compute the computer graphics of the Julia set of $Q_c(z) = z^2 + c$, where c is a complex parameter, that works well when Q_c has an attracting periodic orbit.

Julia set: bifurcation, 111. For $c > 1/4$, the Julia set of Q_c is totally disconnected, but when $-2 \leq c \leq 1/4$, the Julia set is connected.

Julia set: computing error, 82. Small round-off errors made in computing the orbit throws us off the Julia set, and then the orbit tends to 0 or to infinity.

Julia set: escaping orbits, 86. An orbit tends to infinity for quadratic functions of the form $z^2 + c$ with $|c| \leq 2$ if any point on the orbit of z_0 lies outside the circle of radius 2, i.e., has modulus that exceeds 2.

Julia set is invariant under T, 82. If a point starts on the Julia set of T, complex squaring function, its orbit necessarily remains there forever.

Julia set of a polynomial, 83. The boundary of the set of points that escape to infinity. A point in the Julia set has an orbit that does not escape to infinity, but arbitrarily nearby there are points whose orbits do escape. The Julia set is precisely the set of unstable orbits of a complex dynamical system.

Julia set of T, 81. Points with modulus exactly 1, i.e., the circle of radius 1 in the complex plane.

Julia set: other algorithms, 97. Alternative methods for displaying the Julia set of a polynomial function.

Julia sets of higher-degree polynomials, 159. Although every polynomial has a Julia set, the easiest polynomials with which to work are those that have the special form $P_c(z) = z^n + c$ with $n = 3, 4, 5, \ldots$. The exponent n is called the degree of this polynomial. Like the quadratic functions, these polynomials have a single critical point at 0, since 0 is the only point whose image is c. To compute the Julia set of P_c, you may use any of the three methods discussed in Chapters 6 and 7, but some modifications are necessary. Most polynomials have more than one critical point (critical points are simply the points at which the derivative vanishes).

Julia sets of higher-degree polynomials: properties, 160. The polynomials P_c have basically the same properties as the quadratic functions. For example,

a. If the critical orbit escapes, the Julia set is fractal dust.

b. If P_c has an attracting cycle, then the critical orbit is attracted to it.

c. If P_c has an attracting cycle, then the Julia set of P_c is connected.

Julia sets of transcendental functions, 167. Functions such as the exponential, the sine, and the cosine are called transcendental functions. For transcendental functions, we concentrate on the set of points whose orbits escape. For technical reasons, this set contains the Julia set. This may appear to contradict the definition given for polynomials (the Julia set was the boundary of the set of escaping points for polynomials), but actually it does not. For transcendental functions such as the sine or cosine, any point whose orbit escapes is actually on the boundary of the set of escaping orbits.

-K-

Koch snowflake, 137. The Koch snowflake is generated by an infinite succession of "additions." Beginning with the boundary of an equilateral triangle with side of length 1, replace each middle third with two pieces of equal length, giving a star-shaped region. Iterating this process, the ultimate result is a curve that is infinitely wiggly--there are no straight lines in it whatsoever.

Koch snowflake: fact, 138. This fractal has an amazing geometric property: It has finite area but its perimeter is infinite!

Koch snowflake: self-similar, 138. At each stage of construction, magnification by a factor of 3 yields the previous image.

-L-

logistic function, 21. The quadratic function of the form $F(x) = cx(1-x)$ where c is a constant.

-M-

Mandelbrot set, 4. A dictionary, or picture book, of all possible quadratic Julia sets.

Mandelbrot set: algorithm, 116. The algorithm to compute the Mandelbrot set states: simply compute the orbit of 0

$$0, c, c^2 + c, (c^2 + c)^2 + c, \ldots$$

and check whether any point on this orbit has modulus larger than 2. Once this occurs, we are guaranteed that the critical orbit escapes and c does not belong to the Mandelbrot set.

Mandelbrot set: baby, 121. All the complexity that occurs in the "main body" of M occurs as well within smaller copies inside certain magnified windows. So there are baby Mandelbrot sets within these small copies of Mandelbrot set, and so on.

Mandelbrot set: bulb periods, 126. We can assign to each bulb in M a number that corresponds to the period of the attracting cycle for each Q_c with c in that bulb.

Mandelbrot set: connected, 123. The Mandelbrot set is a connected set despite appearances to the contrary.

Mandelbrot set: construction, 115. We need simply to understand the orbit of 0 under Q_c for each different c-value. To be precise, the Mandelbrot set, M, is the set of c-values for which the critical orbit of Q_c does not tend to infinity. The Mandelbrot set is a picture in the c-plane, not in the z-plane, where the Julia sets live. (Mandelbrot set: fix $z_0 = 0$ and vary c; Julia set: fix c and vary z_0).

Mandelbrot set: description, 113. A compilation of all of the Julia sets of the quadratic function $Q_c(z) = z^2 + c$ that explains how all of these different structures and shapes are related.

Mandelbrot set: image, 119. Unlike the situation for Julia sets, where we get a different picture for each different c-value, here there is only one image. In M, each pixel that is colored black corresponds to a quadratic function whose Julia set is connected. The Mandelbrot set tells us which c-values correspond to Julia sets with the same basic shape and which quadratic functions have more or less the same dynamics.

Mandelbrot set: orbit diagram, 126. There is a direct relationship between the orbit diagram and the horizontal slice through the middle of M since this horizontal line represents what happens to the critical orbit for real values of c.

Mandelbrot set: symmetry, 117. Unlike the Julia sets of Q_c, the Mandelbrot set is not symmetric about the origin, but it is symmetric about the x-axis.

Mandelbrot sets for higher-degree polynomials, 163. Since the polynomials P_c have a single critical orbit, there is an analogue of the Mandelbrot set for each integer n. As for the quadratic functions, this set is the collection of c-values for which the critical orbit does not escape. This is precisely the set of c-values for which the Julia set is connected. This set is called the <u>degree</u> n <u>bifurcation</u> <u>set</u>.

modulus $|x + iy|$, 76. The distance from x + iy to the origin, i.e.,

$$|x + iy| = \sqrt{x^2 + y^2}$$

-N-

neutral point, 55. Fixed point that is neither attracting or repelling.

nonlinear function, 57. A function not of the form ax + b.

-O-

orbit analysis, 15. The process of understanding all of the orbits of a given dynamical system.

orbit analysis of complex squaring function, 81. Orbits of points under T fall into three categories:

a. Points with modulus less than 1 tend to 0, a fixed point.

b. Points with modulus larger than 1 escape to infinity.

c. Points with modulus exactly 1 remain forever on the circle of radius 1.

orbit diagram, 70. A diagram that plots the orbit of a particular point versus the parameter value c.

orbit of a point, 13. The list of successive iterates of a point or number.

-P-

parameter, 28. An arbitrary constant. For each different parameter value, we get a new function to iterate, e.g., the parameter c of the logistic function.

period-doubling bifurcation, 67. A period N cycle spawns a new attracting cycle of period 2N, which then doubles and gives an orbit of period 4N, and so forth.

periodic orbit or cycle, 17. An orbit that eventually returns to where it began; i.e., the orbit of x_0 is periodic if there is a positive integer N such that $F^N(x_0) = x_0$.

periodic point, 17. The point x_0 of a periodic orbit is called a periodic point of period N.

polar representation of a complex number, 97. A complex number $z = x + iy$ with Cartesian coordinates (x,y) has polar coordinates (r, θ), where r is the modulus and θ is the polar angle.

polar transformations of a complex number, 98. Given the polar coordinates (r, θ) of a complex number z, then the Cartesian coordinates are $x = r \cos \theta$ and $y = r \sin \theta$. Conversely, given the Cartesian coordinates (x, y) of a complex number z, then the polar coordinates are

$$r = \sqrt{x^2 + y^2} \text{ and } \theta = \arctan (y/x) \text{ if } x > 0$$
$$= \arctan (y/x) + \pi \text{ if } x < 0$$

prime period, 17. The least period N of a cycle.

principal themes, 50. The notions of stability and instability.

program FRACTAL, 142. A BASIC program to compute a random orbit of any point (M,N). No matter which M and N are input, the same figure results.

program ITERATE1, 23. A BASIC program to iterate the function $2x(1-x)$ 25 times on a given initial input x_0. Note that this function is one of the logistic functions with $c = 2$.

program ITERATE2, 34. A BASIC program to display the first 200 points on the orbit of x_0 under $F(x) = 4x(1-x)$ where $0 < x_0 < 1$.

program ITERATE3, 37. A BASIC program that accepts as input an initial value of x_0 together with the endpoints of an interval $1 \leq x_0 \leq r$ in which the orbit is plotted.

program ITERATE4, 78. A BASIC program to iterate the complex squaring function $T(z) = z^2$ or

$$T(x + iy) = x^2 - y^2 + i(2xy)$$

program ITERATE5, 83. A BASIC program to iterate the quadratic polynomial of the form $Q_c(z) = z^2 + c$, where both z and c are complex numbers.

program JULIA1, 87. A BASIC program for the function $Q_c(z) = z^2 + c$ when $|c| \leq 2$ to display in black the set of points within the square $|x|, |y| \leq 2$ whose orbit has not escaped beyond the circle of radius 2 centered at the origin before the 20th iteration of Q_c.

program JULIA2, 102. A BASIC program to produce a rough picture of the Julia set of $Q_c(z) = z^2 + c$, using the backward iteration method.

program JULIA3, 112. A BASIC program to produce a very sharp image of the Julia set of Q_c via the boundary-scanning method.

program JULIACOS, 171. A BASIC program which computes the Julia set of c cos z. This program displays the Julia set in white, not black.

program JULIAEXP, 168. A BASIC program to compute the Julia set of $E_c(z) = ce^z$, where c is a complex parameter. The algorithm produces a picture whose white points lie in the Julia set. The Julia sets for exponential functions look quite different from those of polynomials.

program JULIASIN, 171. A BASIC program which computes the Julia set of sin z. This program displays the Julia set in white, not black.

program MANDELBROT1, 116. A BASIC program to draw the Mandelbrot set. This program is similar to JULIA1; the difference being to treat each grid point as a c-value.

program MANDELBROT2, 121. A BASIC program that allows the user to select a square or window from the output of MANDELBROT1.

program MANDELBROT3, 124. A BASIC program to generate only the boundary of the Mandelbrot set by the boundary-scanning method that was used to display Julia sets.

program MANDELBROT4, 124. A BASIC program to understand the relationship between the Julia sets for Q_c and the corresponding c-values in M. The program builds on MANDELBROT2 and allows the user to select a particular c-value from the image displayed by MANDELBROT2. Then the user is given the choice of selecting one of the previous programs, ITERATE5 or JULIA2, as applied to Q_c for the chosen c-value. The screen is then cleared and the appropriate orbit or Julia set displayed.

program ORBITDGM, 70. A BASIC program which produces the orbit diagram. The first 200 points on the orbit of 0 are calculated for each c, but only the last 150 points are plotted to see the eventual, or asymptotic, behavior of the orbit.

-Q-

quadratic family, 57. The family of quadratic functions

$$Q_c(x) = x^2 + c, \text{ where c is a parameter.}$$

quadratic function: attracting fixed point, 62. For $-3/4 \leq c \leq 1/4$, all orbits appear to be attracted to an attracting fixed point.

quadratic function: non-escaping orbits, 59. All the "interesting" dynanmics of Q_c are confined to the interval $-p \leq x \leq p$ when $c \leq 1/4$ where

$$p = \frac{1 + \sqrt{1-4c}}{2}$$

quadratic function: period 2 cycle, 63. For $-5/4 \leq c < -3/4$, most orbits appear to be attracted to a period 2 cycle.

quadratic function: period 4 and higher cycles, 63. As c decreases below -5/4, most orbits are attracted to a period 4 cycle, then a period 8 cycle, then a period 16 cycle, and so forth.

-R-

repelling fixed point, 48. The fixed point p is called repelling if there is an interval $a < x < b$ containing p in which all points (except p) have orbits that leave the interval.

-S-

saddle-node, or tangent, bifurcation, 59. A fixed point first appears and then suddenly splits into two fixed points as c varies.

"seen", 50. Attracting fixed points can be "seen" using the computer, whereas repelling fixed points cannot.

self-similarity under magnification, 93. A basic property of fractals. The further we delve into the filled-in Julia set, the more decorations we see, but the pictures generated in smaller windows bear a strong resemblance to our original picture.

sensitive dependence on initial conditions, 154. The essential ingredient of a chaotic system. This means that, no matter how close two orbits start out, after just a few iterations they will be very far apart. The orbits separate from each other exponentially. A small error in initial conditions makes a big difference in the results. This is sensitive dependence on initial conditions and is the typical behavior in a chaotic system.

Sierpinski triangle, 130. The Sierpinski triangle is generated by an infinite succession of "removals." Beginning with an equilateral triangle, remove the middle upside-down triangle. Iterate this procedure: from each remaining equilateral triangle, remove the middle triangle, leaving three smaller triangles behind.

Sierpinski triangle: self-similarity, 132. The portion of the triangle contained within a triangle at level n, when magnified by a factor of 2^n, is exactly the same as the whole triangle. That is, small portions of the triangle, when magnified, are similar to the whole triangle.

square root function, 9. Iteration of $\sqrt{x}$ eventually yields the number 1, which then remains unchanged or fixed under subsequent iterations.

square roots of a complex number, 100. Given the polar coordinates (r, θ) of a complex number z, then the two square roots of z are given by

$$w_1 = \sqrt{r}\cos(\theta/2) + i\sqrt{r}\sin(\theta/2)$$

$$w_2 = -\sqrt{r}\cos(\theta/2) - i\sqrt{r}\sin(\theta/2)$$

stable orbit, 50. An orbit that, if you change the initial input slightly, the resulting orbit behaves similarly.

-T-

totally disconnected set, 107. A subset of the plane such that, given any two points in the set, there is always a closed curve that does not intersect the set and that surrounds one of the points but not the other.

triangle inequality, 77. A geometric interpretation of addition of complex numbers gives $|z + w| \leq |z| + |w|$.

-U-

unstable orbit, 50. An orbit whose nearby orbits have vastly different behaviors. Note that an attracting fixed or periodic point is always stable, whereas a repelling point is never stable (nearby initial conditions tend far away).

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