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Statistical Methodology for Analysis of Vehicle Encounter Rates in a National Park

by

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This report is preliminary and has not been reviewed for conformity with U.S. Geological Survey editorial standards and stratigraphic nomenclature.

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INTRODUCTION

The National Park Service is about to embark on a major data collection effort at all of the national parks to address an important problem in visitor use management.

For example, each summer the National Park Service needs to sample and compute vehicle encounter rates on a section of the Salt Valley Road in the Arches National Park, southeastern Utah. Their only concern is with use that occurs during the mid-day peak hours.

Each observation consists of a round-trip drive to Klondike Bluffs. The number of motorized vehicles encountered (met or passed) are counted and recorded. A vehicle is defined as anything that can be steered and has a motor. A motorcycle, RV, vehicle pulling a trailer, NPS vehicle, highway maintenance equipment, ranch truck and normal passenger vehicle are each counted as one vehicle. Because the measured indicator is encounters per hour, it is critical to know the *exact* amount of time spent actually driving on this particular route. An example of the calculation of the encounters per hour is the following: 4 vehicles encountered in trip ÷ 0.75 hours of observation = 5.33 vehicles encountered per hour.

The National Park Service has established that an encounter rate of 5 or more vehicles per hour is unacceptable. However, it is impractical to avoid this condition 100% of the time. Therefore, the goal is to have less than 5 encounters per hour at least 90% of the time during peak hours. We need to determine with confidence whether existing conditions are less than 5 encounters per hour at least 90% of the peak hours. Statistical methodology developed for making these determinations follows.

MODELS AND METHODS

Let random variable V : Vehicle encounters per hour.

V is a continuous random variable that can take on the values $0 \leq v < \infty$.

The National Park Service has data which suggests the encounter rate V has approximately a normal distribution with the lower tail truncated. When an area receives moderate or heavy use, they would expect use to be normally distributed. But in low use areas such as Salt Valley Road, the left tail of the distribution is often truncated.

Let event E : Vehicle encounters per hour is less than 5, i.e., $V < 5$.

Let probability $p = P(E) = P(V < 5)$, i.e., the probability that the vehicle encounters per hour is less than 5.

Define the indicator random variable W by

$$W = \begin{cases} 1 & \text{if } E \text{ occurs} \\ 0 & \text{if } E \text{ does not occur.} \end{cases}$$

Then W is distributed as Bernoulli with parameter p (proportion of vehicle encounter rates less than 5), where success is E <u>occurs</u> and failure is E <u>does not occur</u>.

Let random variable X : Number of successes in n trials.

X is a discrete random variable that can take on the values $x = 0, 1, 2, \ldots, n$.

Then X is distributed as binomial with parameters n and p.

A point estimator of the proportion p is given by the statistic $\hat{P} = X / n$.

Therefore, the sample proportion $\hat{p} = x / n$ will be used as the point estimate of the parameter p.

During the summer of 1994, 22 observations of encounter rates were made on the Salt Valley Road. The actual observations were: 0, 2, 1, 3, 5.17, 0, 2, 2, 2, 2.63, 2, 3, 8, 1, 0, 1.50, 2, 2, 3, 0, 2.67, 0. This sample data has a mean of 2.0 with a standard deviation of 1.8. Since the sample size $n = 22$ and the number of successes $x = 20$, the sample proportion $\hat{p} = 20/22 = 0.91$ is a point estimate of the population proportion p which is the probability that the vehicle encounters per hour is less than 5.

The statistical methodology for dealing with a proportion is well known, e.g., Walpole and Myers (1993).

DETERMINATION OF SAMPLE SIZE

Let us determine how large a sample is necessary to ensure that the error in estimating p will be less than a specified amount e.

> **Theorem 1** *If $\hat{p}$ is used as an estimate of p, and $\hat{q} = 1 - \hat{p}$, we can be $(1-\alpha)100\%$ confident that the error will be less than a specified amount e when the sample size is approximately*
>
> $$n = \frac{z_{\alpha/2}^2 \hat{p}\hat{q}}{e^2}.$$
>
> *where $z_{\alpha/2}$ is the z-value of the standard normal curve leaving an area of $\alpha/2$ to the right.*

Theorem 1 is somewhat misleading in that we must use $\hat{p}$ to determine the sample size n, but $\hat{p}$ is computed from the sample. If a crude estimate of p can be made without taking a sample, we could use this value for $\hat{p}$ and then determine n. Lacking such an estimate, we could take a preliminary sample to provide an estimate of p. Then, using Theorem 1 we could determine approximately how many observations are needed to provide the desired degree of accuracy. All fractional values of n are rounded up to the next whole number.

Example 1

How large a sample is required if we want to be 95% confident that our estimate of p is within 0.05?

Solution. Let us treat the 22 observations as a preliminary sample providing an estimate $\hat{p} = 0.91$. Using a standard normal table, we find $z_{0.025} = 1.96$. Then, by Theorem 1,

$$n = \frac{(1.96)^2(0.91)(0.09)}{(0.05)^2} = 126.$$

Therefore, if we base our estimate of p on a random sample of size 126, we can be 95% confident that our sample proportion will not differ from the true proportion by more than 0.05.

Occasionally, it will be impractical to obtain an estimate of p to be used in determining the sample size for a specified degree of confidence. If this happens, an upper bound for n is established by noting that $\hat{p}\hat{q} = \hat{p}(1-\hat{p})$, which must be at most equal to ¼ when $\hat{p} = ½$, since $\hat{p}$ must lie between 0 and 1.

Theorem 2 *If $\hat{p}$ is used as an estimate of p, we can be* ***at least*** $(1-\alpha)100\%$ *confident that the error will not exceed a specified amount e when the sample size is*

$$n = \frac{z_{\alpha/2}^2}{4e^2}.$$

Example 2

How large a sample is required in Example 1 if we want to be at least 95% confident that our estimate of p is within 0.05?

Solution. Unlike Example 1, we shall now assume that no preliminary sample has been taken to provide an estimate of p. Consequently, we can be at least 95% confident that our sample proportion will not differ from the true proportion by more than 0.05 if we choose a sample of size

$$n = \frac{(1.96)^2}{(4)(0.05)^2} = 385.$$

Comparing the results of Examples 1 and 2, we see that information concerning p, provided by a preliminary sample, or perhaps from past experience, enables us to choose a smaller sample while maintaining our required degree of accuracy.

LARGE-SAMPLE CONFIDENCE INTERVAL FOR p

Large-Sample Confidence Interval for p	*If $\hat{p}$ is the proportion of successes in a random sample of size n, and $\hat{q} = 1 - \hat{p}$, an approximate $(1-\alpha)100\%$ confidence interval for the binomial parameter p is given by* $$\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}\hat{q}}{n}} < p < \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}\hat{q}}{n}},$$ *where $z_{\alpha/2}$ is the z-value of the standard normal curve leaving an area of $\alpha/2$ to the right.*

Example 3

Suppose in a random sample of $n = 126$ observations of vehicle encounter rates it was found that $x = 111$ encounter rates were less than 5. Find a 95% confidence interval for the population proportion of vehicle encounters per hour less than 5.

Solution. The point estimate of p is $\hat{p} = 111/126 = 0.88$. Using a standard normal table, we find $z_{0.025} = 1.96$. Therefore, the 95% confidence interval for p is

$$0.88 - 1.96\sqrt{\frac{(0.88)(0.12)}{126}} < p < 0.88 + 1.96\sqrt{\frac{(0.88)(0.12)}{126}},$$

which simplifies to

$$0.88 - 0.06 < p < 0.88 + 0.06$$

$$0.82 < p < 0.94.$$

Hence, we are 95% confident that the population proportion of vehicle encounter rates less than 5 lies between 0.82 and 0.94.

TESTS OF HYPOTHESES FOR p

Small Sample Test

The steps for testing a null hypothesis about a proportion against various alternatives using the binomial probabilities are as follows:

1. H_0: $p = p_0$. (p_0 is the hypothesized proportion.)
2. H_1: Alternatives are $p < p_0$, $p > p_0$, or $p \neq p_0$.
3. Choose a level of significance equal to α.
4. Critical region:
   (a) All x values such that $P(X \leq x \mid H_0 \text{ is true}) \leq \alpha$ for the alternative $p < p_0$.
   (b) All x values such that $P(X \geq x \mid H_0 \text{ is true}) \leq \alpha$ for the alternative $p > p_0$.
   (c) All x values such that $P(X \leq x \mid H_0 \text{ is true}) \leq \alpha/2$ when $x < np_0$, and all x values such that $P(X \geq x \mid H_0 \text{ is true}) \leq \alpha/2$ when $x > np_0$, for the alternative $p \neq p_0$.
5. Computations: Find x, the number of successes and compute the appropriate P-value.
6. Decision: Draw appropriate conclusions based on the P-value.

Definition: A P-value is the lowest level (of significance) at which the observed value of the test statistic is significant (and the null hypothesis is rejected).

Example 4

Test the null hypothesis that the proportion of vehicle encounter rates less than 5 is at least 0.90 (within standard), against the alternative hypothesis that the proportion is less than 0.90 (outside standard). Suppose in a random sample of $n = 20$ observations of vehicle encounter rates it was found that $x = 16$ encounter rates were less than 5. Use a 0.05 level of significance.

Solution. We follow the six-step procedure for a left-tailed test:

1. H_0: $p = 0.9$ (within standard).
2. H_1: $p < 0.9$ (outside standard).
3. $\alpha = 0.05$.
4. Critical region: All x values such that $P(X \le x \mid H_0 \text{ is true}) \le 0.05$.
5. Computations: We have $x = 16$, $n = 20$, and $\hat{p} = 0.80$. Therefore, using a binomial probability table, the computed P-value is

$$P = P(X \le 16 \mid p = 0.9) = \sum_{x=0}^{16} b(x; 20, 0.9) = 0.1330 > 0.05.$$

6. Decision: Do not reject H_0. We are unable to conclude that the Salt Valley Road is outside standard.

Large Sample Test

To test a hypothesis about a proportion using the normal curve approximation, we proceed as follows:

1. H_0: $p = p_0$.
2. H_1: Alternatives are $p < p_0$, $p > p_0$, or $p \neq p_0$.
3. Choose a level of significance equal to α.
4. Critical region:
   (a) $z < -z_\alpha$ for the alternative $p < p_0$.
   (b) $z > z_\alpha$ for the alternative $p > p_0$.
   (c) $z < -z_{\alpha/2}$ and $z > z_{\alpha/2}$ for the alternative $p \neq p_0$.

5. Computations: Find x from a sample of size n, and then compute

$$z = \frac{x - np_0}{\sqrt{np_0 q_0}} = \frac{\hat{p} - p_0}{\sqrt{p_0 q_0 / n}}$$

6. Decision: Reject H_0 if z falls in the critical region; otherwise, do not reject H_0.

Example 5

Test the null hypothesis that the proportion of vehicle encounter rates less than 5 is at least 0.90 (within standard), against the alternative hypothesis that the proportion is less than 0.90 (outside standard). Suppose in a random sample of $n = 126$ observations of vehicle encounter rates it was found that $x = 111$ encounter rates were less than 5. Use a 0.05 level of significance.

Solution. As usual, we follow the six-step procedure for a left-tailed test:

1. H_0: $p = 0.9$ (within standard).
2. H_1: $p < 0.9$ (outside standard).
3. $\alpha = 0.05$.
4. Critical region: $z < -1.645$.
5. Computations: $x = 111$, $n = 126$, $\hat{p} = 0.88$, $np_0 = (126)(0.90) = 113.4$, and

$$z = \frac{111 - 113.4}{\sqrt{(126)(0.90)(0.10)}} = -0.71$$

$$P = P(Z < -0.71) = 0.2389 > 0.05.$$

6. Decision: Do not reject H_0. We have not found significant statistical evidence to conclude that the Salt Valley Road is outside standard.

Note. $\alpha = P(\text{Type I error})$
$= P(\text{Reject } H_0 \mid H_0 \text{ is true})$
$= P(\text{Conclude outside standard when actually within standard})$

Computing β

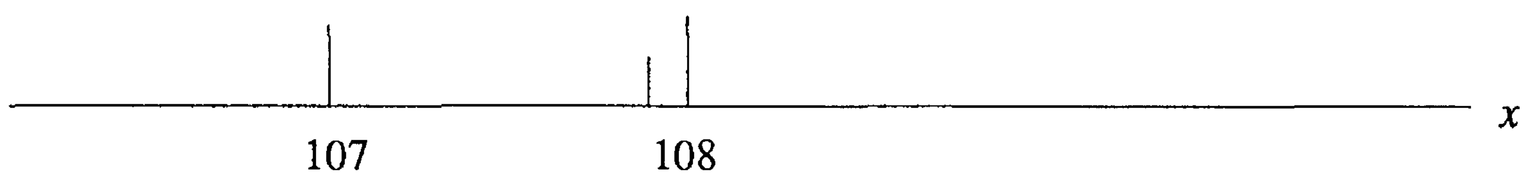
$\alpha = 0.05$ and $z = -1.645$

$$z = \frac{x - np_0}{\sqrt{np_0 q_0}}$$

$$\begin{aligned} x &= np_0 + z\sqrt{np_0 q_0} \\ &= 126(0.9) + (-1.645)\sqrt{126(0.9)(0.1)} \\ &= 113.4 - 5.5 = 107.9 \end{aligned}$$

Reject H_0 if $X \le 107$

Do not reject H_0 if $X > 107$



$\alpha = P(X \le 107 \text{ when } p = 0.9)$

$$\begin{aligned} \beta &= P(\text{Type II error}) \\ &= P(\text{Do not reject } H_0 | H_0 \text{ is false}) \\ &= P(\text{Do not conclude outside standard when actually outside standard}) \\ &= P(X > 107 \text{ when } p = 0.8 \text{ (say)}) \end{aligned}$$

$$\begin{aligned} z &= \frac{x - np}{\sqrt{npq}} = \frac{107.5 - (126)(0.8)}{\sqrt{126(0.8)(0.2)}} \\ &= \frac{107.5 - 100.8}{4.49} = 1.49 \end{aligned}$$

$\beta = P(Z > 1.49) = 1 - 0.9319 = 0.0681$

The probability of not concluding that the Salt Valley Road is outside standard when, in fact, p is as small as 0.8 equals 0.0681.

Power of a Test

One very important concept that relates to error probabilities is the notion of the power of a test.

Definition: The **power** of a test is the probability of rejecting H_0 given that a specific alternative is true.

$$\boxed{\text{Power} = 1 - \beta}$$

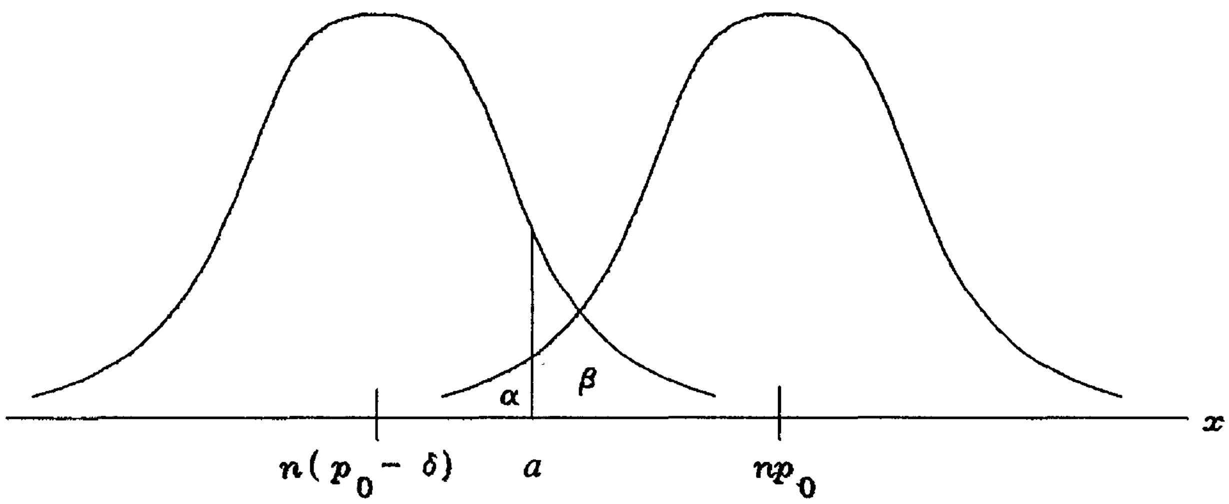
The power of a test can be computed as $1 - \beta$. Often different types of tests are compared by contrasting power properties. Consider the previous illustration in which we were testing H_0: $p = 0.9$ and H_1: $p < 0.9$. As before, suppose we are interested in assessing the sensitivity of the test. The test is governed by the rule that we do not reject if $x > 107$. We seek the capability of the test for properly rejecting H_0 when indeed $p = 0.8$. We have seen that the probability of a type II error is given by $\beta = 0.0681$. Thus the **power** of the test is $1 - 0.0681 = 0.9319$. In a sense, the power is a more succinct measure in being descriptive of how sensitive the test is for "detecting differences" between a proportion of 0.9 and 0.8. In this case, if p is truly 0.8, the test as described will *properly reject* H_0 *93.19% of the time.* As a result, the test would be a good one if it is important that the analyst have a reasonable chance of truly distinguishing between a proportion of 0.9 (specified by H_0) and a proportion of 0.8. From the foregoing, it is clear that in general to produce an increase in the power of a test to a desirable level (say, greater than 0.8), one must either increase α or increase the sample size.

The probability of concluding that the Salt Valley Road is outside standard when, in fact, p is as small as 0.8 equals 0.9319.

CHOICE OF SAMPLE SIZE FOR TESTING A PROPORTION

$H_0: p = p_0$

$H_1: p < p_0 \qquad p = p_0 - \delta \qquad (\delta > 0)$



We define a such that

$$\alpha = P(X < a \text{ when } p = p_0)$$

$$Z = \frac{X - np_0}{\sqrt{np_0 q_0}}$$

$$\alpha = P(Z < -z_\alpha)$$

$$\alpha = P\left(Z < \frac{a - np_0}{\sqrt{np_0 q_0}} \right)$$

Equating, we get

$$\frac{a - np_0}{\sqrt{np_0 q_0}} = -z_\alpha$$

Solving for a, we obtain

$$a = np_0 - z_\alpha \sqrt{np_0 q_0}$$

$$\beta = P(X > a \text{ when } p = p_0 - \delta)$$

$$= P\left(\frac{X - n(p_0 - \delta)}{\sqrt{n(p_0 - \delta)[1 - (p_0 - \delta)]}} > \frac{a - n(p_0 - \delta)}{\sqrt{n(p_0 - \delta)[1 - (p_0 - \delta)]}} \text{ when } p = p_0 - \delta \right)$$

$$= P(Z > z_\beta)$$

Equating, we get

$$
\begin{aligned}
z_\beta &= \frac{a - n(p_0 - \delta)}{\sqrt{n(p_0-\delta)[1-(p_0-\delta)]}} \\
&= \frac{\left(np_0 - z_\alpha\sqrt{np_0q_0}\right) - np_0 + n\delta}{\sqrt{n(p_0-\delta)[1-(p_0-\delta)]}} \\
&= \frac{n\delta - z_\alpha\sqrt{np_0q_0}}{\sqrt{n(p_0-\delta)[1-(p_0-\delta)]}} \\
&= \frac{\delta\sqrt{n} - z_\alpha\sqrt{p_0q_0}}{\sqrt{(p_0-\delta)[1-(p_0-\delta)]}}
\end{aligned}
$$

Solving for *n*, we have

$$
\begin{aligned}
\delta\sqrt{n} &= z_\alpha\sqrt{p_0q_0} + z_\beta\sqrt{(p_0-\delta)[1-(p_0-\delta)]} \\
\sqrt{n} &= \left(\frac{z_\alpha}{\delta}\right)\sqrt{p_0q_0} + \left(\frac{z_\beta}{\delta}\right)\sqrt{(p_0-\delta)[1-(p_0-\delta)]}
\end{aligned}
$$

$$
\boxed{n = \left[\left(\frac{z_\alpha}{\delta}\right)\sqrt{p_0q_0} + \left(\frac{z_\beta}{\delta}\right)\sqrt{(p_0-\delta)[1-(p_0-\delta)]}\right]^2}
$$

Example 6

Suppose that we wish to test the hypothesis

H_0: $p = 0.9$ (within standard)

H_1: $p < 0.9$ (outside standard)

for the proportion of vehicle encounter rates less than 5 at the Salt Valley Road using an $\alpha = 0.05$ level of significance. Find the sample size required if the power of our test is to be 0.95 when the true proportion is 0.8.

Solution.

$$p_0 = 0.9 \qquad p_0 - \delta = 0.8 \qquad \delta = 0.1$$
$$\alpha = 0.05 \qquad z_\alpha = 1.645$$
$$\beta = 0.05 \qquad z_\beta = 1.645$$
$$n = \left[\frac{1.645}{0.1}\sqrt{(0.9)(0.1)} + \frac{1.645}{0.1}\sqrt{(0.8)(0.2)}\right]^2$$
$$= (4.935 + 6.58)^2 = 132.6$$

Therefore, it requires 133 observations if the test is to reject the null hypothesis 95% of the time when, in fact, p is as small as 0.8.

Example 7

Suppose that we wish to test the hypothesis

H_0: $p = 0.9$ (within standard)

H_1: $p < 0.9$ (outside standard)

for the proportion of vehicle encounter rates less than 5 at the Salt Valley Road using an $\alpha = 0.10$ level of significance. Find the sample size required if the power of our test is to be 0.90 when the true proportion is 0.8.

Solution.

$$p_0 = 0.9 \qquad p_0 - \delta = 0.8 \qquad \delta = 0.1$$
$$\alpha = 0.10 \qquad z_\alpha = 1.28$$
$$\beta = 0.10 \qquad z_\beta = 1.28$$
$$n = \left[\frac{1.28}{0.1}\sqrt{(0.9)(0.1)} + \frac{1.28}{0.1}\sqrt{(0.8)(0.2)}\right]^2$$
$$= (3.84 + 5.12)^2 = 80.3$$

Therefore, it requires 81 observations if the test is to reject the null hypothesis 90% of the time when, in fact, p is as small as 0.8.

The preceding examples illustrate the following important properties:

1. The type I error and type II error are related. A decrease in the probability of one generally results in an increase in the probability of the other.
2. The size of the critical region, and therefore the probability of committing a type I error, can always be reduced by adjusting the critical value(s).
3. An increase in the sample size n will reduce α and β simultaneously.
4. If the null hypothesis is false, β is a maximum when the true value of a parameter approaches the hypothesized value. The greater the distance between the true value and the hypothesized value, the smaller β will be.

REFERENCE

Walpole, R.E. and Myers, R.H., 1993, Probability and statistics for engineers and scientists: New York, Macmillan Publishing Company, 5th ed., 784 p.