ATTACHMENT 1.—DERIVATION OF ANALYTICAL SOLUTIONS

STEP RESPONSE FOR FLOW FROM A SEMI-INFINITE CONFINED OR LEAKY AQUIFER

The following is a derivation of the analytical solutions for flow to a fully penetrating stream from a semi-infinite confined or leaky aquifer.

Head Distribution Due to a Step Change in Stream Stage

The governing differential equation of ground-water flow in the aquifer is

$$\frac{\partial^2 h}{\partial x^2} = \frac{S_s}{K_x}\frac{\partial h}{\partial t} + q' \qquad x_0 \le x \quad , \tag{A1.1}$$

where $q' = -\frac{K'}{K_x b}\left(\frac{\partial h'}{\partial z}\right)_{z=b}$.

In equation (A1.1), q' is equal to zero for a confined aquifer. The aquifer is of constant thickness b and is underlain by an impermeable base.

Initial and boundary conditions for the aquifer are

$$h(x, 0) = h_i \tag{A1.2}$$

$$h(\infty, t) = h_i \tag{A1.3}$$

$$h(x_0, t) = h_0 \quad . \tag{A1.4}$$

The governing differential equation of ground-water flow in the aquitard is

$$\frac{\partial^2 h'}{\partial z^2} = \frac{S_s'}{K'}\frac{\partial h'}{\partial t} \qquad b \le z \le b + b' \quad . \tag{A1.5}$$

Initial and boundary conditions for the aquitard are

$$h'(z, 0) = h_i \qquad b \le z \le b + b' \tag{A1.6}$$

$$h'(b, t) = h \quad . \tag{A1.7}$$

The boundary condition at the top of the aquitard for the condition of a constant head overlying the aquitard (case 1) is

$$h'(b + b', t) = h_i \quad ; \tag{A1.8a}$$

for the condition of an impermeable layer overlying the aquitard (case 2), the boundary condition is

$$\frac{\partial h'}{\partial z}(b + b', t) = 0 \quad ; \tag{A1.8b}$$

and for the condition of a water-table aquitard (case 3) the boundary condition is

$$\frac{\partial h'}{\partial z}(b + b', t) = -\frac{S_y'}{K'}\frac{\partial h'}{\partial t}(b + b', t) \quad . \tag{A1.8c}$$

Substituting the dimensionless variables listed in table 1 into equations (A1.1) through (A1.8) results in the following dimensionless boundary-value problem. For the aquifer, the governing equation is

$$\frac{\partial^2 h_D}{\partial x_D^2} = \frac{\partial h_D}{\partial t_D} - \gamma_1^2 \left.\frac{\partial h'_D}{\partial z'_D}\right|_{z_D = 0} \qquad 1 \le x_D < \infty \quad . \tag{A1.9}$$

Initial and boundary conditions are

$$h_D(x_D, 0) = 0 \tag{A1.10}$$

$$h_D(\infty, t_D) = 0 \tag{A1.11}$$

$$h_D(1, t_D) = 1 \quad . \tag{A1.12}$$

For the aquitard, the governing equation is

$$\frac{\partial^2 h'_D}{\partial z'^2_D} = \frac{\sigma_1}{\gamma_1^2}\frac{\partial h'_D}{\partial t_D} \qquad 0 \le z'_D \le 1 \quad , \tag{A1.13}$$

with initial and boundary conditions

$$h'_D(z'_D, 0) = 0 \tag{A1.14}$$

$$h'_D(0, t_D) = h_D \tag{A1.15}$$

$$\left\{\begin{array}{lll} h'_D(1, t_D) = 0 & \text{constant head overlying the aquitard (case 1)} & \text{(A1.16a)} \\[2em] \dfrac{\partial h'_D}{\partial z'_D}(1, t_D) = 0 & \text{impermeable layer overlying the aquitard (case 2)} & \text{(A1.16b)} \\[2em] \dfrac{\partial h'_D}{\partial z'_D}(1, t_D) = -\dfrac{1}{\sigma'\gamma_1^2}\dfrac{\partial h'_D}{\partial t_D} & \text{water-table aquitard (case 3)} & \text{(A1.16c)} \end{array}\right.$$

After performing Laplace transformations, the subsidiary boundary-value problem for the aquifer is

$$\frac{\partial^2 \bar{h}_D}{\partial x_D^2} = p\bar{h}_D - \gamma_1^2 \left.\frac{\partial \bar{h}_D'}{\partial z_D'}\right|_{z_D = 0} \qquad 1 \le x_D < \infty \quad , \tag{A1.17}$$

with boundary conditions

$$\bar{h}_D(\infty, p) = 0 \tag{A1.18}$$

$$\bar{h}_D(1, p) = \frac{1}{p} \quad . \tag{A1.19}$$

The subsidiary boundary-value problem for the aquitard is

$$\frac{\partial^2 \bar{h}_D'}{\partial z_D'^2} = m\bar{h}_D' \qquad 0 \le z_D' \le 1 \quad , \tag{A1.20}$$

where $m = \dfrac{\sigma_1 p}{\gamma_1^2}$.

Boundary conditions are

$$\bar{h}_D'(0, p) = \bar{h}_D \tag{A1.21}$$

$$\bar{h}_D'(1, p) = 0 \qquad \text{constant head overlying the aquitard (case 1)} \tag{A1.22a}$$

$$\frac{\partial \bar{h}_D'}{\partial z_D'}(1, p) = 0 \qquad \text{impermeable layer overlying the aquitard (case 2)} \tag{A1.22b}$$

$$\frac{\partial \bar{h}_D'}{\partial z_D'}(1, p) = -\frac{\bar{h}_D' p}{\sigma' \gamma_1^2} \qquad \text{water-table aquitard (case 3)} \tag{A1.22c}$$

A solution to (A1.20) is

$$\bar{h}'_D = A\cosh(z'_D\sqrt{m}) + B\sinh(z'_D\sqrt{m}) \quad . \tag{A1.23}$$

Applying (A1.21) at $z'_D = 0$ gives

$$A = \bar{h}_D \quad . \tag{A1.24}$$

Applying (A1.22a) at $z'_D = 1$ gives for case 1

$$B = -\frac{\bar{h}_D}{\tanh(\sqrt{m})} \quad ; \tag{A1.25a}$$

or, applying (A1.22b) at $z'_D = 1$ gives for case 2

$$B = \frac{-\bar{h}_D}{\coth(\sqrt{m})} \quad ; \tag{A1.25b}$$

or, applying (A1.22c) at $z'_D = 1$ gives for case 3

$$A\sqrt{m}\sinh(\sqrt{m}) + B\sqrt{m}\cosh(\sqrt{m}) = -p\frac{[A\cosh(\sqrt{m}) + B\sinh(\sqrt{m})]}{\sigma'\gamma_1^2} \quad ,$$

which leads to

$$A = -B\frac{[\sqrt{m}(\sigma'\gamma_1^2)\cosh(\sqrt{m}) + p\sinh(\sqrt{m})]}{[\sqrt{m}(\sigma'\gamma_1^2)\sinh(\sqrt{m}) + p\cosh(\sqrt{m})]} \quad . \tag{A1.25c}$$

Hence, for case 1 from (A1.24) and (A1.25a),

$$\bar{h}'_D = \bar{h}_D\cosh(z'_D\sqrt{m}) - \bar{h}_D\frac{\sinh(z'_D\sqrt{m})}{\tanh(\sqrt{m})} = \bar{h}_D\sinh[\sqrt{m}\,(1 - z'_D)]/\sinh(\sqrt{m}) \quad ; \tag{A1.26a}$$

or, for case 2 from (A1.24) and (A1.25b),

$$\bar{h}'_D = \bar{h}_D\cosh[\sqrt{m}\,(1 - z'_D)]/\cosh(\sqrt{m}) \quad ; \tag{A1.26b}$$

or, for case 3 from (A1.24) and (A1.25c),

$$\bar{h}'_D = \bar{h}_D\cosh(z'_D\sqrt{m}\,) - \bar{h}_D\sinh(z'_D\sqrt{m}\,)\frac{[\sqrt{m}(\sigma'\gamma_1^2)\sinh(\sqrt{m}) + p\cosh(\sqrt{m})]}{[\sqrt{m}(\sigma'\gamma_1^2)\cosh(\sqrt{m}) + p\sinh(\sqrt{m})]} \quad . \tag{A1.26c}$$

Substituting (A1.26a) into (A1.17) yields for case 1

$$\frac{\partial^2 \bar{h}_D}{\partial x_D^2} = p\bar{h}_D + \gamma_1^2 \sqrt{m}\ \bar{h}_D \coth(\sqrt{m})$$

$$= p\bar{h}_D + \bar{q}_D \bar{h}_D \quad , \tag{A1.27a}$$

where $\bar{q}_D = \gamma_1^2 \sqrt{m} \coth(\sqrt{m})$.

Substituting (A1.26b) into (A1.17) yields for case 2

$$\frac{\partial^2 \bar{h}_D}{\partial x_D^2} = p\bar{h}_D + \gamma_1^2 \sqrt{m}\ \bar{h}_D \tanh(\sqrt{m})$$

$$= p\bar{h}_D + \bar{q}_D , \tag{A1.27b}$$

where $\bar{q}_D = \gamma_1^2 \sqrt{m} \tanh(\sqrt{m})$.

Substituting (A1.26c) into (A1.17) yields for case 3

$$\frac{\partial^2 \bar{h}_D}{\partial x_D^2} = p\bar{h}_D + \gamma_1^2 \bar{h}_D \sqrt{m} \frac{[\sqrt{m}(\sigma'\gamma_1^2)\sinh(\sqrt{m}) + p\cosh(\sqrt{m})]}{[\sqrt{m}(\sigma'\gamma_1^2)\cosh(\sqrt{m}) + p\sinh(\sqrt{m})]}$$

$$= p\bar{h}_D + \bar{q}_D \bar{h}_D \quad , \tag{A1.27c}$$

where $\bar{q}_D = \gamma_1^2 \sqrt{m} \dfrac{[\sqrt{m}(\sigma'\gamma_1^2)\tanh(\sqrt{m}) + p]}{[\sqrt{m}(\sigma'\gamma_1^2) + p\tanh(\sqrt{m})]}$.

Now, the subsidiary boundary-value problem for the aquifer becomes

$$\frac{\partial^2 \bar{h}_D}{\partial x_D^2} = \bar{h}_D(p + \bar{q}_D) \quad , \tag{A1.28}$$

with boundary conditions

$$\bar{h}_D(\infty, p) = 0 \tag{A1.29}$$

$$\bar{h}_D(1, p) = \frac{1}{p} \quad . \tag{A1.30}$$

A solution to (A1.28) is

$$\bar{h}_D = C\exp(x_D\sqrt{p+\bar{q}_D}) + D\exp(-x_D\sqrt{p+\bar{q}_D}) \quad . \tag{A1.31}$$

Because of (A1.29), $C = 0$ and because of (A1.30), $D = \frac{1}{p}\exp(\sqrt{p+\bar{q}_D})$. Thus, (A1.31) becomes,

$$\bar{h}_D = \frac{1}{p}\exp[-\sqrt{p+\bar{q}_D}\,(x_D - 1)] \quad , \tag{A1.32}$$

where for a confined aquifer

$$\bar{q}_D = 0 \quad ;$$

for a leaky aquifer with constant head overlying the aquitard (case 1)

$$\bar{q}_D = \gamma_1^2\sqrt{m}\coth(\sqrt{m}) \quad ;$$

for a leaky aquifer with impermeable layer overlying the aquitard (case 2)

$$\bar{q}_D = \gamma_1^2\sqrt{m}\tanh(\sqrt{m}) \quad ;$$

and for a leaky aquifer overlain by a water-table aquitard (case 3)

$$\bar{q}_D = \gamma_1^2\sqrt{m}\frac{[\sqrt{m}(\sigma'\gamma_1^2)\tanh(\sqrt{m}) + p]}{[\sqrt{m}(\sigma'\gamma_1^2) + p\tanh(\sqrt{m})]} \quad .$$

Dimensionless Seepage at Streambank Due to Step Change in Stream Stage

Dimensionless seepage at $x = x_0$ (the streambank) due to a unit-step change in stream stage is

$$\bar{Q}_D = -\frac{d\bar{h}_D}{dx_D} \quad , \tag{A1.33}$$

evaluated at $x_D = 1$. Differentiating (A1.32) with respect to x_D gives

$$\frac{d\bar{h}_D}{dx_D} = -\frac{\sqrt{p+\bar{q}_D}}{p}\exp[-\sqrt{p+\bar{q}_D}(x_D - 1)] \quad . \tag{A1.34}$$

Now, evaluating (A1.34) at $x_D = 1$, (A1.33) becomes

$$\bar{Q}_D = \frac{\sqrt{p+\bar{q}_D}}{p} \quad . \tag{A1.35}$$

STEP RESPONSE FOR FLOW FROM A FINITE-WIDTH CONFINED OR LEAKY AQUIFER WITH A SEMIPERVIOUS STREAMBANK

The following is a derivation of the analytical solutions for flow to a fully penetrating stream with a semipervious streambank from a finite-width confined or leaky aquifer.

Head Distribution Due to a Step Change in Stream Stage

The governing differential equation of ground-water flow in the aquifer is

$$\frac{\partial^2 h}{\partial x^2} = \frac{S_s}{K_x}\frac{\partial h}{\partial t} + q' \qquad \mathrm{x}_0 \le x \le x_L \quad , \tag{A1.36}$$

where $q' = -\frac{K'}{K_x b}\left(\frac{\partial h'}{\partial z}\right)_{z=b}$. For a confined aquifer, q' is equal to zero. The aquifer is of constant thickness b and is underlain by an impermeable base.

Initial and boundary conditions for the aquifer are

$$h(x, 0) = h_i \tag{A1.37}$$

$$\frac{\partial h}{\partial x}(x_L, t) = 0 \tag{A1.38}$$

$$\frac{\partial h}{\partial x}(x_0, t) = \frac{1}{a}[h_0 - h(x_0, t)] \quad . \tag{A1.39}$$

In the same way as derived for a semi-infinite aquifer, the dimensionless subsidiary boundary value problem for the finite-width aquifer becomes

$$\frac{\partial^2 \bar{h}_D}{\partial x_D^2} = \bar{h}_D(p + \bar{q}_D) \quad , \tag{A1.40}$$

with boundary conditions

$$\frac{\partial \bar{h}_D}{\partial x_D}(x_{LD}, t_D) = 0 \tag{A1.41}$$

$$\frac{\partial \bar{h}_D}{\partial x_D}(1, p) = \frac{1}{A}\left(\bar{h}_D - \frac{1}{p}\right) \tag{A1.42}$$

and where $\bar{q}_D$ is defined for case 1, case 2, and case 3 following equation (A1.32).

A solution to (A1.40) is

$$\bar{h}_D = C\exp(x_D\sqrt{p + \bar{q}_D}) + D\exp(-x_D\sqrt{p + \bar{q}_D}) \quad . \tag{A1.43}$$

Applying boundary condition (A1.41) to (A1.43) yields

$$\frac{\partial \bar{h}_D}{\partial x_D} = C\sqrt{p + \bar{q}_D}\exp(x_{LD}\sqrt{p + \bar{q}_D}) - D\sqrt{p + \bar{q}_D}\exp(-x_{LD}\sqrt{p + \bar{q}_D}) = 0 \quad . \tag{A1.44}$$

Let $r_1 = \sqrt{p + \bar{q}_D}$. Then, from (A1.44)

$$C = D\exp(-2r_1 x_{LD}) \quad . \tag{A1.45}$$

Substituting (A1.45) into (A1.43) gives

$$\bar{h}_D = D\exp(-r_1 x_D)\{\exp[-2r_1(x_{LD} - x_D)] + 1\} \quad . \tag{A1.46}$$

Applying (A1.42) to (A1.46) yields

$$D = \frac{\exp(r_1)}{p[Ar_1\{1 - \exp[-2r_1(x_{LD} - 1)]\} + \exp[-2r_1(x_{LD} - 1)] + 1]}$$

$$= \frac{\exp[r_1]}{p\left\{Ar_1\left[\frac{1 - \exp[-2r_1(x_{LD} - 1)]}{1 + \exp[-2r_1(x_{LD} - 1)]}\right] + 1\right\}\{1 + \exp[-2r_1(x_{LD} - 1)]\}} \quad .$$

The Laplace transform solution for head in a finite-width leaky aquifer with a semipervious streambank is obtained upon substitution of D into (A1.46). Thus,

$$\bar{h}_D = \frac{\exp[-r_1(x_D - 1)]}{p\{1 + Ar_1\tanh[r_1(x_{LD} - 1)]\}}\left\{\frac{\exp[-2r_1(x_{LD} - x_D)] + 1}{\exp[-2r_1(x_{LD} - 1)] + 1}\right\} \quad ; \tag{A1.47}$$

or, substituting for r_1

$$\bar{h}_D = \frac{W\exp[-\sqrt{p + \bar{q}_D}(x_D - 1)]}{p\{1 + \sqrt{p + \bar{q}_D}A\tanh[\sqrt{p + \bar{q}_D}(x_{LD} - 1)]\}} \quad , \tag{A1.48}$$

where

$$W = \frac{\exp[-2\sqrt{p + \bar{q}_D}(x_{LD} - x_D)] + 1}{\exp[-2\sqrt{p + \bar{q}_D}(x_{LD} - 1)] + 1} \quad , \tag{A1.48a}$$

and $\bar{q}_D$ is defined following equation (A1.32).

Dimensionless Seepage at Streambank Due to Step Change in Stream Stage

Dimensionless seepage at $x = x_0$ (the streambank) due to a unit-step change in stream stage is

$$\bar{Q}_D = -\frac{d\bar{h}_D}{dx_D} \quad , \tag{A1.49}$$

evaluated at $x_D = 1$.

Letting $r_1 = \sqrt{p + \bar{q}_D}$, $r_2 = \{\exp[-2r_1(x_{LD} - 1)] + 1\}$, and $r_3 = \{1 + r_1 A \tanh[r_1(x_{LD} - 1)]\}$, equation (A1.47) becomes

$$\bar{h}_D = \frac{\{\exp[-2r_1(x_{LD} - x_D)] + 1\}}{pr_2r_3}\{\exp[-r_1(x_D - 1)]\}$$

$$= \frac{1}{pr_2r_3}\{\exp[-2r_1(x_{LD} - x_D) - r_1(x_D - 1)] + \exp[-r_1(x_D - 1)]\} \quad . \tag{A1.50}$$

Differentiating (A1.50) with respect to x_D gives

$$\frac{d\bar{h}_D}{dx_D} = \frac{1}{pr_2r_3}\{r_1 \exp[-2r_1(x_{LD} - x_D) - r_1(x_D - 1)] - r_1 \exp[-r_1(x_D - 1)]\} \quad . \tag{A1.51}$$

Now, evaluating (A1.51) at $x_D = 1$ gives

$$\left.\frac{d\bar{h}_D}{dx_D}\right|_{x_D = 1} = \frac{r_1}{pr_2r_3}\{\exp[-2r_1(x_{LD} - 1)] - 1\} \tag{A1.52}$$

and (A1.49) becomes

$$\bar{Q}_D = \frac{-\sqrt{p + \bar{q}_D}}{p\{1 + \sqrt{p + \bar{q}_D}A \tanh[\sqrt{p + \bar{q}_D}(x_{LD} - 1)]\}}\left\{\frac{\exp[-2\sqrt{p + \bar{q}_D}(x_{LD} - 1)] - 1}{\exp[-2\sqrt{p + \bar{q}_D}(x_{LD} - 1)] + 1}\right\} \quad . \tag{A1.53}$$

The solutions reduce to semi-infinite aquifer solutions and/or solutions without a semipervious streambank with appropriate substitution of $x_{LD} \to \infty$ and/or $A = 0$.

STEP RESPONSE FOR FLOW FROM A SEMI-INFINITE WATER-TABLE AQUIFER

The following is a derivation of the analytical solutions for flow to a fully penetrating stream from a semi-infinite water-table aquifer.

Head Distribution Due to a Step Change in Stream Stage

The governing differential equation of ground-water flow in the aquifer is

$$\frac{\partial^2 h}{\partial x^2} + \frac{K_z}{K_x}\frac{\partial^2 h}{\partial z^2} = \frac{S_s}{K_x}\frac{\partial h}{\partial t} \qquad \begin{cases} x_0 \le x < \infty \\ 0 \le z \le b \end{cases} . \tag{A1.54}$$

Initial and boundary conditions for the aquifer are

$$h(x, z, 0) = h_i \tag{A1.55}$$

$$\frac{\partial h}{\partial z}(x, b, t) = -\frac{S_y}{K_z}\frac{\partial h}{\partial t} \tag{A1.56}$$

$$\frac{\partial h}{\partial z}(x, 0, t) = 0 \tag{A1.57}$$

$$h(\infty, z, t) = h_i \tag{A1.58}$$

$$h(x_0, t) = h_0 \tag{A1.59}$$

Substituting the dimensionless variables in table 2 into equations (A1.54) through (A1.59) results in the following dimensionless boundary-value problem. The governing equation is

$$\frac{\partial^2 h_D}{\partial x_D^2} + \beta_0 \frac{\partial^2 h_D}{\partial z_D^2} = \frac{\partial h_D}{\partial t_D} \qquad \begin{cases} 1 \le x_D < \infty \\ 0 \le z_D \le 1 \end{cases} . \tag{A1.60}$$

Initial and boundary conditions are

$$h_D(x_D, z_D, 0) = 0 \tag{A1.61}$$

$$\frac{\partial h_D}{\partial z_D}(x_D, 1, t_D) = -\frac{1}{\sigma\beta_0}\frac{\partial h_D}{\partial t_D} \tag{A1.62}$$

$$\frac{\partial h_D}{\partial z_D}(x_D, 0, t_D) = 0 \tag{A1.63}$$

$$h_D(\infty, z_D, t_D) = 0 \tag{A1.64}$$

$$h_D(1, t_D) = 1 \tag{A1.65}$$

After performing Laplace transformations, the subsidiary boundary-value problem for the aquifer is

$$\frac{\partial^2 \bar{h}_D}{\partial x_D^2} + \beta_0 \frac{\partial^2 \bar{h}_D}{\partial z_D^2} = p\bar{h}_D \qquad \begin{cases} 1 \le x_D < \infty \\ 0 \le z_D \le 1 \end{cases} \tag{A1.66}$$

with boundary conditions

$$\frac{\partial \bar{h}_D}{\partial z_D}(x_D, 1) = -\frac{p\bar{h}_D}{\sigma\beta_0} \tag{A1.67}$$

$$\frac{\partial \bar{h}_D}{\partial z_D}(x_D, 0) = 0 \tag{A1.68}$$

$$\bar{h}_D(\infty, z_D) = 0 \tag{A1.69}$$

$$\bar{h}_D(1, p) = \frac{1}{p} \quad . \tag{A1.70}$$

A solution to (A1.66) that satisfies (A1.67) and (A1.68) is

$$\bar{h}_D = \sum_{n=0}^{\infty} \bar{g}_n(x_D, p)\cos(\varepsilon_n z_D) \quad , \tag{A1.71}$$

where $n = 0, 1, 2, \ldots$ and ε_n are the roots of

$$\varepsilon_n \tan(\varepsilon_n) = \frac{p}{\sigma\beta_0} \quad . \tag{A1.72}$$

Substitution of (A1.71) into (A1.66) yields

$$\sum_{n=0}^{\infty} [\bar{g}_n'' - (\varepsilon_n{}^2\beta_0 + p)\bar{g}_n] \cos(\varepsilon_n z_D) = 0 \quad . \tag{A1.73}$$

Hence, g_n must satisfy

$$\bar{g}_n'' - (\varepsilon_n{}^2\beta_0 + p)\bar{g}_n = 0 \quad , \tag{A1.74}$$

the solution of which can be written as

$$\bar{g}_n = A_n \exp(q_n x_D) + B_n \exp(-q_n x_D) \quad , \tag{A1.75}$$

where

$$q_n = (\varepsilon_n{}^2\beta_0 + p)^{\frac{1}{2}} \quad . \tag{A1.76}$$

Because of boundary condition (A1.69), $A_n = 0$. Hence,

$$\bar{g}_n = B_n \exp(-q_n x_D) \quad . \tag{A1.77}$$

Substitution of (A1.77) into (A1.71) gives

$$\bar{h}_D = \sum_{n=0}^{\infty} B_n \exp(-q_n x_D) \cos(\varepsilon_n z_D) \quad . \tag{A1.78}$$

The coefficients B_n can be obtained as follows: Apply boundary condition (A1.70) to obtain

$$\sum_{n=0}^{\infty} B_n \exp(-q_n) \cos(\varepsilon_n z_D) = \frac{1}{p} \quad . \tag{A1.79}$$

Multiply both sides of (A1.79) by $\cos(\varepsilon_m z_D)$, where m is an integer. Then

$$\sum_{n=0}^{\infty} B_n \exp(-q_n) \cos(\varepsilon_n z_D) \cos(\varepsilon_m z_D) = \frac{1}{p} \cos(\varepsilon_m z_D) \quad . \tag{A1.80}$$

It is now possible to take advantage of the orthogonality of the set $\{\cos(\varepsilon_n z_D)\}$ by integrating over the interval $z_D = 0$ to $z_D = 1$,

$$\sum_{n=0}^{\infty} B_n \exp(-q_n) \int_0^1 \cos(\varepsilon_n z_D) \cos(\varepsilon_m z_D) dz_D = \frac{1}{p} \int_0^1 \cos(\varepsilon_m z_D) dz_D \quad . \tag{A1.81}$$

By use of trigonometric relations in combination with (A1.72), all terms on the left hand side of (A1.81) can be shown by direct integration to be zero except those for which $n = m$. (For a discussion on the topic of orthogonality, see Hildebrand, 1976.) Hence,

$$B_n \exp(-q_n) \int_0^1 \cos^2(\varepsilon_n z_D) dz_D = \frac{1}{p} \int_0^1 \cos(\varepsilon_n z_D) dz_D \quad , \tag{A1.82}$$

or

$$B_n \exp(-q_n) \left[0.5 + \frac{\sin(2\varepsilon_n)}{4\varepsilon_n} \right] = \frac{\sin(\varepsilon_n)}{p\varepsilon_n} \quad . \tag{A1.83}$$

Then,

$$B_n = \frac{2\exp(q_n)\sin(\varepsilon_n)}{p[\varepsilon_n + 0.5\sin(2\varepsilon_n)]} \quad . \tag{A1.84}$$

Substitution of (A1.84) into (A1.78) yields

$$\bar{h}_D = 2\sum_{n=0}^{\infty}\frac{\exp[-q_n(x_D-1)]\sin(\varepsilon_n)\cos(\varepsilon_n z_D)}{p[\varepsilon_n + 0.5\sin(2\varepsilon_n)]} \quad , \tag{A1.85}$$

where q_n are defined by (A1.76) and ε_n are the roots of equation (A1.72).

Equation (A1.85) is the Laplace transform solution for head at a point x_D, z_D in the aquifer. For the case of a partially penetrating observation well, screened over the interval z_{D1} to z_{D2}, equation (A1.85) is integrated over the interval z_{D1} to z_{D2} to give the average head in a partially penetrating observation well ($\bar{h}_D^*$):

$$\bar{h}_D^* = \frac{1}{(z_{D2}-z_{D1})}\int_{z_{D1}}^{z_{D2}} \bar{h}_D dz_D$$

$$= \frac{2}{(z_{D2}-z_{D1})}\sum_{n=0}^{\infty}\frac{\exp[-q_n(x_D-1)]\sin(\varepsilon_n)}{p[\varepsilon_n+0.5\sin(2\varepsilon_n)]}\int_{z_{D1}}^{z_{D2}}\cos(\varepsilon_n z_D)dz_D$$

$$= \frac{2}{(z_{D2}-z_{D1})}\sum_{n=0}^{\infty}\frac{\exp[-q_n(x_D-1)]\sin(\varepsilon_n)[\sin(\varepsilon_n z_{D2})-\sin(\varepsilon_n z_{D1})]}{p\ \varepsilon_n[\varepsilon_n+0.5\sin(2\varepsilon_n)]} \quad . \tag{A1.86}$$

By setting $z_{D1} = 0$ and $z_{D2} = 1$, one obtains the average head in a fully penetrating observation well ($\hat{\bar{h}}_D$):

$$\hat{\bar{h}}_D = 2\sum_{n=0}^{\infty}\frac{\exp[-q_n(x_D-1)]\sin^2(\varepsilon_n)}{p\ \varepsilon_n[\varepsilon_n+0.5\sin(2\varepsilon_n)]} \quad . \tag{A1.87}$$

Note that if specific yield becomes zero, $\sigma \to \infty$, and, from (A1.72), $\varepsilon_n \to n\pi$. Equation (A1.87) thus becomes zero for all terms other than that for which $n = 0$ (that is, $n = 1, 2, 3, \ldots$). For $n = 0$, one can take note of the fact that

$$\lim_{u\to 0}\frac{\sin u}{u} \to 1 \quad .$$

Also, for $n = 0$, and from (A1.76), $q_n = \sqrt{p}$. Hence, (A1.87) becomes

$$\hat{\bar{h}}_D = \frac{\exp[-\sqrt{p}(x_D-1)]}{p} \quad , \tag{A1.88}$$

which is the solution for a confined aquifer equal to (A1.32), with $\bar{q}_D = 0$.

Dimensionless Seepage at Streambank Due to Step Change in Stream Stage

Dimensionless seepage at $x = x_0$ (the streambank) due to a unit-step change in stream stage is

$$\bar{Q}_D = -\frac{d\hat{\bar{h}}_D}{dx_D} \quad , \tag{A1.89}$$

evaluated at $x_D = 1$. Differentiating (A1.87) with respect to x_D gives

$$\frac{d\hat{\bar{h}}_D}{dx_D} = -2\sum_{n=0}^{\infty} \frac{q_n \exp[-q_n(x_D - 1)]\sin^2(\varepsilon_n)}{p\ \varepsilon_n[\varepsilon_n + 0.5\sin(2\varepsilon_n)]} \quad . \tag{A1.90}$$

Now, evaluating (A1.90) at $x_D = 1$, (A1.89) becomes

$$\bar{Q}_D = 2\sum_{n=0}^{\infty} \frac{q_n \sin^2(\varepsilon_n)}{p\ \varepsilon_n[\varepsilon_n + 0.5\sin(2\varepsilon_n)]} \quad . \tag{A1.91}$$

STEP RESPONSE FOR FLOW FROM A FINITE-WIDTH WATER-TABLE AQUIFER WITH A SEMIPERVIOUS STREAMBANK

The following is a derivation of the analytical solutions for flow to a fully penetrating stream with a semipervious streambank from a finite-width water-table aquifer.

Head Distribution Due to a Step Change in Stream Stage

The governing differential equation of ground-water flow in the aquifer is

$$\frac{\partial^2 h}{\partial x^2} + \frac{K_z}{K_x}\frac{\partial^2 h}{\partial z^2} = \frac{S_s}{K_x}\frac{\partial h}{\partial t} \qquad \begin{cases} x_0 \le x < x_L \\ 0 \le z \le b \end{cases} \quad . \tag{A1.92}$$

Initial and boundary conditions for the aquifer are

$$h(x, z, 0) = h_i \tag{A1.93}$$

$$\frac{\partial h}{\partial z}(x, b, t) = -\frac{S_y}{K_z}\frac{\partial h}{\partial t} \tag{A1.94}$$

$$\frac{\partial h}{\partial z}(x, 0, t) = 0 \tag{A1.95}$$

$$\frac{\partial h}{\partial x}(x_L, z, t) = 0 \tag{A1.96}$$

$$\frac{\partial h}{\partial x}(x_0, t) = \frac{1}{a}[h_0 - h(x_0, t)] \quad . \tag{A1.97}$$

Substituting the dimensionless variables in table 2 into equations (A1.92) through (A1.97) results in the following dimensionless boundary-value problem. The governing equation is

$$\frac{\partial^2 h_D}{\partial x_D^2} + \beta_0 \frac{\partial^2 h_D}{\partial z_D^2} = \frac{\partial h_D}{\partial t_D} \qquad \begin{cases} 1 \le x_D < x_{LD} \\ 0 \le z_D \le 1 \end{cases} . \tag{A1.98}$$

Initial and boundary conditions are

$$h_D(x_D, z_D, 0) = 0 \tag{A1.99}$$

$$\frac{\partial h_D}{\partial z_D}(x_D, 1, t_D) = -\frac{1}{\sigma\beta_0}\frac{\partial h_D}{\partial t_D} \tag{A1.100}$$

$$\frac{\partial h_D}{\partial z_D}(x_D, 0, t_D) = 0 \tag{A1.101}$$

$$\frac{\partial h_D}{\partial x_D}(x_{LD}, z_D, t_D) = 0 \tag{A1.102}$$

$$\frac{\partial h_D}{\partial x_D}(1, t_D) = \frac{h_D - 1}{A} \tag{A1.103}$$

After performing Laplace transformations, the subsidiary boundary-value problem for the aquifer is

$$\frac{\partial^2 \bar{h}_D}{\partial x_D^2} + \beta_0 \frac{\partial^2 \bar{h}_D}{\partial z_D^2} = p\bar{h}_D \qquad \begin{matrix} 1 \le x_D < x_{LD} \\ 0 \le z_D \le 1 \end{matrix} \tag{A1.104}$$

with boundary conditions

$$\frac{\partial \bar{h}_D}{\partial z_D}(x_D, 1) = -\frac{p\bar{h}_D}{\sigma\beta_0} \tag{A1.105}$$

$$\frac{\partial \bar{h}_D}{\partial z_D}(x_D, 0) = 0 \tag{A1.106}$$

$$\frac{\partial \bar{h}_D}{\partial x_D}(x_{LD}, z_D) = 0 \tag{A1.107}$$

$$\frac{\partial \bar{h}_D}{\partial x_D}(1, p) = \frac{\bar{h}_D}{A} - \frac{1}{Ap} \tag{A1.108}$$

A solution to (A1.104) that satisfies (A1.105) and (A1.106) is

$$\bar{h}_D = \sum_{n=0}^{\infty} \bar{g}_n(x_D, p)\cos(\varepsilon_n z_D) \quad , \tag{A1.109}$$

where $n = 0, 1, 2, \ldots$ and ε_n are the roots of

$$\varepsilon_n \tan(\varepsilon_n) = \frac{p}{\sigma\beta_0} \quad . \tag{A1.110}$$

Substitution of (A1.109) into (A1.104) yields

$$\sum_{n=0}^{\infty} [\bar{g}_n'' - (\varepsilon_n^2\beta_0 + p)\bar{g}_n] \cos(\varepsilon_n z_D) = 0 \quad . \tag{A1.111}$$

Hence, g_n must satisfy

$$\bar{g}_n'' - (\varepsilon_n^2\beta_0 + p)\bar{g}_n = 0 \quad , \tag{A1.112}$$

the solution of which can be written as

$$\bar{g}_n = A_n \exp(q_n x_D) + B_n \exp(-q_n x_D) \quad , \tag{A1.113}$$

where

$$q_n = (\varepsilon_n^2\beta_0 + p)^{\frac{1}{2}} \quad . \tag{A1.114}$$

The solution (A1.109) satisfies (A1.107) if

$$\frac{\partial \bar{g}_n}{\partial x_D}(x_{LD}, p) = 0 \quad . \tag{A1.115}$$

Thus, substituting (A1.113) into (A1.115) and letting $x_D = x_{LD}$

$$A_n q_n \exp(q_n x_{LD}) - B_n q_n \exp(-q_n x_{LD}) = 0 \quad . \tag{A1.116}$$

Hence,

$$A_n = B_n \exp(-2q_n x_{LD}) \quad . \tag{A1.117}$$

Now, (A1.113) becomes

$$\bar{g}_n = B_n\left[\frac{\exp(2q_n x_D)\exp(-2x_{LD}q_n)}{\exp(q_n x_D)} + \exp(-q_n x_D)\right]$$

$$= B_n \exp(-q_n x_D)\{\exp[-2q_n(x_{LD} - x_D)] + 1\} \quad . \tag{A1.118}$$

Substitution of (A1.118) into (A1.109) yields

$$\bar{h}_D = \sum_{n=0}^{\infty} B_n \exp(-q_n x_D)\{\exp[-2q_n(x_{LD} - x_D)] + 1\}\cos(\varepsilon_n z_D) \quad . \tag{A1.119}$$

The coefficients B_n can be obtained as follows:

Apply boundary condition (A1.108) to obtain

$$\sum_{n=0}^{\infty} B_n \exp(-q_n)\{\exp[-2q_n(x_{LD} - 1)] + 1\}\cos(\varepsilon_n z_D)$$

$$-\sum_{n=0}^{\infty} B_n A q_n \exp(-q_n)\{\exp[-2q_n(x_{LD} - 1)] - 1\}\cos(\varepsilon_n z_D) = \frac{1}{p} \quad , \tag{A1.120}$$

or, after rearranging terms, (A1.120) becomes

$$\sum_{n=0}^{\infty} B_n \exp(-q_n)\{1 + A q_n \tanh[q_n(x_{LD} - 1)]\}\cos(\varepsilon_n z_D) = \frac{1}{p\{1 + \exp[-2q_n(x_{LD} - 1)]\}} \quad . \tag{A1.121}$$

Now, as demonstrated previously, by multiplying both sides of (A1.121) by $\cos(\varepsilon_m z_D)$, where m is an integer, and integrating over the interval $z_D = 0$ to $z_D = 1$, one can make use of the property of orthogonality of the set $\{\cos(\varepsilon_n z_D)\}$ over the interval $0, 1$, where ε_n are the roots of (A1.110). Thus, the terms for which $m \neq n$ are zero and one obtains

$$B_n \exp(-q_n)\{1 + A q_n \tanh[q_n(x_{LD} - 1)]\}\int_0^1 \cos^2(\varepsilon_n z_D) dz_D$$

$$= \frac{1}{p\{1 + \exp[-2q_n(x_{LD} - 1)]\}}\int_0^1 \cos(\varepsilon_n z_D) dz_D \quad . \tag{A1.122}$$

Thus,

$$B_n = \frac{2\exp(q_n)\sin(\varepsilon_n)}{pR\{1 + \exp[-2q_n(x_{LD} - 1)]\}[\varepsilon_n + 0.5\sin(2\varepsilon_n)]} \quad , \tag{A1.123}$$

where

$$R = 1 + A q_n \tanh[q_n(x_{LD} - 1)] \quad . \tag{A1.124}$$

Substitution of (A1.123) into (A1.119) yields

$$\bar{h}_D = 2\sum_{n=0}^{\infty}\frac{W_n \exp[-q_n(x_D-1)]\sin(\varepsilon_n)\cos(\varepsilon_n z_D)}{Rp[\varepsilon_n + 0.5\sin(2\varepsilon_n)]} \quad , \qquad \text{(A1.125)}$$

where

$$W_n = \frac{\exp[-2q_n(x_{LD}-x_D)]+1}{\exp[-2q_n(x_{LD}-1)]+1} \quad . \qquad \text{(A1.126)}$$

The q_n are defined by (A1.114) and ε_n are the roots of (A1.110).

Equation (A1.125) reduces to the Laplace transform solution previously derived (A1.85) for the step response for flow from a semi-infinite aquifer with no semipervious streambank if $x_{LD} \to \infty$ (that is, $W_n \to 1$) and $A = 0$ (that is, $R = 1$).

Equation (A1.125) is the Laplace transform solution for head at a point x_D, z_D in the aquifer. For the case of a partially penetrating observation well screened over the interval z_{D1} to z_{D2}, equation (A1.125) is integrated over the interval z_{D1} to z_{D2} to give the average head in a partially penetrating observation well ($\bar{h}_D^*$):

$$\bar{h}_D^* = \frac{1}{(z_{D2}-z_{D1})}\int_{z_{D1}}^{z_{D2}} \bar{h}_D dz_D$$

$$= \frac{2}{(z_{D2}-z_{D1})}\sum_{n=0}^{\infty}\frac{W_n \exp[-q_n(x_D-1)]\sin(\varepsilon_n)}{Rp[\varepsilon_n + 0.5\sin(2\varepsilon_n)]}\int_{z_{D1}}^{z_{D2}} \cos(\varepsilon_n z_D) dz_D$$

$$= \frac{2}{(z_{D2}-z_{D1})}\sum_{n=0}^{\infty}\frac{W_n \exp[-q_n(x_D-1)]\sin(\varepsilon_n)[\sin(\varepsilon_n z_{D2}) - \sin(\varepsilon_n z_{D1})]}{Rp\varepsilon_n[\varepsilon_n + 0.5\sin(2\varepsilon_n)]} \quad . \qquad \text{(A1.127)}$$

For a fully penetrating observation well screened over the interval $z_{D1} = 0$ to $z_{D2} = 1$, let $z_{D1} = 0$ and $z_{D2} = 1$ and one obtains the average head in a fully penetrating observation well ($\hat{\bar{h}}_D$):

$$\hat{\bar{h}}_D = 2\sum_{n=0}^{\infty}\frac{W_n \exp[-q_n(x_D-1)]\sin^2(\varepsilon_n)}{Rp\ \varepsilon_n[\varepsilon_n + 0.5\sin(2\varepsilon_n)]} \quad . \qquad \text{(A1.128)}$$

Dimensionless Seepage at Streambank Due to Step Change in Stream Stage

Dimensionless seepage at $x = x_0$ (the streambank) due to a unit-step change in stream stage is

$$\bar{Q}_D = -\frac{d\hat{\bar{h}}_D}{dx_D} \quad , \qquad \text{(A1.129)}$$

evaluated at $x_D = 1$.

Letting

$$r_1 = \frac{\sin^2(\varepsilon_n)}{Rp\varepsilon_n[\varepsilon_n + 0.5\sin(2\varepsilon_n)]}$$

and

$$r_2 = \{\exp[-2q_n(x_{LD} - 1)] + 1\} \quad ,$$

equation (A1.128) becomes

$$\hat{\bar{h}}_D = 2\sum_{n=0}^{\infty}\frac{r_1}{r_2}[\exp[-q_n(x_D - 1)]\{\exp[-2q_n(x_{LD} - x_D)] + 1\}]$$

$$= 2\sum_{n=0}^{\infty}\frac{r_1}{r_2}\{\exp[-q_n(x_D - 1) - 2q_n(x_{LD} - x_D)]\} + \frac{r_1}{r_2}\{\exp[-q_n(x_D - 1)]\} \quad . \qquad \text{(A1.130)}$$

Differentiating (A1.130) with respect to x_D gives

$$\frac{d\hat{\bar{h}}_D}{dx_D} = 2\sum_{n=0}^{\infty}\frac{r_1}{r_2}\{q_n\exp[-q_n(x_D - 1) - 2q_n(x_{LD} - x_D)] - q_n\exp[-q_n(x_D - 1)]\} \quad . \qquad \text{(A1.131)}$$

Now, evaluating (A1.131) at $x_D = 1$

$$\left.\frac{d\hat{\bar{h}}_D}{dx_D}\right|_{x_D = 1} = 2\sum_{n=0}^{\infty}\frac{r_1}{r_2}q_n\{\exp[-2q_n(x_{LD} - 1)] - 1\} \qquad \text{(A1.132)}$$

and

$$\bar{Q}_D = -2\sum_{n=0}^{\infty}\frac{r_1}{r_2}q_n\{\exp[-2q_n(x_{LD} - 1)] - 1\} \quad . \qquad \text{(A1.133)}$$

Substituting definitions of r_1 and r_2 into (A1.133) gives the solution for dimensionless seepage at the streambank

$$\bar{Q}_D = -2\sum_{n=0}^{\infty}\frac{q_n\sin^2(\varepsilon_n)}{Rp\ \varepsilon_n[\varepsilon_n + 0.5\sin(2\varepsilon_n)]}\left\{\frac{\exp[-2q_n(x_{LD} - 1)] - 1}{\exp[-2q_n(x_{LD} - 1)] + 1}\right\} \quad . \qquad \text{(A1.134)}$$

For conditions in which a semipervious streambank is absent, the factor R becomes unity.