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Analytical Solutions and Computer Programs for Hydraulic Interaction of Stream-Aquifer Systems

By PAUL M. BARLOW and ALLEN F. MOENCH

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BRUCE BABBITT, *Secretary*

U.S. GEOLOGICAL SURVEY
Charles G. Groat, *Director*

For additional information write to:

Chief, Massachusetts-Rhode Island District
U.S. Geological Survey
Water Resources Division
28 Lord Road, Suite 280
Marlborough, MA 01752

Copies of this report can be purchased from:

U.S. Geological Survey
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PREFACE

This report presents analytical solutions to the ground-water-flow equation for hydraulic interaction of stream-aquifer systems. The solutions are applicable to confined, leaky, and water-table aquifers. Two computer programs are described that were written to evaluate these analytical solutions. The programs have been successfully tested for a variety of applications. Users are requested to notify the USGS if errors are found in this report or the programs.

CONTENTS

FIGURES

TABLES

CONVERSION FACTORS AND DEFINITION OF SYMBOLS

CONVERSION FACTORS

Multiply	By	To Obtain
cubic foot per day (ft^3/d)	0.02832	cubic meter per day
cubic foot per foot of stream length (ft^2)	0.09290	cubic meter per meter of stream length
foot (ft)	0.3048	meter
foot per day (ft/d)	0.3048	meter per day
square foot per day (ft^2/d)	0.09290	square meter per day

DEFINITION OF SYMBOLS

[L, length; T, time; --, dimensionless]

Symbol	Dimension	Definition
a	L	Streambank leakance
A	--	Dimensionless streambank leakance
b	L	Thickness of aquifer (or saturated thickness for water-table aquifer)
b'	L	Thickness of aquitard (or saturated thickness for water-table aquitard)
c	L	Instantaneous step change in water level of stream
cos	--	Cosine function
coth	--	Hyperbolic cotangent
d	L	Width of semipervious streambank material
erfc	--	Complementary error function
exp	--	Exponential function
F	L	System input
F'	L/T	Time rate of change of system input
h	L	Head in aquifer
h'	L	Head in aquitard
h_D	--	Dimensionless unit-step response solution for head in aquifer
$\bar{h}_D$	--	Dimensionless Laplace transform unit-step response solution for head in aquifer
$\bar{h}_D^*$	--	Dimensionless average head in a partially penetrating observation well
$\hat{\bar{h}}_D$	--	Dimensionless average head in a fully penetrating observation well
h_D'	--	Dimensionless unit-step response solution for head in aquitard

Symbol	Dimension	Definition
$\bar{h}'_D$	--	Dimensionless Laplace transform unit-step response solution for head in aquitard
h_i	L	Initial water level (or potentiometric surface) in stream-aquifer system
h_0	L	Water level in stream after step change
j	--	Upper limit of time integration
k	--	Time variable of integration (time step)
K_D	--	Dimensionless ratio of vertical to horizontal hydraulic conductivity
K_s	L/T	Hydraulic conductivity of semipervious streambank material
K_x, K_z	L/T	Horizontal and vertical hydraulic conductivity of aquifer, respectively
K'	L/T	Vertical hydraulic conductivity of aquitard
l_s	L	Length of stream reach
m	--	Dimensionless aquifer-aquitard Laplace transform variable
n	--	Integer counter in infinite summations
p	--	Laplace transform variable
q_n	--	Terms in the Laplace transform solutions for water-table aquifers
q'	1/T	Volumetric flow rate to or from aquifer per unit volume of aquifer
$\bar{q}_D$	--	Dimensionless Laplace transform leakage between aquifer and aquitard
Q	L^2/T	Seepage rate per unit length of stream
Q_D	--	Dimensionless seepage between stream and aquifer
$\bar{Q}_D$	--	Dimensionless Laplace transform seepage between stream and aquifer
Q_T	L^3/T	Total seepage rate
R	L	Total recharge
S	--	Storativity (storage coefficient) of aquifer
S_s	1/L	Specific storage of aquifer
S_s'	1/L	Specific storage of aquitard
S_y	--	Specific yield of aquifer
S'_y	--	Specific yield of aquitard

Symbol	Dimension	Definition
sin	--	Sine function
t	T	Time
t_D	--	Dimensionless time
t_{Dy}	--	Dimensionless time with respect to specific yield
t_e	T	Time of end of recharge
t_i	T	Time at start of simulation
t_p	T	Time of flood peak
t_s	T	Time of start of flood wave or recharge
T	L^2/T	Transmissivity of aquifer
tan	--	Tangent function
tanh	--	Hyperbolic tangent
Δt	T	Time-step size
V	L^2	Bank-storage volume per unit length of stream
V_T	L^3	Total volume of bank storage
W	--	Term for aquifer width in Laplace transform solutions for confined and leaky aquifers
W_n	--	Term for aquifer width in Laplace transform solutions for water-table aquifers
x	L	Horizontal coordinate
x_D	--	Dimensionless horizontal coordinate
x_L	L	Width of aquifer
x_{LD}	--	Dimensionless width of aquifer
x_0	L	Distance from middle of stream to stream-aquifer boundary (half-width of stream)
x_{0D}	--	Dimensionless distance to streambank
z	L	Vertical coordinate
z_1	L	Vertical coordinate of bottom of screened interval of observation well
z_2	L	Vertical coordinate of top of screened interval of observation well
z_D	--	Dimensionless vertical coordinate in aquifer

Symbol	Dimension	Definition
z_{D1}	--	Dimensionless vertical coordinate of bottom of screened interval of observation well
z_{D2}	--	Dimensionless vertical coordinate of top of screened interval of observation well
z'_D	--	Dimensionless vertical coordinate in aquitard
z_p	L	Vertical coordinate of observation piezometer opening
β_0	--	Dimensionless product of anisotropic ratio of vertical to horizontal hydraulic conductivity and square of dimensionless distance to streambank
ε, ε_n	--	Roots of equations in Laplace transform solutions for water-table aquifers
γ_1	--	Dimensionless ratio of aquitard to aquifer hydraulic conductivity
π	--	Pi (3.141592654)
τ	T	Time variable of integration (delay time)
Σ	--	Summation
σ	--	Dimensionless ratio of aquifer storativity to aquifer specific yield
σ_1	--	Dimensionless ratio of aquitard to aquifer storativity
σ'	--	Dimensionless ratio of aquifer storativity to aquitard specific yield
$\int$	--	Integral
∞	--	Infinity
d	--	Derivative
∂	--	Partial derivative

Analytical Solutions and Computer Programs for Hydraulic Interaction of Stream-Aquifer Systems

By Paul M. Barlow *and* Allen F. Moench

Abstract

Analytical solutions to the ground-water-flow equation are derived for ten cases of hydraulic interaction between a stream and a confined, leaky, or water-table aquifer. The ten aquifer types for which analytical solutions are derived are: a semi-infinite or finite-width confined aquifer; a semi-infinite or finite-width leaky aquifer with constant head overlying the aquitard; a semi-infinite or finite-width leaky aquifer with an impermeable layer overlying the aquitard; a semi-infinite or finite-width leaky aquifer overlain by a water-table aquitard; and a semi-infinite or finite-width water-table aquifer. All aquifer types allow for the presence or absence of a uniform semipervious streambank. Of primary interest are newly derived solutions for water-table aquifers and for leaky aquifers overlain by water-table aquitards.

Two computer programs are described that evaluate the analytical solutions for time-varying stream-stage or recharge stresses that are specified by the user. The programs can simulate the effects of stream-stage fluctuations for all aquifer types. However, simulation of basin-wide recharge or evapotranspiration at the water table is permitted only for water-table aquifers and leaky aquifers overlain by a water-table aquitard. For these aquifer types, effects of recharge or evapotranspiration can be simulated alone or in combination with stream-stage fluctuations. The computer programs use the convolution relation to calculate changes in ground-water levels at an observation well or observation piezometer, seepage rates at the stream-aquifer boundary, and bank storage. The program designated STLK1 was developed for application to confined and leaky aquifers, and the program designated STWT1 was developed for application to water-table aquifers. The programs can be applied to the analysis of a passing flood wave, determination of ground-water discharge rates in response to recharge, determination of aquifer hydraulic properties, design of stream-aquifer data-collection networks, and testing of numerical-model computer codes. Instructions are provided for constructing the necessary data-input files for the programs, and three sample problems are described to provide examples of the uses of the programs.

INTRODUCTION

The hydraulic interaction of ground water with adjoining streams, canals, and drains is an important aspect of many hydrogeologic systems. Ground-water discharge supports stream base flow during periods of little to no precipitation; bank storage can attenuate flood waves and dampen overall flood impacts; and ground-water discharge to drains can lower water tables to maintain favorable root-zone salinity levels and prevent water logging of soil. Methods for evaluating the hydraulic interaction of stream-aquifer systems include field experiments, analytical models, and numerical models. Analytical models are often advantageous because of their simplicity. They are more general than site-specific field experiments, yet are easier to implement for a particular site than numerical models.

Several analytical solutions have been published for evaluation of the interaction of ground-water systems and hydraulically connected surface-water features such as streams, lakes, reservoirs, drains, and canals. These solutions can be useful for understanding base-flow processes, determining aquifer hydraulic properties, and predicting responses of aquifers to changing stream stage. The solutions have not received widespread use, however, particularly in comparison to solutions that have been developed for problems in well hydraulics. One explanation for this is that, for most practical problems in stream-aquifer hydraulics, stream-stage and recharge boundary conditions continuously change, in contrast to problems in well hydraulics in which a constant rate of pumping often can be specified. Because of this, the analytical solutions must be used in combination with the convolution integral to account for continuously changing stream-stage and recharge conditions. To date, computer programs that link these analytical solutions with the convolution method have not been widely available.

In this report, existing and newly derived analytical solutions for transient, hydraulic interaction of stream-aquifer systems are presented and documented. These solutions assume one-dimensional, horizontal flow in confined and leaky aquifers and two-dimensional, horizontal and vertical flow in water-table aquifers. In all cases, ground-water flow is assumed to be in the plane perpendicular to a single, fully penetrating stream that bounds the aquifer. For each aquifer type, solutions are derived for conditions in which semipervious streambank material may be present between the stream and aquifer and for conditions in which the lateral extent of the aquifer is either semi-infinite or of finite width. Solutions are written in terms of ground-water heads as a function of location in the aquifer and time, seepage rates at the stream-aquifer boundary as a function of time, and bank-storage volumes into and out of the aquifer as a function of time.

Two computer programs are provided that implement the analytical solutions for time-varying stream-stage or recharge inputs by use of the method of convolution. The programs calculate head changes, streambank seepage rates, and bank-storage volumes as a function of time for various confined, leaky, and water-table aquifer types in response to changing stream-stage conditions. They also can be used to determine the response of water-table aquifers to time-varying recharge or evapotranspiration (ET). The programs can be applied to the analysis of a passing flood wave, determination of ground-water discharge rates in response to recharge, determination of aquifer hydraulic properties, design of stream-aquifer data-collection networks, and testing of numerical-model computer codes. The computer programs can be used without a detailed understanding of the derivation of the analytical solutions; however, the reader should be familiar with the assumptions on which the analytical solutions are based.

Purpose and Scope

This report describes the derivation and evaluation of new analytical solutions to the ground-water flow equation for the transient, hydraulic interaction between a stream and a confined, leaky, or water-table aquifer. A description of the physical characteristics of the stream-aquifer systems evaluated in this report and of stream-aquifer hydraulic interaction is provided as background for the derivations. The solutions are derived for the condition of an instantaneous step change in stream stage so that they can be readily applied in the convolution relation. The solutions also are applicable to the condition of an instantaneous regional rise or decline in the altitude of the water table. The new analytical solutions are compared graphically to several previously published solutions.

Two computer programs (STLK1 and STWT1) are described that are based on the analytical solutions and method of convolution. These programs can be used to calculate the response of confined, leaky, and water-table aquifers to arbitrary, time-varying stream-stage and (or) recharge conditions that are specified by program users. The program designated STLK1 was developed for application to leaky aquifers (including the confined case) and the program designated STWT1 was developed for application to water-table aquifers (also including the confined case). The programs calculate time-varying ground-water heads at observation wells or piezometers, seepage rates at the stream-aquifer boundary, and bank-storage volumes into and out of the aquifer. Instructions are provided for constructing the necessary data-input files for the programs, and three sample problems are described to provide examples of the uses of the programs.

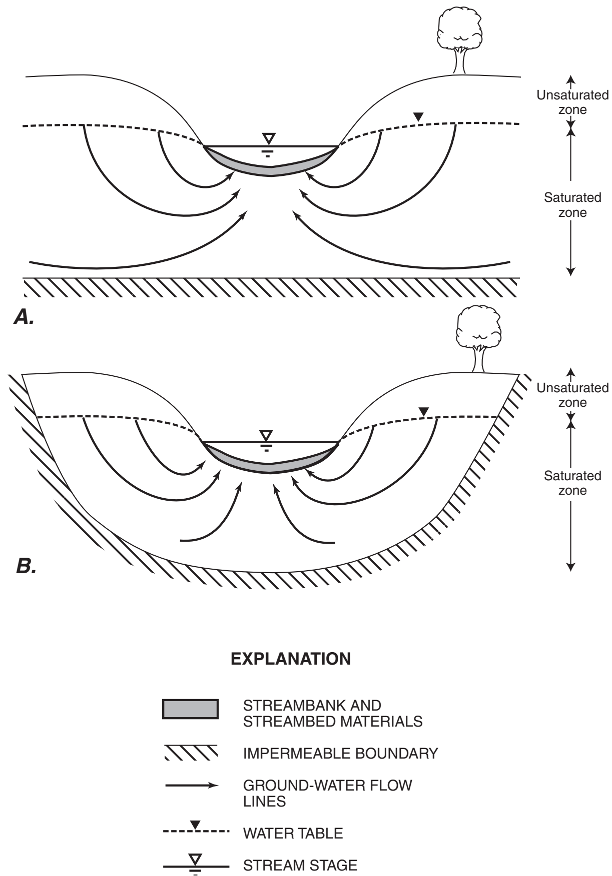


Figure 1. Ground-water discharge from a water-table aquifer to a partially penetrating, hydraulically connected stream: (*A*) laterally extensive (semi-infinite) aquifer; and (*B*) narrow aquifer of finite width.

Description of Stream-Aquifer Systems

Figure 1 illustrates ground-water discharge from a water-table aquifer to a shallow stream. The stream and aquifer are in hydraulic connection, which means that water is able to move freely between them. In the illustration, ground-water heads are greater than the elevation of the stream stage and, hence, ground water discharges to the stream. In this instance, the stream is referred to as a gaining stream. When the elevation of the stream stage is greater than ground-water heads in the immediate vicinity of the stream, seepage occurs from the stream to the aquifer. In this instance, the stream is referred to as a losing stream. The rate at which water moves between a stream and aquifer depends upon the type, lateral extent, and hydraulic properties of the adjoining aquifer system; the depth of penetration of the stream into the aquifer; the hydraulic properties of the streambanks and streambed; and the hydraulic gradient between the stream and aquifer.

Three general types of aquifers are considered in this report—confined, leaky, and water table (or unconfined). A confined aquifer (fig. 2*A*) is one that is overlain by a layer of geologic material (a confining layer) that prevents ground-water flow to or from the underlying aquifer. A leaky aquifer is one that is overlain by a layer of geologic material (an aquitard) with a much lower hydraulic conductivity and usually a greater specific storage than that of the underlying aquifer; the aquitard hinders but does not prevent ground-water flow (leakage) to or from the underlying aquifer. Flow across the aquitard-aquifer boundary is called leakage. Three types of leaky aquifers are evaluated: those in which a source bed with a constant head overlies the aquitard (leaky aquifer case 1, fig. 2*B*); those in which an impermeable layer overlies the aquitard (leaky aquifer case 2, fig. 2*C*); and those that are overlain by an aquitard that is unconfined (a water-table aquitard; leaky aquifer case 3, fig. 2*D*). Finally, a water-table aquifer (fig. 2*E*) is one in which the water table forms the upper boundary to the aquifer and is overlain by an unsaturated zone.

All stream-aquifer systems evaluated in this report are assumed to be underlain by an impermeable boundary, across which no ground-water flow occurs (figs. 1 and 2). In addition, in all cases ground-water flow is assumed to be perpendicular to the stream. For the confined and leaky aquifer types, ground-water flow is one dimensional (horizontal); for the

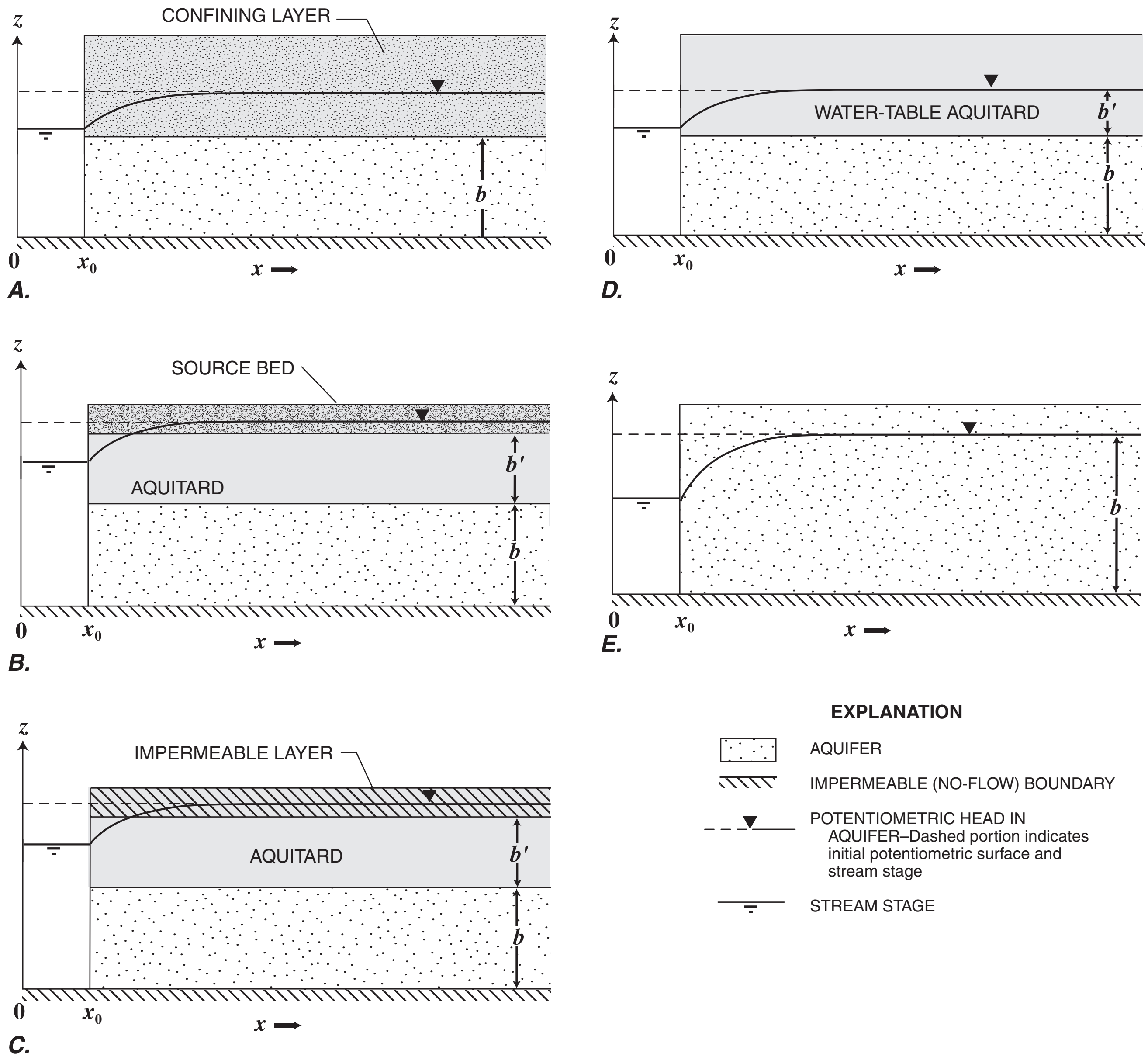


Figure 2. Types of aquifers for which analytical solutions are derived: (*A*) confined; (*B*) leaky, with a constant head overlying the aquitard; (*C*) leaky, with an impermeable layer overlying the aquitard; (*D*) leaky, overlain by a water-table aquitard; and (*E*) water table (unconfined). (*b*, thickness or saturated thickness of aquifer; *b*', thickness or saturated thickness of aquitard; *x,z*, horizontal and vertical coordinate directions, respectively; x_0, distance from middle of stream to stream-aquifer boundary.)

water-table aquifer types, ground-water flow is two dimensional (that is, horizontal and vertical). For each of the aquifer types shown in figure 2, analytical solutions are derived for conditions in which the aquifer is laterally extensive (figs. 1*A* and 2) and for conditions in which the aquifer is relatively narrow (fig. 1*B*). Aquifers that are laterally extensive are referred to as semi-infinite aquifers, whereas narrow aquifers that are bounded laterally by impermeable geologic features are referred to as finite-width aquifers.

Many streams are shallow relative to the thickness of the aquifer in which they lie (fig. 1). Such streams are referred to as partially penetrating and seepage between them and the contiguous aquifer occurs both horizontally and vertically through streambank and streambed materials. Because of the added mathematical difficulties that arise for a partially-penetrating stream, most analytical solutions of stream-aquifer systems have been derived with the assumption that the stream fully penetrates the aquifer (fig. 2). This approach also is taken here. As a consequence of this approach, all seepage between the stream and aquifer is assumed to be one dimensional in the horizontal direction through the streambank. This approximation in the analytical treatment appears to have few detrimental consequences as long as the points of interest (observation wells) are at least 1.5 times as far from the stream as the aquifer is thick (Hantush, 1965).

Several hydraulic properties of the aquifer and of the semipervious streambank material affect ground-water heads and seepage rates. In the simplest case, that for confined aquifers, the relevant aquifer properties are horizontal hydraulic conductivity (K_x), thickness (b), and specific storage (S_s). For leaky aquifers, the hydraulic properties of the overlying aquitard also must be considered. These are vertical hydraulic conductivity (K'), specific storage (S_s'), thickness (b'), and (for water-table aquitards) specific yield (S_y'). For water-table aquifers, the relevant aquifer properties are vertical (K_z) and horizontal (K_x) hydraulic conductivity, specific storage (S_s), and specific yield (S_y). The transmissivity (T) and storativity (or storage coefficient) (S) of confined, leaky, and water-table aquifers often are used in place of horizontal hydraulic conductivity and specific storage. Transmissivity is equal to the product of the horizontal hydraulic conductivity and thickness (or saturated thickness for water-table conditions) of the aquifer ($T = K_x b$); storativity is equal to the product of the specific storage and thickness (or saturated thickness for water-table conditions) of the aquifer ($S = S_s b$).

When streambank materials are present that impede seepage between the stream and aquifer, it is necessary to include the hydraulic conductivity K_s and width d of the semipervious streambank material in the analytical solution. These properties are accounted for by streambank leakance (a, see equation 14). Streambank leakance also may be used to loosely account for constricted flow at the stream-aquifer interface due to the fact that the stream may not penetrate the full saturated thickness of the aquifer. For mathematical simplicity, the streambank is assumed to have negligible storage capacity. Hantush (1965) describes streambank leakance as the effective width of aquifer required to cause the same head loss between the aquifer and the stream channel.

Seepage occurs when there is a hydraulic gradient between the stream and adjoining aquifer. Hydraulic gradients are caused by flood waves, ground-water recharge, ground-water recession, and evapotranspiration. Figure 3 illustrates the response of a stream-aquifer system to a passing flood wave. Prior to the flood wave (times prior to t_s, fig. 3), hydraulic gradients are toward the (gaining) stream and ground water discharges to the stream. As the stream stage rises (figs. 3*A,B*), seepage occurs from the stream to the aquifer (fig. 3*C*), and ground-water heads near the stream increase (fig. 3*D*). Seepage that enters the aquifer adjacent to the stream is referred to as bank storage, and the total volume of bank storage held by the aquifer continues to increase until shortly after the time of the flood peak (t_p, fig. 3*E*). After the flood-wave passes and stream stage falls, water in bank storage is discharged back to the stream, and ground-water heads return to pre-flood wave conditions.

The response of a stream-aquifer system to gradual recharge that occurs uniformly over a ground-water basin is shown schematically in figure 4. Here, for purposes of illustration, it is assumed that the stream stage remains constant during the recharge event. A total amount of recharge that arrives at the water table equal to R (units of length) occurs between times t_s and t_e (figs. 4*A,B*). During the recharge event, ground-water head (illustrated in figure 4*C*) may rise by the amount Δh (R/S_y), and ground-water discharge increases over ambient conditions (fig. 4*D*). After recharge ends at time t_e, ground-water heads and discharge rates gradually return to pre-recharge levels. The falling limb of the ground-water discharge graph is referred to as the ground-water recession curve (fig. 4*D*). Daniel (1976) provides a discussion of the effects of basin-wide recharge and/or evapotranspiration on recession curves using the Rorabaugh (1964) model.

Hydraulic gradients between the stream and adjoining aquifer also may be caused by evapotranspiration from the water table. In such cases, streamflow can be drawn into the aquifer by hydraulic gradients that are toward the aquifer. Evapotranspiration from the water table also can cause recession curves (fig. 4*D*) to diverge from and lie below those that occur in the absence of evapotranspiration (Daniel, 1976). Because evapotranspiration has an opposite effect on hydraulic gradients between a stream and aquifer than does recharge, it can be viewed as negative recharge.

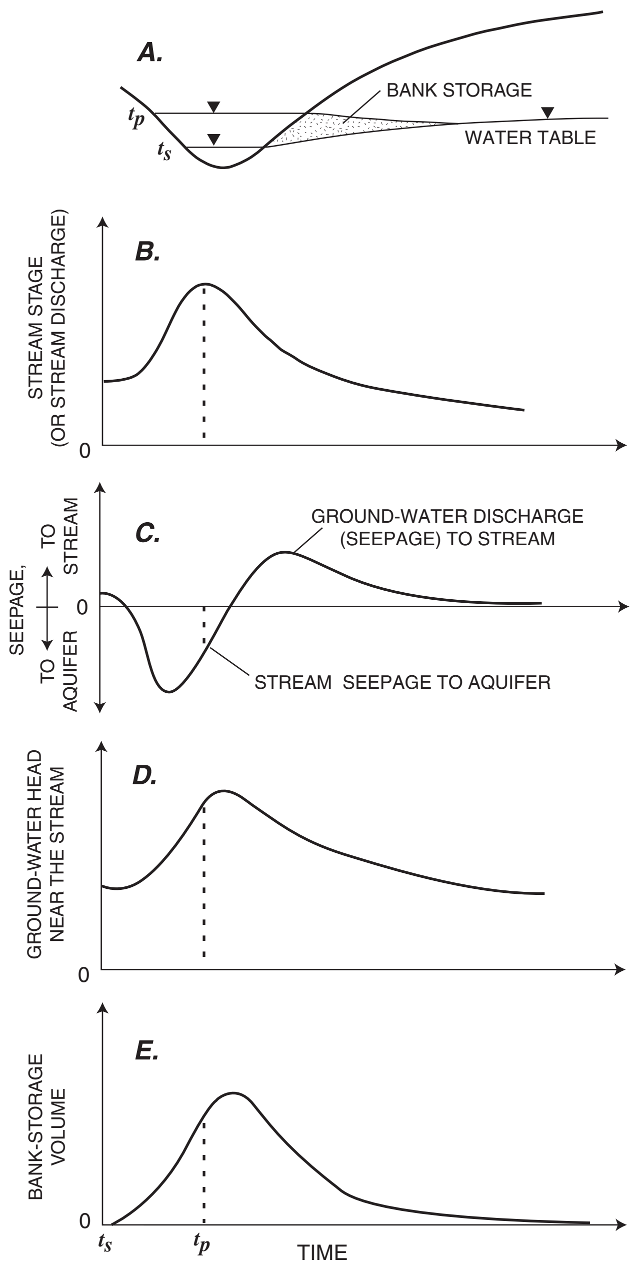


Figure 3. Response of stream-aquifer system to flood wave: (*A*) rise of stream stage and seepage of streamflow into aquifer as bank storage; (*B*) stream-stage hydrograph; (*C*) seepage hydrograph; (*D*) ground-water-head hydrograph; and (*E*) bank-storage-volume hydrograph (t_s, start of flood wave; t_p, time of flood peak). (Adapted from Freeze and Cherry, 1979, p. 227.)

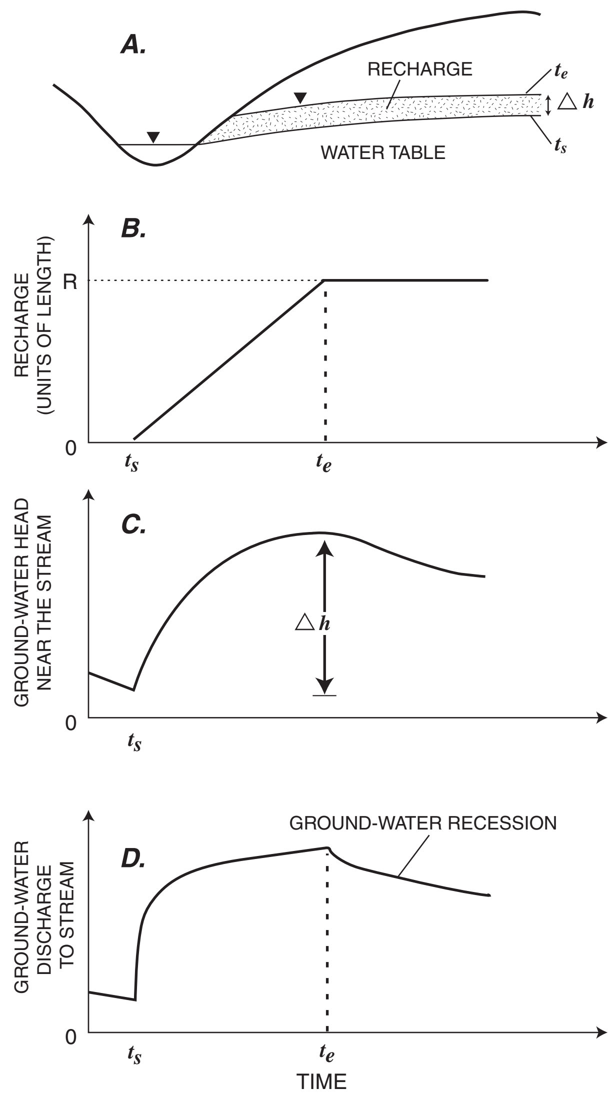


Figure 4. Response of stream-aquifer system to a gradual recharge event: (*A*) rise of water table; (*B*) recharge hydrograph; (*C*) ground-water-head hydrograph; and (*D*) ground-water-discharge hydrograph (t_s, start of recharge; t_e, end of recharge; *R*, total recharge; Δh, maximum rise of water table).

Previous Studies

Several analytical solutions can be found in the literature to evaluate the interaction of confined, leaky, and water-table aquifers in hydraulic connection with an adjoining stream. The majority of analytical solutions developed for stream-aquifer hydraulic interaction have been for the case of one-dimensional, horizontal ground-water flow in confined aquifers bounded by a single, fully penetrating stream. These confined solutions also are frequently used for water-table aquifers under the assumptions that specific yield replaces storativity and that changes in the height of the water table are small in comparison with the saturated thickness of the aquifer (see, for example, Cooper and Rorabaugh, 1963; Hall and Moench, 1972; Sahuquillo, 1986; Workman and others, 1997).

Analytical solutions for confined aquifers have been developed for several types of boundary conditions at the stream. The most widely applied solutions have been for the cases of an instantaneous unit impulse or unit step change in stream-stage elevation in a stream bounding a semi-infinite or finite-width aquifer (Stallman, 1962; Glover, 1966 and 1974; Pinder and others, 1969; Singh, 1969; Venetis, 1970; Hall and Moench, 1972). Rowe (1960) and Hantush (1961a) developed solutions for ground-water head changes in semi-infinite aquifers in response to changes in stream stage that vary linearly with time. A solution for the case of sinusoidal water-level fluctuations in a surface-water body bounding a semi-infinite aquifer was presented by Ferris (1963). Cooper and Rorabaugh (1963) extended this work by developing solutions for a symmetric or asymmetric (damped) sinusoidal-type flood-wave oscillation of a single cycle in either semi-infinite or finite-width aquifers. Workman and others (1997) developed an analytical solution for water-table fluctuations in a finite-width aquifer resulting from changes in stream stage and a mean recharge rate to the aquifer.

Theoretical solutions for ideal boundary conditions such as step, linear, or sinusoidal stream-stage fluctuations are useful for understanding the transient response of ground-water systems to stream-stage changes. However, for applicability to realistic field conditions in which stream stage varies arbitrarily with time, the solutions must be linked with the convolution method. A comprehensive discussion of the use of the convolution method in stream-aquifer studies is provided by Hall and Moench (1972), who showed how the method can be used to compute time-varying heads and seepage rates in response to time-varying stream-stage fluctuations. An additional aspect of their work was that they provided analytical solutions for seepage at the stream-aquifer interface in addition to solutions for ground-water heads.

Most applications of the convolution method in stream-aquifer studies have been for the purposes of determining ground-water-level fluctuations and aquifer diffusivity (the ratio of transmissivity to storage), for conditions in which it was assumed that semipervious streambank material was absent (Bedinger and Reed, 1964; Pinder and others, 1969; Grubb and Zehner, 1973; Reynolds, 1987; Workman and others, 1997; Serrano and Workman, 1998). Moench and others (1974), however, applied the method to the problem of streamflow routing modified by bank storage. They compared measured streamflow hydrographs of the North Canadian River in central Oklahoma to hydrographs calculated using the semi-infinite confined-aquifer solutions for conditions with and without semipervious streambank material. They found that the inclusion of a streambank leakance term improved the match between measured and calculated hydrographs. Moench and Kisiel (1970) developed an analytical solution and an inverse convolution method to estimate ground-water recharge from a transient ground-water mound induced by a flood wave in a finite-width stream under ephemeral flow conditions. They applied the method to the determination of ground-water recharge to the water-table aquifer underlying the Rillito River in Tucson, Arizona.

Mathematically, the response of a ground-water basin containing a stream to recharge, irrigation, or evapotranspiration occurring uniformly over the basin can be determined using the same analytical solutions that are used to determine the response to a rise or fall in stream stage of perennial streams. On this basis, several investigators (Kraijenhoff van de Leur, 1958; Rorabaugh, 1964; Singh, 1969; Singh and Stall, 1971; Daniel, 1976) have applied analytical solutions for one-dimensional, horizontal flow to the problem of base-flow recession (the discharge of stored ground water to streams). Reviews of mathematical approaches for evaluating base-flow recession are provided by Hall (1968), Singh (1969), Rutledge (1993), and Tallaksen (1995). In addition, Rutledge (1993, 1997) provides computer programs for estimating ground-water recharge and evapotranspiration based on the methods of Rorabaugh (1964) and Daniel (1976).

Fewer analytical solutions are available for leaky aquifers than are available for confined aquifers because of the additional complications brought about by the presence of an overlying aquitard. Hantush (1961b) derived transient solutions for ground-water head and streambank seepage in a leaky aquifer with a nonstorative aquitard overlain by a constant-head source bed. His solutions are extensions of the steady-state solutions for similar aquifer-aquitard conditions developed by Peterson (1961). Kabala and Thorne (1997) also assumed no storage in the aquitard, but unlike previous investigators they used a constant-discharge boundary condition at the stream; they also provide solutions for both fully-penetrating and partially-penetrating streams. Spiegel (1962) developed several solutions for leaky aquifers found in the Rio Grande drainage basin of Colorado and New Mexico. Zhang (1992) developed solutions for a leaky aquifer overlain by a water-table aquitard that included a storage term (specific yield) for the aquitard. Zhang's solutions are for a step change in stream stage and for linearly increasing stream stage.

Three approaches have been used to derive analytical solutions for flow in water-table (unconfined) aquifers. In the first approach, described previously, solutions for confined aquifers are applied to water-table conditions under the assumptions that specific yield can be substituted for storativity, that changes in the height of the water table are small in comparison with the saturated thickness of the aquifer, and, hence, that the saturated thickness of the aquifer can be assumed to remain constant. This approach for the use of confined solutions for water-table aquifer conditions presumes one-dimensional, horizontal flow in a homogeneous and isotropic aquifer.

The second approach also assumes one-dimensional, horizontal flow in a homogeneous and isotropic aquifer with specific yield substituted for storativity. However, in this approach, the saturated thickness of the aquifer is taken to be a function of the height of the water table, which varies with time. Under these assumptions, ground-water flow is described by the nonlinear Boussinesq equation. Solutions based on the Boussinesq equation are presented by Singh (1969), Marino (1973), Govindaraju and Koelliker (1994), Guo (1997), and Serrano and Workman (1998). Applications of these solutions to base-flow-recession analyses are given by Brutsaert and Nieber (1977), Vogel and Kroll (1992), and Szilagyi and Parlange (1998), among others.

The third approach for deriving analytical solutions for water-table aquifers is to treat ground-water flow as two dimensional in the *x,z* plane. This approach was taken by Streltsova (1975), Higgins (1980), Gill (1985), Neuman (1981), and van de Giesen and others (1994). Streltsova (1975) derived a solution for the average ground-water head in a vertical section of a semi-infinite, water-table aquifer by accounting for vertical flow at the water table through a vertical-diffusivity parameter, which is composed of the vertical hydraulic conductivity, specific yield, and thickness of the vertical zone through which the water table falls. Unlike Streltsova, Higgins (1980) provides a solution for head at any point (*x,z*) in the domain of a semi-infinite, water-table aquifer. Higgins' solution, however, is based on the assumptions of a single, isotropic value of hydraulic conductivity and ignores elastic-storage properties of the aquifer. Gill (1985) and van de Giesen and others (1994) took approaches that were similar to Higgins' (1980) for a finite-width, water-table aquifer, in which they assumed isotropic conditions and ignored elastic storage. A two-dimensional solution also was developed by van de Giesen and others (1994). They compared the results from their solution with those derived from the Boussinesq equation and found that, because the Boussinesq equation neglects vertical flow, the resulting solution overestimated seepage rates immediately after a sudden change in stream stage and underestimated seepage rates at later times. Neuman (1981) extended the work of Higgins (1980) by accounting for anisotropic hydraulic conductivity (K_x and K_z), elastic storage, and drainage at the water table. Consequently, Neuman's analytical solution is the most comprehensive of those that have been published for water-table aquifers and is the planar flow analog to the solution he developed for radial flow to a fully penetrating well in a water-table aquifer (Neuman, 1972).

Two additional topics are closely related to the preceding discussions. First, because of the similarity between stream-aquifer hydraulic interaction and ground-water flow to drains and canals, some of the solutions for stream-aquifer hydraulic interaction also have been applied to problems in irrigation and drainage. Discussions of the application of these solutions to problems concerning transient flow to drains and canals are given by Spiegel (1962), Glover (1966, 1974), van Schilfgaarde (1970), Marino and Luthin (1982), Gill (1984), van de Giesen and others (1994), and Khan and Rushton (1996). Second, analytical solutions have been derived for aquifers bounded by more than one stream. Papadopulos (1963) and Stallman and Papadopulos (1966) developed solutions for two-dimensional (planar), wedge-shaped aquifers bounded by two streams, and Brown (1963) developed solutions for two-dimensional rectangular aquifers bounded by four streams (or canals).

Some situations in stream-aquifer interaction, such as the presence of complicated aquifer boundary conditions, aquifer heterogeneity, or complicated stream discharge and stage relations, are not handled easily by use of analytical methods. In these cases, it may be necessary to use numerical-modeling methods that couple open-channel flow equations and the ground-water flow equation to simultaneously solve for stream stage and ground-water heads (Pinder and Sauer, 1971; Zitta and Wiggert, 1971; Prudic, 1989; Hunt, 1990; Swain and Wexler, 1996; and Perkins and Koussis, 1996). Numerical-modeling methods that can be applied to such situations are outside the scope of this work.

GENERAL THEORETICAL BACKGROUND

The analytical solutions presented in this report are based on the mathematical theory of ground-water flow in confined, leaky, and water-table aquifers bounded by a single, fully penetrating stream. These solutions are derived for the condition of an instantaneous step change (or step input) in the water level of the bounding stream relative to the water level in the adjacent aquifer. These step-input solutions are then implemented in computer programs STLK1 and STWT1 for time-varying stream-stage and recharge inputs by use of convolution relations, which are a form of mathematical superposition. This section provides a general theoretical background on the mathematical techniques that are used in the derivation of the analytical solutions and development of the two computer programs.

Governing Differential Equation and Initial and Boundary Conditions

Analytical solutions derived in this report are mathematical models of stream-aquifer hydraulic interaction. The solutions are based on the governing partial differential equation of transient ground-water flow in a saturated, homogeneous, slightly compressible, and anisotropic aquifer in which the principal directions of hydraulic conductivity are oriented parallel to the coordinate axes. This equation derives from Darcy's law and the law of conservation of mass (continuity equation), which states that the net rate of fluid mass flow into any elemental volume of aquifer is equal to the time rate of change of fluid mass storage within the element (Freeze and Cherry, 1979). For the most general case considered in this report, the equation is written in two space dimensions as

$$K_x \frac{\partial^2 h}{\partial x^2} + K_z \frac{\partial^2 h}{\partial z^2} = S_s \frac{\partial h}{\partial t} + q' \quad , \qquad (1)$$

where

- h is ground-water head (units of length);
- K_x, K_z are horizontal and vertical hydraulic conductivity of the aquifer, respectively (units of length per time);
- S_s is specific storage of the aquifer (units of inverse length);
- q' is a volumetric flow rate to or from the aquifer per unit volume of aquifer, and represents sources or sinks of water to the aquifer (units of inverse time);
- x, z are horizontal and vertical coordinate directions, respectively (units of length); and
- t is time (units of time).

For leaky-aquifer systems, the flow equation has only one (horizontal) space dimension but is coupled with a similar equation for vertical flow in the aquitard. Detailed assumptions used in the derivations of the analytical solutions are provided in the section "Presentation of Analytical Solutions." The dependent variable in equation 1 for which solutions are derived is the head distribution throughout the aquifer, h, which is a function of space (x, z) and time (t), and can be written as $h(x, z, t)$.

Particular solutions to equation 1 are determined by defining a specific set of boundary and initial conditions. These conditions are mathematical statements that describe the head or flow conditions of the aquifer along its boundaries at a particular time. The combination of equation 1 with the set of boundary and initial conditions is known as a boundary-value problem. Three general types of boundary conditions are used in the derivations—specified head, specified flux, and head-dependent flux. In the boundary-value problems described in this report, the stream is modeled either as a specified-head boundary (for conditions in which a semipervious streambank is absent) or as a head-dependent flux boundary (for conditions in which a semipervious streambank is present). The set of mathematical boundary conditions used for each stream-aquifer system is described in detail in Attachment 1 and summarized in the section "Presentation of Analytical Solutions." A single initial condition is used for all derivations, which states that the water level in the stream is at the same elevation as the water level (ground-water head) everywhere in the aquifer at $t = 0$ (that is, the system is initially in static equilibrium).

All of the stream-aquifer system parameters are assumed to be time invariant, which means that the hydraulic properties of the aquifer and semipervious streambank material remain constant with time. The systems also are linear, because the governing partial differential equation of ground-water flow and all of the boundary and initial conditions used in the derivations are linear. The linearity of the systems allows for the use of convolution.

Analytical solutions to the boundary-value problems are derived by use of the Laplace transform method. This method involves the elimination of the time variable by an integral transform of the original boundary-value problem; it results in a subsidiary boundary-value problem in the Laplace domain. The subsidiary problem is solved in the Laplace domain and the resulting solution is then numerically inverted back to the time domain using a numerical-inversion method described by Stehfest (1970). Moench and Ogata (1984) discuss the application of the Laplace transform and Stehfest numerical-inversion method for boundary-value problems in ground-water flow.

All of the analytical solutions derived in this report are for the condition of an instantaneous step change in the water level of the stream relative to the water level in the adjacent aquifer. Such solutions are referred to mathematically as unit-step responses of the aquifer. Alternatively, one could use an impulse-response function. There does not appear to be a distinct advantage of using one approach over the other. Unit-step response solutions used here are dimensionless ground-water head functions that describe the ratio of the change in head of the aquifer at a given location x, z and at time t to the instantaneous step change in water level of the stream

$$h_D(x, z, t) = \frac{h_i - h(x, z, t)}{c} , \qquad (2)$$

where

$h_D(x, z, t)$ is the dimensionless unit-step response solution;

c is the instantaneous step change in water level of the stream $(h_i - h_0)$ (units of length);

h_i is the initial water level in the stream-aquifer system (units of length); and

h_0 is the water level in the stream after the step change (units of length).

A different unit-step response solution is derived for each specific aquifer type and set of boundary conditions. The unit-step response solutions are derived in the Laplace domain and referred to as Laplace transform unit-step response solutions. These solutions then form the basis for the superposition methods described below.

Although the definition of the unit-step response solutions in equation 2 was made in reference to a sudden rise or fall in the water level of the stream, the solutions are mathematically equivalent to a step rise or fall in the water level of the aquifer relative to that of the stream, caused, for example, by area-wide recharge, irrigation, or evapotranspiration (see, for example, Rorabaugh, 1960 and 1964). The only difference between the two types of stresses is the direction of seepage at the stream-aquifer boundary. A rise in stream stage will result in surface-water seepage to the aquifer and a rise in the water level of the aquifer (caused by recharge or irrigation) will result in ground-water discharge (seepage) to the stream.

Convolution Relations

Inasmuch as the boundary-value problems are linear, the total response of a ground-water system to a time series of individual step changes in stream stage or water level of the aquifer can be determined by summation (superposition) of the unit-step response solutions for the individual step changes. Mathematically, the individual responses are summed by use of the convolution integral (or convolution equation), which relates a time series of step changes (system input stresses) to a time-series of ground-water head changes (system output responses):

$$h(x, z, t) = h_i + \int_0^t F'(\tau) h_D(x, z, t-\tau) d\tau \quad , \qquad (3)$$

where

$F'(\tau)$ is the time rate of change of the system stress (change in either stream stage or water level of the aquifer due to recharge or ET) (units of length per time); and

τ is the time variable of integration (delay time) (units of time).

Use of the convolution integral is valid for time-invariant linear systems. For linearity to hold here, changes in heads must be relatively small in comparison to the saturated thickness of the aquifer.

Convolution also is used to determine time-varying seepage rates between a stream and aquifer and bank-storage volumes. Seepage rates are determined from the head gradient at the stream-aquifer boundary ($x = x_0$) according to Darcy's law (Hall and Moench, 1972):

$$Q(t) = \frac{K_x b}{x_0} \int_0^t F'(\tau) \frac{\partial h_D(x_0, z, t-\tau)}{\partial x_D} d\tau \quad , \qquad (4)$$

where

$Q(t)$ is the seepage rate per unit length of stream from (or to) one side of the stream (units of volume per time per length of stream);

x_D is the dimensionless distance x/x_0; and

x_0 is the distance from the middle of the stream to the stream-aquifer boundary (units of length).

Parameter x_0, whose definition is illustrated in figure 2, is used only to define non-dimensional parameters in the derivations of the analytical solutions. Its specific value is immaterial to the seepage-rate or head determinations.

As used in this report, seepage is negative when flow is from the stream to the aquifer and positive when flow is from the aquifer to the stream. The total seepage rate, $Q_T(t)$, from both sides of a stream over a stream reach of length l_s is calculated by multiplying equation 4 by $2l_s$:

$$Q_T(t) = 2Q(t)l_s \quad , \qquad (5)$$

where $Q_T(t)$ has units of volume per time and l_s has units of length.

Bank storage occurs when water flows from the stream to the aquifer in response to an increase in the water level of the stream relative to that of the aquifer. Bank storage, $V(t)$, is defined as the cumulative volume of water per unit length of stream that has entered the aquifer from one side of the stream over time t (Cooper and Rorabaugh, 1963, p. 349):

$$V(t) = -\int_0^t Q(t) dt \quad . \qquad (6)$$

The negative sign is introduced because bank storage is taken to be a positive quantity, and seepage is negative when flow is from the stream to the aquifer. A total volume of bank storage that enters the aquifer from both sides of a stream over a reach l_s is calculated from:

$$V_T(t) = 2V(t)l_s \quad , \qquad (7)$$

where $V_T(t)$ has units of volume.

PRESENTATION OF ANALYTICAL SOLUTIONS

This section describes the simplifying assumptions and boundary and initial conditions that were used to develop boundary-value problems of stream-aquifer hydraulic interaction for each of the confined, leaky, and water-table aquifers for which Laplace transform step-response analytical solutions are derived. Complete derivations of the Laplace transform solutions for all aquifer types are given in Attachment 1; the resulting solutions for head and seepage also are presented in this section for convenience and discussion. Solutions for confined and leaky aquifers are presented simultaneously because of the similarity of the aquifer types and resulting solutions.

Confined and Leaky Aquifers

Figures 5-12 are diagrammatic cross sections through part of several idealized semi-infinite and finite-width, confined and leaky aquifer types for which analytical solutions are derived. For each aquifer type, solutions are provided for conditions in which semipervious streambank material is absent and for conditions in which it is present. In the figures, the semipervious streambank material extends only to the top of each aquifer because it is assumed that there is no direct interaction (seepage) between the stream and overlying confining layer or aquitard. Solutions are derived for confined aquifers (figs. 5 and 6) and for three types of leaky aquifers: those in which a source bed with a constant head overlies the aquitard (leaky aquifer case 1, figs. 7 and 8); those in which an impermeable layer overlies the aquitard (leaky aquifer case 2, figs. 9 and 10); and those that are overlain by a water-table aquitard (leaky aquifer case 3, figs. 11 and 12).

Each aquifer is bounded by a stream that extends from the impermeable boundary underlying the aquifer ($z = 0$) to a position lying above the thickness of the aquifer at $z = b$. The figures also show the location of the origin of the coordinate system at the middle of the stream. As described in the previous section, although the variable x_0 enters into the derivations, the analytical solutions do not depend on its specific value.

Ground-water flow is assumed to be horizontal (one dimensional) in the direction perpendicular to the stream for each of the confined and leaky aquifer types. In addition, for the leaky aquifers, flow is assumed to be strictly vertical through the overlying aquitard. For this to be the case, hydraulic conductivity of the aquitard must be small compared with hydraulic conductivity of the aquifer. Neuman and Witherspoon (1969) have evaluated this assumption by use of a finite-element model for the case of flow to a pumping well in a leaky-aquifer system. They found that the errors introduced by this assumption are usually less than 5 percent when the hydraulic conductivity of the aquifer is more than 100 times the hydraulic conductivity of the aquitard. As a practical matter, such a large contrast in hydraulic conductivity may not be essential. Because the cone of depression around a pumping well is much more pronounced for a given discharge rate than the head distribution near a stream for that given discharge, diagonal flow components in an aquitard will tend to be greater near a pumping well than for the stream-aquifer case.

Figure 5*A* includes a schematic drawing of an observation well at which ground-water-level measurements could be made. Because ground-water flow is assumed to be horizontal in the confined and leaky aquifers, equipotentials in each aquifer are vertical and, therefore, ground-water heads are uniform throughout the thickness of each aquifer. Thus, the head is independent of vertical location.

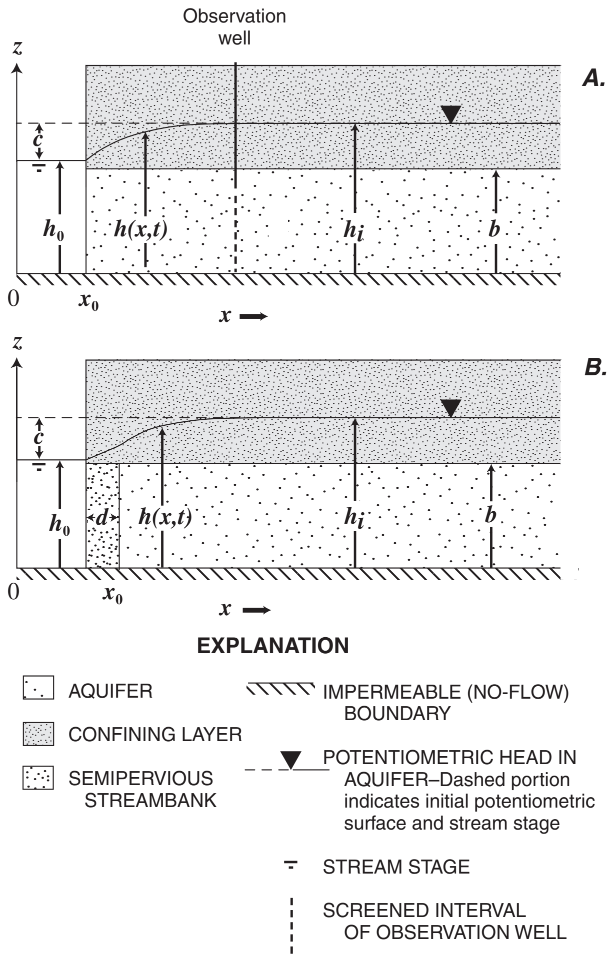


Figure 5. Semi-infinite, confined aquifer (*A*) without semipervious streambank material and (*B*) with semipervious streambank material. (b, aquifer thickness; c, instantaneous step change in water level of stream; d, width of semipervious streambank material; $h(x,t)$, potentiometric head in aquifer, which is a function of distance from middle of stream (x) and time (t); h_i, initial potentiometric surface and stream stage; h_0, water level in stream after step change; x_0, distance from middle of stream to stream-aquifer boundary.)

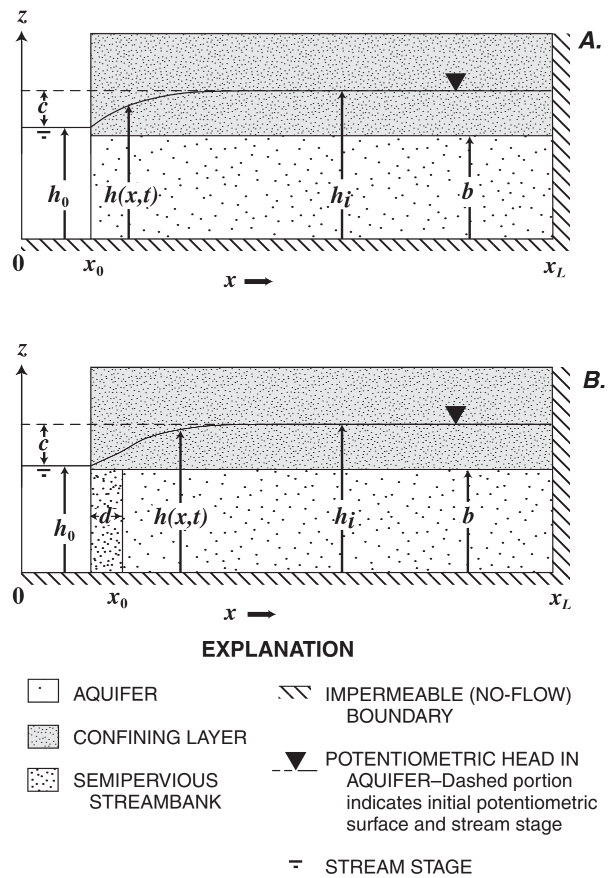


Figure 6. Finite-width, confined aquifer (*A*) without semipervious streambank material and (*B*) with semipervious streambank material (b, aquifer thickness; c, instantaneous step change in water level of stream; d, width of semipervious streambank material; $h(x,t)$, potentiometric head in aquifer, which is a function of distance from middle of stream (x) and time (t); h_i, initial potentiometric surface and stream stage; h_0, water level in stream after step change; x_0, distance from middle of stream to stream-aquifer boundary; x_L, aquifer width.)

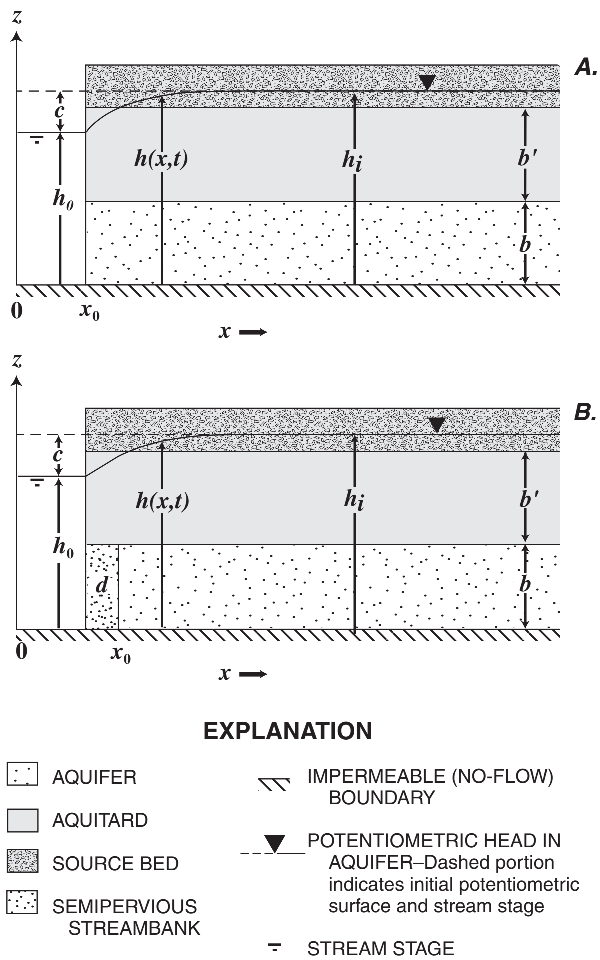


Figure 7. Semi-infinite, leaky aquifer with constant head overlying the aquitard (case 1) (*A*) without semipervious streambank material and (*B*) with semipervious streambank material. (*b*, aquifer thickness; *b*′, aquitard thickness; *c*, instantaneous step change in water level of stream; *d*, width of semipervious streambank material; $h(x,t)$, potentiometric head in aquifer, which is a function of distance from middle of stream (x) and time (t); h_i, initial potentiometric surface and stream stage; h_0, water level in stream after step change; x_0, distance from middle of stream to stream-aquifer boundary.)

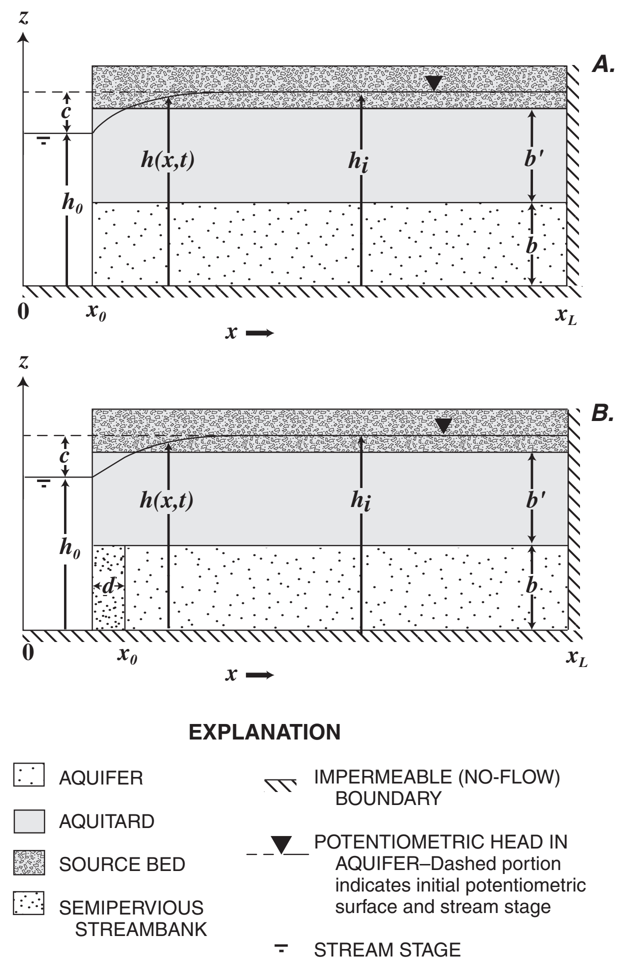


Figure 8. Finite-width, leaky aquifer with constant head overlying the aquitard (case 1) (*A*) without semipervious streambank material and (*B*) with semipervious streambank material. (*b*, aquifer thickness; *b*′, aquitard thickness; *c*, instantaneous step change in water level of stream; *d*, width of semipervious streambank material; $h(x,t)$, potentiometric head in aquifer, which is a function of distance from middle of stream (x) and time (t); h_i, initial potentiometric surface and stream stage; h_0, water level in stream after step change; x_0, distance from middle of stream to stream-aquifer boundary; x_L, aquifer width.)

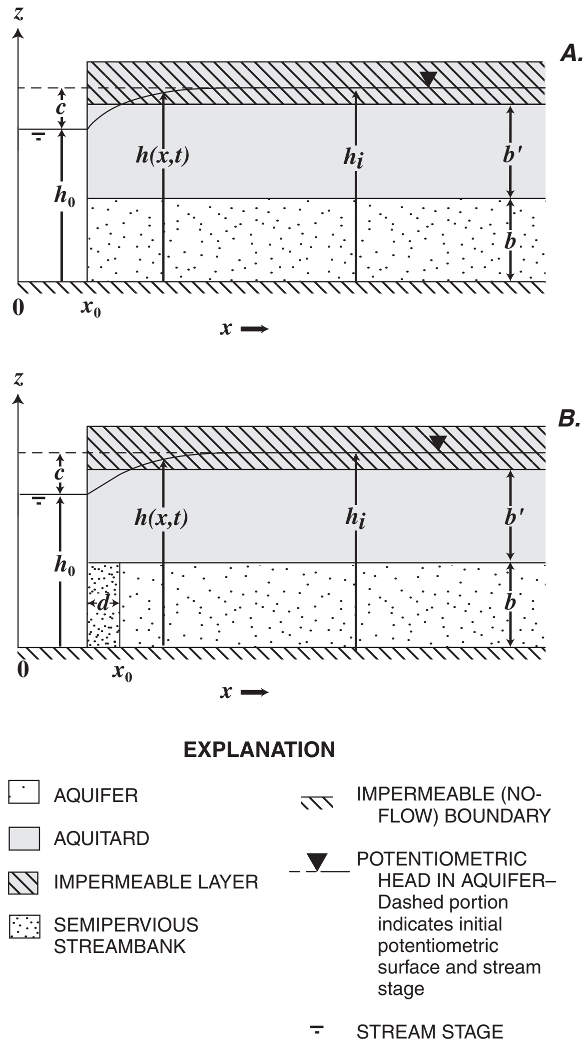


Figure 9. Semi-infinite, leaky aquifer with impermeable layer overlying the aquitard (case 2) (*A*) without semipervious streambank material and (*B*) with semipervious streambank material. (*b*, aquifer thickness; *b′*,aquitard thickness; *c*, instantaneous step change in water level of stream; *d*, width of semipervious streambank material; *h*(*x,t*), potentiometric head in aquifer, which is a function of distance from middle of stream (*x*) and time (*t*); h_i, initial potentiometric surface and stream stage; h_0, water level in stream after step change; x_0, distance from middle of stream to stream-aquifer boundary.)

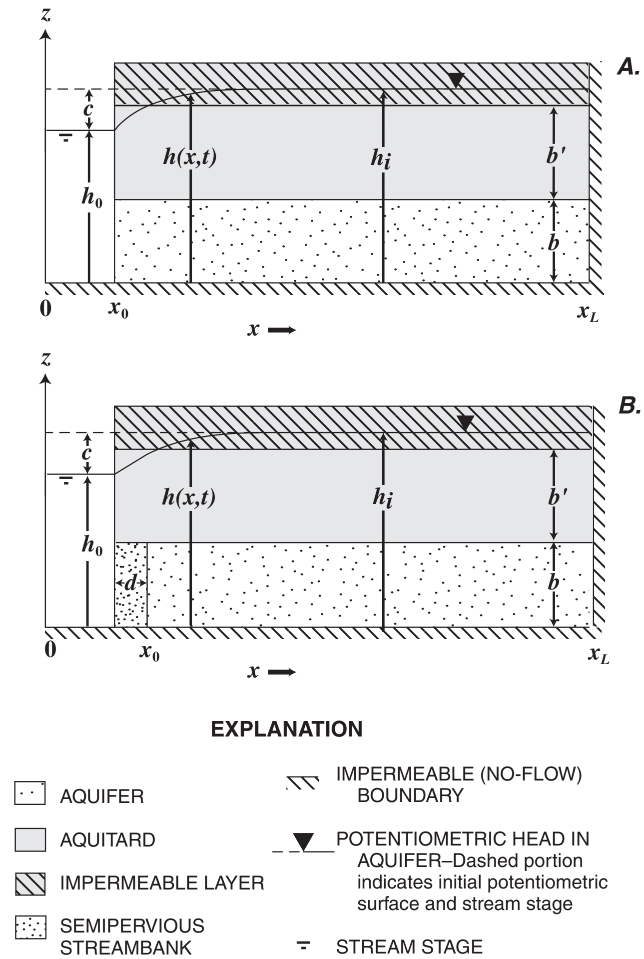


Figure 10. Finite-width, leaky aquifer with impermeable layer overlying the aquitard (case 2) (*A*) without semipervious streambank material and (*B*) with semipervious streambank material. (*b*, aquifer thickness; *b′*, aquitard thickness; *c*, instantaneous step change in water level of stream; *d*, width of semipervious streambank material; *h*(*x,t*), potentiometric head in aquifer, which is a function of distance from middle of stream (*x*) and time (*t*); h_i, initial potentiometric surface and stream stage; h_0, water level in stream after step change; x_0, distance from middle of stream to stream-aquifer boundary; x_L, aquifer width.)

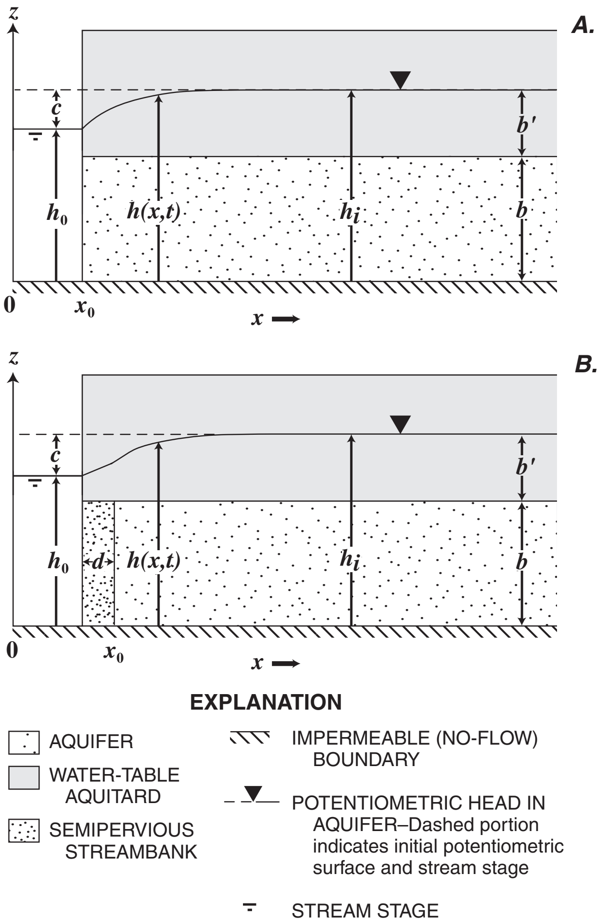


Figure 11. Semi-infinite, leaky aquifer overlain by a water-table aquitard (case 3) (*A*) without semipervious streambank material and (*B*) with semipervious streambank material. (*b*, aquifer thickness; *b*′,saturated thickness of aquitard; *c*, instantaneous step change in water level of stream; *d*, width of semipervious streambank material; *h*(*x,t*), potentiometric head in aquifer, which is a function of distance from middle of stream (*x*) and time (*t*); h_i, initial potentiometric surface and stream stage; h_0, water level in stream after step change; x_0, distance from middle of stream to stream-aquifer boundary.)

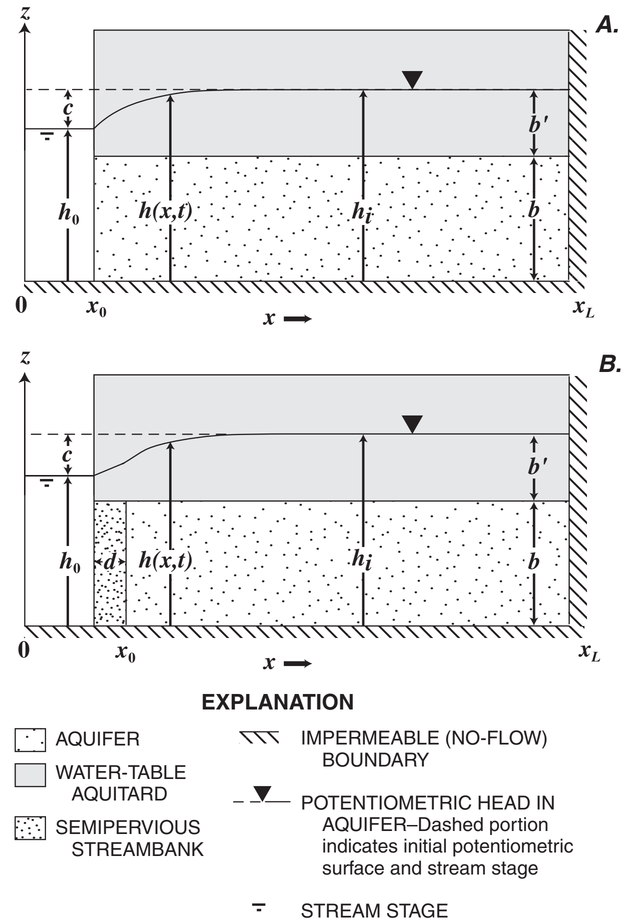


Figure 12. Finite-width, leaky aquifer overlain by a water-table aquitard (case 3) (*A*) without semipervious streambank material and (*B*) with semipervious streambank material. (*b*, aquifer thickness; *b*′,saturated thickness of aquitard; *c*, instantaneous step change in water level of stream; *d*, width of semipervious streambank material; *h*(*x,t*), potentiometric head in aquifer, which is a function of distance from middle of stream (*x*) and time (*t*); h_i, initial potentiometric surface and stream stage; h_0, water level in stream after step change; x_0, distance from middle of stream to stream-aquifer boundary; x_L, aquifer width.)

Assumptions

In addition to the assumptions of horizontal flow in the aquifers and strictly vertical flow in the aquitards, several other simplifying assumptions were necessary to represent each stream-aquifer system mathematically. These assumptions are as follows:

Assumptions for both confined and leaky aquifer types—

1. Each aquifer is homogeneous, isotropic, and of uniform thickness.
2. The lower boundary of each aquifer type is horizontal and impermeable.
3. Hydraulic properties of the aquifers do not change with time.
4. The porous medium and fluid are slightly compressible.
5. Observation wells or piezometers are infinitesimal in diameter and respond instantly to pressure changes in the aquifer.
6. The stream that forms a boundary to the aquifer is straight and fully penetrates the aquifer.
7. Initially, the water level in the stream is at the same elevation as the water level everywhere in the aquifer and aquitard. At time $t = 0$, the water level in the stream is suddenly lowered (or raised) to a new position lying a distance of one unit below (or above) the original one (that is, a unit-step excitation).
8. The semipervious streambank material, if present, has negligible capacity to store water.

Additional assumptions for leaky aquifer types—

9. The aquitard is homogeneous, isotropic, and of uniform thickness.
10. The hydraulic conductivity of the aquitard is small compared to the hydraulic conductivity of the underlying aquifer.
11. Hydraulic properties of the aquitard do not change with time.
12. For a leaky aquifer overlain by a water-table aquitard, water in the aquitard is released (or taken up) instantaneously in a vertical direction from (or into) the zone above the water table in response to a decline (or rise) in the elevation of the water table. Also, the change in saturated thickness of the water-table aquitard due to stream-stage fluctuations or recharge is small compared with the initial saturated thickness of the aquitard. Finally, pressure changes caused by a recharge event are propagated instantaneously through the water-table aquitard to the underlying aquifer.

Boundary-Value Problems

The governing partial differential equation describing one-dimensional, horizontal ground-water flow in a confined or leaky aquifer based on equation 1 is

$$\frac{\partial^2 h}{\partial x^2} = \frac{S_s}{K_x}\frac{\partial h}{\partial t} + q' , \qquad (8)$$

where

$q' = -\dfrac{K'}{K_x b}\left(\dfrac{\partial h'}{\partial z}\right)_{z=b}$;

K' is the vertical hydraulic conductivity of the aquitard (units of length per time); and

h' is the head in the aquitard (units of length).

For confined aquifers, $K' = 0$, hence $q' = 0$. The domain for equation 8 for semi-infinite aquifers is $x_0 \le x < \infty$ and for finite-width aquifers is $x_0 \le x \le x_L$ (where x_L is the width of a finite-width aquifer). In equation 8, h is a function of x and t and h' is a function of z and t.

The initial condition for all boundary-value problems is

$$h(x, 0) = h_i \tag{9}$$

where h_i is the initial water level (or potentiometric surface) in the stream-aquifer system.

Several boundary conditions are used for the confined and leaky aquifers; the particular set of boundary conditions used for each system depends on the conditions being modeled. For a semi-infinite aquifer, the boundary condition as x approaches infinity is

$$h(\infty, t) = h_i \ , \tag{10}$$

whereas for a finite-width aquifer, the boundary condition at $x = x_L$ is

$$\frac{\partial h}{\partial x}(x_L, t) = 0. \tag{11}$$

The boundary condition used at the stream-aquifer interface depends upon the presence or absence of semipervious streambank material. For conditions in which there is no semipervious streambank material, a specified-head boundary condition is used at x_0

$$h(x_0, t) = h_0 \ , \tag{12}$$

where h_0 is the water level in the stream after the instantaneous step change. For conditions in which semipervious streambank material is present, a head-dependent flux boundary condition is used at x_0

$$\frac{\partial h(x_0, t)}{\partial x} = -\frac{1}{a}[h_0 - h(x_0, t)] \ , \tag{13}$$

where a is streambank leakance and $[h_0 - h(x_0, t)]$ is the change in head across the semipervious streambank material. Streambank leakance is defined as

$$a = \frac{K_x d}{K_s} \ , \tag{14}$$

where

- d is the width of the semipervious streambank material (units of length); and
- K_s is the hydraulic conductivity of the semipervious streambank material (units of length per time).

The ratio K_s/d can be considered a single fluid-transfer parameter.

For leaky-aquifer conditions, a governing partial differential equation describing one-dimensional, vertical flow in the overlying aquitard must be solved with appropriate boundary conditions and coupled with equation 8. This equation is

$$\frac{\partial^2 h'}{\partial z^2} = \frac{S_s'}{K'}\frac{\partial h'}{\partial t} \ , \tag{15}$$

where S_s' is the specific storage of the aquitard. The domain for which equation 15 is applicable is $b \le z \le b + b'$.

The initial condition for head in the aquitard for all boundary-value problems is

$$h'(z, 0) = h_i \ . \tag{16}$$

The boundary condition at the aquitard-aquifer boundary ($z = b$) is

$$h'(b, t) = h \ . \tag{17}$$

Alternative boundary conditions are used for the top of the aquitard ($z = b + b'$) that depend upon the presence and hydraulic conditions of the overlying bed. For the condition of constant head overlying the aquitard (case 1; figs. 7 and 8), the boundary condition at the top of the aquitard is

$$h'(b + b', t) = h_i \ . \tag{18}$$

For the condition of an impermeable layer overlying the aquitard (case 2; figs. 9 and 10), the boundary condition is

$$\frac{\partial h'}{\partial z}(b + b', t) = 0 \ . \tag{19}$$

For the condition in which the overlying material is unsaturated, the aquitard is under water-table conditions (case 3; figs. 11 and 12). In this case, the boundary condition at the water table is

$$\frac{\partial h'}{\partial z}(b + b', t) = -\frac{S_y'}{K'}\frac{\partial h'}{\partial t}(b + b', t) \ , \qquad (20)$$

where S_y' is the specific yield of the aquitard.

Laplace Transform Analytical Solutions

The dimensional boundary-value problems described by equations 8–20 are made dimensionless by substituting the dimensionless variables and variable groupings shown in table 1. The Laplace transform solutions for all confined and leaky aquifer types can be written in the most general form as (equation A1.48 in Attachment 1)

$$\bar{h}_D = \frac{W \exp[-\sqrt{p + \bar{q}_D}(x_D - 1)]}{p\{1 + \sqrt{p + \bar{q}_D} A \tanh[\sqrt{p + \bar{q}_D}(x_{LD} - 1)]\}} \ , \qquad (21)$$

where $\bar{h}_D$ is the dimensionless Laplace transform unit-step response solution at each point (x_D) in a vertical section of the aquifer. The bar over the unit step response (h_D) represents the Laplace transform. The Laplace transform variable, p, is inversely related to dimensionless time t_D. For the semi-infinite aquifers, x_{LD} goes to infinity and the hyperbolic tangent in equation 21 is unity.

Parameter W is a function of the width of the aquifer perpendicular to the stream and is defined as

$$W = \frac{\exp[-2\sqrt{p + \bar{q}_D}(x_{LD} - x_D)] + 1}{\exp[-2\sqrt{p + \bar{q}_D}(x_{LD} - 1)] + 1} \ .$$

W equals 1 for semi-infinite conditions.

Table 1. Dimensionless variables and variable groupings for confined and leaky aquifers

Dimensionless variable or grouping	Definition
x_D	$\frac{x}{x_0}$
x_{LD}	$\frac{x_L}{x_0}$
x_{0D}	$\frac{x_0}{b}$
z_D'	$\frac{z - b}{b'}$ $\begin{cases} z_D' = 0 \text{ at } z = b \\ z_D' = 1 \text{ at } z = b + b' \end{cases}$
h_D	$\frac{h_i - h}{c}$
h_D'	$\frac{h_i - h'}{c}$
t_D	$\frac{K_x t}{S_s x_0^2}$
A	$\frac{K_x d}{K_s x_0}$
σ_1	$\frac{S_s' b'}{S_s b}$
σ'	$\frac{S_s b}{S_y'}$
γ_1	$\frac{x_0}{b'}\sqrt{\frac{K' b'}{K_x b}}$
m	$\frac{\sigma_1 p}{\gamma_1^2}$

Parameter A is dimensionless streambank leakance

$$A = \frac{a}{x_0} ,$$

where a, streambank leakance, was defined previously. For conditions in which there is no semipervious streambank material, $A = 0$.

Parameter $\bar{q}_D$ accounts for leakage between the aquifer and overlying aquitard. For a confined aquifer with no overlying aquitard

$$\bar{q}_D = 0 \quad ;$$

for a leaky aquifer with constant head overlying the aquitard (case 1)

$$\bar{q}_D = \gamma_1^2 \sqrt{m} \coth(\sqrt{m}) \quad ;$$

for a leaky aquifer with an impermeable layer overlying the aquitard (case 2)

$$\bar{q}_D = \gamma_1^2 \sqrt{m} \tanh(\sqrt{m}) \quad ;$$

and for a leaky aquifer overlain by a water-table aquitard (case 3)

$$\bar{q}_D = \gamma_1^2 \sqrt{m} \frac{[\sqrt{m}(\sigma'\gamma_1^2)\tanh(\sqrt{m}) + p]}{[\sqrt{m}(\sigma'\gamma_1^2) + p\tanh(\sqrt{m})]} \quad .$$

Parameters γ_1, m, σ' are defined in table 1.

Equation 21 is the general solution for all of the confined and leaky aquifer types. For example, for a semi-infinite, confined aquifer with no semipervious streambank material between the aquifer and stream, $W = 1$, $A = 0$, and $\bar{q}_D = 0$. Under these conditions, equation 21 becomes

$$\bar{h}_D = \frac{\exp[-\sqrt{p}(x_D - 1)]}{p} , \qquad (22)$$

which can be analytically inverted from the Laplace domain and written in the real-time domain as

$$h_D = erfc\left[\frac{(x_D - 1)}{(4t_D)^{1/2}}\right] . \qquad (23)$$

Equation 23 is the form most often cited in the literature for the condition in which the origin of the coordinate system is at $x_0 = 0$ (Hall and Moench, 1972, equation 8, p. 489; Neuman, 1981, equation 12, p. 409).

The Laplace transform solution for seepage between the stream and aquifer can be determined by finding the gradient of the unit-step response solution at the stream-aquifer boundary (that is, at $x_D = 1$). This gradient is found by differentiation of equation 21 with respect to x_D and evaluation of the resulting solution at $x_D = 1$

$$\bar{Q}_D = -\left.\frac{d\bar{h}_D}{dx_D}\right|_{x_D = 1} , \qquad (24)$$

where $\bar{Q}_D$ is dimensionless seepage in the Laplace domain. As described in Attachment 1, the gradient at the stream-aquifer boundary for the confined and leaky aquifers, based on equation 21, is

$$\bar{Q}_D = \frac{-\sqrt{p + \bar{q}_D}}{p\{1 + \sqrt{p + \bar{q}_D}A \tanh[\sqrt{p + \bar{q}_D}(x_{LD} - 1)]\}}\left\{\frac{\exp[-2\sqrt{p + \bar{q}_D}(x_{LD} - 1)] - 1}{\exp[-2\sqrt{p + \bar{q}_D}(x_{LD} - 1)] + 1}\right\} . \qquad (25)$$

For a semi-infinite, confined aquifer with no semipervious streambank material between the aquifer and stream, $A = 0$, $\bar{q}_D = 0$, and the exponential terms in the brackets equal -1. Under these conditions, equation 25 becomes

$$\bar{Q}_D = \frac{\sqrt{p}}{p}\,, \qquad (26)$$

which can be analytically inverted from the Laplace domain and written in the real-time domain as

$$Q_D = -\left(\frac{1}{(\pi t_D)^{1/2}}\right), \qquad (27)$$

where Q_D is dimensionless seepage in the real-time domain. Equation 27 is identical to that given by Hall and Moench (1972, equation 10, p. 489) except for the difference in coordinate systems between that used here and that used by Hall and Moench.

Water-Table Aquifers

Figures 13 and 14 are diagrammatic cross sections through part of idealized semi-infinite (fig. 13) and finite-width (fig. 14) water-table aquifers for which new analytical solutions are derived. For each aquifer type, solutions are provided for conditions in which semipervious streambank material is absent and for conditions in which they are present. Each aquifer is bounded by a stream that initially extends from the impermeable boundary underlying the aquifer ($z = 0$) to the top of the saturated thickness of the aquifer at $z = b$. The figures show the location of the origin of the coordinate system. As with the confined and leaky aquifers, the distance from the middle of the stream to the aquifer boundary is x_0.

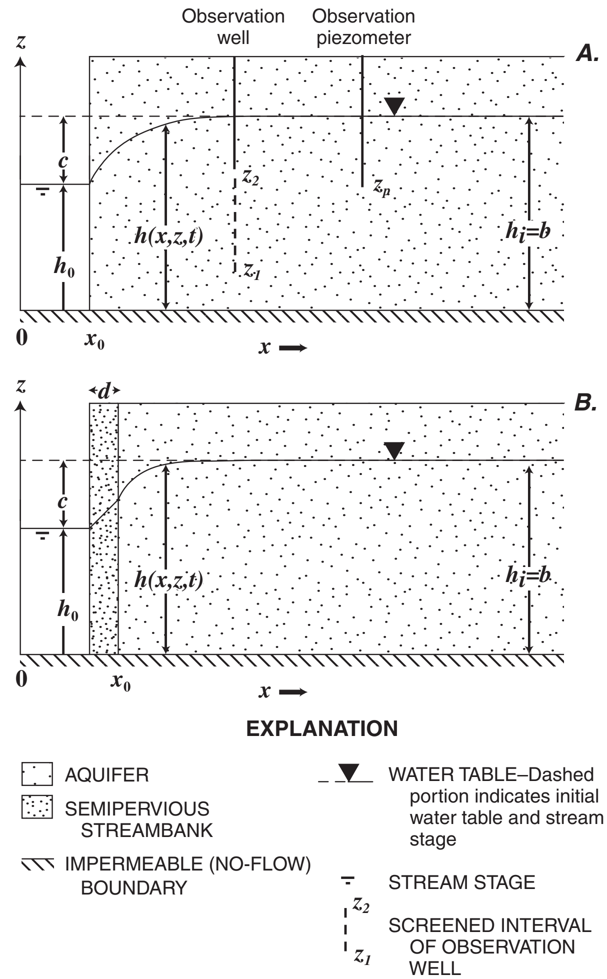


Figure 13. Semi-infinite, water-table aquifer (*A*) without semipervious streambank material and (*B*) with semipervious streambank material. (*b*, saturated thickness of aquifer; *c*, instantaneous step change in water level of stream; *d*, width of semipervious streambank material; $h(x,z,t)$, head in aquifer, which is a function of distance from middle of stream (x), vertical coordinate (z), and time (t); h_i, initial head and stream stage; h_0, water level in stream after step change; x_0, distance from middle of stream to stream-aquifer boundary; z_p, observation piezometer opening; z_1 bottom of screened interval; z_2, top of screened interval.)

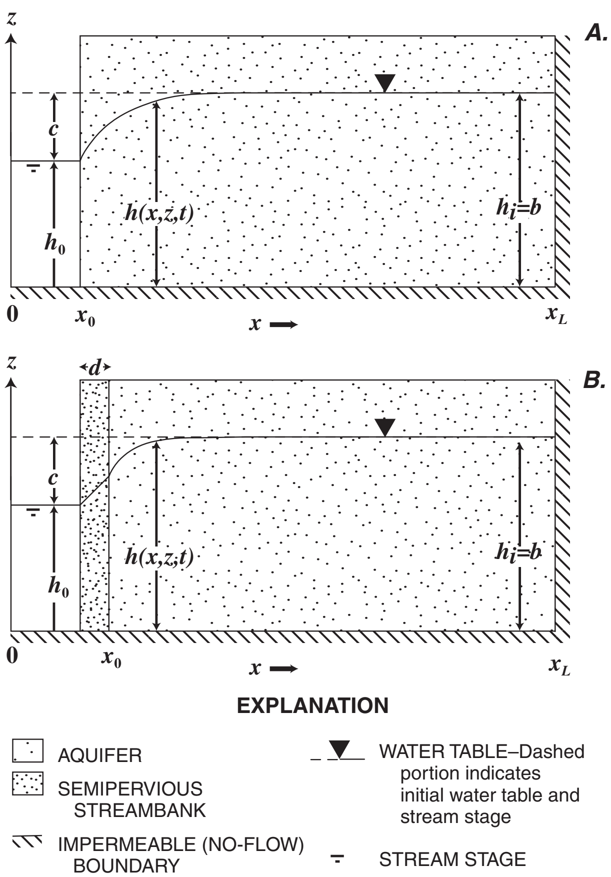


Figure 14. Finite-width, water-table aquifer (*A*) without semipervious streambank material and (*B*) with semipervious streambank material. (*b*, saturated thickness of aquifer; *c*, instantaneous step change in water level of stream; *d*, width of semipervious streambank material; $h(x,z,t)$, head in aquifer, which is a function of distance from middle of stream (x), vertical coordinate (z), and time (t); h_i, initial head and stream stage; h_0, water level in stream after step change; x_0, distance from middle of stream to stream-aquifer boundary; x_L, aquifer width.)

Ground-water flow is assumed to be two dimensional in the x,z plane perpendicular to the stream for each of the water-table aquifers. Hence, ground-water heads can vary in both the x and z directions and are not necessarily uniform over the thickness of each aquifer. Figure 13*A* also shows schematic drawings of a partially penetrating observation well and an observation piezometer at which ground-water-level measurements could be made. Though only shown in figure 13*A*, the observation well and observation piezometer could be located in any of the aquifers shown in figures 13 and 14. The head measured at the observation well is the average head that exists over the screened interval of the well. Because ground-water heads can vary over the thickness of the aquifer, it is likely that heads measured in an observation piezometer and in a partially penetrating observation well located the same distance from the stream would not be equivalent. The only condition under which the heads would be equivalent is that in which a uniform head distribution occurred over the full saturated thickness of the aquifer, such as might occur far from the stream where flow may be essentially horizontal.

Assumptions

In addition to the assumption of two-dimensional flow in each aquifer, several other simplifying assumptions were necessary to represent each stream-aquifer system mathematically. These assumptions are as follows:

1. Each aquifer is homogeneous and of uniform thickness.
2. Each aquifer can be anisotropic provided that the principal directions of the hydraulic conductivity tensor are parallel to the x, z coordinate axes.
3. The lower boundary of each aquifer type is horizontal and impermeable.
4. Hydraulic properties of the aquifers do not change with time.
5. Water is released (or taken up) instantaneously in a vertical direction from (or into) the zone above the water table in response to a decline (or rise) in the elevation of the water table.
6. The change in saturated thickness of the aquifer due to stream-stage fluctuations or recharge is small compared with the initial saturated thickness.
7. The porous medium and fluid are slightly compressible.
8. Observation wells or piezometers are infinitesimal in diameter and respond instantly to pressure changes in the aquifer.
9. The stream that forms a boundary to the aquifer is straight and fully penetrates the aquifer.

10. Seepage and ground-water head at the stream-aquifer boundary are independent of depth.
11. Initially, the water level in the stream is at the same elevation as the water level everywhere in the aquifer. At time $t = 0$, the water level in the stream is suddenly lowered (or raised) to a new position lying a distance of one unit below (or above) the original one.
12. The semipervious streambank material, if present, has negligible capacity to store water.

With regard to the zone above the water table where water is held under tension, assumption 5 implies that the equilibrium profile of soil moisture versus depth in the unsaturated and nearly-saturated zones moves instantaneously in the vertical direction by an amount equal to the change in altitude of the water table. Assumption 5 also implies that there is no hysteresis in the relation between the soil-moisture profile and soil-matric potential as the water table fluctuates in response to stream-stage variations. Hysteresis causes the soil-moisture profile to have different shapes when soils are wetting and drying (Freeze and Cherry, 1979) and is more apparent for coarse-grained soils than for fine-grained soils.

Boundary-Value Problems

The governing partial differential equation describing two-dimensional, cross-sectional (*x,z*) flow in a water-table aquifer based on equation 1 is

$$\frac{\partial^2 h}{\partial x^2} + \frac{K_z}{K_x}\frac{\partial^2 h}{\partial z^2} = \frac{S_s}{K_x}\frac{\partial h}{\partial t} \quad , \qquad (28)$$

where K_z is the vertical hydraulic conductivity of the aquifer (units of length per time). The *x*-domain for equation 28 for semi-infinite aquifers is $x_0 \le x < \infty$ and for finite-width aquifers is $x_0 \le x < x_L$. The *z*-domain for all water-table aquifers is $0 \le z \le b$. In equation 28, h is a function of x, z, and t.

The initial condition for all solutions is

$$h(x, z, 0) = h_i \ , \qquad (29)$$

where h_i is the initial head in the aquifer.

Several boundary conditions are used for each of the water-table aquifers; the particular set of boundary conditions used for each system depends on the conditions being modeled. For a semi-infinite aquifer, the boundary condition as x approaches infinity is

$$h(\infty, z, t) = h_i \ , \qquad (30)$$

whereas for a finite-width aquifer, the boundary condition at $x = x_L$ is

$$\frac{\partial h}{\partial x}(x_L, z, t) = 0 \ . \qquad (31)$$

The boundary condition used at the stream-aquifer interface depends upon the presence or absence of semipervious streambank material. For conditions in which there is no semipervious streambank material, a specified-head boundary condition is used at x_0

$$h(x_0, z, t) = h_0 \ , \qquad (32)$$

where h_0 is the water level in the stream after the instantaneous step change. For conditions in which semipervious streambank material is present, a head-dependent flux boundary condition is used at x_0

$$\frac{\partial h}{\partial x}(x_0, z, t) = -\frac{1}{a}[h_0 - h(x_0, z, t)] \ , \qquad (33)$$

where a, streambank leakance, is defined in equation 14 and $[h_0 - h(x_0, t)]$ is the change in head across the semipervious streambank material.

The boundary condition at the water table ($z = b$) is

$$\frac{\partial h}{\partial z}(x, b, t) = -\frac{S_y}{K_z}\frac{\partial h}{\partial t} \ , \qquad (34)$$

where S_y is the specific yield of the aquifer.

The boundary condition at the impermeable (no-flow) lower boundary ($z = 0$) is

$$\frac{\partial h}{\partial z}(x, 0, t) = 0 \ . \qquad (35)$$

Laplace Transform Analytical Solutions

The dimensional boundary-value problems described by equations 28–35 are made dimensionless by substituting the dimensionless variables and variable groupings shown in table 2. The solutions for all water-table aquifer types can be written in the most general form as (equation A1.125 in Attachment 1)

$$\bar{h}_D = 2\sum_{n=0}^{\infty} \frac{W_n \exp[-q_n(x_D-1)]\sin(\varepsilon_n)\cos(\varepsilon_n z_D)}{\{1 + A q_n \tanh[q_n(x_{LD}-1)]\} p[\varepsilon_n + 0.5\sin(2\varepsilon_n)]} \quad ,(36)$$

where

$$q_n = (\varepsilon_n^2 \beta_0 + p)^{\frac{1}{2}} \quad (37)$$

and ε_n are the roots of

$$\varepsilon_n \tan(\varepsilon_n) = \frac{p}{\sigma\beta_0} \quad . \quad (38)$$

In equation 36, $\bar{h}_D$ is the Laplace transform unit-step response solution at each point (x_D, z_D) of a water-table aquifer. The bar over the unit step response (h_D) represents the Laplace transform. The Laplace transform variable, p, is inversely related to dimensionless time t_D. For the semi-infinite aquifers, x_{LD} goes to infinity and the hyperbolic tangent in equation 36 is unity.

Parameter W_n is a function of the width of the aquifer perpendicular to the stream and is defined as

$$W_n = \frac{\exp[-2q_n(x_{LD}-x_D)] + 1}{\exp[-2q_n(x_{LD}-1)] + 1} \quad .$$

W_n equals 1 for semi-infinite conditions.

As with the confined and leaky aquifer types, parameter A is dimensionless streambank leakance

$$A = \frac{a}{x_0} .$$

For conditions in which there is no semipervious streambank material, $A = 0$.

Table 2. Dimensionless variables and variable groupings for water-table aquifers

Dimensionless variable or grouping	Definition
x_D	$\frac{x}{x_0}$
x_{LD}	$\frac{x_L}{x_0}$
x_{0D}	$\frac{x_0}{b}$
z_D	$\frac{z}{b}$
z_{D1}	$\frac{z_1}{b}$
z_{D2}	$\frac{z_2}{b}$
h_D	$\frac{h_i - h}{c}$
t_D	$\frac{K_x t}{S_s x_0^2} = \frac{Tt}{S x_0^2}$
t_{Dy}	$\frac{Tt}{S_y x_0^2}$
A	$\frac{K_x d}{K_s x_0}$
σ	$\frac{S_s b}{S_y}$
K_D	$\frac{K_z}{K_x}$
β_0	$K_D\, x_{0D}^2$

Equation 36 is the Laplace transform solution for head at each point in a water-table aquifer, such as at an observation piezometer (fig. 13*A*). For a partially penetrating observation well (fig. 13*A*), the average head in the well ($\bar{h}_D^*$) is found by integrating equation 36 over the screened interval z_{D1} to z_{D2}. The result is

$$\bar{h}_D^* = \frac{2}{(z_{D2} - z_{D1})} \sum_{n=0}^{\infty} \frac{W_n \exp[-q_n(x_D - 1)] \sin(\varepsilon_n)[\sin(\varepsilon_n z_{D2}) - \sin(\varepsilon_n z_{D1})]}{\{1 + A q_n \tanh[q_n(x_{LD} - 1)]\} p \varepsilon_n[\varepsilon_n + 0.5 \sin(2\varepsilon_n)]} \quad . \tag{39}$$

By setting $z_{D1} = 0$ and $z_{D2} = 1$, one obtains the average head in a fully penetrating observation well ($\hat{\bar{h}}_D$):

$$\hat{\bar{h}}_D = 2 \sum_{n=0}^{\infty} \frac{W_n \exp[-q_n(x_D - 1)] \sin^2(\varepsilon_n)}{\{1 + A q_n \tanh[q_n(x_{LD} - 1)]\} p \, \varepsilon_n[\varepsilon_n + 0.5 \sin(2\varepsilon_n)]} \quad . \tag{40}$$

Equations 36–40 are general solutions for all of the water-table aquifer types. For example, for a semi-infinite, water-table aquifer with no semipervious streambank material, $W_n = 1$ and $A = 0$. Under these conditions, and the additional condition in which head is measured in a fully penetrating observation well, equation 40 becomes

$$\hat{\bar{h}}_D = 2 \sum_{n=0}^{\infty} \frac{\exp[-q_n(x_D - 1)] \sin^2(\varepsilon_n)}{p \, \varepsilon_n[\varepsilon_n + 0.5 \sin(2\varepsilon_n)]} \quad . \tag{41}$$

As demonstrated in Attachment 1 [following equation (A1.87)], equation 41 reduces to the solution for a confined aquifer (eq. 22) if specific yield is set equal to zero.

The Laplace transform solution for seepage between the stream and aquifer can be determined by finding the gradient of the unit-step response solution at the stream-aquifer boundary (that is, at $x_D = 1$). This gradient is found by differentiation of equation 40 with respect to x_D and evaluation of the resulting solution at $x_D = 1$

$$\bar{Q}_D = -\left.\frac{d\hat{\bar{h}}_D}{dx_D}\right|_{x_D = 1} \quad , \tag{42}$$

where $\bar{Q}_D$ is dimensionless seepage in the Laplace domain. The general solution for dimensionless seepage at the streambank, derived in Attachment 1, is

$$\bar{Q}_D = -2 \sum_{n=0}^{\infty} \frac{q_n \sin^2(\varepsilon_n)}{\{1 + A q_n \tanh[q_n(x_{LD} - 1)]\} p \, \varepsilon_n[\varepsilon_n + 0.5 \sin(2\varepsilon_n)]} \left\{ \frac{\exp[-2q_n(x_{LD} - 1)] - 1}{\exp[-2q_n(x_{LD} - 1)] + 1} \right\} \quad . \tag{43}$$

EVALUATION OF ANALYTICAL SOLUTIONS FOR STEP INPUT

In this section, the analytical solutions are evaluated for hypothetical confined, leaky, and water-table aquifers for a 1.0 ft unit-step increase (input) in the elevation of stream stage relative to that of piezometric head in the adjoining aquifer. The evaluation demonstrates the influence of aquifer type, aquifer extent, and aquifer and streambank hydraulic properties on ground-water heads and seepage rates. The solutions also are compared graphically to several previously published solutions.

From equation 2, changes in ground-water heads are related to a unit-step increase according to:

$$h_i - h(x, t) = -h_D c \quad , \qquad (44)$$

where c is the step increase in water level of the stream relative to the water level in the aquifer (c = 1.0 ft in this evaluation) and the negative sign is introduced so that changes in ground-water heads are positive for a rise in stream stage. Dimensional seepage rates are determined from equation 42, Darcy's law, and the definitions of h_D and x_D given in table 1:

$$Q(t) = \frac{K_x bc}{x_0} Q_D \quad , \qquad (45)$$

where $Q(t)$ is seepage rate per unit stream length at time t and Q_D is the dimensionless seepage in the real-time domain.

Confined and Leaky Aquifers

Parameters and dimensions of the hypothetical confined and leaky aquifers and overlying aquitards used in the evaluation are shown in table 3. Changes in ground-water heads were calculated at a hypothetical observation well 100 ft from the middle of the stream (75 ft from the stream-aquifer boundary).

Figures 15 and 16 show changes in ground-water heads and seepage rates for a semi-infinite (fig. 15) and finite-width (fig. 16) confined aquifer with and without semipervious streambank material. Heads and seepage rates were calculated by use of the Laplace-transform analytical solutions and by use of the real-time domain solutions reported by Hall and Moench (1972) for the same parameters and dimensions shown in table 3. Negative seepage rates indicate that water flows from the stream to the adjoining aquifer. Results for two streambed-leakance values are shown in the figures, a = 100 ft and a = 1,000 ft. For a hydraulic conductivity of the aquifer equal to 200 ft/d (table 3), values of a = 100 ft and a = 1,000 ft correspond to a 5 ft thick streambank with hydraulic conductivity of 10 ft/d and 1 ft/d, respectively. Matches between the Laplace-transform solutions and real-time domain solutions of Hall and Moench (1972) for both heads and seepage rates for all of the semi-infinite and finite-width aquifer conditions are excellent (figs. 15 and 16).

Table 3. Parameters and dimensions of the hypothetical confined and leaky aquifers

Parameter	Value
Aquifer	
Horizontal hydraulic conductivity (K_x)	200 ft/d
Specific storage (S_s)	1 x 10^{-5} ft^{-1}
Thickness (b)	25 ft
Width of aquifer[1] (x_L)	500 ft
Distance from middle of stream to stream-aquifer boundary(x_0)	25 ft
Aquitard[2]	
Vertical hydraulic conductivity (K')	2 ft/d
Specific storage (S_s')	1 x 10^{-4} ft^{-1}
Specific yield[3] (S_y')	2.5 x 10^{-1}
Thickness or saturated thickness (b')	25 ft

[1]For finite-width aquifers.
[2]For leaky aquifers.
[3]For leaky aquifers overlain by a water-table aquitard.

Both sets of head solutions asymptotically approach the unit-step stream-stage increase of 1.0 ft (figs. 15*A*, 16*A*). Initially, for a = 0, seepage rates from the stream to adjoining aquifer are large (figs. 15*B*, 16*B*). With increased time, ground-water heads near the stream approach the stream-stage level and, as a result, hydraulic gradients and seepage rates at the stream-aquifer boundary approach zero. The inclusion of a streambank leakance term delays the increase in ground-water heads at the observation well and reduces seepage rates to the aquifer. As the streambank leakance term is increased from 100 to 1,000 ft, seepage rates at the stream-aquifer interface are greatly diminished by the increased hydraulic resistance at the streambank.

The response of semi-infinite and finite-width confined aquifers without semipervious streambank material are compared for several values of aquifer width in figure 17. At early-time periods (less than

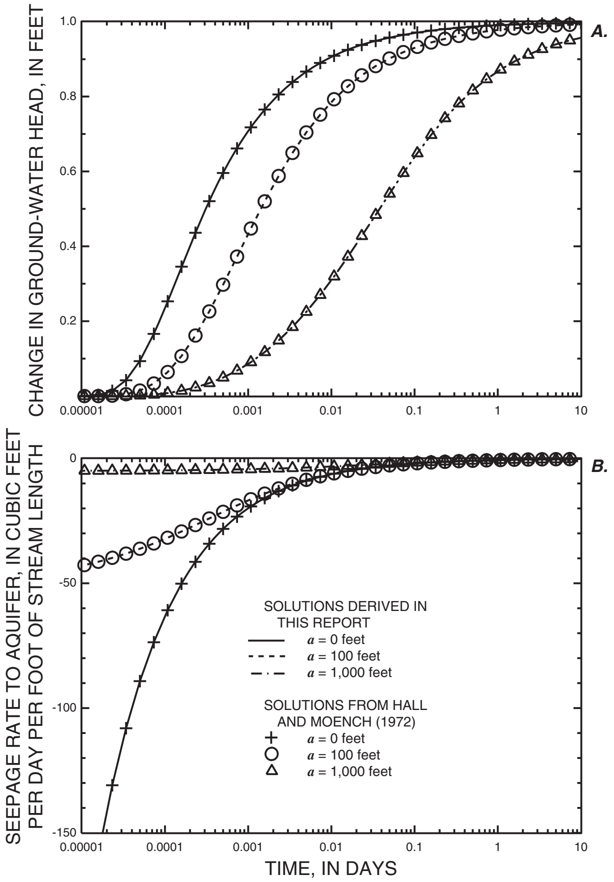


Figure 15. (*A*) Change in ground-water head and (*B*) seepage rate to aquifer, for 1-foot increase in stream stage, semi-infinite confined aquifer with and without semipervious streambank material. Observation well 75 feet from stream-aquifer interface; *a*, streambank leakance; other model parameters and dimensions in table 3.

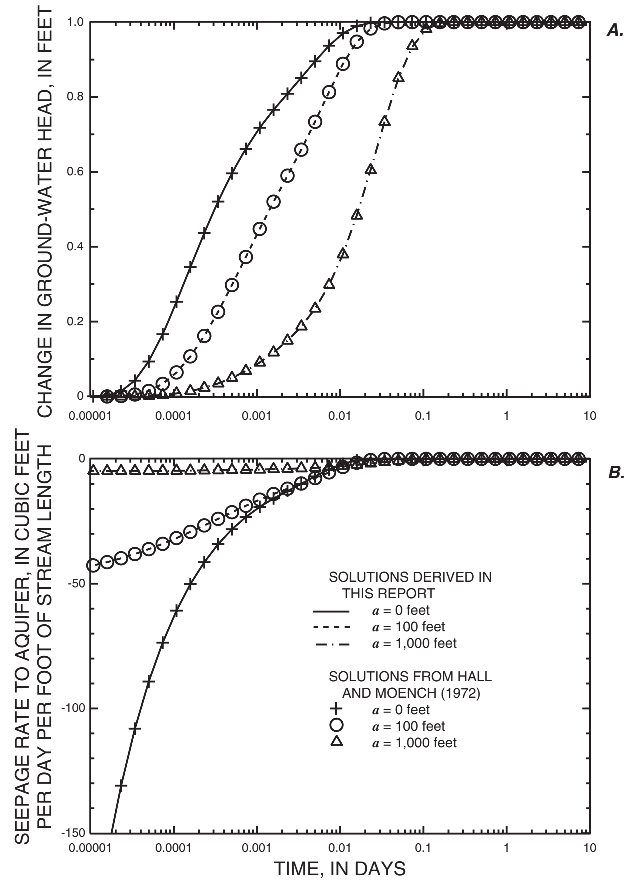


Figure 16. (*A*) Change in ground-water head and (*B*) seepage rate to aquifer, for 1-foot increase in stream stage, finite-width confined aquifer with and without semipervious streambank material. Observation well 75 feet from stream-aquifer interface; *a*, streambank leakance; other model parameters and dimensions in table 3.

about 4 x 10^{-4} days), the semi-infinite and finite-width aquifers respond similarly. At later times, the narrower aquifers (x_L small) cause ground-water heads to rise more quickly and seepage rates to approach zero more rapidly than do those for the wider aquifers (x_L large) because of the overall smaller storage capacity available in the narrower aquifers. As the width of the finite-width aquifer is increased, the finite-width aquifer solutions approach the semi-infinite aquifer solutions, as would be expected.

Solutions for a semi-infinite leaky aquifer with constant head overlying the aquitard (leaky aquifer case 1) without semipervious streambank material are shown in figure 18 for several values of the specific storage of the aquitard (S_s'). Also shown in the figure are the solutions for a semi-infinite confined aquifer with a storativity (S) of 2.5 x 10^{-4}. Each of the leaky-aquifer solutions asymptotically approaches a constant (steady-state) value of ground-water head that is smaller, and a constant rate of seepage that is larger, than the confined-aquifer solutions. These result from the constant-head boundary condition that overlies the aquitard and provides an infinite source (or sink) of ground-water storage to the aquifer/aquitard system. The figure shows that the response of the leaky-aquifer system is delayed relative to the confined aquifer, and that the delay is increased as the specific storage of the aquitard increases. The real-time domain solutions of Hantush (1961b) for similar leaky-aquifer conditions also are shown in figure 18. Hantush's solutions do not consider storage in the aquitard; consequently, those solutions are equivalent to the solutions derived in this

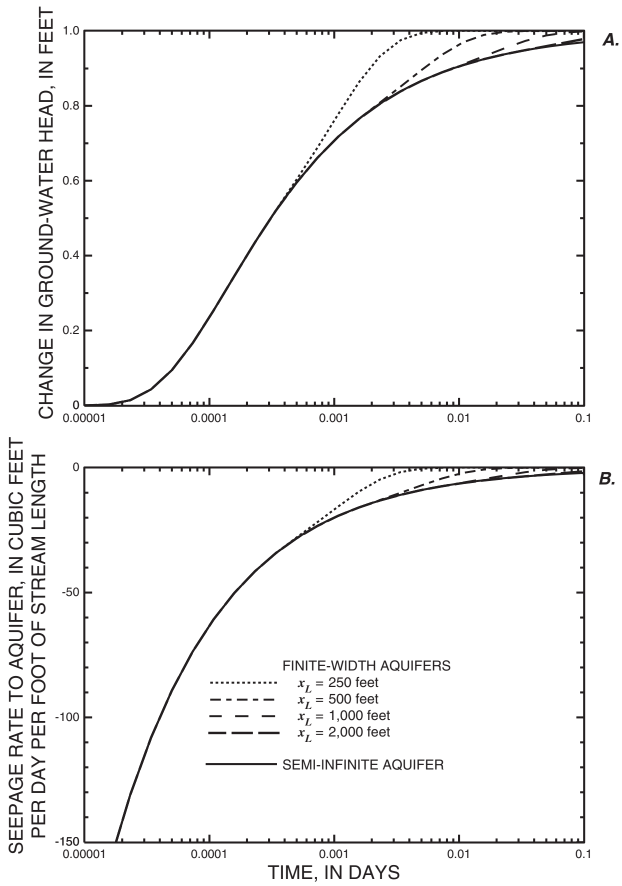


Figure 17. (*A*) Change in ground-water head and (*B*) seepage rate to aquifer, for 1-foot increase in stream stage, finite-width and semi-infinite confined aquifers. Observation well 75 feet from stream-aquifer interface; x_L, aquifer width; other model parameters and dimensions in table 3.

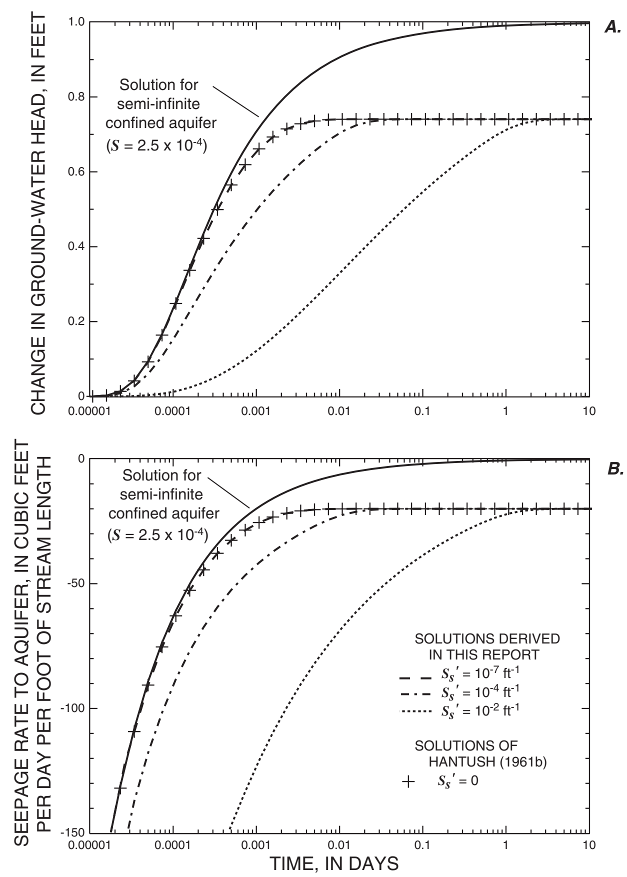


Figure 18. (*A*) Change in ground-water head and (*B*) seepage rate to aquifer, for 1-foot increase in stream stage, semi-infinite leaky aquifer with constant head overlying the aquitard. Observation well 75 feet from stream-aquifer interface; S_s', specific storage of aquitard; S, storativity of aquifer; other model parameters and dimensions in table 3.

report only when the specific storage of the aquitard is very small, such as the value of 10^{-7} ft^{-1} shown in the figure.

Solutions for all three types of leaky aquifers without semipervious streambank material are compared in figure 19. Also shown in the figure are solutions for a semi-infinite confined aquifer with a storativity of 2.5 x 10^{-4} and 2.5 x 10^{-1}. These two storativities are limiting values for the confined/leaky systems modeled here: the value 2.5 x 10^{-4} is that of the confined aquifer (no aquitard) and the value 2.5 x 10^{-1} equals the specific yield of the water-table aquitard. At early times the leaky-aquifer head solutions quickly depart from the confined-aquifer solution with S = 2.5 x 10^{-4} (fig. 19*A*). The solutions for the three aquifer types yield identical drawdowns up to a time of about 0.01 days, when they begin to diverge from one another because of the influence of the upper boundary condition of the aquitard.

At late time, the solutions for case 1 (aquitard overlain by constant-head boundary) asymptotically approach steady-state values of head and seepage (as also shown in fig. 18) because of the constant-head boundary condition that overlies the aquitard. Solutions for case 2 (aquitard overlain by an impermeable boundary) asymptotically approach the confined-aquifer solutions but are shifted in time relative to the confined-aquifer solutions by a factor of $1 + \frac{1}{\sigma_1}$. The shift is analogous to that which occurs in

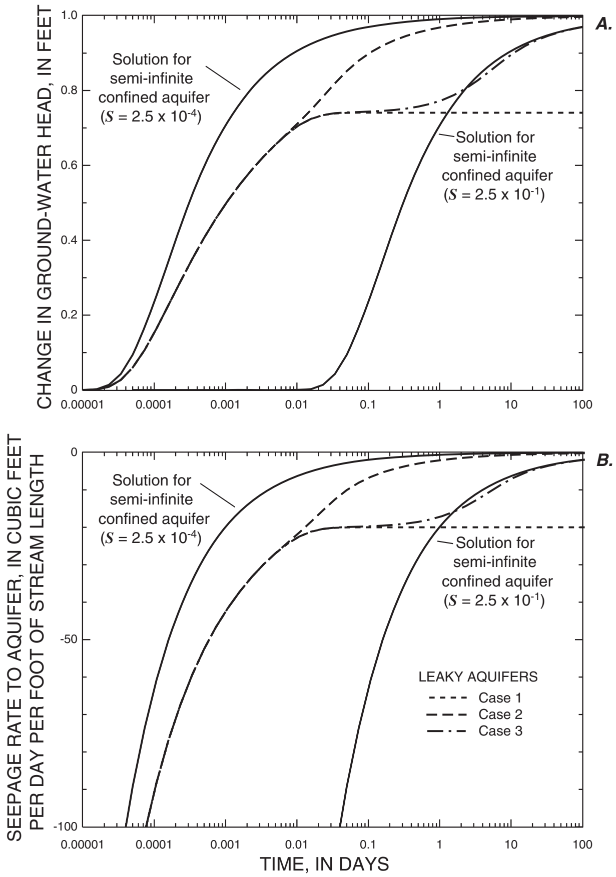


Figure 19. (*A*) Change in ground-water head and (*B*) seepage rate to aquifer, for 1-foot increase in stream stage, semi-infinite leaky aquifers. Case 1, constant head overlies the aquitard; Case 2, impermeable layer overlies the aquitard; Case 3, water-table aquitard. Observation well 75 feet from stream-aquifer interface; *S*, storativity (dimensionless); other model parameters and dimensions in table 3.

flow to a well in leaky aquifers (see Moench, 1985, p. 1129). The leaky-aquifer solutions approach the confined-aquifer solutions because the impermeable boundary condition at the top of the aquitard prevents any additional source (or sink) of leakage to the aquifer at late time.

Solutions for case 3 (water-table aquitard) are identical to those of case 1 up to a time of about 0.1 days because the large storage capacity provided by the water-table boundary causes the system to respond as it would to a constant-head boundary overlying the aquitard. At late times, the solutions for case 3 lie between those of cases 1 and 2 because the rate of flow into storage at the water table slows. Eventually, head changes and seepage rates for the water-table aquitard system approach those of a confined aquifer with storativity equal to the specific yield of the aquitard (2.5 x 10^{-1}).

Water-Table Aquifers

Parameters and dimensions of the hypothetical water-table aquifer used in the evaluation are shown in table 4. Changes in ground-water heads were calculated at a hypothetical observation well 100 ft from the middle of the stream (75 ft from the stream-aquifer boundary).

Figure 20 shows changes in ground-water heads and seepage rates for a semi-infinite water-table aquifer without semipervious streambank material for three values of K_D (dimensionless ratio of vertical to horizontal hydraulic conductivity) calculated by use of the Laplace-transform analytical solutions. Also shown in figure 20*A* are heads calculated by use of the real-time domain solution reported by Neuman (1981). Ground-water heads shown in the figure are the average head over the full saturated thickness of the aquifer at the hypothetical observation well. As with the confined and leaky solutions, negative seepage rates indicate that water flows from the stream to the adjoining aquifer in response to the unit-step increase in stream stage. Also shown in the figure are solutions for a semi-infinite confined aquifer with a storativity of 2.5 x 10^{-4} and 2.5 x 10^{-1}. These are the limiting storativities for the hypothetical water-table aquifer: the value of 2.5 x 10^{-4} represents the hypothetical condition in which there is no water table present (that is, specific yield equals zero); the value of 2.5 x 10^{-1} equals the specific yield of the aquifer and represents

Table 4. Parameters and dimensions of the hypothetical water-table aquifer

Parameter	Value
Horizontal hydraulic conductivity (K_x)	200 ft/d
Vertical hydraulic conductivity (K_z)	40 ft/d
Specific storage (S_s)	1 x 10^{-5} ft^{-1}
Specific yield (S_y)	2.5 x 10^{-1}
Saturated thickness (b)	25 ft
Distance from middle of stream to stream-aquifer boundary (x_0)	25 ft

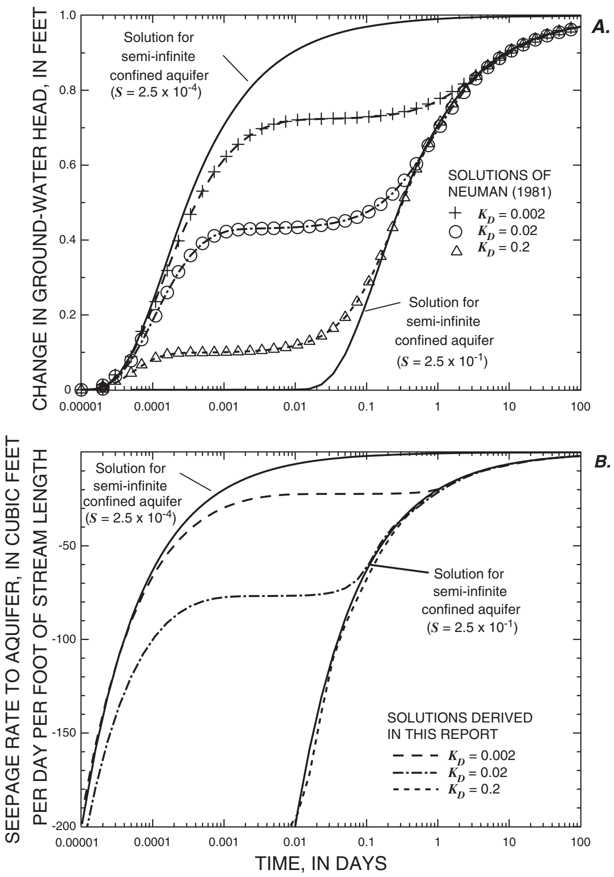


Figure 20. (*A*) Change in ground-water head and (*B*) seepage rate to aquifer, for 1-foot increase in stream stage, semi-infinite water-table aquifer without semipervious streambank material. Observation well 75 feet from stream-aquifer interface; K_D, ratio of vertical to horizontal hydraulic conductivity of aquifer; other model parameters and dimensions in table 4.

the hypothetical condition in which the aquifer is rigid and the water is incompressible (that is, specific storage equals zero).

Neuman's (1981) solution for ground-water flow to a fully penetrating stream in a water-table aquifer is very similar to one that he developed for the comparable problem of flow to a fully penetrating pumped well in a water-table aquifer (Neuman, 1972). Neuman developed a computer program (DELAY2) to calculate ground-water heads based on his solutions for flow to a fully penetrating or partially penetrating pumped well. DELAY2 was modified by the authors to calculate ground-water heads based on Neuman's (1981) solution for flow to a fully penetrating stream. Because of the similarity between Neuman's solutions for flow to a fully penetrating pumped well (Neuman, 1972) and to a fully penetrating stream (Neuman, 1981), few modifications were needed to the DELAY2 program. As shown in figure 20*A*, matches between the Laplace-transform solution for ground-water head derived in this report and Neuman's (1981) real-time solution are excellent for the three values of K_D evaluated.

Ground-water heads in figure 20*A* for any particular value of K_D show the three characteristic segments of the response of water-table aquifers to a step change in the stream stage. Physical explanations for these three segments have been described by several authors for the case of ground-water flow to a pumped well (see for example discussions by Neuman, 1972 and 1974), and the explanations are similar for the response of a water-table aquifer to stream-stage fluctuations. During the early-time segment, the aquifer responds as would a strictly confined aquifer with storativity equal to 2.5×10^{-4} (fig. 20*A*). That is, water goes into elastic storage by expansion of the aquifer materials and compression of the pore water. Effects of vertical flow into the zone above the water table are not prevalent during the early-time segment where horizontal flow dominates. The length of time during which elastic-storage effects are prominent is increased as the ratio of vertical to horizontal hydraulic conductivity (K_D) is decreased. This is due to increased resistance to vertical flow in the aquifer because of the smaller values of vertical hydraulic conductivity. Although not shown in figure 20, the length of time during which elastic-storage effects are prominent also decreases as the ratio of storativity to specific yield (σ, table 2) decreases (Neuman, 1972); that is, as the aquifer becomes more rigid.

During the intermediate-time segment, upward flow into the unsaturated zone becomes important and the rate of change of ground-water heads is slowed (fig. 20*A*). The delayed response of the water table is similar to the response of the leaky-aquifer systems shown in figure 19. Vertical-flow components are important during this segment as the water table rises. Finally, during the late-time segment, the aquifer again responds as would a strictly confined aquifer and ground-water heads converge on the solution for a confined aquifer with storativity equal to 2.5×10^{-1} (fig. 20*A*), which equals the specific yield of the aquifer. Water goes into storage only by an increase in the elevation of the water table. Horizontal ground-water flow dominates during this time segment, as it did during the early-time segment.

Figure 21 shows ground-water heads at three vertical positions in the aquifer and the average head over the full saturated thickness of the aquifer for $K_D = 0.2$. Vertical variations in ground-water heads over the saturated thickness of the aquifer result in upward flow into the zone above the water table. The results shown in the figure are similar to those presented by Neuman (1972, fig. 4, p. 1037) for the case of ground-water flow to a well. Ground-water heads below the water table ($z_D < 1.0$) respond quickly to the change in head at the stream-aquifer boundary as a result of elastic storage of the aquifer. An equivalent head change at the water table ($z_D = 1.0$) is delayed relative to head changes deeper in the aquifer in response to saturation of the pores as the water table rises. The average head change over the thickness of the aquifer responds more quickly than that at the water table but lags behind those for $z_D = 0.0$ and $z_D = 0.5$. At late time, all of the curves approach the solution for the confined aquifer with storativity equal to 2.5 x 10^{-1}, which implies that heads are uniform over the thickness of the aquifer and that horizontal ground-water flow dominates. As noted by Neuman (1972), the convergence of the curves to the single, uniform solution is consistent with the Dupuit-Forchheimer theory of horizontal ground-water flow in a water-table

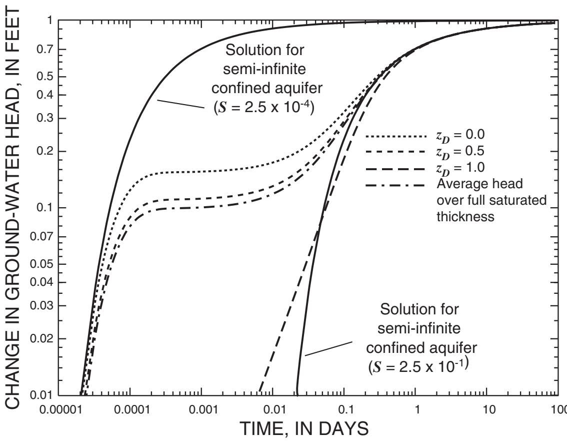


Figure 21. Change in ground-water head for 1-foot increase in stream stage at several vertical positions in a semi-infinite water-table aquifer. Observation well 75 feet from stream-aquifer interface; z_D, vertical distance from bottom of aquifer to observation piezometer divided by saturated thickness of aquifer (dimensionless); other model parameters and dimensions in table 4.

aquifer. It is only after this point in time that the use of the confined-aquifer solution with storativity equal to the specific yield of the aquifer is truly justified.

Figure 22 shows a comparison of the response in a water-table aquifer to that of an aquifer overlain by a water-table aquitard. As noted by Boulton and Streltsova (1975) for the case of flow to a pumped well, because the boundary condition used at the water table in a water-table aquifer is the same as that used for the water table in a water-table aquitard, ground-water heads (and seepage rates) calculated for the two aquifer types should approach one another as the thickness of the water-table aquitard becomes zero. That this is also true for stream-aquifer settings is confirmed by the results shown in figure 22, in which simulations were made for several values of aquitard thickness for the hypothetical leaky aquifer overlain by a water-table

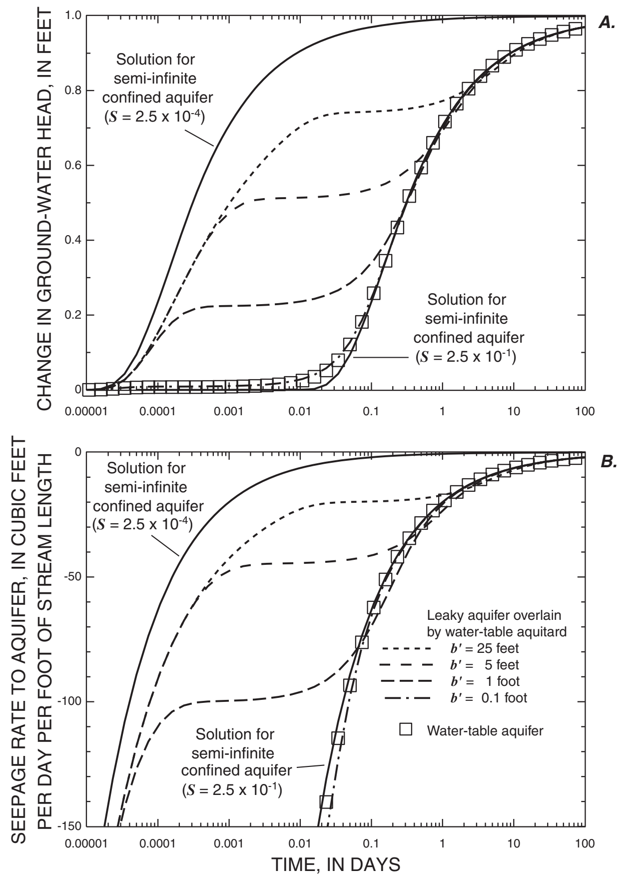


Figure 22. (*A*) Change in ground-water head and (*B*) seepage rate to aquifer, for 1-foot increase in stream stage, semi-infinite water-table aquifer and leaky aquifer overlain by water-table aquitard. Observation well 75 feet from stream-aquifer interface; *b′*, saturated thickness of aquitard; *S*, storativity of aquifer; other model parameters and dimensions in tables 2 and 4.

aquitard (table 3) and a single simulation for the water-table aquifer (table 4) in which $K_D = 1.0$. As shown in the figure, ground-water heads and seepage rates for the water-table aquitard condition approach those of the water-table aquifer as the thickness of the aquitard is reduced from 25 ft to 0.1 ft.

COMPUTER PROGRAMS STLK1 AND STWT1—IMPLEMENTATION OF ANALYTICAL SOLUTIONS FOR TIME-VARYING INPUTS

Two computer programs written in the FORTRAN-77 computer language were developed to determine ground-water heads, seepage rates, and bank-storage volumes for arbitrary, time-varying stream-stage and/or recharge stresses that are specified by program users. Program STLK1 is used for confined and leaky aquifers (figs. 5–12) and program STWT1 is used for water-table aquifers (figs. 13 and 14). To avoid having to create two separate data-input files for analysis of confined and water-table aquifers, program STWT1 also can be used for confined aquifers.

The programs implement the convolution relations described previously (see "General Theoretical Background"). For a given set of input conditions, the programs calculate ground-water head at an observation well or observation piezometer (equation 3), seepage rates at the stream-aquifer boundary (equations 4 and 5), and bank storage (equations 6 and 7). The programs can simulate the response to stream-stage fluctuations for all aquifer types. Simulation of the response to recharge is permitted only for water-table aquifers and leaky aquifers overlain by a water-table aquitard. For these aquifer types, the aquifer response to recharge can be simulated alone or simultaneously with the response to stream-stage fluctuations. Recharge can be positive or negative. Negative recharge occurs in response to regional evapotranspiration from the water table.

The following sections describe discretization of the convolution integrals for use in STLK1 and STWT1, instructions for preparing data-input files required for program execution, the result and plot files generated by the programs, and three sample problems that illustrate applications of the programs. Descriptions of the computer codes are provided in Attachment 2.

Discretization of Convolution Relations

For computational purposes, the integrals in equations 3, 4, and 6 are written in discretized forms for implementation in programs STLK1 and STWT1. The discretized forms are

$$h(x, z, j) = h_i + \sum_{k=2}^{J} F'(k-1) h_D(x, z, j-k+1) \Delta t \quad , (46)$$

$$Q(j) = \frac{K_x b}{x_0} \sum_{k=2}^{J} F'(k-1) \frac{\partial h_D(x_0, z, j-k+1)}{\partial x_D} \Delta t \quad , (47)$$

and

$$V(j) = -\sum_{k=2}^{J} Q(k)\Delta t \quad , \qquad (48)$$

where

- j is the upper limit of time integration (dimensionless);
- k is the time variable of integration (time step) (dimensionless);
- Δt is the time-step size (units of time); and
- $F'(k-1)$ is the time rate of change of the system input (units of length per time).

In equations 48–50, time is calculated from

$$t = \Delta t(k-1) + t_i \ , \qquad (49)$$

where t_i is the time at the start of the simulation, which is specified by program users. Time-step size (Δt) also is specified by program users and must be a constant length during each simulation.

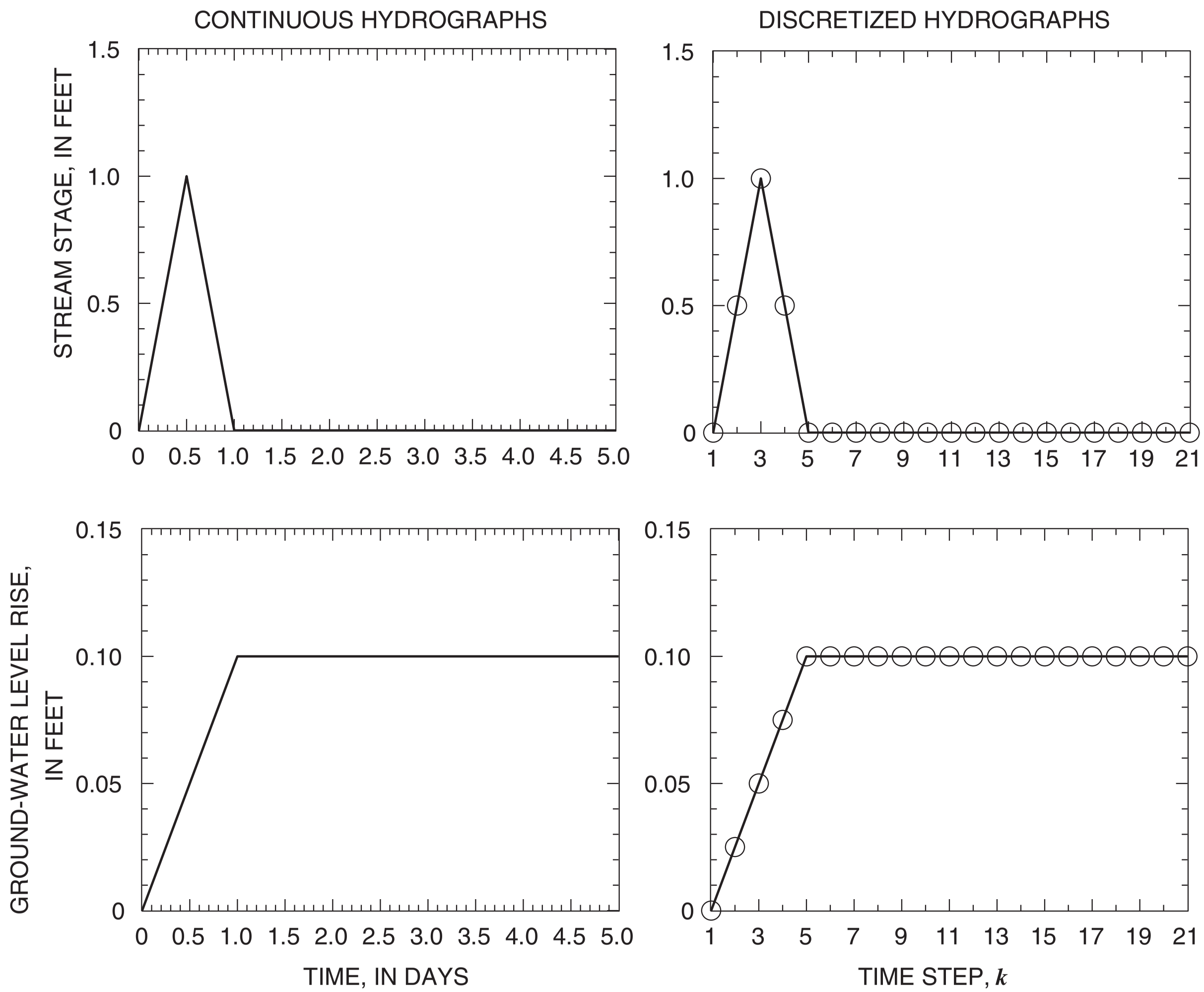
The programs require approximation of input hydrographs (continuous records of stream-stage, recharge, or evapotranspiration) into a time series of discrete step changes during each time step. The time rate of change of the system input is calculated from

$$F'(k-1) = \frac{F(k) - F(k-1)}{\Delta t} \quad , \qquad (50)$$

where $F(k-1)$ and $F(k)$ are the system inputs (stream stage or recharge) (units of length) at time steps $k-1$ and k, respectively. As with all discretization schemes, the accuracy of the convolution method, and therefore of the programs, is improved by use of smaller time steps. Discretization issues are further discussed with Sample Problem 1.

Heads, seepage rates, and bank storage are calculated at the end of each time step. At the end of the first time step ($k = 1$ and $t = t_i$), $h(x, z) = h_i$, $Q = 0$, and $V = 0$. The first calculations for head, seepage rate, and bank storage made by the programs are at the end of the second time step ($k = 2$), and use $F'(1)$.

Examples of how continuous stream-stage or recharge inputs are discretized for use in convolution equations 48–50 are shown in figure 23. On the left side of the figure are continuous, 5-day hydrographs for hypothetical stream-stage (fig. 23*A*) and recharge (fig. 23*B*) stresses; on the right side are equivalent hydrographs that have been discretized into $k = 21$ time steps. The constant time interval (time-step size) between each set of adjoining time steps is $\Delta t = 0.25$ days. Twenty-one time steps are required for the 5-day hydrographs because the first time step is at $t = 0$ days. For each pair of adjacent time steps, there is an associated time rate of change of the system input, $F'(k-1)$, which equals the slope of the hydrograph over the interval Δt (see equation 50). There are 20 values of F' for the 21 time steps of each discretized hydrograph. Recharge applied to the system results in a uniform ground-water level rise; it is the ground-water level rise that is actually specified to the model. The ground-water level rise remains constant once recharge stops at the end of the first day; the slope of the ground-water level hydrograph (F') therefore equals zero after the first day. Further discussion of the relation between recharge and ground-water level rise is provided with Sample Problem 3. The two tables on figure 23 show the discretized hydrographs in tabular format (for brevity, time steps 7–19 are not shown).



A.

Time (day)	Time step (k)	Stream stage (feet)	$F'(k\text{-}1)$
0.00	1	0.000	0.0
0.25	2	0.500	2.0
0.50	3	1.000	2.0
0.75	4	0.500	-2.0
1.00	5	0.000	-2.0
1.25	6	0.000	0.0
.	.	.	.
.	.	.	.
.	.	.	.
4.75	20	0.000	0.0
5.00	21	0.000	0.0

B.

Time (day)	Time step (k)	Ground-water level rise (feet)	$F'(k\text{-}1)$
0.00	1	0.000	0.0
0.25	2	0.025	0.1
0.50	3	0.050	0.1
0.75	4	0.075	0.1
1.00	5	0.100	0.1
1.25	6	0.100	0.0
.	.	.	.
.	.	.	.
.	.	.	.
4.75	20	0.100	0.0
5.00	21	0.100	0.0

Figure 23. Continuous and discretized (*A*) stream-stage and (*B*) recharge hydrographs. Open circles on discretized hydrographs indicate time steps, which are each 0.25 days in length. ($F'(k\text{-}1)$, time rate of change of stream stage or ground-water level rise.)

Instructions for Preparing Data-Input Files

Programs STLK1 and STWT1 each require a data-input file for execution. The data-input files contain information on the types and hydraulic parameters of the aquifer, aquitard (if present), and semipervious streambank material (if present) being simulated; initial conditions in the aquifer; stress inputs to the aquifer; and solution parameters. All input data are read using free-format style, which means that data values in each line of input do not have to be in specific columns; however, data values must be separated by one or more blank spaces. A consistent set of length and time units must be used throughout the input file—for example, feet and days. All real-valued variables are double-precision format in STLK1 and STWT1; consequently, double-precision format should be used for real-valued variables in the input file. For example, the value 1.33×10^{-3} could be entered as 1.33D-3 or as 0.00133.

Program STLK1

Line-by-line instructions for creating a data-input file for program STLK1 follow. Variable names that are used in the input file and computer program are shown in upper-case text.

Line 1:

TITLE1—First line of title, which can be up to 70 characters in length. Leave this line blank if no title is specified.

Line 2:

TITLE2—Second line of title, which can be up to 70 characters in length. Leave this line blank if no title is specified.

Line 3:

ISTRESS—Type of stress being simulated. Three options are provided:
- ISTRESS = 0: Stream-stage fluctuations are simulated.
- ISTRESS = 1: Recharge/ET are simulated. Valid only for leaky aquifer overlain by a water-table aquitard (IAQ = 3).
- ISTRESS = 2: Both stream-stage fluctuations and recharge/ET are simulated. Valid only for leaky aquifer overlain by a water-table aquitard (IAQ = 3).

DELT—Time-step size. A uniform time-step size must be used throughout the simulation. Note that the value of DELT will affect solution accuracy. Smaller time steps will improve solution accuracy but increase the amount of time required for the program to run a particular simulation (see Sample Problem 1).

IPRINT—An option to print or suppress the printing of stress data to the results file:
- IPRINT = 0: Do not print stress data.
- IPRINT = 1: Print stress data.

Line 4:

IXL—Extent of aquifer being simulated. Two options are provided:
- IXL = 0: Semi-infinite aquifer.
- IXL = 1: Finite-width aquifer.

IAQ—Type of aquifer being simulated. Four options are provided:
- IAQ = 0: Confined aquifer.
- IAQ = 1: Leaky aquifer, with constant head overlying the aquitard (leaky aquifer case 1).
- IAQ = 2: Leaky aquifer, with an impermeable layer overlying the aquitard (leaky aquifer case 2).
- IAQ = 3: Leaky aquifer overlain by a water-table aquitard (leaky aquifer case 3).

IXA—Streambank code. Two options are provided:
- IXA = 0: semipervious streambank material is absent.
- IXA = 1: semipervious streambank material is present.

Line 5:

XZERO—Half-width of stream, in units of length. Must be greater than 0.0D0. Note: the half-width of the stream does not need to be known for the solution. The variable XZERO is simply used to nondimensionalize some of the parameters in the analytical solutions. Therefore, an arbitrary value of XZERO may be used; however, all distances from the center of the stream channel used in the input file must be consistent with the value of XZERO that is selected.

XLL—Width of aquifer, in units of length. Use for finite-width aquifers. Enter 0.0D0 if IXL = 0.

XAA—Streambank leakance, in units of length. Streambank leakance is defined in equation 14. Enter 0.0D0 if IXA = 0.

XSTREAM—Length of stream reach, in units of length. Must be greater than 0.0D0. XSTREAM is used to calculate total seepage and bank-storage volume over the stream reach of interest.

Line 6:

AK—Horizontal hydraulic conductivity of aquifer, in units of length per time.

AS—Specific storage of aquifer, in units of inverse length. The program will calculate the storativity of the aquifer by multiplying specific storage (AS) by the thickness of the aquifer at the beginning of the simulation (AB).

AB—Thickness of aquifer at beginning of simulation, in units of length.

Line 7:

AKT—Vertical hydraulic conductivity of aquitard, in units of length per time. Enter 0.0D0 if IAQ = 0.

AST—Specific storage of aquitard, in units of inverse length. The program will calculate the storativity of the aquitard by multiplying specific storage (AST) by the saturated thickness of the aquitard at the beginning of the simulation (ABT). Enter 0.0D0 if IAQ = 0.

ABT—Thickness or saturated thickness of aquitard at beginning of simulation, in units of length. Enter 0.0D0 if IAQ = 0.

ASYT—Specific yield of aquitard, dimensionless. Enter 0.0D0 if IAQ ≠ 3.

Line 8:

X—Distance to observation well from stream-channel center, in units of length.

HINIT—Initial head at observation well, in units of length. Heads calculated by the program are added to or subtracted from HINIT.

TINIT—Simulation start time, in units of time. A start time to which simulation results are referenced.

Line 9:

NS—Number of terms used in the Stehfest algorithm. This must be an even number, the value of which depends upon computer precision. If the computer holds 16 significant figures in double precision, let NS = 8 to 12. A value of 8 is often sufficient. If numerical results for head and seepage are unstable, NS can be reduced to 6 (or even 4). Precision will be reduced, however, and results should be checked for accuracy. The user can compare simulation results using alternative values of NS (such as 6, 8, and 12) to determine if numerically stable results have been attained.

Line 10:

NT—Number of time steps. Program STLK1 is dimensioned to handle up to 1,000 time steps. If NT > 1,000, parameter IMAXX must be increased to a value of at least NT in the main routine and in subroutine DATAIO of program STLK1.

Lines 11 to NT+10:

Lines 11 to NT + 10 contain the stress data for each simulation.

XTIME(I)—Time of stream-stage and/or recharge/ET input for time step I.

STAGE(I)—Stream stage for time step I.

RECH(I)—Recharge/ET for time step I.

A summary of the data-input requirements for STLK1 is shown in table 5. An example data-input file, which is named "input.prob1a" and is used in sample problem 1 below, is shown in figure 24. The data-input file is based on the hypothetical confined aquifer described in table 3. Variable names are shown for convenience on the right side of each line of the example data-input file (fig. 24).

Table 5. Input data format for program STLK1

[Free-format input style--each variable in a line must be separated by at least one space; for real variables use double precision values, such as 1.33D-3 or 0.00133; ET, evapotranspiration]

Line	Variable name	Type	Explanation
1	TITLE1	Character	First line of title (up to 70 characters)
2	TITLE2	Character	Second line of title (up to 70 characters)
3	ISTRESS	Integer	Stress type: ISTRESS = 0 : stream-stage fluctuations ISTRESS = 1: recharge/ET ISTRESS = 2 : stream-stage fluctuations and recharge/ET
	DELT	Real	Time-step size. A uniform time-step size must be used.
	IPRINT	Integer	Option for printing stress data to result file: IPRINT = 0: do not print stress data IPRINT = 1: print stress data
4	IXL	Integer	Aquifer extent: IXL = 0: semi-infinite IXL = 1: finite width
	IAQ	Integer	Aquifer type: IAQ = 0: confined IAQ = 1: leaky, with constant head IAQ = 2: leaky, with impermeable layer IAQ = 3: leaky, with water-table aquitard
	IXA	Integer	Streambank code: IXA = 0: semipervious streambank material absent IXA = 1: semipervious streambank material present
5	XZERO	Real	Half width of stream. Must be > 0.0D0
	XLL	Real	Width of aquifer. Enter 0.0D0 if IXL = 0
	XAA	Real	Streambank leakance. Enter 0.0D0 if IXA = 0
	XSTREAM	Real	Length of stream reach
6	AK	Real	Horizontal hydraulic conductivity of aquifer
	AS	Real	Specific storage of aquifer
	AB	Real	Thickness of aquifer
7	AKT	Real	Vertical hydraulic conductivity of aquitard. Enter 0.0D0 if IAQ = 0
	AST	Real	Specific storage of aquitard. Enter 0.0D0 if IAQ = 0
	ABT	Real	Thickness or saturated thickness of aquitard. Enter 0.0D0 if IAQ = 0
	ASYT	Real	Specific yield of aquitard. Enter 0.0D0 if IAQ $\neq$ 3
8	X	Real	Distance to observation well from stream-channel center.
	HINIT	Real	Initial head at observation well.
	TINIT	Real	Simulation start time.
9	NS	Integer	Number of Stehfest terms. Must be an even integer. 8 terms are usually sufficient
10	NT	Integer	Number of time steps: If NT > 1,000, increase parameter IMAXX in program STLK1 to a value of at least NT
11 to (NT+10)	XTIME(I)	Real	Time of stream-stage and/or recharge/ET input for time step I
	STAGE(I)	Real	Stream stage for time step I
	RECH(I)	Real	Recharge/ET for time step I

```
Sample problem 1a. Sample input file for program STLK1                TITLE1
One-day stream-stage flood event. Confined aquifer. Delt is 0.25 days.  TITLE2
    0    0.25D+0  1                                          ISTRESS  DELT IPRINT
    0      0      0                                                   IXL  IAQ  IXA
 25.0D0  0.0D0   0.0D0  1.0D3                              XZERO  XLL  XAA  XSTREAM
  2.0D2  1.0D-5 25.0D0                                                    AK  AS  AB
  0.0D0  0.0D0   0.0D0  0.0D0                                  AKT  AST  ABT  ASYT
  1.0D3  0.0D0   0.0D0                                              X  HINIT  TINIT
    8                                                                            NS
   21                                                                            NT
  0.00     0.0000     0.0000                           XTIME(I)  STAGE(I)  RECH(I)
  0.25     0.5000     0.0000
  0.50     1.0000     0.0000
  0.75     0.5000     0.0000
  1.00     0.0000     0.0000
  1.25     0.0000     0.0000
  1.50     0.0000     0.0000
  1.75     0.0000     0.0000
  2.00     0.0000     0.0000
  2.25     0.0000     0.0000
  2.50     0.0000     0.0000
  2.75     0.0000     0.0000
  3.00     0.0000     0.0000
  3.25     0.0000     0.0000
  3.50     0.0000     0.0000
  3.75     0.0000     0.0000
  4.00     0.0000     0.0000
  4.25     0.0000     0.0000
  4.50     0.0000     0.0000
  4.75     0.0000     0.0000
  5.00     0.0000     0.0000
```

Figure 24. Example data-input file for program STLK1.

Program STWT1

Line-by-line instructions for creating a data-input file for program STWT1 follow. Variable names that are used in the input file and computer program are shown in upper-case text.

Line 1:

TITLE1—First line of title, which can be up to 70 characters in length. Leave this line blank if no title is specified.

Line 2:

TITLE2—Second line of title, which can be up to 70 characters in length. Leave this line blank if no title is specified.

Line 3:

ISTRESS—Type of stress being simulated. Three options are provided:

ISTRESS = 0: Stream-stage fluctuations are simulated.

ISTRESS = 1: Recharge/ET are simulated. Valid only for water-table aquifer (IAQ = 1).

ISTRESS = 2: Both stream-stage fluctuations and recharge/ET are simulated. Valid only for water-table aquifer (IAQ = 1).

DELT—Time-step size. A uniform time-step size must be used throughout the simulation. Note that the value of DELT will affect solution accuracy. Smaller time steps will improve solution accuracy but increase the amount of time required for the program to run a particular simulation (see Sample Problem 1).

IPRINT—An option to print or suppress the printing of stress data to the results file:
IPRINT = 0: Do not print stress data.
IPRINT = 1: Print stress data.

Line 4:

IXL—Extent of aquifer being simulated. Two options are provided:
IXL = 0: Semi-infinite aquifer.
IXL = 1: Finite-width aquifer.

IAQ—Type of aquifer being simulated. Two options are provided:
IAQ = 0: Confined aquifer.
IAQ = 1: Water-table aquifer.

IXA—Streambank code. Two options are provided:
IXA = 0: semipervious streambank material is absent.
IXA = 1: semipervious streambank material is present.

Line 5:

XZERO—Half-width of stream, in units of length. Must be greater than 0.0D0. Note: the half-width of the stream does not need to be known for the solution. The variable XZERO is simply used to nondimensionalize some of the parameters in the analytical solutions. Therefore, an arbitrary value of XZERO may be used; however, all distances from the center of the stream channel used in the input file must be consistent with the value of XZERO that is selected.

XLL—Width of aquifer, in units of length. Use for finite-width aquifers. Enter 0.0D0 if IXL = 0.

XAA—Streambank leakance, in units of length. Streambank leakance is defined in equation 14. Enter 0.0D0 if IXA = 0.

XSTREAM—Length of stream reach, in units of length. Must be greater than 0.0D0. XSTREAM is used to calculate total seepage and bank-storage volume over the stream reach of interest.

Line 6:

AKX—Horizontal hydraulic conductivity of aquifer, in units of length per time.

XKD—Ratio of vertical to horizontal hydraulic conductivity of aquifer, dimensionless. Enter 0.0D0 if IAQ=0.

AS—Specific storage of aquifer, in units of inverse length. The program will calculate the storativity of the aquifer by multiplying specific storage (AS) by the saturated thickness of the aquifer at the beginning of the simulation (AB).

ASY—Specific yield of aquifer, dimensionless. Enter 0.0D0 if IAQ=0.

AB—Saturated thickness of aquifer at beginning of simulation, in units of length.

Line 7: See figure 13A for definitions of ZP, Z1, and Z2.

X—Distance to observation well from stream-channel center, in units of length.

IOWS—Type of observation well:
IOWS = 0: Partially penetrating observation well.
IOWS = 1: Fully penetrating observation well.
IOWS = 2: Observation piezometer.

Z1—Vertical distance from bottom of aquifer to bottom of screened interval of observation well. Use for IOWS = 0 or 1. Enter 0.0D0 if IOWS = 2.

Z2—Vertical distance from bottom of aquifer to top of screened interval of observation well. Use for IOWS = 0 or 1. Enter 0.0D0 if IOWS = 2.

ZP—Vertical distance from bottom of aquifer to observation piezometer. Use for IOWS = 2. Enter 0.0D0 if IOWS = 0 or 1.

Line 8:

HINIT—Initial head at observation well, in units of length. Heads calculated by the program are added to or subtracted from HINIT.

TINIT—Simulation start time, in units of time. A start time to which simulation results are referenced.

Line 9: Variables NS, RERRNR, and XTRMS are program-solution variables that are used in the numerical-inversion algorithm. Suggested values are provided for each variable. Relatively smaller values of RERRNR will increase solution precision and time.

NS—Number of terms used in the Stehfest algorithm. This must be an even number, the value of which depends upon computer precision. If the computer holds 16 significant figures in double precision, let NS = 8 to 12. A value of 8 is often sufficient. If numerical results for head and seepage are unstable, NS can be reduced to 6 (or even 4). Precision will be reduced, however, and results should be checked for accuracy. The user can compare simulation results using alternative values of NS (such as 6, 8, and 12) to determine if numerically stable results have been attained.

RERRNR—Relative error for Newton-Raphson iteration and summation. Use for IAQ = 1. A value of 1.D-10 is suggested. Enter 0.0D0 for IAQ = 0. If RERRNR is exceeded after 100 Newton-Raphson iterations, then a message is printed to the result file and the program is stopped.

XTRMS—Factor used to determine number of terms in the finite sums for head and seepage. Suggested values are 20.0D0 or 30.0D0. The user should ensure that a sufficient number of terms are being used in the summations by making multiple runs in which XTRMS is increased from one simulation to the next (for example, doubled), continuing until simulation results do not vary substantially when XTRMS is increased.

Line 10:

NT—Number of time steps. Program STWT1 is dimensioned to handle up to 1,000 time steps. If NT > 1,000, parameter IMAXX must be increased to a value of at least NT in the main routine and in subroutine DATAIO of program STWT1.

Lines 11 to NT+10:

Lines 11 to NT + 10 contain the stress data for each simulation.

XTIME(I)—Time of stream-stage and/or recharge/ET input for time step I.

STAGE(I)—Stream stage for time step I.

RECH(I)—Recharge/ET for time step I.

A summary of the data-input requirements for STWT1 is shown in table 6. An example data-input file, which is named "input.prob2a" and is used in sample problem 1 below, is shown in figure 25. The data-input file is based on the hypothetical water-table aquifer described in table 4. Variable names are shown for convenience on the right side of each line of the example data-input file (fig. 25).

Result and Plot Files

Example result and plot files for program STLK1 (files "result.prob1a" and "plot.prob1a," respectively) are shown in figures 26 and 27. The example files were created by the program using data-input file "input.prob1a" (fig. 24). After the program banner (fig. 26), the result file first gives the title of the simulation and a listing of the parameters that were specified in the data-input file. The program then gives a summary of the stress data that were specified in the data-input file. Several dimensionless parameters that are defined in table 1 and are calculated by STLK1 are printed in the next block of program output. Small, nonzero values are shown for parameters SIGMA1 and GAMMA1 (0.1D-03), which are variables that are used for leaky-aquifer conditions. These small values are used to prevent division by zero in the computer program; they do not affect the confined-aquifer solutions. The final block of output data shows the calculated results for the simulation. Several quantities are listed (time, head, seepage, total seepage, bank storage, and bank-storage volume); definitions of the quantities can be found in the “General Theoretical Background” section of this report. Negative values of seepage and total seepage indicate streamflow seepage to the aquifer; positive values of seepage and total seepage indicate ground-water discharge to the stream. Positive values of bank storage and bank-storage volume indicate a net flow of stream water into the aquifer during the simulation; negative values of bank storage and bank-storage volume indicate a net discharge of ground water out of the aquifer.

Table 6. Input data format for program STWT1

[Free-format input style--each variable in a line must be separated by at least one space; for real variables use double precision values, such as 1.33D-3 or 0.00133; ET, evapotranspiration]

Line	Variable name	Type	Explanation
1	TITLE1	Character	First line of title (up to 70 characters)
2	TITLE2	Character	Second line of title (up to 70 characters)
3	ISTRESS	Integer	Stress type: ISTRESS = 0 : stream-stage fluctuations ISTRESS = 1 : recharge/ET ISTRESS = 2 : stream-stage fluctuations and recharge/ET
	DELT	Real	Time-step size. A uniform time-step size must be used.
	IPRINT	Integer	Option for printing stress data to result file: IPRINT = 0: do not print stress data IPRINT = 1: print stress data
4	IXL	Integer	Aquifer extent: IXL = 0: semi-infinite IXL = 1: finite width
	IAQ	Integer	Aquifer type: IAQ = 0: confined IAQ = 1: water table
	IXA	Integer	Streambank code: IXA = 0: semipervious streambank material absent IXA = 1: semipervious streambank material present
5	XZERO	Real	Half width of stream. Must be > 0.0D0
	XLL	Real	Width of aquifer. Enter 0.0D0 if IXL = 0
	XAA	Real	Streambank leakance. Enter 0.0D0 if IXA = 0
	XSTREAM	Real	Length of stream reach
6	AKX	Real	Horizontal hydraulic conductivity of aquifer
	XKD	Real	Ratio of vertical to horizontal hydraulic conductivity of aquifer. Enter 0.0D0 if IAQ = 0
	AS	Real	Specific storage of aquifer
	ASY	Real	Specific yield of aquifer. Enter 0.0D0 if IAQ = 0
	AB	Real	Thickness or saturated thickness of aquifer
7	X	Real	Distance to observation well from stream-channel center.
	IOWS	Integer	Type of observation well: IOWS = 0: Partially penetrating observation well IOWS = 1: Fully penetrating observation well IOWS = 2: Observation piezometer
	Z1	Real	Use for IOWS = 0 or 1. Vertical distance from bottom of aquifer to bottom of screened interval of observation well. Enter 0.0D0 if IOWS = 2
	Z2	Real	Use for IOWS = 0 or 1. Vertical distance from bottom of aquifer to top of screened interval of observation well. Enter 0.0D0 if IOWS = 2
	ZP	Real	Use for IOWS = 2. Vertical distance from bottom of aquifer to observation piezometer. Enter 0.0D0 if IOWS = 0 or 1
8	HINIT	Real	Initial head at observation well.
	TINIT	Real	Simulation start time.
9	NS	Integer	Number of Stehfest terms. Must be an even integer. 8 terms are usually sufficient
	RERRNR	Real	Relative error for Newton-Raphson iteration and summation. Suggested value is 1.D-10
	XTRMS	Real	Factor used to determine number of terms in the finite sums for head and seepage. Suggested values are 20.D0 or 30.D0
10	NT	Integer	Number of time steps: If NT > 1,000, increase parameter IMAXX in program STWT1 to a value of at least NT
11 to (NT+10)	XTIME(I)	Real	Time of stream-stage and/or recharge/ET input for time step I
	STAGE(I)	Real	Stream stage for time step I
	RECH(I)	Real	Recharge/ET for time step I

```
Sample problem 2a. Sample input file for program STWT1.                 TITLE1
One-day stream-stage flood event. Water-table aquifer. DELT=0.25days.   TITLE2
    0    0.25D+0  1                                         ISTRESS DELT IPRINT
    0      1      0                                                IXL IAQ  IXA
 25.0D0  0.0D0   0.0D0   1.0D3                          XZERO  XLL  XAA  XSTREAM
  2.0D2  2.0D-1  1.0D-5  2.5D-1 25.0D0                     AKX  XKD  AS  ASY  AB
  1.0D2    1     0.0D0   25.0D0  0.0D0                       X  IOWS  Z1  Z2  ZP
  0.0D0  0.0D0                                                     HINIT  TINIT
    8  1.0D-10  30.0D0                                        NS  RERRNR  XTRMS
   21                                                                        NT
  0.00      0.0000     0.0000                       XTIME(I)  STAGE(I)  RECH(I)
  0.25      0.5000     0.0000
  0.50      1.0000     0.0000
  0.75      0.5000     0.0000
  1.00      0.0000     0.0000
  1.25      0.0000     0.0000
  1.50      0.0000     0.0000
  1.75      0.0000     0.0000
  2.00      0.0000     0.0000
  2.25      0.0000     0.0000
  2.50      0.0000     0.0000
  2.75      0.0000     0.0000
  3.00      0.0000     0.0000
  3.25      0.0000     0.0000
  3.50      0.0000     0.0000
  3.75      0.0000     0.0000
  4.00      0.0000     0.0000
  4.25      0.0000     0.0000
  4.50      0.0000     0.0000
  4.75      0.0000     0.0000
  5.00      0.0000     0.0000
```

Figure 25. Example data-input file for program STWT1

```
                *****************************************************
                *                                                   *
                *         ****  U.S. GEOLOGICAL SURVEY  ****        *
                *                                                   *
                *          **** STLK1: PROGRAM OUTPUT ****          *
                *                                                   *
                * ONE-DIMENSIONAL MODEL OF STREAM-AQUIFER HYDRAULIC *
                *                                                   *
                *    INTERACTION FOR CONFINED AND LEAKY AQUIFERS    *
                *                                                   *
                *       BOUNDED BY A FULLY PENETRATING STREAM       *
                *                                                   *
                *           VERSION CURRENT AS OF 09/01/98          *
                *                                                   *
                *****************************************************


     Sample problem 1a. Sample input file for program STLK1
     One-day stream-stage flood event. Confined aquifer. Delt is 0.25 days.


                    SUMMARY OF INPUT DATA
                    ---------------------

   STRESS TYPE (ISTRESS):               0 (stream-stage fluctuations)
   TIME-STEP SIZE (DELT):               0.250D+00 (units of time)
   PRINTING CODE (IPRINT):              1 (stress data printed)

 AQUIFER AND STREAMBANK CHARACTERISTICS (INPUT LINES 4 AND 5)
   AQUIFER EXTENT (IXL):                0 (semi infinite)
   AQUIFER TYPE (IAQ):                  0 (confined)
   STREAMBANK CODE (IXA):               0 (semipervious streambank absent)
   STREAM HALF WIDTH (XZERO):           0.250D+02 (units of length)
   LENGTH OF STREAM (XSTREAM):          0.100D+04 (units of length)

 AQUIFER PROPERTIES (INPUT LINE 6)
   HYDRAULIC CONDUCTIVITY (AK):         0.200D+03 (units of length per time)
   SPECIFIC STORAGE (AS):               0.100D-04 (units of inverse length)
   SATURATED THICKNESS (AB):            0.250D+02 (units of length)

 AQUITARD PROPERTIES (INPUT LINE 7)
   HYDRAULIC CONDUCTIVITY (AKT):        0.000D+00 (units of length per time)
   SPECIFIC STORAGE (AST):              0.000D+00 (units of inverse length)
   SATURATED THICKNESS (ABT):           0.000D+00 (units of length)
   SPECIFIC YIELD (ASYT):               0.000D+00 (dimensionless)

 OBSERVATION WELL DATA AND INITIAL CONDITIONS (INPUT LINE 8)
   DISTANCE TO OBSERVATION WELL (X):  0.100D+04 (units of length)
   INITIAL HEAD AT WELL (HINIT):      0.000D+00 (units of length)
   START TIME OF SIMULATION (TINIT):  0.000D+00 (units of time)
```

Figure 26. Example result file for program STLK1

```
PROGRAM SOLUTION VARIABLES (INPUT LINE 9)
  NUMBER OF STEHFEST TERMS (NS):                 8


                  SUMMARY OF STRESS DATA
                  ----------------------

  NUMBER OF SPECIFIED TIME STEPS (NT):              21

        TIME        STAGE         RECH
        ----        -----         ----
     0.0000D+00   0.0000D+00   0.0000D+00
     0.2500D+00   0.5000D+00   0.0000D+00
     0.5000D+00   0.1000D+01   0.0000D+00
     0.7500D+00   0.5000D+00   0.0000D+00
     0.1000D+01   0.0000D+00   0.0000D+00
     0.1250D+01   0.0000D+00   0.0000D+00
     0.1500D+01   0.0000D+00   0.0000D+00
     0.1750D+01   0.0000D+00   0.0000D+00
     0.2000D+01   0.0000D+00   0.0000D+00
     0.2250D+01   0.0000D+00   0.0000D+00
     0.2500D+01   0.0000D+00   0.0000D+00
     0.2750D+01   0.0000D+00   0.0000D+00
     0.3000D+01   0.0000D+00   0.0000D+00
     0.3250D+01   0.0000D+00   0.0000D+00
     0.3500D+01   0.0000D+00   0.0000D+00
     0.3750D+01   0.0000D+00   0.0000D+00
     0.4000D+01   0.0000D+00   0.0000D+00
     0.4250D+01   0.0000D+00   0.0000D+00
     0.4500D+01   0.0000D+00   0.0000D+00
     0.4750D+01   0.0000D+00   0.0000D+00
     0.5000D+01   0.0000D+00   0.0000D+00


            DIMENSIONLESS PARAMETERS (CALCULATED BY PROGRAM)
            ------------------------------------------------

  DIMENSIONLESS DISTANCE TO WELL (XD):                         0.400D+02
  DIMENSIONLESS DISTANCE TO STREAMBANK (XZEROD):               0.100D+01
  DIMENSIONLESS WIDTH OF AQUIFER (XLLD):                        INFINITE
  DIMENSIONLESS STREAMBANK LEAKANCE (XAAD):                    0.000D+00
  DIMENSIONLESS RATIO OF AQUITARD TO AQUIFER
     STORATIVITY (SIGMA1):                                     0.100D-03
  DIMENSIONLESS RATIO OF AQUITARD TO AQUIFER HYDRAULIC
     CONDUCTIVITY (GAMMA1):                                    0.100D-03
  DIMENSIONLESS RATIO OF AQUIFER STORATIVITY
     TO AQUITARD SPECIFIC YIELD (SIGMAP):                      0.000D+00
```

Figure 26. Example result file for program STLK1—*Continued*.

RESULTS

TIME (T)	HEAD (L)	SEEPAGE (L**2/T)	TOTAL SEEPAGE (L**3/T)	BANK STORAGE (L**2)	BANK-STORAGE VOLUME (L**3)
0.000000D+00	0.00000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00
0.250000D+00	0.37891D+00	-.6308D+00	-.1262D+04	0.1577D+00	0.3154D+03
0.500000D+00	0.79261D+00	-.1077D+01	-.2154D+04	0.4269D+00	0.8539D+03
0.750000D+00	0.46414D+00	-.1794D+00	-.3589D+03	0.4718D+00	0.9436D+03
0.100000D+01	0.75475D-01	0.3973D+00	0.7946D+03	0.3725D+00	0.7450D+03
0.125000D+01	0.40842D-01	0.2127D+00	0.4255D+03	0.3193D+00	0.6386D+03
0.150000D+01	0.26984D-01	0.1400D+00	0.2800D+03	0.2843D+00	0.5686D+03
0.175000D+01	0.19625D-01	0.1016D+00	0.2031D+03	0.2589D+00	0.5178D+03
0.200000D+01	0.15131D-01	0.7819D-01	0.1564D+03	0.2394D+00	0.4787D+03
0.225000D+01	0.12139D-01	0.6266D-01	0.1253D+03	0.2237D+00	0.4474D+03
0.250000D+01	0.10024D-01	0.5170D-01	0.1034D+03	0.2108D+00	0.4215D+03
0.275000D+01	0.84628D-02	0.4362D-01	0.8724D+02	0.1999D+00	0.3997D+03
0.300000D+01	0.72703D-02	0.3746D-01	0.7491D+02	0.1905D+00	0.3810D+03
0.325000D+01	0.63349D-02	0.3262D-01	0.6525D+02	0.1823D+00	0.3647D+03
0.350000D+01	0.55848D-02	0.2875D-01	0.5750D+02	0.1752D+00	0.3503D+03
0.375000D+01	0.49723D-02	0.2559D-01	0.5118D+02	0.1688D+00	0.3375D+03
0.400000D+01	0.44644D-02	0.2297D-01	0.4594D+02	0.1630D+00	0.3260D+03
0.425000D+01	0.40377D-02	0.2077D-01	0.4154D+02	0.1578D+00	0.3157D+03
0.450000D+01	0.36749D-02	0.1890D-01	0.3780D+02	0.1531D+00	0.3062D+03
0.475000D+01	0.33635D-02	0.1730D-01	0.3459D+02	0.1488D+00	0.2976D+03
0.500000D+01	0.30938D-02	0.1591D-01	0.3181D+02	0.1448D+00	0.2896D+03

Figure 26. Example result file for program STLK1—*Continued*.

T	H	SEEP	SEEPT	BANK	BANKV
0.000000E+00	0.00000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
0.250000E+00	0.37891E+00	-.6308E+00	-.1262E+04	0.1577E+00	0.3154E+03
0.500000E+00	0.79261E+00	-.1077E+01	-.2154E+04	0.4269E+00	0.8539E+03
0.750000E+00	0.46414E+00	-.1794E+00	-.3589E+03	0.4718E+00	0.9436E+03
0.100000E+01	0.75475E-01	0.3973E+00	0.7946E+03	0.3725E+00	0.7450E+03
0.125000E+01	0.40842E-01	0.2127E+00	0.4255E+03	0.3193E+00	0.6386E+03
0.150000E+01	0.26984E-01	0.1400E+00	0.2800E+03	0.2843E+00	0.5686E+03
0.175000E+01	0.19625E-01	0.1016E+00	0.2031E+03	0.2589E+00	0.5178E+03
0.200000E+01	0.15131E-01	0.7819E-01	0.1564E+03	0.2394E+00	0.4787E+03
0.225000E+01	0.12139E-01	0.6266E-01	0.1253E+03	0.2237E+00	0.4474E+03
0.250000E+01	0.10024E-01	0.5170E-01	0.1034E+03	0.2108E+00	0.4215E+03
0.275000E+01	0.84628E-02	0.4362E-01	0.8724E+02	0.1999E+00	0.3997E+03
0.300000E+01	0.72703E-02	0.3746E-01	0.7491E+02	0.1905E+00	0.3810E+03
0.325000E+01	0.63349E-02	0.3262E-01	0.6525E+02	0.1823E+00	0.3647E+03
0.350000E+01	0.55848E-02	0.2875E-01	0.5750E+02	0.1752E+00	0.3503E+03
0.375000E+01	0.49723E-02	0.2559E-01	0.5118E+02	0.1688E+00	0.3375E+03
0.400000E+01	0.44644E-02	0.2297E-01	0.4594E+02	0.1630E+00	0.3260E+03
0.425000E+01	0.40377E-02	0.2077E-01	0.4154E+02	0.1578E+00	0.3157E+03
0.450000E+01	0.36749E-02	0.1890E-01	0.3780E+02	0.1531E+00	0.3062E+03
0.475000E+01	0.33635E-02	0.1730E-01	0.3459E+02	0.1488E+00	0.2976E+03
0.500000E+01	0.30938E-02	0.1591E-01	0.3181E+02	0.1448E+00	0.2896E+03

Figure 27. Example plot file for program STLK1.

The plot file (fig. 27) provides tabulated simulation results only, which can be used for graphing packages. The results are listed by column in the same order in which they were printed in the output file: T (time), H (head), SEEP (seepage), SEEPT (total seepage), BANK (bank storage), and BANKV (bank-storage volume).

Result and plot files for program STWT1 are very similar to those for program STLK1, and examples of these files are not provided here. The dimensionless parameters listed in the result file for program STWT1 are defined in Table 2.

Sample Problems

Three sample problems are provided to demonstrate application of programs STLK1 and STWT1 to time-varying stream-stage and recharge inputs. The sample problems also illustrate the effect of time-step size on simulation results. It is suggested that program users read both Sample Problems 1 and 2 before using program STWT1 because the concepts related to discretization of the stream-stage hydrograph discussed in Sample Problem 1 also are applicable to water-table aquifers.

Sample Problem 1—Response of a Confined Aquifer to a Sinusoidal Flood Wave

Cooper and Rorabaugh (1963) developed analytical solutions for ground-water heads, seepage rates, and bank storage in a semi-infinite confined aquifer in response to a sinusoidal variation of stream stage. Their closed-form solutions are exact and therefore do not require discretization of the stream-stage hydrograph or convolution method. The solutions are used here to test the convolution equations used in programs STLK1 and STWT1 and to demonstrate the effect of discretization of the stream-stage hydrograph on simulation results.

A semi-infinite confined aquifer with the hydraulic properties and dimensions listed in table 3 was simulated by use of program STLK1 and by use of the analytical solutions of Cooper and Rorabaugh (1963). The effects of a one-day sinusoidal flood wave with a peak stream stage of 1.0 ft (fig. 28, inset) were simulated over a 5-day period. Ground-water heads

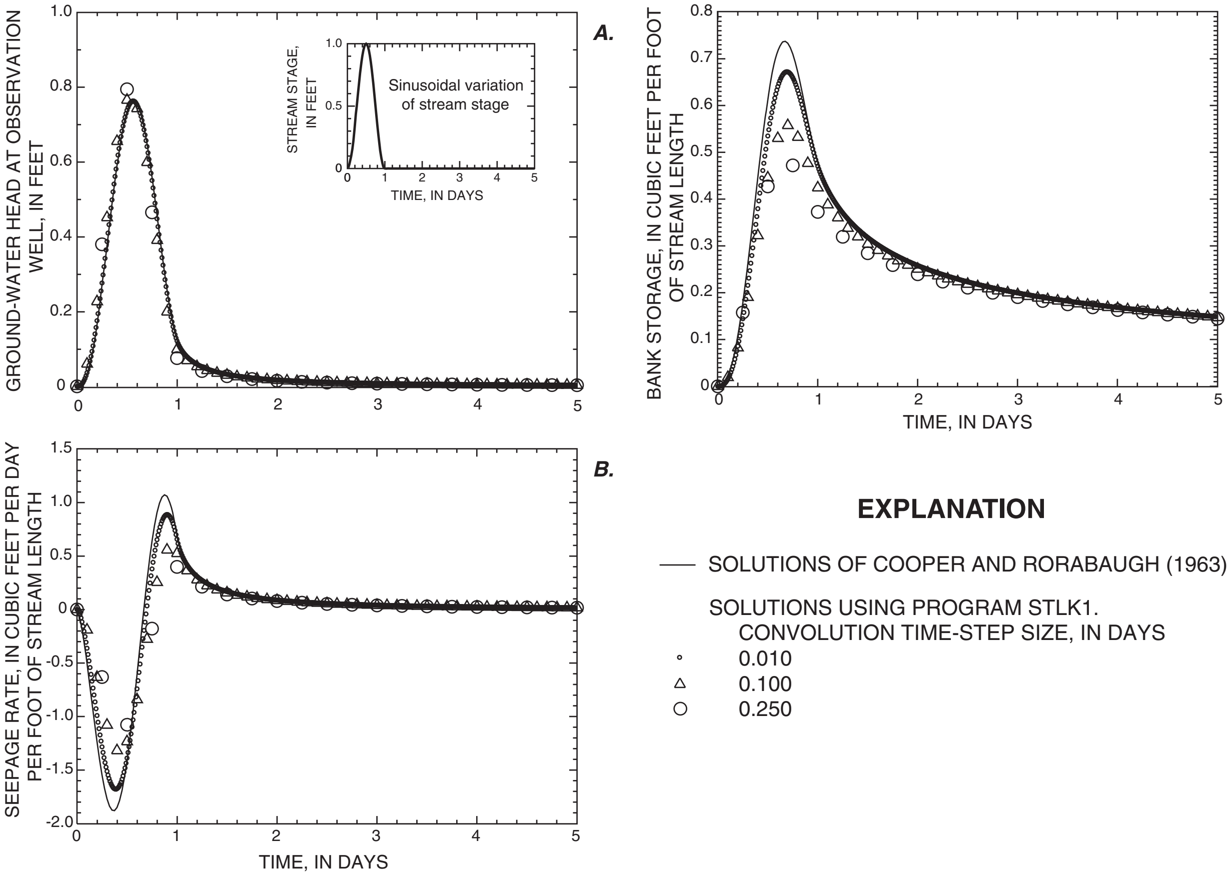


Figure 28. (*A*) Ground-water head at observation well, (*B*) seepage rate between stream and aquifer, and (*C*) bank storage in aquifer, for a one-day sinusoidal flood wave (inset), semi-infinite confined aquifer. Observation well is 975 feet from stream-aquifer boundary. Model parameters and dimensions given in table 3.

were calculated at an observation well 1,000 ft from the middle of the stream (975 ft from the stream-aquifer boundary). Three separate simulations were made with STLK1 using three values of the time-step size (input variable DELT, table 5): 0.010 days, 0.100 days, and 0.250 days. The number of time steps (input variable NT, table 5) required for each simulation were: 501 (DELT=0.010 days), 51 (DELT=0.100 days), and 21 (DELT=0.250 days, fig. 24). Figure 24, described previously, shows the data-input file for STLK1 for this problem using a time-step size of 0.250 days. A schematic diagram of the discretization scheme used for stream-stage fluctuations was previously described (fig. 23*A*).

Figure 28 shows calculated ground-water heads, seepage rates, and bank storage for the simulated conditions. The match between ground-water heads calculated by use of the solution of Cooper and Rorabaugh (1963) and those calculated by use of program STLK1 (fig. 28*A*) improves as the time-step size is decreased from 0.250 days to 0.010 days, as would be expected. Whereas differences in calculated heads between the closed-form solution and convolution equations are insignificant for a time-step size of 0.010 days, differences between the two solution methods for seepage rates (fig. 28*B*) and bank storage (fig. 28*C*) can be significant even when using a relatively small time-step size, particularly at the times of maximum and minimum seepage rates. These results point to the necessity of using a relatively fine discretization of the stream-stage hydrograph for accurate calculations of seepage rates and bank storage.

Sample Problem 2—Response of a Water-Table Aquifer to a Sinusoidal Flood Wave

A semi-infinite water-table aquifer with the hydraulic properties and dimensions listed in table 4 was simulated by use of program STWT1. The effects of a one-day sinusoidal flood wave with a peak stream stage of 1.0 ft (fig. 29, inset) were simulated over a 5-day period. No closed-form analytical solution is available to which the results of the STWT1 simulation can be compared. Ground-water heads were calculated at a fully penetrating observation well (variable IOWS=1, data-input line 7, fig. 25) 100 ft from the middle of the stream (75 ft from the stream-aquifer boundary). Three separate simulations were made with STWT1 for water-table conditions using values of K_D (the ratio of vertical to horizontal hydraulic conductivity) of 0.2, 0.02, and 0.002, respectively, and a time-step size (variable DELT) of 0.010 days. Figure 25, described previously, shows the data-input file for STWT1 for this same problem using a DELT of 0.250 days.

Results for the water-table aquifer conditions were compared to those for a confined aquifer with the same hydraulic properties, aquifer dimensions, and observation-well location as were used for the water-table aquifer, but using two values of aquifer storativity (2.5×10^{-4} and 2.5×10^{-1}), in two separate simulations. These are the limiting storativities for the hypothetical water-table aquifer: the value of 2.5×10^{-4} represents the hypothetical condition in which there is no water table present (that is, specific yield equals zero); the value of 2.5×10^{-1} equals the specific yield of the aquifer and represents the hypothetical condition in which the aquifer is rigid and the water is incompressible (that is, specific storage equals zero). The confined results also were determined using STWT1.

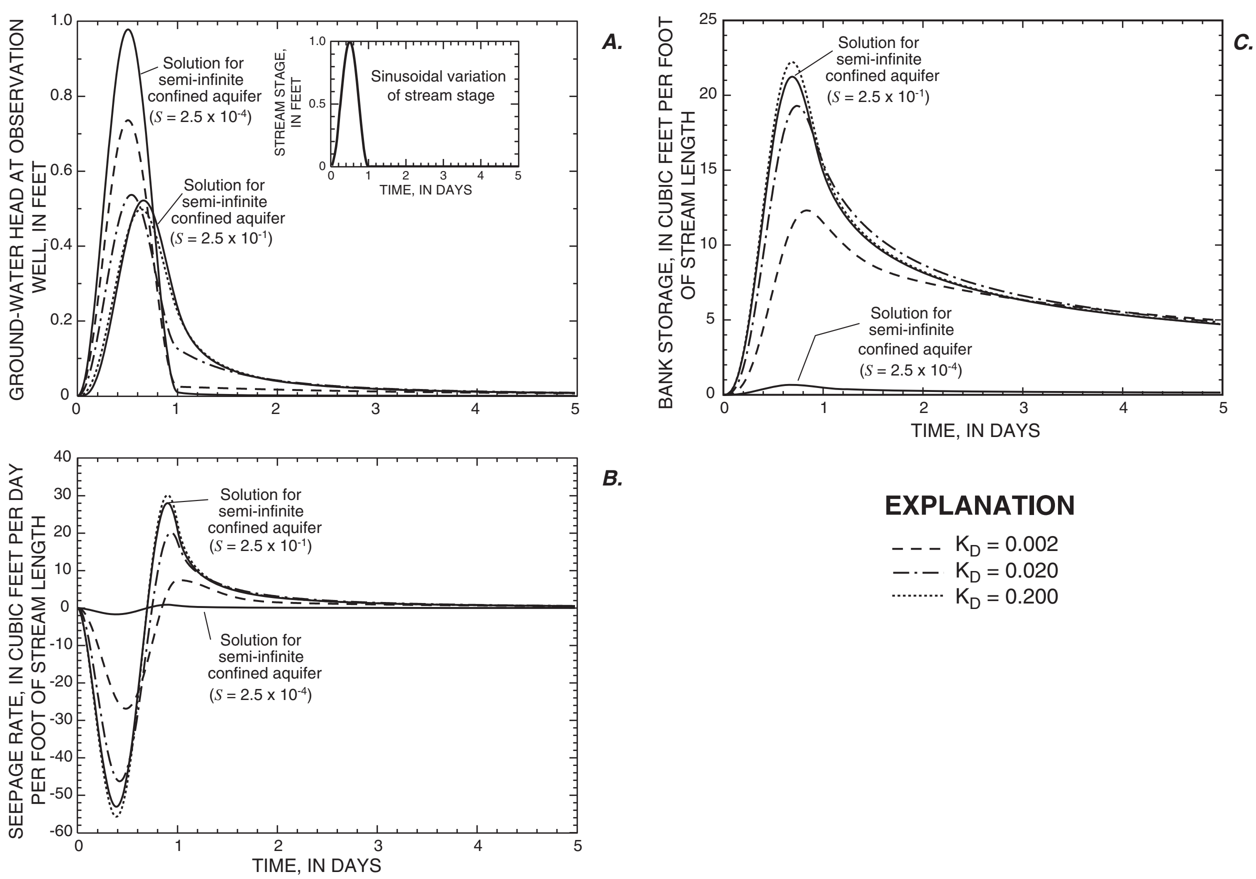


Figure 29. (*A*) Ground-water head at observation well, (*B*) seepage rate between stream and aquifer, and (*C*) bank storage in aquifer, for a one-day sinusoidal flood wave (inset), semi-infinite water-table aquifer. Observation well is 75 feet from stream-aquifer boundary; time-step size is 0.010 days; K_D, ratio of vertical to horizontal hydraulic conductivity of aquifer; *S*, storativity of aquifer; other model parameters and dimensions given in table 4.

Figure 29 shows calculated ground-water heads, seepage rates, and bank storage for the simulated conditions. The two solutions for confined-aquifer conditions are shown by solid lines in the figure. Calculated heads at the observation well for water-table aquifer conditions approach those for the confined-aquifer condition with $S = 2.5 \times 10^{-4}$ as the value of K_D is decreased (fig. 29*A*). Also, calculated seepage rates (fig. 29*B*) and bank storage (fig. 29*C*) for the water-table aquifer decrease as the value of K_D is decreased. These trends are caused by the increased resistance to vertical movement of the water table that results from the smaller values of vertical hydraulic conductivity. Figures 29*B* and 29*C* also demonstrate that seepage rates and bank storage that occur in water-table aquifers are substantially larger than those of the confined aquifer with only elastic storage (that is, $S = 2.5 \times 10^{-4}$).

Sample Problem 3—Response of a Water-Table Aquifer to Recharge

In this sample problem, program STWT1 is used to simulate a 1-day period of constant-rate recharge to a finite-width water-table aquifer with the stream stage held constant. The aquifer is 2,000 ft in width as measured from the center of the stream to an impermeable boundary at the edge of the hypothetical river valley (see figure 14A for aquifer conditions). The aquifer has a hydraulic conductivity of 200 ft/day, a ratio of vertical to horizontal hydraulic conductivity of 0.2, a specific storage of 1×10^{-4} ft^{-1}, a specific yield of 0.3, and a saturated thickness of 25 ft. A maximum ground-water-level increase of 0.1 ft (fig. 30, inset) occurs by the end of the 1-day recharge. The ground-water-level increase that is specified in the model is calculated by dividing the recharge rate to the aquifer (0.03 ft during the one day, for a recharge rate of 0.03 ft/day) by the specific yield of the aquifer (0.3). After the recharge event, the ground-water-level is held constant at 0.1 ft for four additional days (fig. 30, inset; fig. 31, data-input lines 15–31). This specified increase is that which is assumed to occur under ideal conditions; the actual change in ground-water level resulting from a recharge event will depend on antecedent conditions, the thickness of the unsaturated zone, the height of the capillary fringe, and variations in specific yield due to aquifer heterogeneity.

Three separate simulations were made with STWT1 using three values of the time-step size: 0.010 days, 0.100 days, and 0.250 days. Figure 31 shows the data-input file for this problem using DELT=0.250 days. A schematic diagram of the discretization scheme used for recharge events was previously described (fig. 23*B*).

Figure 30 shows calculated ground-water discharge from the aquifer to the stream for the simulated conditions. The peak discharge rate increases as the time-step size is decreased from 0.250 days to 0.010 days. Note that ground-water discharge rates decrease after the recharge event ends at 1.0 days; this is the recession limb of the ground-water-discharge hydrograph.

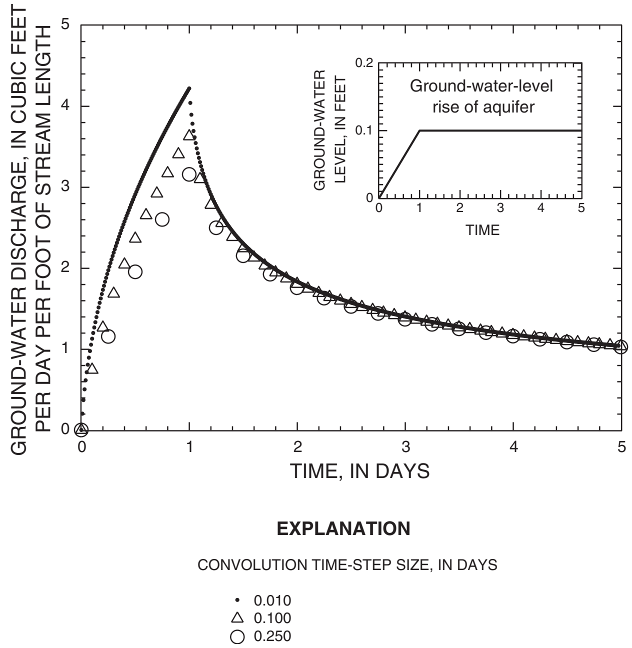


Figure 30. Ground-water discharge for a one-day recharge event (inset), finite-width water-table aquifer. Model conditions described in text.

```
Sample problem 3a. Sample input file for program STWT1.                TITLE1
One-day recharge event. Water-table aquifer. Delt=0.250 days.          TITLE2
    1    0.25D+0  1                                        ISTRESS  DELT IPRINT
    1      1      0                                                  IXL  IAQ  IXA
 25.0D0  2.0D3  0.0D0   1.0D3                          XZERO  XLL  XAA  XSTREAM
  2.0D2  0.2D0  1.0D-4  0.3D0  25.0D0                        AKX  XKD  AS  ASY  AB
  1.0D2    1     0.0D0   25.0D0  0.0D0                         X  IOWS  Z1  Z2  ZP
  0.0D0  0.0D0                                                        HINIT  TINIT
    8  1.0D-10  30                                                 NS RERRNR XTRMS
   21                                                                           NT
   0.0000    0.0000    0.0000                       XTIME(I)  STAGE(I)  RECH(I)
   0.2500    0.0000    0.0250
   0.5000    0.0000    0.0500
   0.7500    0.0000    0.0750
   1.0000    0.0000    0.1000
   1.2500    0.0000    0.1000
   1.5000    0.0000    0.1000
   1.7500    0.0000    0.1000
   2.0000    0.0000    0.1000
   2.2500    0.0000    0.1000
   2.5000    0.0000    0.1000
   2.7500    0.0000    0.1000
   3.0000    0.0000    0.1000
   3.2500    0.0000    0.1000
   3.5000    0.0000    0.1000
   3.7500    0.0000    0.1000
   4.0000    0.0000    0.1000
   4.2500    0.0000    0.1000
   4.5000    0.0000    0.1000
   4.7500    0.0000    0.1000
   5.0000    0.0000    0.1000
```

Figure 31. Example data-input file for program STWT1 for recharge event (sample problem 3).

SUMMARY

The hydraulic interaction of ground water with adjoining streams, canals, and drains is an important aspect of many hydrogeologic systems. Because of their relative simplicity, analytical solutions of stream-aquifer hydraulic interaction combined with the method of convolution (a superposition method) are an advantageous means for determining ground-water head variations, seepage rates, and bank-storage volumes that result from time-varying fluctuations in the water level of a bounding stream or from recharge and evapotranspiration from the water table.

This report describes the derivation and evaluation of analytical solutions to the ground-water flow equation for ten cases of transient, hydraulic interaction between a fully penetrating stream and a confined, leaky, or water-table aquifer. These solutions assume one-dimensional, horizontal flow in confined and leaky aquifers and two-dimensional, horizontal and vertical flow in water-table aquifers. The ten aquifer types for which analytical solutions are derived are: a semi-infinite or finite-width confined aquifer; a semi-infinite or finite-width leaky aquifer with constant head overlying the aquitard; a semi-infinite or finite-width

leaky aquifer with an impermeable layer overlying the aquitard; a semi-infinite or finite-width leaky aquifer overlain by a water-table aquitard; and a semi-infinite or finite-width water-table aquifer. All aquifer types allow for the presence or absence of a uniform semipervious streambank.

The solutions are based on the governing differential equation of transient ground-water flow in a saturated, homogeneous, slightly compressible, and anisotropic aquifer. All of the solutions are derived for the condition of an instantaneous step change (input) in stream stage and are equally applicable to the condition of an instantaneous regional rise or decline in the altitude of the water table or piezometric surface of an aquifer, caused, for example, by area-wide recharge, irrigation, or evapotranspiration.

Of primary interest are newly derived solutions for water-table aquifers and for leaky aquifers overlain by water-table aquitards. For these aquifers, it is assumed that water is released (or taken up) instantaneously in a vertical direction from (or into) the zone above the water table in response to a decline (or rise) in the elevation of the water table. This assumption implies that the equilibrium profile of soil moisture versus depth in the unsaturated and nearly-saturated zones moves instantaneously in the vertical direction by an amount equal to the change in altitude of the water table. The general aspects of the response of water-table aquifers and water-table aquitards to changes in the water level of a bounding stream are similar to those that occur in response to the withdrawal or injection of ground water from a well pumping from a water-table aquifer or leaky aquifer overlain by a water-table aquitard, and, consequently, conclusions drawn in this study from an evaluation of the analytical solutions for these aquifer types are similar to previous investigations in the field of well hydraulics.

It is assumed that each of the stream-aquifer systems for which analytical solutions are derived can be described by linear partial differential equations of ground-water flow and by linear boundary and initial conditions. The linearity of the systems allows for the use of the convolution (superposition) equation. For linearity to hold, however, it is necessary that changes in ground-water heads due to stream-stage fluctuations, recharge, or evapotranspiration be small in comparison to the initial saturated thickness of the aquifer.

Two computer programs (STLK1 and STWT1) that are based on the analytical solutions and method of convolution are described in this report. The program designated STLK1 was developed for application to confined or leaky aquifers and the program designated STWT1 was developed for application to water-table aquifers. The programs calculate changes in ground-water levels at an observation well or observation piezometer, seepage rates at the stream-aquifer boundary, and bank storage for time-varying stream-stage and/or recharge stresses that are specified by the user. The programs can simulate the response to stream-stage fluctuations for all aquifer types. Simulation of the response to recharge or evapotranspiration at the water table is permitted only for water-table aquifers and leaky aquifers overlain by a water-table aquitard. For these aquifer types, the response to recharge and evapotranspiration can be simulated alone or in combination with the response to stream-stage fluctuations. The programs require approximation of input hydrographs (continuous records of stream-stage, recharge, or evapotranspiration) as a time series of discrete step changes that occur in time steps that are a constant length. As with all discretization schemes, the accuracy of the convolution method, and therefore of the programs, is improved by use of smaller time steps.

The programs can be applied to the analysis of a passing flood wave, determination of ground-water discharge rates in response to recharge, determination of aquifer hydraulic properties, design of stream-aquifer data-collection networks, and testing of numerical-model computer codes. Instructions are provided for constructing the necessary data-input files. Three sample problems are described to provide examples of the uses of the programs.

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ATTACHMENT 1.—DERIVATION OF ANALYTICAL SOLUTIONS

STEP RESPONSE FOR FLOW FROM A SEMI-INFINITE CONFINED OR LEAKY AQUIFER

The following is a derivation of the analytical solutions for flow to a fully penetrating stream from a semi-infinite confined or leaky aquifer.

Head Distribution Due to a Step Change in Stream Stage

The governing differential equation of ground-water flow in the aquifer is

$$\frac{\partial^2 h}{\partial x^2} = \frac{S_s}{K_x}\frac{\partial h}{\partial t} + q' \qquad x_0 \le x \quad , \tag{A1.1}$$

where $q' = -\frac{K'}{K_x b}\left(\frac{\partial h'}{\partial z}\right)_{z=b}$.

In equation (A1.1), q' is equal to zero for a confined aquifer. The aquifer is of constant thickness b and is underlain by an impermeable base.

Initial and boundary conditions for the aquifer are

$$h(x, 0) = h_i \tag{A1.2}$$

$$h(\infty, t) = h_i \tag{A1.3}$$

$$h(x_0, t) = h_0 \quad . \tag{A1.4}$$

The governing differential equation of ground-water flow in the aquitard is

$$\frac{\partial^2 h'}{\partial z^2} = \frac{S_s'}{K'}\frac{\partial h'}{\partial t} \qquad b \le z \le b + b' \quad . \tag{A1.5}$$

Initial and boundary conditions for the aquitard are

$$h'(z, 0) = h_i \qquad b \le z \le b + b' \tag{A1.6}$$

$$h'(b, t) = h \quad . \tag{A1.7}$$

The boundary condition at the top of the aquitard for the condition of a constant head overlying the aquitard (case 1) is

$$h'(b + b', t) = h_i \quad ; \tag{A1.8a}$$

for the condition of an impermeable layer overlying the aquitard (case 2), the boundary condition is

$$\frac{\partial h'}{\partial z}(b + b', t) = 0 \quad ; \tag{A1.8b}$$

and for the condition of a water-table aquitard (case 3) the boundary condition is

$$\frac{\partial h'}{\partial z}(b + b', t) = -\frac{S_y'}{K'}\frac{\partial h'}{\partial t}(b + b', t) \quad . \tag{A1.8c}$$

Substituting the dimensionless variables listed in table 1 into equations (A1.1) through (A1.8) results in the following dimensionless boundary-value problem. For the aquifer, the governing equation is

$$\frac{\partial^2 h_D}{\partial x_D^2} = \frac{\partial h_D}{\partial t_D} - \gamma_1^2 \left.\frac{\partial h'_D}{\partial z'_D}\right|_{z_D = 0} \qquad 1 \le x_D < \infty \quad . \tag{A1.9}$$

Initial and boundary conditions are

$$h_D(x_D, 0) = 0 \tag{A1.10}$$

$$h_D(\infty, t_D) = 0 \tag{A1.11}$$

$$h_D(1, t_D) = 1 \quad . \tag{A1.12}$$

For the aquitard, the governing equation is

$$\frac{\partial^2 h'_D}{\partial z'^2_D} = \frac{\sigma_1}{\gamma_1^2} \frac{\partial h'_D}{\partial t_D} \qquad 0 \le z'_D \le 1 \quad , \tag{A1.13}$$

with initial and boundary conditions

$$h'_D(z'_D, 0) = 0 \tag{A1.14}$$

$$h'_D(0, t_D) = h_D \tag{A1.15}$$

$$\begin{cases} h'_D(1, t_D) = 0 & \text{constant head overlying the aquitard (case 1)} \quad \text{(A1.16a)} \\[2em] \dfrac{\partial h'_D}{\partial z'_D}(1, t_D) = 0 & \text{impermeable layer overlying the aquitard (case 2)} \quad \text{(A1.16b)} \\[2em] \dfrac{\partial h'_D}{\partial z'_D}(1, t_D) = -\dfrac{1}{\sigma' \gamma_1^2} \dfrac{\partial h'_D}{\partial t_D} & \text{water-table aquitard (case 3)} \quad \text{(A1.16c)} \end{cases}$$

After performing Laplace transformations, the subsidiary boundary-value problem for the aquifer is

$$\frac{\partial^2 \bar{h}_D}{\partial x_D^2} = p\bar{h}_D - \gamma_1^2 \left.\frac{\partial \bar{h}_D'}{\partial z_D'}\right|_{z_D = 0} \qquad 1 \le x_D < \infty \quad , \tag{A1.17}$$

with boundary conditions

$$\bar{h}_D(\infty, p) = 0 \tag{A1.18}$$

$$\bar{h}_D(1, p) = \frac{1}{p} \quad . \tag{A1.19}$$

The subsidiary boundary-value problem for the aquitard is

$$\frac{\partial^2 \bar{h}_D'}{\partial z_D'^2} = m\bar{h}_D' \qquad 0 \le z_D' \le 1 \quad , \tag{A1.20}$$

where $m = \frac{\sigma_1 p}{\gamma_1^2}$.

Boundary conditions are

$$\bar{h}_D'(0, p) = \bar{h}_D \tag{A1.21}$$

$$\bar{h}_D'(1, p) = 0 \qquad \text{constant head overlying the aquitard (case 1)} \tag{A1.22a}$$

$$\frac{\partial \bar{h}_D'}{\partial z_D'}(1, p) = 0 \qquad \text{impermeable layer overlying the aquitard (case 2)} \tag{A1.22b}$$

$$\frac{\partial \bar{h}_D'}{\partial z_D'}(1, p) = -\frac{\bar{h}_D' p}{\sigma' \gamma_1^2} \qquad \text{water-table aquitard (case 3)} \tag{A1.22c}$$

A solution to (A1.20) is

$$\bar{h}'_D = A\cosh(z'_D\sqrt{m}) + B\sinh(z'_D\sqrt{m}) \quad . \tag{A1.23}$$

Applying (A1.21) at $z'_D = 0$ gives

$$A = \bar{h}_D \quad . \tag{A1.24}$$

Applying (A1.22a) at $z'_D = 1$ gives for case 1

$$B = -\frac{\bar{h}_D}{\tanh(\sqrt{m})} \quad ; \tag{A1.25a}$$

or, applying (A1.22b) at $z'_D = 1$ gives for case 2

$$B = \frac{-\bar{h}_D}{\coth(\sqrt{m})} \quad ; \tag{A1.25b}$$

or, applying (A1.22c) at $z'_D = 1$ gives for case 3

$$A\sqrt{m}\sinh(\sqrt{m}) + B\sqrt{m}\cosh(\sqrt{m}) = -p\frac{[A\cosh(\sqrt{m}) + B\sinh(\sqrt{m})]}{\sigma'\gamma_1^2} \quad ,$$

which leads to

$$A = -B\frac{[\sqrt{m}(\sigma'\gamma_1^2)\cosh(\sqrt{m}) + p\sinh(\sqrt{m})]}{[\sqrt{m}(\sigma'\gamma_1^2)\sinh(\sqrt{m}) + p\cosh(\sqrt{m})]} \quad . \tag{A1.25c}$$

Hence, for case 1 from (A1.24) and (A1.25a),

$$\bar{h}'_D = \bar{h}_D\cosh(z'_D\sqrt{m}) - \bar{h}_D\frac{\sinh(z'_D\sqrt{m})}{\tanh(\sqrt{m})} = \bar{h}_D\sinh[\sqrt{m}\,(1 - z'_D)]/\sinh(\sqrt{m}) \quad ; \tag{A1.26a}$$

or, for case 2 from (A1.24) and (A1.25b),

$$\bar{h}'_D = \bar{h}_D\cosh[\sqrt{m}\,(1 - z'_D)]/\cosh(\sqrt{m}) \quad ; \tag{A1.26b}$$

or, for case 3 from (A1.24) and (A1.25c),

$$\bar{h}'_D = \bar{h}_D\cosh(z'_D\sqrt{m}) - \bar{h}_D\sinh(z'_D\sqrt{m})\frac{[\sqrt{m}(\sigma'\gamma_1^2)\sinh(\sqrt{m}) + p\cosh(\sqrt{m})]}{[\sqrt{m}(\sigma'\gamma_1^2)\cosh(\sqrt{m}) + p\sinh(\sqrt{m})]} \quad . \tag{A1.26c}$$

Substituting (A1.26a) into (A1.17) yields for case 1

$$\frac{\partial^2 \bar{h}_D}{\partial x_D^2} = p\bar{h}_D + \gamma_1^2 \sqrt{m}\ \bar{h}_D \coth(\sqrt{m})$$

$$= p\bar{h}_D + \bar{q}_D \bar{h}_D \quad , \tag{A1.27a}$$

where $\bar{q}_D = \gamma_1^2 \sqrt{m} \coth(\sqrt{m})$.

Substituting (A1.26b) into (A1.17) yields for case 2

$$\frac{\partial^2 \bar{h}_D}{\partial x_D^2} = p\bar{h}_D + \gamma_1^2 \sqrt{m}\ \bar{h}_D \tanh(\sqrt{m})$$

$$= p\bar{h}_D + \bar{q}_D , \tag{A1.27b}$$

where $\bar{q}_D = \gamma_1^2 \sqrt{m} \tanh(\sqrt{m})$.

Substituting (A1.26c) into (A1.17) yields for case 3

$$\frac{\partial^2 \bar{h}_D}{\partial x_D^2} = p\bar{h}_D + \gamma_1^2 \bar{h}_D \sqrt{m} \frac{[\sqrt{m}(\sigma'\gamma_1^2)\sinh(\sqrt{m}) + p\cosh(\sqrt{m})]}{[\sqrt{m}(\sigma'\gamma_1^2)\cosh(\sqrt{m}) + p\sinh(\sqrt{m})]}$$

$$= p\bar{h}_D + \bar{q}_D \bar{h}_D \quad , \tag{A1.27c}$$

where $\bar{q}_D = \gamma_1^2 \sqrt{m} \frac{[\sqrt{m}(\sigma'\gamma_1^2)\tanh(\sqrt{m}) + p]}{[\sqrt{m}(\sigma'\gamma_1^2) + p\tanh(\sqrt{m})]}$.

Now, the subsidiary boundary-value problem for the aquifer becomes

$$\frac{\partial^2 \bar{h}_D}{\partial x_D^2} = \bar{h}_D(p + \bar{q}_D) \quad , \tag{A1.28}$$

with boundary conditions

$$\bar{h}_D(\infty, p) = 0 \tag{A1.29}$$

$$\bar{h}_D(1, p) = \frac{1}{p} \quad . \tag{A1.30}$$

A solution to (A1.28) is

$$\bar{h}_D = C\exp(x_D\sqrt{p+\bar{q}_D}) + D\exp(-x_D\sqrt{p+\bar{q}_D}) \quad . \tag{A1.31}$$

Because of (A1.29), $C = 0$ and because of (A1.30), $D = \frac{1}{p}\exp(\sqrt{p+\bar{q}_D})$. Thus, (A1.31) becomes,

$$\bar{h}_D = \frac{1}{p}\exp[-\sqrt{p+\bar{q}_D}\,(x_D - 1)] \quad , \tag{A1.32}$$

where for a confined aquifer

$$\bar{q}_D = 0 \quad ;$$

for a leaky aquifer with constant head overlying the aquitard (case 1)

$$\bar{q}_D = \gamma_1^2\sqrt{m}\,\coth(\sqrt{m}) \quad ;$$

for a leaky aquifer with impermeable layer overlying the aquitard (case 2)

$$\bar{q}_D = \gamma_1^2\sqrt{m}\,\tanh(\sqrt{m}) \quad ;$$

and for a leaky aquifer overlain by a water-table aquitard (case 3)

$$\bar{q}_D = \gamma_1^2\sqrt{m}\frac{[\sqrt{m}(\sigma'\gamma_1^2)\tanh(\sqrt{m}) + p]}{[\sqrt{m}(\sigma'\gamma_1^2) + p\tanh(\sqrt{m})]} \quad .$$

Dimensionless Seepage at Streambank Due to Step Change in Stream Stage

Dimensionless seepage at $x = x_0$ (the streambank) due to a unit-step change in stream stage is

$$\bar{Q}_D = -\frac{d\bar{h}_D}{dx_D} \quad , \tag{A1.33}$$

evaluated at $x_D = 1$. Differentiating (A1.32) with respect to x_D gives

$$\frac{d\bar{h}_D}{dx_D} = -\frac{\sqrt{p+\bar{q}_D}}{p}\exp[-\sqrt{p+\bar{q}_D}(x_D - 1)] \quad . \tag{A1.34}$$

Now, evaluating (A1.34) at $x_D = 1$, (A1.33) becomes

$$\bar{Q}_D = \frac{\sqrt{p+\bar{q}_D}}{p} \quad . \tag{A1.35}$$

STEP RESPONSE FOR FLOW FROM A FINITE-WIDTH CONFINED OR LEAKY AQUIFER WITH A SEMIPERVIOUS STREAMBANK

The following is a derivation of the analytical solutions for flow to a fully penetrating stream with a semipervious streambank from a finite-width confined or leaky aquifer.

Head Distribution Due to a Step Change in Stream Stage

The governing differential equation of ground-water flow in the aquifer is

$$\frac{\partial^2 h}{\partial x^2} = \frac{S_s}{K_x}\frac{\partial h}{\partial t} + q' \qquad x_0 \le x \le x_L \quad , \tag{A1.36}$$

where $q' = -\frac{K'}{K_x b}\left(\frac{\partial h'}{\partial z}\right)_{z=b}$. For a confined aquifer, q' is equal to zero. The aquifer is of constant thickness b and is underlain by an impermeable base.

Initial and boundary conditions for the aquifer are

$$h(x, 0) = h_i \tag{A1.37}$$

$$\frac{\partial h}{\partial x}(x_L, t) = 0 \tag{A1.38}$$

$$\frac{\partial h}{\partial x}(x_0, t) = \frac{1}{a}[h_0 - h(x_0, t)] \quad . \tag{A1.39}$$

In the same way as derived for a semi-infinite aquifer, the dimensionless subsidiary boundary value problem for the finite-width aquifer becomes

$$\frac{\partial^2 \bar{h}_D}{\partial x_D^2} = \bar{h}_D(p + \bar{q}_D) \quad , \tag{A1.40}$$

with boundary conditions

$$\frac{\partial \bar{h}_D}{\partial x_D}(x_{LD}, t_D) = 0 \tag{A1.41}$$

$$\frac{\partial \bar{h}_D}{\partial x_D}(1, p) = \frac{1}{A}\left(\bar{h}_D - \frac{1}{p}\right) \tag{A1.42}$$

and where $\bar{q}_D$ is defined for case 1, case 2, and case 3 following equation (A1.32).

A solution to (A1.40) is

$$\bar{h}_D = C\exp(x_D\sqrt{p + \bar{q}_D}) + D\exp(-x_D\sqrt{p + \bar{q}_D}) \quad . \tag{A1.43}$$

Applying boundary condition (A1.41) to (A1.43) yields

$$\frac{\partial \bar{h}_D}{\partial x_D} = C\sqrt{p + \bar{q}_D}\exp(x_{LD}\sqrt{p + \bar{q}_D}) - D\sqrt{p + \bar{q}_D}\exp(-x_{LD}\sqrt{p + \bar{q}_D}) = 0 \quad . \tag{A1.44}$$

Let $r_1 = \sqrt{p + \bar{q}_D}$. Then, from (A1.44)

$$C = D\exp(-2r_1 x_{LD}) \quad . \tag{A1.45}$$

Substituting (A1.45) into (A1.43) gives

$$\bar{h}_D = D\exp(-r_1 x_D)\{\exp[-2r_1(x_{LD} - x_D)] + 1\} \quad . \tag{A1.46}$$

Applying (A1.42) to (A1.46) yields

$$D = \frac{\exp(r_1)}{p[Ar_1\{1 - \exp[-2r_1(x_{LD} - 1)]\} + \exp[-2r_1(x_{LD} - 1)] + 1]}$$

$$= \frac{\exp[r_1]}{p\left\{Ar_1\left[\frac{1 - \exp[-2r_1(x_{LD} - 1)]}{1 + \exp[-2r_1(x_{LD} - 1)]}\right] + 1\right\}\{1 + \exp[-2r_1(x_{LD} - 1)]\}} \quad .$$

The Laplace transform solution for head in a finite-width leaky aquifer with a semipervious streambank is obtained upon substitution of D into (A1.46). Thus,

$$\bar{h}_D = \frac{\exp[-r_1(x_D - 1)]}{p\{1 + Ar_1\tanh[r_1(x_{LD} - 1)]\}}\left\{\frac{\exp[-2r_1(x_{LD} - x_D)] + 1}{\exp[-2r_1(x_{LD} - 1)] + 1}\right\} \quad ; \tag{A1.47}$$

or, substituting for r_1

$$\bar{h}_D = \frac{W\exp[-\sqrt{p + \bar{q}_D}(x_D - 1)]}{p\{1 + \sqrt{p + \bar{q}_D}A\tanh[\sqrt{p + \bar{q}_D}(x_{LD} - 1)]\}} \quad , \tag{A1.48}$$

where

$$W = \frac{\exp[-2\sqrt{p + \bar{q}_D}(x_{LD} - x_D)] + 1}{\exp[-2\sqrt{p + \bar{q}_D}(x_{LD} - 1)] + 1} \quad , \tag{A1.48a}$$

and $\bar{q}_D$ is defined following equation (A1.32).

Dimensionless Seepage at Streambank Due to Step Change in Stream Stage

Dimensionless seepage at $x = x_0$ (the streambank) due to a unit-step change in stream stage is

$$\bar{Q}_D = -\frac{d\bar{h}_D}{dx_D} \quad , \tag{A1.49}$$

evaluated at $x_D = 1$.

Letting $r_1 = \sqrt{p + \bar{q}_D}$, $r_2 = \{\exp[-2r_1(x_{LD} - 1)] + 1\}$, and $r_3 = \{1 + r_1 A \tanh[r_1(x_{LD} - 1)]\}$, equation (A1.47) becomes

$$\bar{h}_D = \frac{\{\exp[-2r_1(x_{LD} - x_D)] + 1\}}{pr_2r_3}\{\exp[-r_1(x_D - 1)]\}$$

$$= \frac{1}{pr_2r_3}\{\exp[-2r_1(x_{LD} - x_D) - r_1(x_D - 1)] + \exp[-r_1(x_D - 1)]\} \quad . \tag{A1.50}$$

Differentiating (A1.50) with respect to x_D gives

$$\frac{d\bar{h}_D}{dx_D} = \frac{1}{pr_2r_3}\{r_1\exp[-2r_1(x_{LD} - x_D) - r_1(x_D - 1)] - r_1\exp[-r_1(x_D - 1)]\} \quad . \tag{A1.51}$$

Now, evaluating (A1.51) at $x_D = 1$ gives

$$\left.\frac{d\bar{h}_D}{dx_D}\right|_{x_D = 1} = \frac{r_1}{pr_2r_3}\{\exp[-2r_1(x_{LD} - 1)] - 1\} \tag{A1.52}$$

and (A1.49) becomes

$$\bar{Q}_D = \frac{-\sqrt{p + \bar{q}_D}}{p\{1 + \sqrt{p + \bar{q}_D}A\tanh[\sqrt{p + \bar{q}_D}(x_{LD} - 1)]\}}\left\{\frac{\exp[-2\sqrt{p + \bar{q}_D}(x_{LD} - 1)] - 1}{\exp[-2\sqrt{p + \bar{q}_D}(x_{LD} - 1)] + 1}\right\} \quad . \tag{A1.53}$$

The solutions reduce to semi-infinite aquifer solutions and/or solutions without a semipervious streambank with appropriate substitution of $x_{LD} \rightarrow \infty$ and/or $A = 0$.

STEP RESPONSE FOR FLOW FROM A SEMI-INFINITE WATER-TABLE AQUIFER

The following is a derivation of the analytical solutions for flow to a fully penetrating stream from a semi-infinite water-table aquifer.

Head Distribution Due to a Step Change in Stream Stage

The governing differential equation of ground-water flow in the aquifer is

$$\frac{\partial^2 h}{\partial x^2} + \frac{K_z}{K_x}\frac{\partial^2 h}{\partial z^2} = \frac{S_s}{K_x}\frac{\partial h}{\partial t} \qquad \begin{cases} x_0 \le x < \infty \\ 0 \le z \le b \end{cases} . \tag{A1.54}$$

Initial and boundary conditions for the aquifer are

$$h(x, z, 0) = h_i \tag{A1.55}$$

$$\frac{\partial h}{\partial z}(x, b, t) = -\frac{S_y}{K_z}\frac{\partial h}{\partial t} \tag{A1.56}$$

$$\frac{\partial h}{\partial z}(x, 0, t) = 0 \tag{A1.57}$$

$$h(\infty, z, t) = h_i \tag{A1.58}$$

$$h(x_0, t) = h_0 \tag{A1.59}$$

Substituting the dimensionless variables in table 2 into equations (A1.54) through (A1.59) results in the following dimensionless boundary-value problem. The governing equation is

$$\frac{\partial^2 h_D}{\partial x_D^2} + \beta_0 \frac{\partial^2 h_D}{\partial z_D^2} = \frac{\partial h_D}{\partial t_D} \qquad \begin{cases} 1 \le x_D < \infty \\ 0 \le z_D \le 1 \end{cases} . \tag{A1.60}$$

Initial and boundary conditions are

$$h_D(x_D, z_D, 0) = 0 \tag{A1.61}$$

$$\frac{\partial h_D}{\partial z_D}(x_D, 1, t_D) = -\frac{1}{\sigma\beta_0}\frac{\partial h_D}{\partial t_D} \tag{A1.62}$$

$$\frac{\partial h_D}{\partial z_D}(x_D, 0, t_D) = 0 \tag{A1.63}$$

$$h_D(\infty, z_D, t_D) = 0 \tag{A1.64}$$

$$h_D(1, t_D) = 1 \tag{A1.65}$$

After performing Laplace transformations, the subsidiary boundary-value problem for the aquifer is

$$\frac{\partial^2 \bar{h}_D}{\partial x_D^2} + \beta_0 \frac{\partial^2 \bar{h}_D}{\partial z_D^2} = p\bar{h}_D \qquad \begin{cases} 1 \le x_D < \infty \\ 0 \le z_D \le 1 \end{cases} \tag{A1.66}$$

with boundary conditions

$$\frac{\partial \bar{h}_D}{\partial z_D}(x_D, 1) = -\frac{p\bar{h}_D}{\sigma\beta_0} \tag{A1.67}$$

$$\frac{\partial \bar{h}_D}{\partial z_D}(x_D, 0) = 0 \tag{A1.68}$$

$$\bar{h}_D(\infty, z_D) = 0 \tag{A1.69}$$

$$\bar{h}_D(1, p) = \frac{1}{p} \quad . \tag{A1.70}$$

A solution to (A1.66) that satisfies (A1.67) and (A1.68) is

$$\bar{h}_D = \sum_{n=0}^{\infty} \bar{g}_n(x_D, p)\cos(\varepsilon_n z_D) \quad , \tag{A1.71}$$

where $n = 0, 1, 2, \ldots$ and ε_n are the roots of

$$\varepsilon_n \tan(\varepsilon_n) = \frac{p}{\sigma\beta_0} \quad . \tag{A1.72}$$

Substitution of (A1.71) into (A1.66) yields

$$\sum_{n=0}^{\infty} [\bar{g}_n'' - (\varepsilon_n^2\beta_0 + p)\bar{g}_n] \cos(\varepsilon_n z_D) = 0 \quad . \tag{A1.73}$$

Hence, g_n must satisfy

$$\bar{g}_n'' - (\varepsilon_n^2\beta_0 + p)\bar{g}_n = 0 \quad , \tag{A1.74}$$

the solution of which can be written as

$$\bar{g}_n = A_n \exp(q_n x_D) + B_n \exp(-q_n x_D) \quad , \tag{A1.75}$$

where

$$q_n = (\varepsilon_n^2\beta_0 + p)^{\frac{1}{2}} \quad . \tag{A1.76}$$

Because of boundary condition (A1.69), $A_n = 0$. Hence,

$$\bar{g}_n = B_n \exp(-q_n x_D) \quad . \tag{A1.77}$$

Substitution of (A1.77) into (A1.71) gives

$$\bar{h}_D = \sum_{n=0}^{\infty} B_n \exp(-q_n x_D) \cos(\varepsilon_n z_D) \quad . \qquad \text{(A1.78)}$$

The coefficients B_n can be obtained as follows: Apply boundary condition (A1.70) to obtain

$$\sum_{n=0}^{\infty} B_n \exp(-q_n) \cos(\varepsilon_n z_D) = \frac{1}{p} \quad . \qquad \text{(A1.79)}$$

Multiply both sides of (A1.79) by $\cos(\varepsilon_m z_D)$, where m is an integer. Then

$$\sum_{n=0}^{\infty} B_n \exp(-q_n) \cos(\varepsilon_n z_D) \cos(\varepsilon_m z_D) = \frac{1}{p} \cos(\varepsilon_m z_D) \quad . \qquad \text{(A1.80)}$$

It is now possible to take advantage of the orthogonality of the set $\{\cos(\varepsilon_n z_D)\}$ by integrating over the interval $z_D = 0$ to $z_D = 1$,

$$\sum_{n=0}^{\infty} B_n \exp(-q_n) \int_0^1 \cos(\varepsilon_n z_D) \cos(\varepsilon_m z_D) dz_D = \frac{1}{p} \int_0^1 \cos(\varepsilon_m z_D) dz_D \quad . \qquad \text{(A1.81)}$$

By use of trigonometric relations in combination with (A1.72), all terms on the left hand side of (A1.81) can be shown by direct integration to be zero except those for which $n = m$. (For a discussion on the topic of orthogonality, see Hildebrand, 1976.) Hence,

$$B_n \exp(-q_n) \int_0^1 \cos^2(\varepsilon_n z_D) dz_D = \frac{1}{p} \int_0^1 \cos(\varepsilon_n z_D) dz_D \quad , \qquad \text{(A1.82)}$$

or

$$B_n \exp(-q_n) \left[0.5 + \frac{\sin(2\varepsilon_n)}{4\varepsilon_n}\right] = \frac{\sin(\varepsilon_n)}{p\varepsilon_n} \quad . \qquad \text{(A1.83)}$$

Then,

$$B_n = \frac{2\exp(q_n)\sin(\varepsilon_n)}{p[\varepsilon_n + 0.5\sin(2\varepsilon_n)]} \quad . \qquad \text{(A1.84)}$$

Substitution of (A1.84) into (A1.78) yields

$$\bar{h}_D = 2\sum_{n=0}^{\infty} \frac{\exp[-q_n(x_D - 1)]\sin(\varepsilon_n)\cos(\varepsilon_n z_D)}{p[\varepsilon_n + 0.5\sin(2\varepsilon_n)]} \quad , \tag{A1.85}$$

where q_n are defined by (A1.76) and ε_n are the roots of equation (A1.72).

Equation (A1.85) is the Laplace transform solution for head at a point x_D, z_D in the aquifer. For the case of a partially penetrating observation well, screened over the interval z_{D1} to z_{D2}, equation (A1.85) is integrated over the interval z_{D1} to z_{D2} to give the average head in a partially penetrating observation well ($\bar{h}_D^*$):

$$\bar{h}_D^* = \frac{1}{(z_{D2} - z_{D1})}\int_{z_{D1}}^{z_{D2}} \bar{h}_D dz_D$$

$$= \frac{2}{(z_{D2} - z_{D1})}\sum_{n=0}^{\infty} \frac{\exp[-q_n(x_D - 1)]\sin(\varepsilon_n)}{p[\varepsilon_n + 0.5\sin(2\varepsilon_n)]}\int_{z_{D1}}^{z_{D2}} \cos(\varepsilon_n z_D) dz_D$$

$$= \frac{2}{(z_{D2} - z_{D1})}\sum_{n=0}^{\infty} \frac{\exp[-q_n(x_D - 1)]\sin(\varepsilon_n)[\sin(\varepsilon_n z_{D2}) - \sin(\varepsilon_n z_{D1})]}{p\ \varepsilon_n[\varepsilon_n + 0.5\sin(2\varepsilon_n)]} \quad . \tag{A1.86}$$

By setting $z_{D1} = 0$ and $z_{D2} = 1$, one obtains the average head in a fully penetrating observation well ($\hat{\bar{h}}_D$):

$$\hat{\bar{h}}_D = 2\sum_{n=0}^{\infty} \frac{\exp[-q_n(x_D - 1)]\sin^2(\varepsilon_n)}{p\ \varepsilon_n[\varepsilon_n + 0.5\sin(2\varepsilon_n)]} \quad . \tag{A1.87}$$

Note that if specific yield becomes zero, $\sigma \to \infty$, and, from (A1.72), $\varepsilon_n \to n\pi$. Equation (A1.87) thus becomes zero for all terms other than that for which $n = 0$ (that is, $n = 1, 2, 3, \ldots$). For $n = 0$, one can take note of the fact that

$$\lim_{u \to 0} \frac{\sin u}{u} \to 1 \quad .$$

Also, for $n = 0$, and from (A1.76), $q_n = \sqrt{p}$. Hence, (A1.87) becomes

$$\hat{\bar{h}}_D = \frac{\exp[-\sqrt{p}(x_D - 1)]}{p} \quad , \tag{A1.88}$$

which is the solution for a confined aquifer equal to (A1.32), with $\bar{q}_D = 0$.

Dimensionless Seepage at Streambank Due to Step Change in Stream Stage

Dimensionless seepage at $x = x_0$ (the streambank) due to a unit-step change in stream stage is

$$\bar{Q}_D = -\frac{d\hat{\bar{h}}_D}{dx_D} \quad , \tag{A1.89}$$

evaluated at $x_D = 1$. Differentiating (A1.87) with respect to x_D gives

$$\frac{d\hat{\bar{h}}_D}{dx_D} = -2\sum_{n=0}^{\infty}\frac{q_n \exp[-q_n(x_D-1)]\sin^2(\varepsilon_n)}{p\ \varepsilon_n[\varepsilon_n + 0.5\sin(2\varepsilon_n)]} \quad . \tag{A1.90}$$

Now, evaluating (A1.90) at $x_D = 1$, (A1.89) becomes

$$\bar{Q}_D = 2\sum_{n=0}^{\infty}\frac{q_n \sin^2(\varepsilon_n)}{p\ \varepsilon_n[\varepsilon_n + 0.5\sin(2\varepsilon_n)]} \quad . \tag{A1.91}$$

STEP RESPONSE FOR FLOW FROM A FINITE-WIDTH WATER-TABLE AQUIFER WITH A SEMIPERVIOUS STREAMBANK

The following is a derivation of the analytical solutions for flow to a fully penetrating stream with a semipervious streambank from a finite-width water-table aquifer.

Head Distribution Due to a Step Change in Stream Stage

The governing differential equation of ground-water flow in the aquifer is

$$\frac{\partial^2 h}{\partial x^2} + \frac{K_z}{K_x}\frac{\partial^2 h}{\partial z^2} = \frac{S_s}{K_x}\frac{\partial h}{\partial t} \qquad \begin{cases} x_0 \le x < x_L \\ 0 \le z \le b \end{cases} \quad . \tag{A1.92}$$

Initial and boundary conditions for the aquifer are

$$h(x, z, 0) = h_i \tag{A1.93}$$

$$\frac{\partial h}{\partial z}(x, b, t) = -\frac{S_y}{K_z}\frac{\partial h}{\partial t} \tag{A1.94}$$

$$\frac{\partial h}{\partial z}(x, 0, t) = 0 \tag{A1.95}$$

$$\frac{\partial h}{\partial x}(x_L, z, t) = 0 \tag{A1.96}$$

$$\frac{\partial h}{\partial x}(x_0, t) = \frac{1}{a}[h_0 - h(x_0, t)] \quad . \tag{A1.97}$$

Substituting the dimensionless variables in table 2 into equations (A1.92) through (A1.97) results in the following dimensionless boundary-value problem. The governing equation is

$$\frac{\partial^2 h_D}{\partial x_D^2} + \beta_0 \frac{\partial^2 h_D}{\partial z_D^2} = \frac{\partial h_D}{\partial t_D} \qquad \begin{cases} 1 \le x_D < x_{LD} \\ 0 \le z_D \le 1 \end{cases} . \tag{A1.98}$$

Initial and boundary conditions are

$$h_D(x_D, z_D, 0) = 0 \tag{A1.99}$$

$$\frac{\partial h_D}{\partial z_D}(x_D, 1, t_D) = -\frac{1}{\sigma\beta_0}\frac{\partial h_D}{\partial t_D} \tag{A1.100}$$

$$\frac{\partial h_D}{\partial z_D}(x_D, 0, t_D) = 0 \tag{A1.101}$$

$$\frac{\partial h_D}{\partial x_D}(x_{LD}, z_D, t_D) = 0 \tag{A1.102}$$

$$\frac{\partial h_D}{\partial x_D}(1, t_D) = \frac{h_D - 1}{A} \tag{A1.103}$$

After performing Laplace transformations, the subsidiary boundary-value problem for the aquifer is

$$\frac{\partial^2 \bar{h}_D}{\partial x_D^2} + \beta_0 \frac{\partial^2 \bar{h}_D}{\partial z_D^2} = p\bar{h}_D \qquad \begin{matrix} 1 \le x_D < x_{LD} \\ 0 \le z_D \le 1 \end{matrix} \tag{A1.104}$$

with boundary conditions

$$\frac{\partial \bar{h}_D}{\partial z_D}(x_D, 1) = -\frac{p\bar{h}_D}{\sigma\beta_0} \tag{A1.105}$$

$$\frac{\partial \bar{h}_D}{\partial z_D}(x_D, 0) = 0 \tag{A1.106}$$

$$\frac{\partial \bar{h}_D}{\partial x_D}(x_{LD}, z_D) = 0 \tag{A1.107}$$

$$\frac{\partial \bar{h}_D}{\partial x_D}(1, p) = \frac{\bar{h}_D}{A} - \frac{1}{Ap} \tag{A1.108}$$

A solution to (A1.104) that satisfies (A1.105) and (A1.106) is

$$\bar{h}_D = \sum_{n=0}^{\infty} \bar{g}_n(x_D, p)\cos(\varepsilon_n z_D) \quad , \tag{A1.109}$$

where $n = 0, 1, 2, \ldots$ and ε_n are the roots of

$$\varepsilon_n \tan(\varepsilon_n) = \frac{p}{\sigma\beta_0} \quad . \tag{A1.110}$$

Substitution of (A1.109) into (A1.104) yields

$$\sum_{n=0}^{\infty} [\bar{g}_n'' - (\varepsilon_n^2\beta_0 + p)\bar{g}_n] \cos(\varepsilon_n z_D) = 0 \quad . \tag{A1.111}$$

Hence, g_n must satisfy

$$\bar{g}_n'' - (\varepsilon_n^2\beta_0 + p)\bar{g}_n = 0 \quad , \tag{A1.112}$$

the solution of which can be written as

$$\bar{g}_n = A_n \exp(q_n x_D) + B_n \exp(-q_n x_D) \quad , \tag{A1.113}$$

where

$$q_n = (\varepsilon_n^2\beta_0 + p)^{\frac{1}{2}} \quad . \tag{A1.114}$$

The solution (A1.109) satisfies (A1.107) if

$$\frac{\partial \bar{g}_n}{\partial x_D}(x_{LD}, p) = 0 \quad . \tag{A1.115}$$

Thus, substituting (A1.113) into (A1.115) and letting $x_D = x_{LD}$

$$A_n q_n \exp(q_n x_{LD}) - B_n q_n \exp(-q_n x_{LD}) = 0 \quad . \tag{A1.116}$$

Hence,

$$A_n = B_n \exp(-2q_n x_{LD}) \quad . \tag{A1.117}$$

Now, (A1.113) becomes

$$\bar{g}_n = B_n\left[\frac{\exp(2q_n x_D)\exp(-2x_{LD} q_n)}{\exp(q_n x_D)} + \exp(-q_n x_D)\right]$$

$$= B_n \exp(-q_n x_D)\{\exp[-2q_n(x_{LD} - x_D)] + 1\} \quad . \tag{A1.118}$$

Substitution of (A1.118) into (A1.109) yields

$$\bar{h}_D = \sum_{n=0}^{\infty} B_n \exp(-q_n x_D)\{\exp[-2q_n(x_{LD} - x_D)] + 1\}\cos(\varepsilon_n z_D) \quad . \tag{A1.119}$$

The coefficients B_n can be obtained as follows:

Apply boundary condition (A1.108) to obtain

$$\sum_{n=0}^{\infty} B_n \exp(-q_n)\{\exp[-2q_n(x_{LD} - 1)] + 1\}\cos(\varepsilon_n z_D)$$

$$-\sum_{n=0}^{\infty} B_n A q_n \exp(-q_n)\{\exp[-2q_n(x_{LD} - 1)] - 1\}\cos(\varepsilon_n z_D) = \frac{1}{p} \quad , \tag{A1.120}$$

or, after rearranging terms, (A1.120) becomes

$$\sum_{n=0}^{\infty} B_n \exp(-q_n)\{1 + A q_n \tanh[q_n(x_{LD} - 1)]\}\cos(\varepsilon_n z_D) = \frac{1}{p\{1 + \exp[-2q_n(x_{LD} - 1)]\}} \quad . \tag{A1.121}$$

Now, as demonstrated previously, by multiplying both sides of (A1.121) by $\cos(\varepsilon_m z_D)$, where m is an integer, and integrating over the interval $z_D = 0$ to $z_D = 1$, one can make use of the property of orthogonality of the set $\{\cos(\varepsilon_n z_D)\}$ over the interval $0, 1$, where ε_n are the roots of (A1.110). Thus, the terms for which $m \neq n$ are zero and one obtains

$$B_n \exp(-q_n)\{1 + A q_n \tanh[q_n(x_{LD} - 1)]\}\int_0^1 \cos^2(\varepsilon_n z_D) dz_D$$

$$= \frac{1}{p\{1 + \exp[-2q_n(x_{LD} - 1)]\}} \int_0^1 \cos(\varepsilon_n z_D) dz_D \quad . \tag{A1.122}$$

Thus,

$$B_n = \frac{2\exp(q_n)\sin(\varepsilon_n)}{pR\{1 + \exp[-2q_n(x_{LD} - 1)]\}[\varepsilon_n + 0.5\sin(2\varepsilon_n)]} \quad , \tag{A1.123}$$

where

$$R = 1 + A q_n \tanh[q_n(x_{LD} - 1)] \quad . \tag{A1.124}$$

Substitution of (A1.123) into (A1.119) yields

$$\bar{h}_D = 2\sum_{n=0}^{\infty}\frac{W_n\exp[-q_n(x_D-1)]\sin(\varepsilon_n)\cos(\varepsilon_n z_D)}{Rp[\varepsilon_n+0.5\sin(2\varepsilon_n)]} \quad , \qquad \text{(A1.125)}$$

where

$$W_n = \frac{\exp[-2q_n(x_{LD}-x_D)]+1}{\exp[-2q_n(x_{LD}-1)]+1} \quad . \qquad \text{(A1.126)}$$

The q_n are defined by (A1.114) and ε_n are the roots of (A1.110).

Equation (A1.125) reduces to the Laplace transform solution previously derived (A1.85) for the step response for flow from a semi-infinite aquifer with no semipervious streambank if $x_{LD}\to\infty$ (that is, $W_n\to 1$) and $A = 0$ (that is, $R = 1$).

Equation (A1.125) is the Laplace transform solution for head at a point x_D, z_D in the aquifer. For the case of a partially penetrating observation well screened over the interval z_{D1} to z_{D2}, equation (A1.125) is integrated over the interval z_{D1} to z_{D2} to give the average head in a partially penetrating observation well ($\bar{h}_D^*$):

$$\bar{h}_D^* = \frac{1}{(z_{D2}-z_{D1})}\int_{z_{D1}}^{z_{D2}}\bar{h}_D dz_D$$

$$= \frac{2}{(z_{D2}-z_{D1})}\sum_{n=0}^{\infty}\frac{W_n\exp[-q_n(x_D-1)]\sin(\varepsilon_n)}{Rp[\varepsilon_n+0.5\sin(2\varepsilon_n)]}\int_{z_{D1}}^{z_{D2}}\cos(\varepsilon_n z_D)dz_D$$

$$= \frac{2}{(z_{D2}-z_{D1})}\sum_{n=0}^{\infty}\frac{W_n\exp[-q_n(x_D-1)]\sin(\varepsilon_n)[\sin(\varepsilon_n z_{D2})-\sin(\varepsilon_n z_{D1})]}{Rp\varepsilon_n[\varepsilon_n+0.5\sin(2\varepsilon_n)]} \quad . \qquad \text{(A1.127)}$$

For a fully penetrating observation well screened over the interval $z_{D1} = 0$ to $z_{D2} = 1$, let $z_{D1} = 0$ and $z_{D2} = 1$ and one obtains the average head in a fully penetrating observation well ($\hat{\bar{h}}_D$):

$$\hat{\bar{h}}_D = 2\sum_{n=0}^{\infty}\frac{W_n\exp[-q_n(x_D-1)]\sin^2(\varepsilon_n)}{Rp\ \varepsilon_n[\varepsilon_n+0.5\sin(2\varepsilon_n)]} \quad . \qquad \text{(A1.128)}$$

Dimensionless Seepage at Streambank Due to Step Change in Stream Stage

Dimensionless seepage at $x = x_0$ (the streambank) due to a unit-step change in stream stage is

$$\bar{Q}_D = -\frac{d\hat{\bar{h}}_D}{dx_D} \quad , \qquad \text{(A1.129)}$$

evaluated at $x_D = 1$.

Letting

$$r_1 = \frac{\sin^2(\varepsilon_n)}{Rp\varepsilon_n[\varepsilon_n + 0.5\sin(2\varepsilon_n)]}$$

and

$$r_2 = \{\exp[-2q_n(x_{LD} - 1)] + 1\} \quad ,$$

equation (A1.128) becomes

$$\hat{\bar{h}}_D = 2\sum_{n=0}^{\infty} \frac{r_1}{r_2}[\exp[-q_n(x_D - 1)]\{\exp[-2q_n(x_{LD} - x_D)] + 1\}]$$

$$= 2\sum_{n=0}^{\infty} \frac{r_1}{r_2}\{\exp[-q_n(x_D - 1) - 2q_n(x_{LD} - x_D)]\} + \frac{r_1}{r_2}\{\exp[-q_n(x_D - 1)]\} \quad . \qquad \text{(A1.130)}$$

Differentiating (A1.130) with respect to x_D gives

$$\frac{d\hat{\bar{h}}_D}{dx_D} = 2\sum_{n=0}^{\infty} \frac{r_1}{r_2}\{q_n \exp[-q_n(x_D - 1) - 2q_n(x_{LD} - x_D)] - q_n \exp[-q_n(x_D - 1)]\} \quad . \qquad \text{(A1.131)}$$

Now, evaluating (A1.131) at $x_D = 1$

$$\left.\frac{d\hat{\bar{h}}_D}{dx_D}\right|_{x_D = 1} = 2\sum_{n=0}^{\infty} \frac{r_1}{r_2} q_n\{\exp[-2q_n(x_{LD} - 1)] - 1\} \qquad \text{(A1.132)}$$

and

$$\bar{Q}_D = -2\sum_{n=0}^{\infty} \frac{r_1}{r_2} q_n\{\exp[-2q_n(x_{LD} - 1)] - 1\} \quad . \qquad \text{(A1.133)}$$

Substituting definitions of r_1 and r_2 into (A1.133) gives the solution for dimensionless seepage at the streambank

$$\bar{Q}_D = -2\sum_{n=0}^{\infty} \frac{q_n \sin^2(\varepsilon_n)}{Rp\ \varepsilon_n[\varepsilon_n + 0.5\sin(2\varepsilon_n)]}\left\{\frac{\exp[-2q_n(x_{LD} - 1)] - 1}{\exp[-2q_n(x_{LD} - 1)] + 1}\right\} \quad . \qquad \text{(A1.134)}$$

For conditions in which a semipervious streambank is absent, the factor R becomes unity.

ATTACHMENT 2.—DOCUMENTATION OF PROGRAMS STLK1 AND STWT1

PROGRAM STLK1

Narrative

Program STLK1 calculates ground-water levels, seepage rates, and bank-storage changes for arbitrarily changing stream-stage and/or recharge stresses applied to confined and leaky aquifers bounded by a fully penetrating stream. The program reads input data provided by the user and then calculates aquifer responses based on the convolution technique. A description of the numbered steps in the code follows:

Main program:

1. Call subroutine OFILE to open input (variable IN), result (variable IO), and plot (variable IP) files.
2. Call subroutine DATAIO to read input data from input file and write input data to result file.
3. Define dimensionless and other program parameters.
4. Call subroutine LINVST to calculate coefficients used for the Stehfest (1970) algorithm.
5. Initialize response and other arrays. These arrays hold the head, seepage, and bank-storage results.
6. Calculate the time rate of change of system stress.
7. Calculate heads, seepage, and bank storage using convolution. This block of instructions, up to step 9, uses the convolution technique to calculate system responses at each time step.
8. Call subroutine LTST1 to calculate dimensionless head (variable HD) and seepage (variable QD) for each value of dimensionless time (variable TD).
9. Write results of convolution for all time steps to result and plot files.
10. Close input, result, and plot files; stop program execution; and end.

Subroutine OFILE:

11. Subroutine OFILE opens input, result, and plot files.

Function LENCHR:

12. Function LENCHR calculates the length of a character string (code from R. S. Regan, USGS, written commun., 1997).

Subroutine DATAIO:

13. Subroutine DATAIO reads input data from the input file and writes this data to the result file.

Subroutine BANNER:

14. Subroutine BANNER writes a program banner to the result file.

Subroutine RDERR:

15. If an error is found in the input data in subroutine DATAIO, subroutine RDERR writes a message to the result file indicating the input line on which the error was found and then stops program execution.

Subroutine LINVST:

16. Subroutine LINVST calculates coefficients used for the Stehfest (1970) algorithm.

Subroutine LTST1:

17. Subroutine LTST1 calculates the Laplace transform solutions for heads and seepage for each dimensionless time (variable TD). A value of TD is passed to LTST1 from the Main program and values of head (variable HDT) and seepage (variable QTD) are returned to the Main program from LTST1.

List of Program Variables

Variable	Definition
AB	Saturated thickness of aquifer.
ABT	Saturated thickness of aquitard.
AK	Hydraulic conductivity of aquifer.
AKT	Hydraulic conductivity of aquitard.
AS	Specific storage of aquifer.
ASC	Storativity (storage coefficient of aquifer).
AST	Specific storage of aquitard.
ASYT	Specific yield of aquitard.
AT	Transmissivity of aquifer.
BANK	DIMENSION (IMAXX), Array of bank-storage volumes per unit stream length at each time step.
BANKV	DIMENSION (IMAXX), Array of bank-storage volumes at each time step.
C1, C2, C3, C4	Coefficients for head and seepage calculations in subroutine LTST1.
CA, CAQ	Coefficients for head and seepage calculations in subroutine LTST1.
DELT	Time-step size.
EXPMAX	Maximum allowable absolute value of exponential arguments.
FH	DIMENSION (IMAXX), Array of the time rate of change of system stresses for heads at each time step.
FQ	DIMENSION (IMAXX), Array of the time rate of change of system stresses for seepage at each time step.
FF	Coefficient for aquifer-width term for calculations of head and seepage in subroutine LTST1.
FI	Counter in subroutine LINVST.
FDEN, FNUM	Coefficients for aquifer-width term for calculations of head and seepage in subroutine LTST1.
G	DIMENSION (20), Array used in the calculation of Stehfest (1970) coefficients in subroutine LINVST.
GAMMA1	Dimensionless ratio of aquitard to aquifer hydraulic conductivity.
GAM1SQ	Square of dimensionless ratio of aquitard to aquifer hydraulic conductivity.
H	DIMENSION (IMAXX), Array of heads at the observation well.
HD	Dimensionless head at observation well at time TD in main program.
HDT	Dimensionless head at observation well at time TD in subroutine LTST1.
HINIT	Initial head at observation well at start of simulation.
HNET	DIMENSION(IMAXX), Array of total stress to system at each time step for calculations of head.
HS	DIMENSION (20), Array used in the calculation of Stehfest (1970) coefficients in subroutine LINVST.
I	Counter in main program and subroutines.
IAQ	Aquifer type: IAQ=0, confined aquifer; IAQ=1, leaky aquifer with constant head overlying aquitard; IAQ=2, leaky aquifer with impervious layer overlying aquitard; IAQ=3, leaky aquifer overlain by water-table aquitard.
ID	Counter in subroutine DATAIO.
IFNAME	Input file name.
ILINE	Integer line counter.
IMAXX	Maximum number of time steps.
IN	Input file unit number.
IO	Result file unit number.
IP	Plot file unit number.
IPLOT	Integer test as to whether or not to write results to plot file: IPLOT=0, results not written to plot file; IPLOT=1, results written to plot file.
IPRINT	Option for printing stress data to result file: IPRINT=0, stress data not printed; IPRINT=1, stress data printed.
IS	Counter in subroutine LINVST.

Variable	Definition
ISTRESS	Stress type: ISTRESS=0, stream-stage fluctuations; ISTRESS=1, recharge/ET; ISTRESS = 2, stream-stage fluctuations and recharge/ET.
IXA	Streambank code: IXA=0, semipervious streambank material absent; IXA=1, semipervious streambank material present.
IXL	Aquifer extent: IXL=0, semi-infinite aquifer; IXL=1, finite-width aquifer.
JT	Counter for upper limit of time in convolution integral.
KS, K1, K2	Counters in subroutine LINVST.
KT	Counter for time in convolution integral.
LENCHR	Length of character string.
MAX	Maximum length of character string.
N	Length of character string.
NH	One half the number of Stehfest (1970) terms.
NS	Number of Stehfest (1970) terms.
NT	Number of stress events (also equals number of time steps).
OFNAME	Result file name.
PDL, PDLQ	Coefficients for calculations of head and seepage in subroutine LTST1.
PFNAME	Plot file name.
PP	Laplace transform variable.
QD	Dimensionless seepage at streambank at time TD in main program and subroutine LTST1.
QNET	DIMENSION(IMAXX), Array of total net stress to system at each time step for calculations of seepage.
QTD	Dimensionless seepage at streambank at time TD in subroutine LTST1.
RECH	DIMENSION(IMAXX), Array of recharge amount at each time step.
RE0, RE0Q	Coefficients for calculations of head and seepage in subroutine LTST1.
SEEP	DIMENSION (IMAXX), Array of seepage per unit stream length at each time step.
SEEPT	DIMENSION (IMAXX), Array of seepage at each time step.
SIGMA1	Dimensionless ratio of aquitard to aquifer storativity.
SIGMAP	Dimensionless ratio of aquifer storativity to aquitard specific yield.
SN	Changes sign of alternating series for array V in subroutine LINVST.
SQRTM	Square root of ratio of aquitard to aquifer hydraulic properties multiplied by Laplace transform variable.
STAGE	DIMENSION (IMAXX), Array of stream-stage level at each time step.
STRING	A character string.
SUMH	Sum of head terms in convolution integral.
SUMQ	Sum of seepage terms in convolution integral.
T	Time.
TD	Dimensionless time.
TINIT	Simulation start time.
TITLE1	First line of title.
TITLE2	Second line of title.
V	DIMENSION (20), Array of Stehfest (1970) coefficients.
VAR	DIMENSION (11), Array of character data listing line-by-line input variables, in subroutine RDERR.
X	Distance to observation well from stream-channel center.
XAA	Streambank leakance.
XAAD	Dimensionless streambank leakance.
XD	Dimensionless distance to observation well.
XDEN	Denominator for seepage term for leaky aquifer overlain by water-table aquitard in subroutine LTST1.

Variable	Definition
XLL	Width of aquifer.
XLLD	Dimensionless width of aquifer.
XLN2	Logarithm of 2.
XM	Ratio of aquitard to aquifer hydraulic properties multiplied by Laplace transform variable.
XNUM	Numerator for seepage term for leaky aquifer overlain by water-table aquitard in subroutine LTST1.
XP, XPQ	Sum of head (XP) and seepage (XPQ) terms in Laplace transform solution.
XSTREAM	Length of stream reach.
XTIME	DIMENSION (IMAXX), Array of times for each stream stage or recharge event.
XZERO	Stream half width.
XZEROD	Dimensionless stream half width.
XZEROSQ	Square of stream half width.

PROGRAM STWT1

Narrative

Program STWT1 calculates ground-water levels, seepage rates, and bank-storage changes for arbitrarily changing stream-stage and/or recharge stresses applied to water-table (or confined) aquifers bounded by a fully penetrating stream. The program reads input data provided by the user and then calculates aquifer responses based on the convolution technique. A description of the numbered steps in the code follows:

Main program:

1. Call subroutine OFILE to open input (variable IN), result (variable IO), and plot (variable IP) files.
2. Call subroutine DATAIO to read input data from input file and write input data to result file.
3. Define dimensionless and other program parameters.
4. Call subroutine LINVST to calculate coefficients used for the Stehfest (1970) algorithm.
5. Initialize response and other arrays. These arrays hold the head, seepage, and bank-storage results.
6. Calculate the time rate of change of system stress.
7. Calculate heads, seepage, and bank storage using convolution. This block of instructions, up to step 9, uses the convolution technique to calculate system responses at each time step.

8. If aquifer is confined, call subroutine LTST1 to calculate dimensionless head (variable HD) and seepage (variable QD) for each value of dimensionless time (variable TD). If aquifer is unconfined, call subroutine LTST2 to calculate dimensionless head and seepage for each value of dimensionless time.
9. Write results of convolution for all time steps to result and plot files.
10. Close input, result, and plot files; stop program execution; and end.

Subroutine OFILE:

11. Subroutine OFILE opens input, result, and plot files.

Function LENCHR:

12. Function LENCHR calculates the length of a character string (code from R.S. Regan, USGS, written commun., 1997).

Subroutine DATAIO:

13. Subroutine DATAIO reads input data from the input file and writes this data to the result file.

Subroutine BANNER:

14. Subroutine BANNER writes a program banner to the result file.

Subroutine RDERR:

15. If an error is found in the input data in subroutine DATAIO, subroutine RDERR writes a message to the result file indicating the input line on which the error was found and then stops program execution.

Subroutine LINVST:

16. Subroutine LINVST calculates coefficients used for the Stehfest (1970) algorithm.

Subroutine LTST1:

17. Subroutine LTST1 calculates the Laplace transform solutions for heads and seepage for confined aquifers for each dimensionless time (variable TD). A value of TD is passed to LTST1 from the Main program and values of head (variable HDT) and seepage (variable QTD) are returned to the Main program from LTST1.

Subroutine LTST2:

18. Subroutine LTST2 calculates the Laplace transform solutions for heads and seepage for water-table aquifers for each dimensionless time (variable TD). A value of TD is passed to LTST2 from the Main program and values of head (variable HD) and seepage (variable QWT) are returned to the Main program from LTST2.

List of Program Variables

Many of the variables used in program STWT1 are equivalent in name and definition to those in program STLK1. For brevity, only those variables that differ in meaning from those of program STLK1 or are only used in program STWT1 are listed here. Definitions for variables in STWT1 that are not listed here can be found in the List of Variables for program STLK1.

Variable	Definition
A1, A2	Coefficients in Newton-Raphson technique in subroutine LTST2.
AKX	Horizontal hydraulic conductivity of aquifer.
ASY	Specific yield of aquifer.
BETA	Product of ratio of vertical to horizontal hydraulic conductivity of aquifer and square of ratio of distance to observation well to saturated thickness of aquifer.
BETA0	Product of ratio of vertical to horizontal hydraulic conductivity of aquifer and square of dimensionless stream half-width.
EPS, EPS0	Current (EPS) and previous (EPS0) values of epsilon in subroutine LTST2.
ERR, ERRQ	Relative error terms for infinite summations of head (ERR) and seepage (ERRQ) in subroutine LTST2.
F, FP	Function (F) and first derivative of function (FP) in Newton-Raphson technique in subroutine LTST2.
FDEN, FNUM	Coefficients for aquifer-width term for calculations of head and seepage in subroutine LTST1 and for head only in subroutine LTST2.
FDENQ, FNUMQ	Coefficients for seepage calculations in subroutine LTST2.
FN	Coefficient for aquifer-width term for calculations of head in subroutine LTST2.
FNQ	Coefficient for seepage term in subroutine LTST2.
HWT	Dimensionless head at observation well at time TD in subroutine LTST2.
IAQ	Aquifer type: IAQ=0, confined aquifer; IAQ=1, water-table aquifer.
IOWS	Type of observation well: IOWS=0, partially penetrating observation well; IOWS=1, fully penetrating observation well; IOWS=2, observation piezometer.
NN	Maximum (integer) number of terms for finite summations in Laplace transform solutions for head and seepage in subroutine LTST2.
NNR	Upper bound on number of Newton-Raphson iterations (100) in subroutine LTST2.
PDL, PDLQ	Coefficients for calculations of head and seepage in subroutines LTST1 and LTST2.

Variable	Definition
PI	The number pi.
QN, QNXD	Terms in the head and seepage Laplace transform solutions in subroutine LTST2.
QWT	Dimensionless seepage at streambank at time TD in subroutine LTST2.
RE0, RE0Q	Coefficients for calculations of head (RE0) and seepage (RE0Q) in subroutines LTST1 and LTST2.
RERRNR	Relative error for Newton-Raphson iteration and summation in subroutine LTST2.
RHS	Right-hand side of function in Newton-Raphson technique in subroutine LTST2.
SIGMA	Ratio of aquifer storativity to specific yield.
SINE	Sine of variable epsilon in subroutine LTST2.
SUMM, SUMT	Summation terms for head in Laplace transform solution in subroutine LTST2.
SUMMQ, SUMTQ	Summation terms for seepage in Laplace transform solution in subroutine LTST2.
XAD	Product of dimensionless streambank leakance and hyperbolic tangent function in Laplace transform solutions for head and seepage in subroutine LTST2.
XDEN	Term in the head and seepage Laplace transform solutions in subroutine LTST2.
XKD	Ratio of vertical to horizontal hydraulic conductivity of aquifer.
XN	Counter for finite summations in Laplace transform solutions for head and seepage in subroutine LTST2.
XN_MAX	Maximum (real) number of terms for finite summations in Laplace transform solutions for head and seepage in subroutine LTST2.
XNUM, XNUMQ	Terms in the head (XNUM) and seepage (XNUMQ) Laplace transform solutions in subroutine LTST2.
XTRMS	Factor used to determine number of terms in the finite summations for Laplace transform solutions for head and seepage in subroutine LTST2.
ZP	Vertical distance from bottom of aquifer to observation piezometer.
Z1	Vertical distance from bottom of aquifer to bottom of screened interval of observation well.
Z2	Vertical distance from bottom of aquifer to top of screened interval of observation well.
ZD	Dimensionless position of observation piezometer.
ZD1	Dimensionless position of bottom of screened interval of observation well.
ZD2	Dimensionless position of top of screened interval of observation well.