

**FUNDAMENTAL CONSIDERATIONS
RELATING TO THE STRENGTH OF ROCK***

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1.0 THE STRENGTH OF ROCK

Whether the aim be to prevent collapse or to promote it, rock strength is an important factor in many practical problems of rock mechanics. The property of ‘strength’ of a rock derives, as with all solids, from various types of inter-atomic and intermolecular bonds, some strong and others relatively weak, that exist within the rock. To break the rock, it is necessary to break enough bonds to separate it into at least two pieces.

A bond between a pair of atoms (or molecules) is formed as a result of two forces — one an attraction, the other a repulsion. The force-intensity versus atomic-separation relationship differs for the two forces such that the resultant force between the atom-pair for various spacings is as shown in Figure 1.

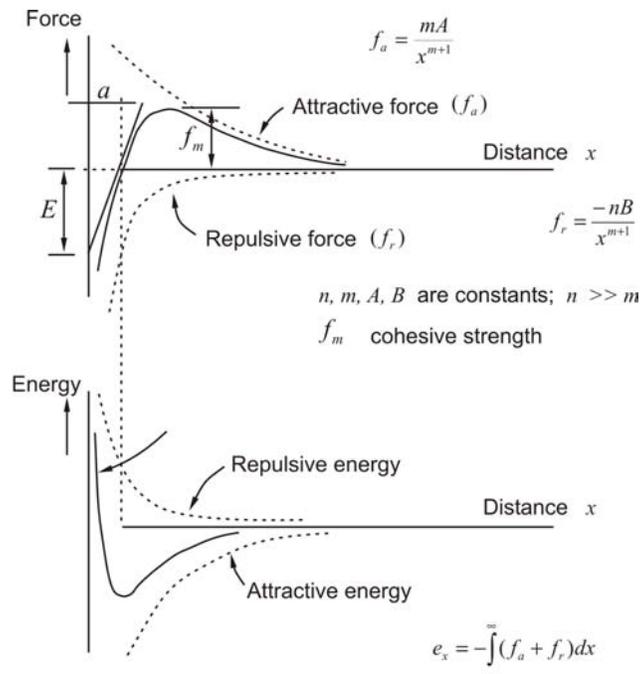


Figure 1 Schematic representation of (i) forces; (ii) potential energy trough between atoms (after Houwink, 1958 p.23)

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Both forces increase as the inter-atomic spacing is reduced, with the compressive (i.e., repulsion) force becoming very dominant at small spacings. The value of the exponents m and n in Figure 1 vary depending on the nature of the bond so that the force and spacing scales will change, but the general shape of the net-force vs. spacing curve remains the same — i.e., there is always a tensile or cohesive limit, a decrease in net force at large spacing, and a rapid increase without limit at small spacings.

Application of external compression forces will reduce the inter-atomic spacing (i.e., cause the solid to contract) in accordance with the net force-spacing relationship; application of external tension will increase the spacing. Release the external loads and the spacing will return to the zero net force value. The slope of the curve in the vicinity of the zero-force value represents the elastic modulus of the material.

Careful distinction must be drawn between the physical reality of discrete bond forces and the mathematical construct of stress. The mathematical, continuum, concept of ‘stress at a point’ has no direct physical meaning. Because it is, however, common to determine stresses, and to compute stress-strain behavior from the physical response of solids to applied forces, it is convenient to indicate the analogy at the atomic level.¹

If we assume that, over a small planar element of area ΔA , in the solid, all n possible bonds are fully active normal to the element, then we may define an average inter-atomic normal ‘stress’ (σ) across the plane ΔA as

$$\sigma = \frac{nF}{\Delta A} \quad \Delta A \rightarrow 0 \quad (1)$$

where F is the inter-atomic force per bond.

We may similarly define an intrinsic² elastic modulus, E , for small displacement (x) about the equilibrium spacing (a).

¹ Filonenko-Borodic explains the situation as follows: “... it is sometimes said that the theory of elasticity is based on the hypothesis of the continuous structure of solids. It must be borne in mind, of course, that this hypothesis is but a working hypothesis; it is dictated by the adopted mathematical method of investigation and does not intrude into the branches of physics that are directly concerned with the problems of body structure.”

² ‘Intrinsic’ in that the deformation from which the strain is computed results entirely from deformation (extension or compression) of the bonds. Macroscopically observed moduli in rocks will include additional deformations [e.g., due to pre-existing cracks (‘zero-modulus bonds’) and will tend to be less than this intrinsic value.

Thus, defining the inter-atomic ‘strain’ ε ,

$$\varepsilon = \frac{x}{a} \quad (2)$$

we obtain

$$E = \frac{\sigma a}{x} \quad (3)$$

2.0 COHESIVE STRENGTH

We may now obtain a rough estimate of the value of the cohesive strength as a (maximum) tensile stress (σ_m). (Joffé et al. 1935). Assuming that the form of the inter-atomic stress-spacing curve is sinusoidal (see Figure 2) for extensional displacements x from the equilibrium position in the range, $0 < x < a$ — i.e.,

$$\sigma = \sigma_m \sin \frac{2\pi x}{\lambda} \quad (4)$$

where σ_m is the inter-atomic cohesive strength, and

λ is a separation parameter in the direction of x , which describes the range of inter-atomic force interaction (see Figure 2).

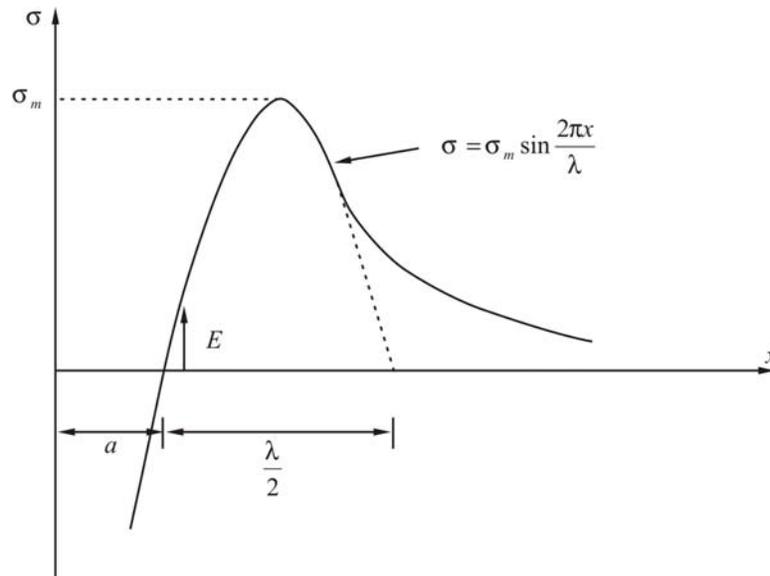


Figure 2 *Inter-atomic stress spacing curve*

Differentiating (4) with respect to x , we have

$$\begin{aligned}\frac{d\sigma}{dx} &= \frac{2\pi}{\lambda} \sigma_m \cos \frac{2\pi x}{\lambda} \\ &= \frac{2\pi}{\lambda} \sigma_m, \text{ for small } x\end{aligned}\tag{5}$$

Substituting in (5) from (3), we have

$$\frac{E}{a} = \frac{2\pi\sigma_m}{\lambda} \quad \text{or} \quad \sigma_m = \frac{E\lambda}{2\pi a}\tag{6}$$

For an ‘order-of-magnitude’ estimate, we will consider $\lambda/2$ to be approximately equal to a . Hence, from (6) we obtain

$$\sigma_m = \frac{E}{3}\tag{7}$$

Alternatively, if we assume that the work done to create unit area of surface

[i.e., to separate the atomic bonds acting across a unit area of the solid to the point where the bonds are ‘broken’, or no longer attract each other. This is effectively to separate them to infinite spacing]

reappears entirely as the ‘surface energy’ 2γ , where γ is the specific energy for each of the two (upper and lower) surfaces created, then we avoid the necessity of assigning a value to γ .

Noting that the work done per unit area of new surface (2γ) equals the shaded area in Figure 2, which we will assume to be roughly twice the area under the half-sine wave, we may then write

$$\int_0^{\frac{\lambda}{2}} \sigma_m \sin \frac{2\pi x}{\lambda} \cdot dx = \gamma \quad \text{or} \quad \frac{\lambda\sigma_m}{\pi} = \gamma\tag{8}$$

Substituting for γ from (6), we obtain

$$\sigma_m = \sqrt{\frac{E\gamma}{2a}}\tag{9}$$

The values for the theoretical cohesive strength indicated by (7) and (9) are orders of magnitude larger than the observed tensile strengths of most solids. For granite, for example, the typical elastic modulus (E) in tension may be of the order of 5×10^6 lb. per sq. in.; according to (7), the cohesive strength should then be of the order of 2×10^6 lb. per sq. in. A tensile strength of 2×10^3 lb. per sq.

in., or three orders of magnitude smaller, is closer to the value usually observed. Similar discrepancies, although not as great, exist for other materials.

The hypothesis presented by Griffith in 1921 was an attempt to explain the discrepancies. It will be discussed in some detail below. Before leaving the topic of atomic interaction, however, several points of interest should be noted.

1. A solid can only be ruptured by exceeding the cohesive strength. This requires “stretching” of the inter-atomic bonds, either by direct extension or by shearing. Tension and shear are thus the two basic modes of causing rupture. There is no fundamental compression mechanism of rupture.
2. Rupture of a solid occurs when all bonds have been broken across (as a minimum) a continuous plane of separation traversing the solid. To ‘break’ a bond, it is necessary to separate the atoms to such a spacing (effectively to ‘infinity’ on an inter-atomic scale) that the atoms no longer attract each other. Each atom forms a new (surface) equilibrium with its neighbors in the absence of the previously unbroken bond. If the rupturing force is released at any value of spacing less than that necessary to break down inter-atomic attraction, even though the extension be beyond that corresponding to that at the cohesive strength, the atoms will attract each other back to the equilibrium spacing. In other words, the complete energy of separation — the total area under the extension portion of the inter-atomic force-spacing curve (shaded in Figure 2) must be supplied before rupture will occur.

In short, rupture involves two conditions. Both are necessary for rupture to occur but, together, they are also sufficient to ensure rupture. The two are:

- (1) a ‘stress’ condition (The inter-atomic cohesive stress must be exceeded.); and
- (2) an energy condition. (The energy necessary to form two distinct surfaces through the solid must be supplied.)

3.0 GRIFFITH THEORY OF RUPTURE

Griffith recognized the need to satisfy the two conditions for fracture. He advanced the hypothesis that the average applied stress did in fact produce stresses to the cohesive-strength value because of intense stress concentrations produced at the tips of the sub-microscopic micro-cracks. He then demonstrated that, once a certain stress level had been reached, the supply of energy necessary to produce rupture would indeed become available. Although Griffith limited his analysis to an explanation of the strength of brittle solids, it will be seen later that the general principles upon which his Theory is based are applicable to a variety of practical situations.

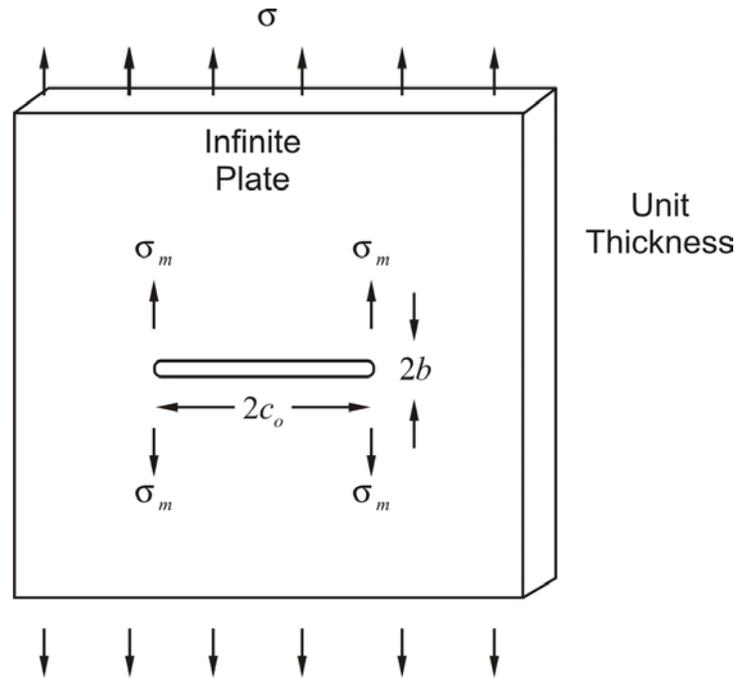


Figure 3 Plate model analyzed by Griffith

Griffith considered the conditions necessary for rupture of a thin, linearly elastic, isotropic, infinite plate loaded in plane stress by a ‘dead-weight’ tension, σ (i.e., the tension did not change with displacement of the boundary), into which was introduced a single thin slit, or ‘flaw’, of length $2c$, and having a radius of curvature ρ at each tip. The stress, σ_m , developed at the tip of the crack (considered as an ellipse of major axis $2c$, minor axis $2b$) due to the applied stress σ is given by the expression

$$\sigma_m = \sigma_o \left(1 + \frac{2c}{b} \right) \quad (10a)$$

or, for sharp cracks,

$$\sigma_m \cong \sigma_o \sqrt{\frac{c}{\rho}} \quad (10b)$$

Thus, provided the ratio c/b or, equivalently $\sqrt{c/\rho}$, is very large (i.e., the cracks are very narrow and considered as degenerate ellipses), the required stress, σ_m , equal to the cohesive strength, may be achieved with only moderate applied stress, σ .

To establish the energy condition necessary for the cracks to start to spread (i.e., for rupture to begin), Griffith invoked the Theorem of Minimum Potential Energy, which may be stated as

The stable equilibrium state of a system is that for which the potential energy of the system is a minimum

For the particular application of this theorem to rupture, Griffith added the statement

[T]he equilibrium position, if equilibrium is possible, must be one in which rupture of the solid has occurred, if the system can pass from the unbroken to the broken condition by a process involving a continuous decrease of potential energy.

The process of passing “from the unbroken to the broken condition” occurs by the progressive lengthening of the crack across the plate. Therefore, in order to proceed with the mathematical formulation of the energy criterion for rupture, we must consider the energy changes that occur when the crack lengthens. We will do this by first considering the energy change produced by introducing a crack into the plate, and then examining the effect of changing the crack length.

There are three parts of the system which contribute to the energy changes due to crack extension:

- (1) potential energy of the applied forces (W);
- (2) strain energy of the loaded plate (U); and
- (3) surface energy of the crack surfaces (S).

Other energies in the system (e.g., the surface energy of the faces and outer (infinite) boundary of the plate) are assumed not to change with crack extension and therefore can be neglected. We will consider the three energies in turn.

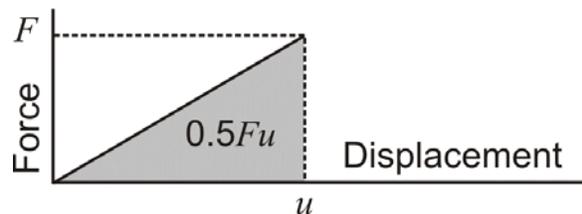
3.1 Potential Energy of Applied Forces (W)

It may be shown (Love 1927) that, when a linearly elastic body is deformed by constant forces applied at the outer boundary, the potential energy of these forces is reduced by an amount (ΔW) equal to twice the increase in strain energy (ΔU) of the body produced by the deformation.³ Thus,

³ An elementary example of this is the energy change that occurs when an elastic solid is loaded in uniaxial tension by a constant force F . The loss in potential energy (ΔW) of the applied force F due to the elastic deformation u is Fu . [See Fig. a1]. The gain in strain energy (ΔU), equal to the area under the linear elastic force-deformation curve, is $1/2 Fu$. Thus,

$$\Delta W = -2 \Delta U \quad (\text{a1})$$

Figure a1 Loss in Potential Energy (ΔP) and Gain in Strain Energy (ΔU) of a Solid Loaded By a Constant Force



$$\Delta W = 2\Delta U \quad (11)$$

3.2 Strain Energy of the Loaded Plate

Griffith demonstrated that the strain energy of the infinite plate increased by an amount $(\pi c^2 \sigma^2 / E)$, where E is the elastic modulus of the plate material by introduction of a crack, length $2c$. This result was confirmed subsequently by Sneddon (1946) and by Irwin (1954)⁴:

$$\Delta U = \frac{\pi c^2 \sigma^2}{E} \quad (12)$$

3.3 Surface Energy of the Crack Surfaces

If we designate the specific surface energy as γ , then, as the crack is of length $2c$, and of unit thickness, the total increase in surface introduced is

$$\Delta S = 2 \cdot 2c \cdot \gamma = 4c\gamma \quad (13)$$

3.4 Change in Potential Energy of the System (ΔP)

The total change in potential energy of the system (ΔP) produced by introduction of the crack may now be calculated. Thus,

$$\Delta P = \Delta W + \Delta U + \Delta S \quad (14)$$

Substituting from equation (11) we obtain

$$\Delta P = \Delta S - \Delta U \quad (15)$$

Substituting from equations (12) and (13), we find

$$\Delta P = 4c\gamma - \frac{\pi c^2 \sigma^2}{E} \quad (16)$$

This relationship is plotted in Figure 4 for various crack lengths ($2c$) and two stress levels, σ_a, σ_b , with σ_b a greater tension than σ_a .

⁴ See the appendix.

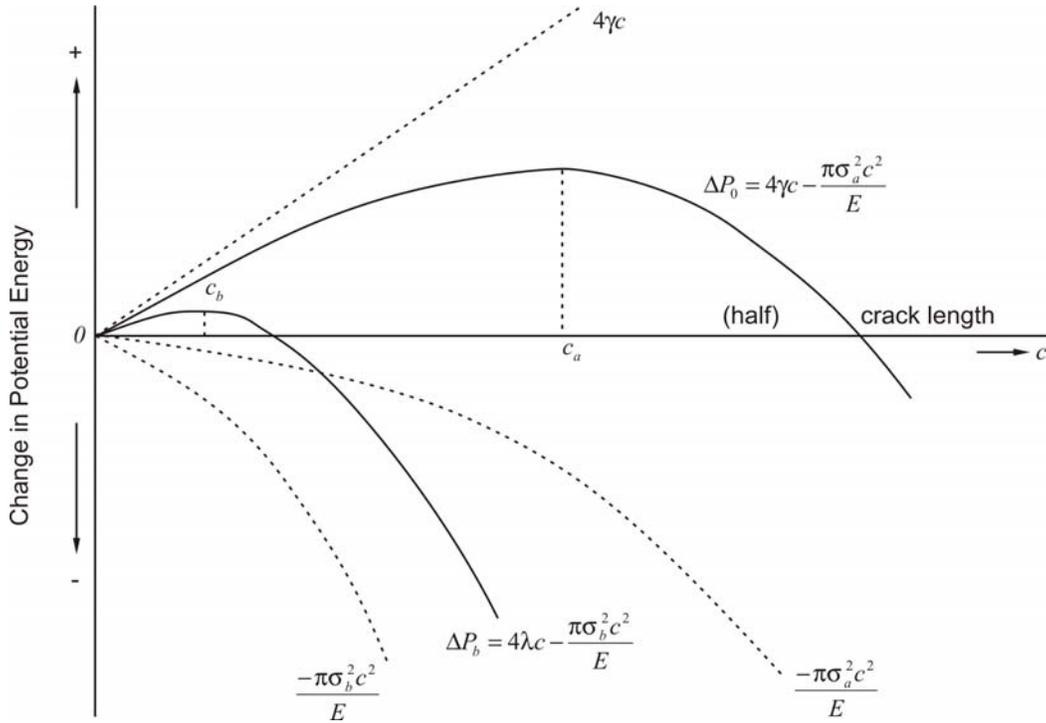


Figure 4 Change in potential energy (ΔP) of the plate-load system shown in Figure 3 due to the introduction of a crack of variable length $2c$ [or, equivalently, variation in ΔP with 'extension' of a crack under constant load]. N.B. The crack would actually 'extend' only on the right-hand (instability) side of ΔP maximum.

According to Griffith's energy instability hypothesis of crack extension, cracks of length less than $2c_a$ would not extend at an applied stress level σ_a , as crack extension would require an increase (ΔP) in the potential energy of the system. The same is true for cracks less than $2c_b$ in length at an applied stress σ_b . Conversely, introduction of a crack longer than $2c_a$ or $2c_b$ would immediately result in crack extension, at stress levels σ_a and σ_b , respectively, since $\partial(\Delta P)/\partial c \leq 0$ for these lengths.

The Griffith energy criterion for tensile rupture is thus satisfied whenever $(\partial/\partial c)(\Delta P) \leq 0$, and rupture will begin when

$$\frac{\partial(\Delta P)}{\partial c} = 0 \quad (17)$$

Applying this criterion to (16), we find the tensile stress (σ_g) at which rupture begins. Thus,

$$4\gamma - \frac{2\pi c \sigma_g^2}{E} = 0 \quad \dots\dots\dots \text{(a)}$$

or (18)

$$\sigma_g = \sqrt{\frac{2E\gamma}{\pi c}} \quad \dots\dots\dots \text{(b)}$$

Thus, according to the Griffith criterion, σ_g is the tensile strength of a plate loaded in uniaxial (plane stress) tension.

Analysis of the corresponding plane-strain problem reveals that E in (18) is replaced by $E/(1-\nu^2)$, where ν is Poisson's ratio for the material. It has also been shown (Sack 1946) that, when the line crack is replaced by a 'penny-shaped' crack in a solid, the criterion is modified such that $2/\pi$ in (18) is replaced by $\pi/4(1-\nu^2)$. In all of these cases, the numerical factors associated with $\sqrt{E\gamma/c_o}$ are of similar magnitude. This indicates that the geometry of the crack is probably not very influential on the tensile strength.

There are several points that should be noted concerning the Griffith tensile strength (σ_g) as represented by (18).

1. There is no fundamental significance to the value σ_g . It is not **the** tensile strength of the plate material, but, rather, the stress at which spontaneous extension of the crack $2c$ will occur in the uniaxially loaded plate of Figure 3. The fundamental criterion for rupture is that given by (17) — i.e., the onset of energy instability. As will be seen later, the stress at which (17) is satisfied may differ appreciably for different loading systems and 'specimen' geometries.
2. The appearance of the length dimension (c_o) in (17) is a consequence of the fact that the change in energy stored in the system [second term in (16)] is proportional to the **volume** of the solid affected by the crack whereas energy absorbed by the system depends on the **area** of the crack. Any linear, elastic, instability criterion will result in a rupture condition of the general form

$$\sigma_t^2 d = \text{const.} \quad (19)$$

where σ_t is the tensile strength of the solid, and

d is a length that characterizes the dimensions of those critical 'flaws' in the solid responsible for the onset of tensile rupture.

The value of the constant in (19) will depend on the geometry of the flaws and the manner in which energy is absorbed during flaw growth.

4.0 INFLUENCE OF STIFFNESS OF APPLIED LOAD ON THE GRIFFITH CRITERION

In the above analysis, it was assumed that the plate was loaded by a perfectly soft system — i.e., the tension forces applied at the boundary were constant independent of displacement of the boundary. This was the situation assumed by Griffith. It may readily be shown, however, that the tensile rupture criterion is not affected by the load-deformation (or ‘stiffness’) characteristics of the applied load.

This is most conveniently done by considering the force and displacement changes that will occur, simultaneously, at the plate boundary when the crack is introduced. We will consider the plate loaded to a force F through a spring attached to a (rigid) end grip, as shown in Figure 5.

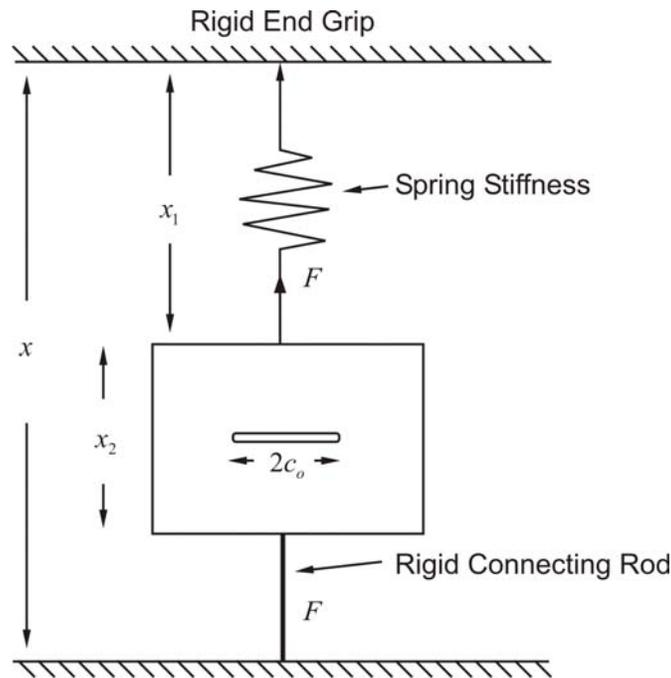


Figure 5 Griffith plate stretched through a non-rigid applied load

The strain energy (V) stored in the system (i.e., plate and spring) at the force level F is

$$V = \int_0^F x.dF = \frac{[M(K) + M(c)]F^2}{2} \quad (20)$$

where $M(K)$ is the (constant) elastic compliance of the applied load (i.e., the reciprocal of the spring stiffness K) (Thus, $x_1 = M(K)F_{x_1}$, where x_1 is the spring extension at load F_{x_1}), and $M(c)$ is the (variable) elastic compliance of the specimen. ($M(c)$ will increase with crack extension ($x_2 = M(c)F$).

The differential dV of the elastic energy when both F and c [or, equivalently, $M(c)$] change simultaneously is

$$dV = \frac{F^2}{2} d[M(c)] + [M(K) + M(c)] \cdot F dF \quad (21)$$

As the crack lengthens (and, hence, x_2 changes), the total extension (x) between the rigid end grips remains constant

$$x = x_1 + x_2 = \mathbf{constant}$$

or

$$x = [M(K) + M(c)] \cdot F = \mathbf{constant} \quad (22)$$

Therefore,

$$dx = [M(K) + M(c)] df + F \cdot d[M(c)] = 0$$

or

$$[M(K) + M(c)] df = -F \cdot d[M(c)] \quad (23)$$

Substitution of (23) into (21) yields

$$dV = \frac{-F^2}{2} \cdot d[M(c)] \quad (24)$$

That is, the change in strain energy of the system (dV) is independent of the compliance $[M(K)]$ or the stiffness of the applied load.

Note that dV is equal to the sum ($\Delta U + \Delta W$) in the earlier derivation of the Griffith criterion.

It is important to recognize that, so far, we have considered the onset, or ‘initiation’, of rupture. Stiffness of the loading system can play a part in fracture **propagation**, as will be shown below. We have established the important point that the tensile strength of a solid that behaves as a Griffith material (i.e., a cracked plate) should not depend on the particular direct tensile testing system used.

5.0 STRESS-STRAIN BEHAVIOR OF A GRIFFITH MATERIAL IN TENSION

It is useful to consider the macroscopic ‘stress-strain’ behavior of the Griffith plate system loaded in tension [see Figure 2]. The analysis below follows that presented by Berry (1961).

The Griffith plate is assumed to be of unit thickness and large cross-sectional area A .

The total strain energy of the plate before introduction of the crack = $\frac{A\sigma^2}{2E}$.

Increase of strain energy of the plate due to introduction of a crack, length $2c$,

$$\frac{\pi c^2 \sigma^2}{E}$$

\therefore Strain energy of the cracked plate = $\frac{\sigma^2}{2E} [A + 2\pi c^2]$.

If we define an **average** elastic modulus E' of the cracked plate such that

$$\frac{\sigma^2}{2E} [A + 2\pi c^2] = \frac{\sigma^2 A}{2E'}$$

we have

$$E' = \frac{AE}{A + 2\pi c^2} + \frac{E}{1 + \frac{2\pi c^2}{A}} \quad (25)$$

From (18) we have

$$\sigma^2 g = \frac{2E\gamma}{\pi c}$$

However, for a linearly elastic material

$$\sigma_g = E' \varepsilon_g \quad (26)$$

where ε_g is the **average**⁵ axial strain in the plate at rupture.

Substituting in (26) for σ_g from 18(a), and for E' from (25), we obtain

$$\varepsilon_g = \frac{\sigma_g}{E} + \frac{8E\gamma^2}{A\pi\sigma_g^3} \quad (27)$$

(27) is the locus in stress-strain space of the Griffith criterion for rupture in direct tension. See Figure (6).

⁵Obviously, the strain will not be uniform in the vicinity of the crack

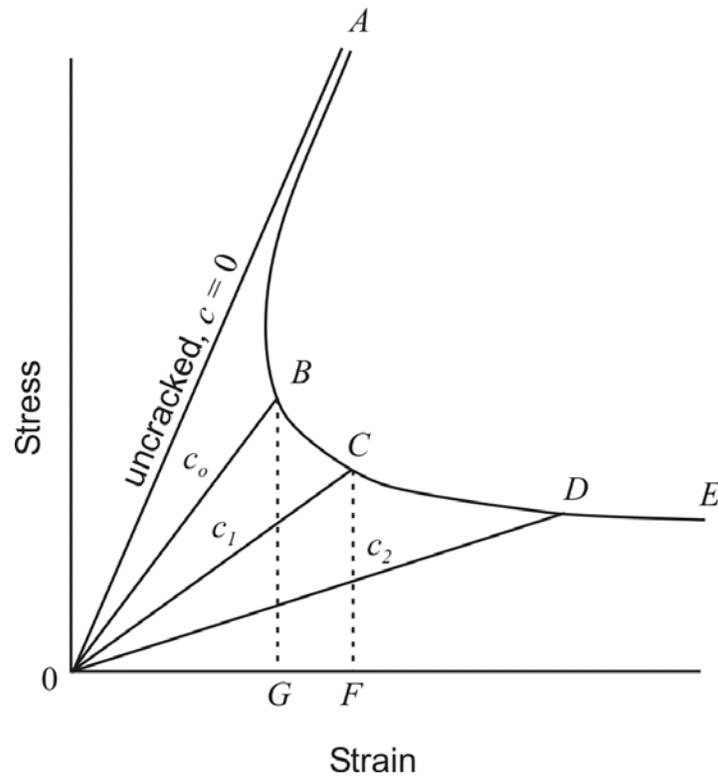


Figure 6 Locus of the Griffith criterion for rupture in direct tension

It is seen that the locus is asymptotic to the slope $\tan^{-1}E$, where E is the modulus of elasticity of the uncracked plate, for which, according to Griffith, the strength would equal the theoretical cohesive strength. For a plate with an initial crack of length $2c$, the loading path would follow OB . At B , the crack would start to lengthen, and the overall plate would ‘strain’ accordingly. If the applied load were reduced with crack extension so as to follow the locus $BCDE$, the crack would be slowly extended to traverse the plate — i.e., the Griffith criterion $\partial P / \partial c = 0$ is continuously just satisfied, with no excess energy to allow onset of the rapid instability condition $\partial P / \partial c < 0$. At loading condition B , the plate contains a crack length $2c$ about to extend, and strain energy proportional to the area OBF . At C , the crack has lengthened to $2c_1$, and the plate contains strain energy proportional to the area OCG . As the energy input to the plate in extending the crack from $2c_0$ to $2c_1$ (i.e., from B to C) is proportional to the area $BCGF$, it follows that the area OBC is proportional to the increase in surface energy, $4\gamma(c_1 - c_0)$, of the crack. Similarly, area OBD is proportional to the new surface energy, $4\gamma(c_2 - c_0)$. It is seen that the total surface energy of a crack is proportional to the area enclosed by the asymptote OA , the locus, and the slope representing the crack length.

6.0 INFLUENCE OF STIFFNESS OF APPLIED LOAD ON CRACK EXTENSION

It has been shown that the stiffness of the load application system does not influence the Griffith criterion for (the onset of) crack extension. The stiffness of the applied load does affect fracture propagation. Assume, for example, as did Griffith, that the applied load is independent of plate boundary displacement when crack extension occurs. If the initial crack length was $2c_0$, then the load, on reaching B, in Figure 7, would follow the path BB'B". At an extension corresponding to OG, then the energy supplied to the plate by the applied load during crack extension is proportional to the area FBB'G; the energy required for slow crack extension to an average plate strain of OG is proportional to FBCG; the excess energy, proportional to BB'C, will produce crack acceleration, the kinetic energy of the system increasing with crack extension, leading rapidly to rupture of the plate. If the plate is loaded through a system of stiffness K , as in Figure 5, then the applied load will decrease with crack extension.

Assume then that the applied load decreases along BHDJ. At a crack extension corresponding to average strain OG, the excess energy is proportional to area BHC, considerably less than for the 'dead-load' system, but also causing crack acceleration. With such a loading system, the crack would continue to accelerate as it extended, generating maximum kinetic energy at the crack length $2c_2$ corresponding to the line OD. The crack would continue to grow as the applied load fell along DJ, but since the energy released by the load in region DJ is less than that required to satisfy the criterion for fracture, kinetic energy is abstracted from the system until the crack eventually stops at condition J, at a length $2c_3$ corresponding to the line OE. At the point J, the kinetic energy BCD has been transformed into the surface energy, DJE, necessary to establish the crack length $2c_3$. For further crack propagation, the load must be raised along OJ to E, adding the strain energy JELK necessary to satisfy the fracture criterion. With such a system, the crack would initially propagate unstably, but would eventually come to rest before rupturing the plate. Rupture would involve successive re-loading of the plate.

With an applied load of high stiffness such as represented, for example, by the slope BM, any crack propagation immediately causes the applied load to fall to a value below that required for further propagation (i.e., points on the locus BCDE). If the load drops to M, for example, an amount of energy proportional to the area BCM must be added to the system before the crack will extend to the length $2c_1$ (point C on the locus). In such a high stiffness system, in effect, the crack will propagate only if the necessary energy is added to the system by outward displacement of the rigid boundary under load. Thus, while the stiffness of the applied load does not influence *initiation* of fracturing in a Griffith material, it can have a marked effect on *propagation* of fracture.

Substituting from (29) for the first term in (28), we obtain

$$\Delta P = \frac{2\pi c_0 c \sigma^2}{E} - \frac{\pi \sigma^2 c^2}{E} + \frac{k \rho c^2 v^2 \sigma^2}{2E^2} \quad (30)$$

A boundary condition for the velocity is

$$v \left(= \frac{dc}{dt} \right) = 0, \text{ when } c = c_0$$

Application of the boundary condition to (30) gives

$$\Delta P = \frac{2\pi \sigma^2 c_0^2}{E} - \frac{\pi \sigma^2 c_0^2}{E} + 0 = \frac{\pi \sigma^2 c_0^2}{E} = \text{constant} \quad (31)$$

Substituting (31) into (30), we obtain

$$\frac{\pi \sigma^2 c_0^2}{E} = \frac{2\pi c_0 c \sigma^2}{E} - \frac{\pi \sigma^2 c^2}{E} + \frac{k \rho c^2 v^2 \sigma^2}{2E^2} \quad (32)$$

Re-arranging, collecting terms, etc., (32) yields the following expression for v:

$$v = \sqrt{\frac{2\pi E}{k\rho}} \left[1 - \frac{c_0}{c} \right] \quad (33)$$

$$v = v_m \left[1 - \frac{c_0}{c} \right] \quad (33)a$$

As the crack length (c) becomes much larger than the initial length (c_0), the velocity tends to the maximum value (v_m), where

$$v_m = \sqrt{\frac{2\pi E}{k\rho}} \quad (34)$$

In this case (34) becomes

$$v_m = 0.38 \sqrt{\frac{E}{\rho}} \quad (35)$$

Thus, a Griffith crack propagating across a plate, as shown in Figure 3, will attain a maximum velocity of the order of one-third of the velocity of sound, $(\sqrt{E/\rho})$, in the medium.

Figure 8 shows the crack velocity as a function of crack half-length, as predicted by 33(a).

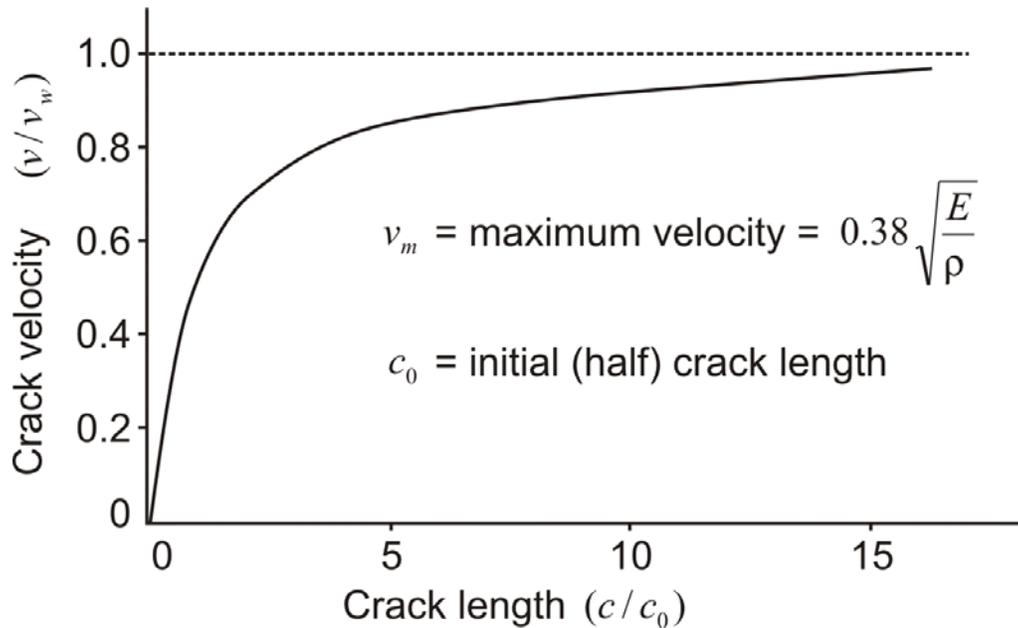


Figure 8 Crack velocity as a function of crack half-length

The crack velocity rapidly approaches a high value (e.g., $v \cong 0.6 v_m$ for $c = 2.5 c_0$). For a material with a sonic velocity around 12,000 ft. per second, the velocity will reach a value of almost 3000 ft. per second when the crack has extended only 2.5 times its original length. At such velocities, the time interval between crack initiation and complete tensile rupture of a specimen of 3-4 inches in diameter would be of the order of 100 microseconds only.

Berry (1960) has also derived (33) for the Griffith crack. He points out (1960, 1961) that there is a fundamental inconsistency in the crack-velocity analysis as outlined above, in that the Griffith criterion 18(a), to (29), represents an (unstable) **equilibrium** condition. The crack velocity, in fact, will remain identically zero for all crack lengths if the applied stress is sufficient only to reach the equilibrium condition. In other words, it is incorrect to assume that the third term of (28) exists unless the Griffith stress is (infinitesimally) exceeded, so as to generate an excess of energy over that required for (unstable) static equilibrium. Berry demonstrates, however, that (33) is the correct limiting expression for the velocity, when the Griffith stress is infinitesimally exceeded. If the load is applied so that the Griffith stress is instantaneously exceeded to a greater degree, then the crack acceleration is somewhat greater — to the same limiting value, v_m , of (35).

It will be shown later that there are important differences between the velocity behavior of the ideal Griffith material and that of many real materials.

8.0 MODIFICATION OF THE GRIFFITH CRITERION FOR REAL MATERIALS

The original Griffith criterion considered the highly idealized situation in which a single, critically stressed, crack begins to extend and propagate throughout the solid. It is implicitly assumed that there are no other cracks in the material or, equivalently, that any other cracks remain elastic (i.e., do not propagate) and, hence, do not contribute to the incremental energy changes of the system.

This assumption does not hold for real materials, and it is necessary to modify the original criterion to obtain a more valid description of actual behavior. Real materials contain many ‘flaws’ and stress raisers that play the role of Griffith cracks (i.e., to raise the stress intensity sufficiently to overcome inter-atomic or intermolecular bonds and to serve as a source of potential energy instability). The onset of unstable crack extension at one location does not necessarily imply that the same crack will continuously extend to total rupture.

Consider, for example, the situation depicted in Figure 9, in which a major crack, AA, is extending under a mean applied tension σ through a material containing a large number of smaller, initially non-propagating cracks. It may be shown from elastic theory that the highly stressed region ahead of the crack tip increases in extent as the main crack extends.

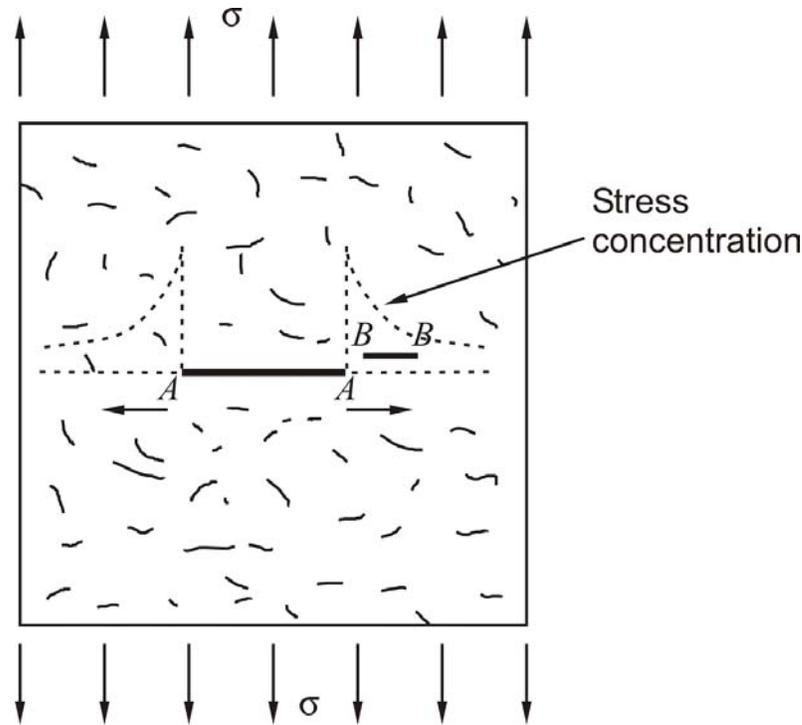


Figure 9 Crack propagation in an extensively pre-cracked material

Thus, small cracks in the path of the main crack are subjected to increasingly high stresses as it approaches. At a given instant, for example, a small crack BB may be subjected to a mean stress, $\bar{\sigma}_B$, high enough to start extension of the crack, both away from and toward the main crack. The small crack may be located off the axis of the main crack, resulting in a ‘step’ in the rupture path

when the two intersect. In addition, there will be many such cracks ahead of the main crack, and their number will increase with growth of the overall rupture path. Each mini-crack instability contributes to the energy balance equation, affecting both the surface-energy-demand term and the strain-energy-release term.

Glücklich and Cohen (1967) have suggested that, in the plane stress case, the two terms are modified as follows:

$$\Delta S_{mc} = 4\gamma c + 4\gamma \sum_{i=1}^{n=sv} b_i \quad (36)$$

$$\Delta U_{mc} = \frac{\pi\sigma^2 c^2}{E} + \frac{\pi\sigma^2}{E} \sum_{i=1}^{n=sv} b_i^2 \quad (37)$$

where ΔS_{mc} , ΔU_{mc} are the changes in surface energy, and stored energy of the loaded (pre-cracked) plate, respectively, for the multiple-crack-growth situation,

b_i is the (typical) half-length of the small cracks that propagate,

n is the number of the small cracks,

v is the volume ahead of the main crack within which the stress is sufficient to cause propagation of the small cracks, and

$s = n/v$ is the density of small cracks.

Other terms are as defined earlier for the simple single-crack Griffith analysis. [E , the overall modulus of elasticity of the plate, will have a value appropriate to that for an extensively cracked plate. It is assumed that this value does not change significantly due to cracking in the region ahead of the main crack.]

The increase in energy release is proportional to $\sum b_n^2$ [not to $(\sum b_n)^2$] — i.e., the sum of the energies released by growth of all the small cracks is much less than the energy released by a single long crack of the same total length. The increase in surface energy demand, being proportional to $\sum b_n$ is the same for the multitude of small cracks as for a single long crack of equivalent length. As the volume, v , over which minor cracking develops, increases with main crack length, the phenomenon of pre-cracking ahead of a main crack results in a rapidly increasing energy demand, while having relatively little effect on the energy released. The energy demand term may be represented approximately in the non-linear form

$$\Delta U_{mc} = 4\sigma c^{\alpha(c)} \quad (38)$$

where the exponent α increases progressively with main-crack growth, from the value $\alpha \cong 1$ at initiation of the main crack. Figure 10 outlines the energy changes for multiple cracking crack-growth under a constant mean applied tension. Comparing Figure 10 with Figure 2, depending on the exponent α , major instability may not immediately result from the start of main crack extension. At the stress level, σ_a , a crack of half-length c_a , will extend unstably to the half length c_a , beyond which it is no longer unstable and will quickly cease to propagate. If the higher mean stress, σ_b is applied, then a crack would remain unstable to a greater half-length (c_b).

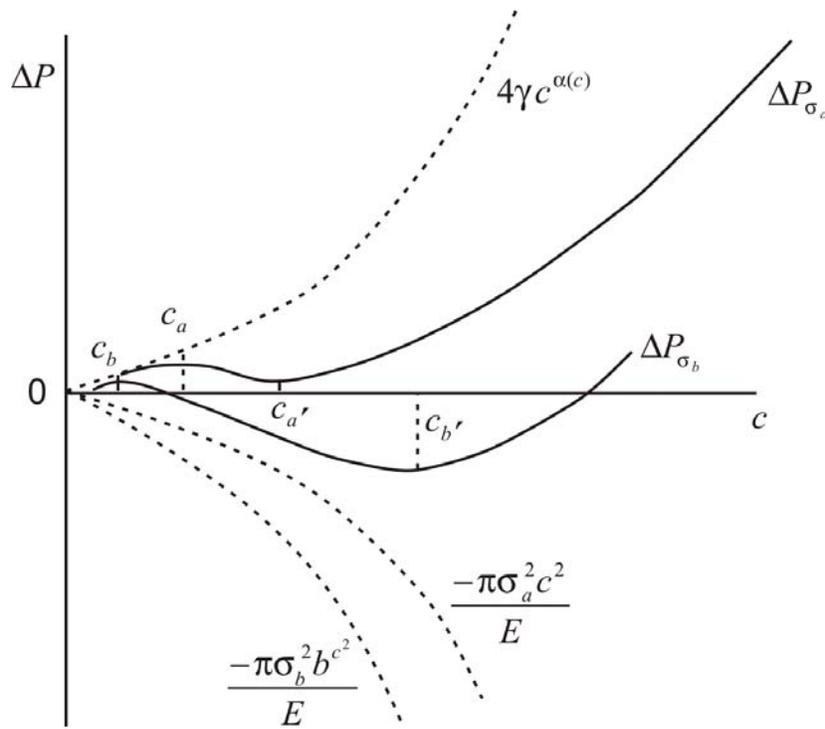


Figure 10 Change in potential energy of a pre-cracked Griffith plate model with growth of a crack, length c

Thus, to continue unstable crack extension beyond c_a , it would be necessary to increase the mean applied stress. A material that is extensively ‘pre-cracked’ (or, equivalently, that contains a high density of stress-raisers) thus can exhibit the phenomenon of ‘stable crack-growth’. The solid does not rupture spontaneously as soon as a critical stress is reached; instead, the crack may grow slowly as a series of transient instabilities under increasing load.

The stable crack-growth model shown in Figure 10 predicts that the applied load must be increased continuously without limit for continued crack growth — i.e., no macroscopic instability will ever occur. In an infinitely large solid, this may be true; in reality, several effects may intervene to cause unstable rupture.

Thus, with a test specimen of finite length, as the crack approaches the outer edges, growth of the pre-cracking volume becomes limited, and the mean stress in the volume rises rapidly. This has the effect of placing an upper limit on the exponent α in (38) and causing a more rapid increase in the energy-release rate. Together, these two effects will produce a major instability.

It may be conjectured that the surface energy term, γ , could exhibit some form of ‘rate-of-loading’ dependence. Thus, as the applied load is increased, the amount of excess energy released during the transient instabilities may tend to increase. In turn, this tends to result in more rapid rate of crack extension during the instability. If the processes of energy absorption during generation of the new surfaces are such that the energy required is the lower the more rapidly created the surfaces, in effect

this will reduce exponent α . Beyond a critical size of main crack and a critical rate of applied-load increase, the exponent could fall to a value where major instability results.

The mechanisms of stable crack growth are not well understood and probably differ for different materials. Nevertheless, all involve the common requirement that the energy absorbed by the processes of crack extension increases more rapidly than the energy released by the same crack extension.

Figure 11 illustrates an alternative representation of the potential energy changes for the extending Griffith crack shown in Figure 4. Whereas Figure 4 represents (16), Figure 11 represents 18(a), the derivative with respect to crack length (c) of (16).

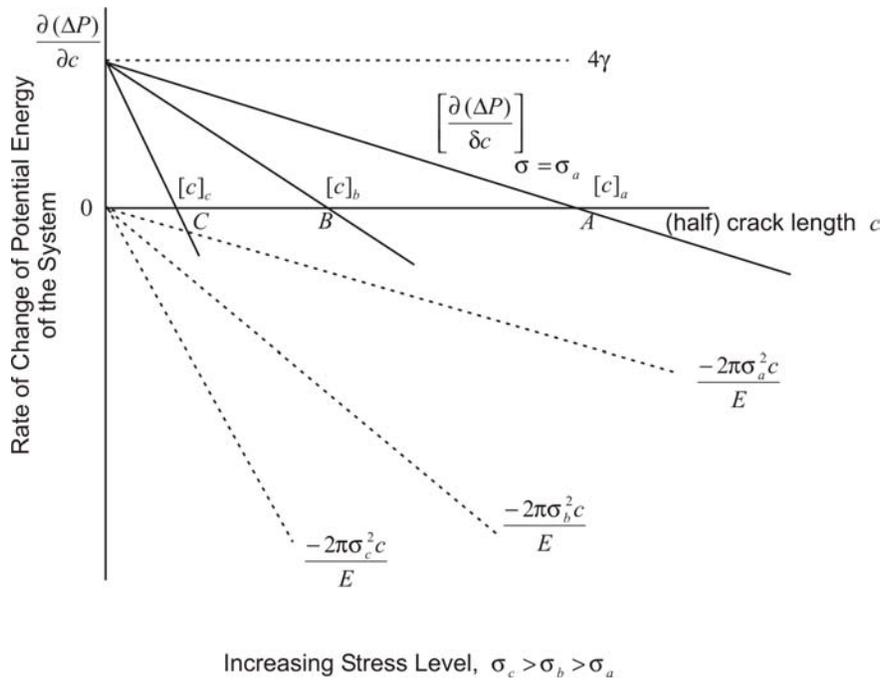


Figure 11 Variation in the rate of change in potential energy $\left[\frac{\partial(\Delta P)}{\partial c}\right]$ of the plate-load system of Figure 3 with variation in crack length $\left[\frac{\partial(\Delta P)}{\partial c}\right]$

The point at which the net curve crosses the abscissa corresponds to the crack length at which instability occurs. The region above the abscissa corresponds to stable conditions, the region below to unstable conditions. Thus, at a constant mean-applied-stress of σ_b , all cracks equal to or greater than $2c_b$ in length will extend unstably. To follow the Griffith crack locus shown in Figure 6 (i.e., to slowly extend a Griffith crack, or to constantly remain on the point of instability) is equivalent to changing the applied stress level such that the crack extension condition continuously follows the abscissa, $\left[\frac{\partial(\Delta P)}{\partial c} = 0\right]$, in Figure 11.

For the stable crack growth situation shown in Figure 10, the zero slope straight line, 4γ , of Figure 11, is changed to a line of continuously increasing slope, up to a (half) crack length, say, of c_d , at which the pre-cracked volume becomes limited, as shown in Figure 12. It is assumed, as in Figure 10, that the strain energy terms are negligibly affected.

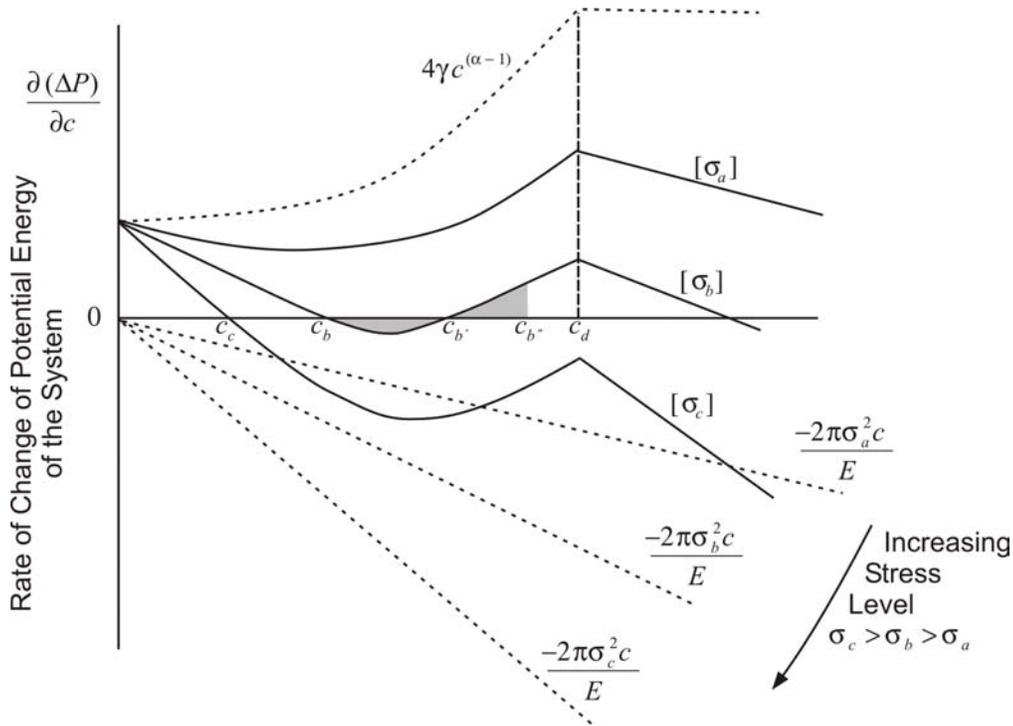


Figure 12 Variation in the rate of change in potential energy with variation in crack length for a pre-cracked plate system

In the case shown in Figure 12, at a stress level of σ_a , the system is stable for all crack lengths (i.e., the solid line marked $[\sigma_a]$ is positive everywhere); hence, no crack growth occurs. At the higher stress level, σ_b , (See solid line marked $[\sigma_b]$), a crack of (half) length c_b would start to extend, accelerating to (half) length $c_{b'}$ and continuing, at a decelerating rate, to (half) length $c_{b''}$. At this point, the crack would stop. [The excess energy released over the length range $(c_b - c_{b'})$, i.e., the shaded areas above and below the abscissa $[\partial(\Delta P)/\partial c = 0]$ are equal.] There is insufficient energy available to extend the crack further — the specimen does not break. At the still higher stress σ_c (see solid line marked $[\sigma_c]$), the system is unstable for all (half) crack lengths greater than c_c . The specimen would rupture ‘spontaneously’ if it contained cracks of (half) length c_c or greater.

9.0 INFLUENCE OF INHOMOGENEITIES ON FRACTURE PROPAGATION

Most rocks consist of more than one mineral constituent, with grains of variable orientation and pores between the grains. Tensile rupture by crack propagation in rocks is consequently considerably more complex than even the pre-cracked model a crack may start at A in Figure 13, for example, and may traverse the specimen along a very irregular path.

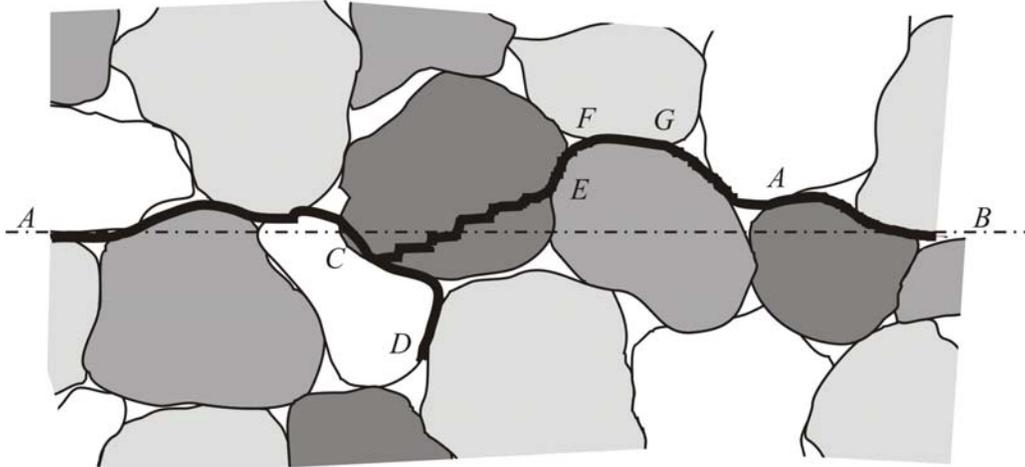


Figure 13 Irregular crack paths in rock loaded in tension

Propagation along the direct path AB would involve cleavage through grains of various orientations, requiring higher ‘surface energy’ than required to propagate the same projected length along a more tortuous path. At C, for example, the crack may tend to follow a grain boundary, say to D. Further propagation along the boundary would require a higher stress than that required to restart the crack at C, traversing the grain along CE. A new crack may then start, say at G, running toward F, leaving the step EF, a pore space may be intersected, etc. In such a complicated fracture process, the term ‘surface energy’ tends to be an inadequate description for the multitude of energy-absorbing mechanisms that can develop as tensile rupture propagates through rock. ‘Work of Fracture’ (W_F), one of the alternative terms used frequently, seems more appropriate. It may be defined as “the energy required to extend rupture by unit length along the mean plane of rupture” — AB in Figure 13.

The rate of variation of W_F per unit advance in the direction AB may appear as shown by the solid line d...M in Figure 14.

The rupture path is assumed to have extended stably to the projected length c_d , at which point it encounters a pore space or pre-existing crack such that no energy is required to extend the length to c_e . With a constant stress or dead weight loading system, the rupture would accelerate quickly from c_d to c_e , generating kinetic energy represented by the shaded area $dc_d c_e$. [The constant applied-stress (energy-release) line is shown dotted.] Immediately beyond the pore space, the rupture encounters a ‘hard’ or ‘strong’ grain. Over this increment, W_F is appreciably in excess (by the shaded area efg) of that released by the constant stress system. The deficiency (efg) is abstracted from the kinetic energy of the rapidly advancing rupture — i.e., the rupture slows down.

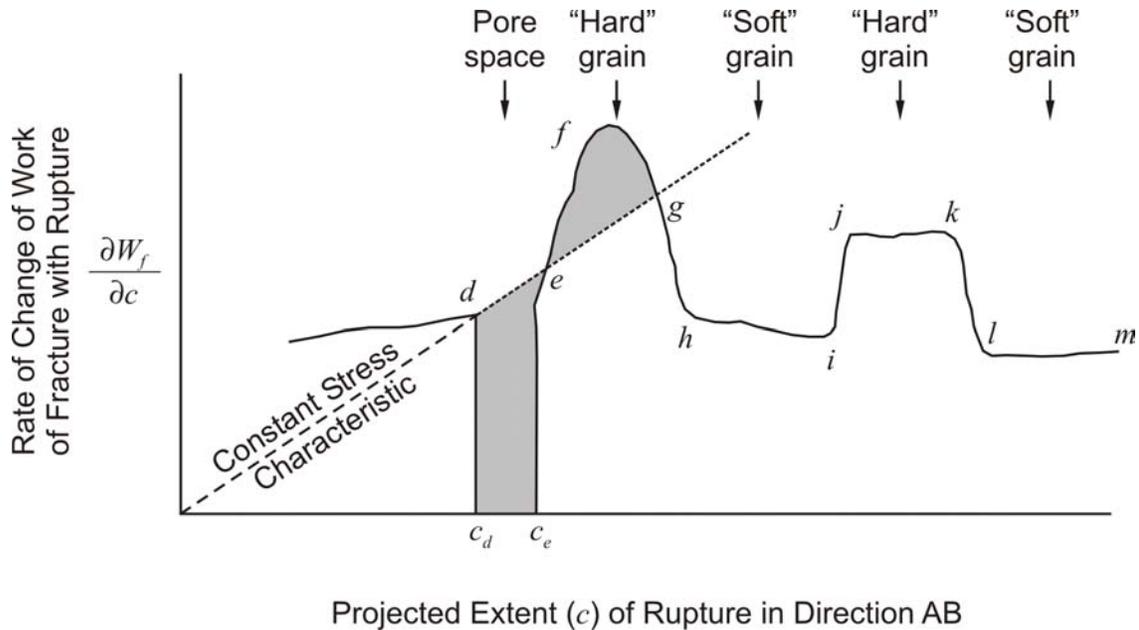


Figure 14 Energy release - Work of Fracture: Interaction during tensile rupture of a rock specimen loaded by a constant stress system

If the deficiency exceeds the total kinetic energy (i.e., if area $dc_d c_e e > \text{area } efg$), then the rupture will stop, and it will be necessary to add energy to the system — i.e., raise the applied stress level to further propagate the fracture. If the reverse is true, then the rupture will slow down prior to accelerating again, toward rapid disintegration over the relatively 'soft' (i.e., low W_F) regions $ghiklm$.

If the work of rupture is supplied through a perfectly rigid loading system, then, by definition, the applied load always will be exactly equal to that required for rupture propagation — i.e., the energy released by the loading system exactly balances the work of fracture at all stages of rupture. In Figure 14, the energy-release-rate curve everywhere overlies the work of the fracture curve. Rupture proceeds at precisely the applied displacement rate of the loading system.

Actual rupture of real systems generally falls between the above two extreme idealizations. Even where a specimen is loaded through rigid grips, for example, the unfractured portions of the specimen release stored energy when rupture is initiated. This can result in significant fracture propagation if, for any reason, W_F tends to decrease after initiation.

The behavior of real systems is illustrated by Figure 15

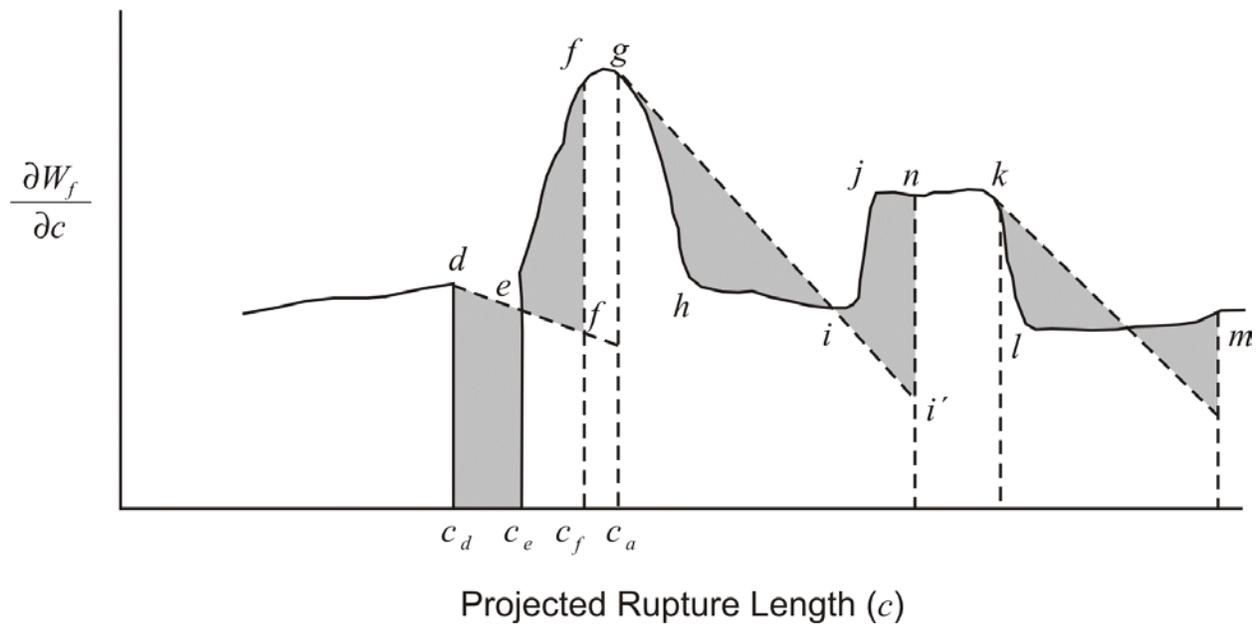


Figure 15 “Energy Release-Work of Fracture” interaction during tensile rupture of a rock specimen under realistic loading conditions [same specimen as in Figure 14]

As with the dead-load system, the rupture encounters a pore-space or pre-existing crack at d . The excess energy release, shaded area $dc_f c_e e$ is now somewhat lower because the applied load decreases with fracture extension, due to the finite stiffness of the applied load. The fracture accelerates to a length c_e , then starts to decelerate, reaching a length c_f at which the deficiency of energy release, the shaded area eff' , balances the excess generated over the extension $c_d - c_e$. The applied force then is increased to further extend the fracture through the hard grain, from c_f to c_g . At this point, the system again becomes temporarily unstable, developing kinetic energy equivalent to the shaded area ghi , as the applied load drops less rapidly with fracture extension than the rate of change in W_F . Again, the fracture decelerates and eventually halts, at length c_n , such that the area ghi equals area $ii'nj$. The sequence is again repeated when the length c_k is reached. Eventually (not shown in Figure 15), as the edges of the specimen are approached, a macroscopic instability ensues, and the specimen ruptures.

The excess energy released by a ‘soft’ loading system may be sufficient to propagate a fracture through ‘hard’ zones, for example, to the point where total disintegration may occur, without increase in the applied load. With a relatively stiff loading system, the corresponding excess energy may be sufficiently lower that the fracture is arrested at a shorter length. The applied load would then need to be further increased to produce final disintegration. Glücklich and Cohen (1967) suggest that this loading stiffness effect could be sufficient to cause significant differences between the ‘strengths’ of a given material measured in a soft and a stiff system. There is yet little evidence to indicate the importance of the effect in rock testing.

10.0 ROCK FRACTURE IN COMPRESSION

Many of the practical problems of rock mechanics involve disintegration or structural collapse of the rock due to the application of compressive loads. As mentioned earlier, however, compressive disintegration results from the action of tension and shear forces generated within the loaded rock by the applied compression.

Griffith (1924) attempted to extend his theory of rupture to explain the observed strength of brittle materials in compression by assuming that tensions equal to the cohesive strength σ_m (Figure 1) were generated locally at the 'flaws' within the material. Accepting the Griffith hypothesis that the uniaxial tensile strength of a brittle solid is determined by flaws within the solid, then if, as in an isotropic material, the strength is the same for all directions through the solid, the flaws must be randomly distributed through it. In direct tension, the most critical orientation of a flaw (assumed to be representable as an elliptical crack) is normal to the applied tension, as this produced the greatest amplification, or 'concentration', of the applied stress, allowing the theoretical cohesive strength to be reached.

If the Griffith crack is located within a compressive stress field, localized tension around the crack is possible only:

- (1) when the crack is oriented at an angle to the principal directions of applied stress;
and
- (2) when the applied stress field is not hydrostatic.

Using the analytical solution by Inglis (1913) for stresses around an elliptical crack in a uniformly loaded plate, Griffith examined the tensions generated around the crack tip. Assuming that the crack remained open in the compressive field, he noted that the magnitude of the tension changed both with position along the crack edge and with orientation (see Figure 16) of the crack axis to the direction of maximum applied compression (σ_1). Specification of the required tension stress (i.e., the cohesive strength σ_m) for fracturing posed a fundamental problem, as both the real value of σ_m and the actual shape of the 'flaws' were unknown. [In the original analysis of the tensile rupture, Griffith had, in effect, merely demonstrated that **any** value of σ_m could be achieved with a realistic value of applied tension, provided the flaw (i.e., crack) was sharp enough. He did not specify any required dimensions or cohesive strengths.]

To overcome this difficulty, he assumed that the [inclined] critical flaw in the compression field was of the same [unknown] shape and size as the [normal] critical flaw in the direct tension field. For this case, the **ratio** between the maximum tension around the compression crack and that around the direct tension crack is independent of the crack shape. Assuming, then, that the same value of σ_m must be achieved in both cases, he could express the condition for the onset of crack initiation in compression in terms of the tensile rupture criterion expressed by 10(a) and 10(b). Proceeding in this way, he derived the two-part condition for the onset of rupture as follows.

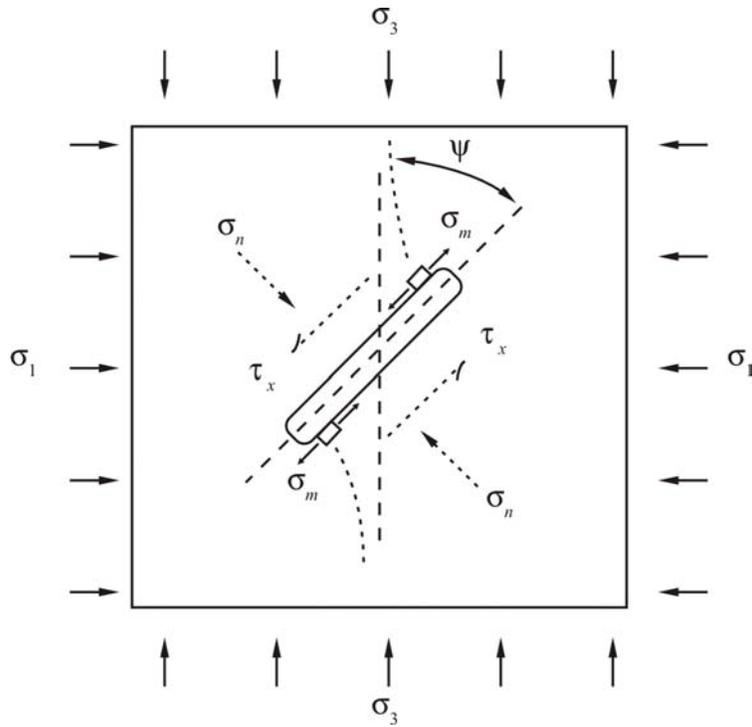


Figure 16 Griffith crack in a compression stress field

Rupture begins

- (i) when $(\sigma_1 - \sigma_3)^2 + 8\sigma_0(\sigma_1 + \sigma_3) = 0$; if $3\sigma_1 + \sigma_3 > 0$

In this case, the plane of the initial crack is oriented to the direction of σ_3 at angle ψ , where

$$\psi = \frac{1}{2} \cos^{-1} \frac{\sigma_1 - \sigma_3}{2(\sigma_1 + \sigma_3)} \quad (39)a$$

- (ii) when $\sigma_1 = \sigma_0$; if $3\sigma_1 + \sigma_3 > 0$

In this case, the initial crack orientation is

$$\psi = \frac{\pi}{2} \quad (39)b$$

In (39), tensile stresses are assumed positive, $\sigma_1 > \sigma_3$; σ_0 is the uniaxial direct tension strength of the material; and ψ is the orientation of the crack (along which rupture starts) to the direction of σ_1 .

The criterion expressed in (39) includes the tensile rupture criterion, as the limiting case of (39b). It also indicates that, for a range where the major principal stress (σ_1) is tensile and the minor principal stress (σ_3) is compressive, but less than one-third of the absolute value of σ_1 , then rupture will occur as in direct tension — i.e., the relatively small compressive stress has no influence. It is interesting that the criterion indicates that the rock will begin to rupture in uniaxial compression ($\sigma_1 = 0$) when the applied compression (σ_3) reaches a value 8 times as high as the direct tensile strength (σ_0). Although somewhat low, the ratio is of the correct order of magnitude. [Compression strength values for rock specimens usually range from 15-20 times the direct tension strength.]

The fact that (unequal) applied compressions produce tensions close to the tips of an inclined crack can be readily understood by noting that the shear stresses (τ_ψ in Figure 16) produced parallel to the crack axis by the compressions will tend to cause displacement of each side of the crack in the opposite sense (left-hand side downward, right-hand side upward in Figure 16). This will result in a tensile ‘tearing tendency’ slightly off each tip.

11.0 CRITICISM OF GRIFFITH COMPRESSION CRITERION

Surprisingly, in deriving the Compression Fracture criterion, Griffith did not consider at all the question of the energetics, or stability, of compression fracture. It seems he implicitly assumed that, once initiated, compression fracture extension would be an unstable process, leading directly to collapse (although he intimated that the situation might need more study when noting that the compression fracture may not propagate along the plane of the major crack).

Subsequent work (Brace and Bombolakis 1963; Hoek 1968) has demonstrated that, indeed, the compression criterion of (30) is a crack **initiation** criterion only, which does not result in energy instability. Thus, failure to examine stability considerations has led to an erroneous fracture criterion — by the very person who introduced the concept of fracture as due to an energy instability!

It must be said, however, that study of energy stability for the compression situation (i.e., as shown in Figure 16) is considerably more complicated than for the direct tension situation. The compressive crack is not initiated along the axis of the pre-existing crack, and it has been found (Brack and Bombolakis 1963; Hoek and Bieniawski 1966) that it changes orientation continuously as it is propagated [under increasing stress difference ($\sigma_1 - \sigma_3$)]. The extending crack tends to align itself parallel to the direction of maximum applied compression (see dotted lines in Fig. 16); this is the most stable orientation (i.e., the one for which the compressive stress field produces the least tension at the extending crack tip) for a crack. Wawersik (1965) has shown that the tensile stress concentration produced by crack extension parallel to an applied principal compression tends, in the limit, to zero. Thus, although tension cracking in a compressive stress field can occur in the manner indicated by Griffith, it does not lead to an unstable energy situation. Compression collapse cannot occur by this mechanism alone. Hoek and Bieniawski (1965) have studied experimentally the propagation of cracks from an open elliptical crack in a glass plate. The results are reproduced in Figure 17.

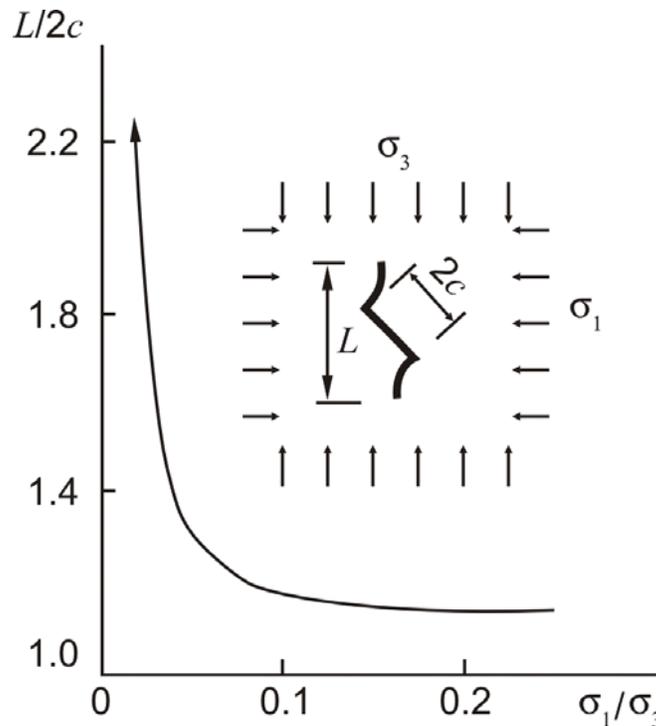


Figure 17 Length of stable crack propagated from an open elliptical flaw under compressive stress conditions (after Hoek and Bieniawski 1965)

These results confirm the conclusion that the crack extension is a stable process. They also reveal how effectively a small minor compressive stress (σ_1) suppresses crack extension due to the compressive stress concentration effect of σ_1 at the tip of the extending crack.

A second important objection to the Griffith compression criterion is the assumption that the original (included) crack will remain open. It seems obvious that it will tend to close in the compressive stress field. McClintock and Walsh (1962) have examined theoretically the significance of crack closure. It is apparent that, if the closed crack is subjected to a normal stress, σ_n (Figure 16), then shear stress τ_ψ , tending to produce the tangential tension σ_m , will be counteracted by a frictional shear resistance ($\tau_r = \mu\sigma_n$), where μ is the coefficient of friction across the closed crack surfaces. The shear stress effective in producing tension will then be $(\tau_\psi - \mu\sigma_n)$. The net result is that, for the compression region of the Griffith criterion, as represented by (39a), the parabolic condition is replaced by a linear criterion, which is essentially equivalent to the Coulomb criterion of failure. The McClintock-Walsh (1962) modification does not consider the problem of stable crack propagation. The crack will still extend essentially as in the Griffith compression criterion, albeit at a still higher stress level.

The Griffith crack is but one example of an inhomogeneity in the rock. Pore spaces, soft inclusions in a hard matrix, and hard grains (inclusions) in a soft matrix are other examples. In all of these, it seems that the stress field produced by the inhomogeneity is such as to promote tensile cracking essentially parallel to the major applied compressive stress. In addition, in all cases, the cracks are stable and tend to propagate for a short distance only from the point of initiation. It is not surprising,

therefore, that extensive short-range cracks, all more or less parallel to each other within grains or from grain boundaries, are frequently observed in rock specimens. The condition is often present in rocks as they are taken from the field before any loads are imposed in the laboratory. If, as in the case of laboratory tests, or in the field where rock is either adjacent to a free surface or confined by compression, the rock is not subjected to an overall tension (σ_1), normal to the major applied compression (σ_3) [Note: tension stresses positive], then the sum of the local tensions (in the direction of σ_3) by the inhomogeneity (or flaw) must be balanced by equal local compression forces acting in the same direction, as shown along AB in Figure 18. (The same is true across any vertical section) in order to be in equilibrium with the free surface stress $\sigma_1 = 0$. [Addition of a confining pressure $\sigma_1 \neq 0$ will merely bias the local stresses so that the resultant force is in equilibrium with σ_1]. This, again, indicates that the tensions induced by inhomogeneities or flaws situated in a compression crack extension result in stable crack extension. In cases where end-loading conditions (i.e., across surfaces CAD, EBF) are non-uniform, the tension region may extend continuously over a large central region of the specimen, the ‘balancing compression’ being concentrated close to the ends. [See, for example, the tangential tension stress distribution for the Brazilian test situation.] In such cases, axial cracking parallel to σ_3 may extend much farther. It could accelerate and penetrate into the compression region before stopping, much as described earlier for unstable transient crack growth in tension. Axial splitting of compression test specimens, which is so frequently observed, is probably the result of the development of non-uniform axial loading during the test.

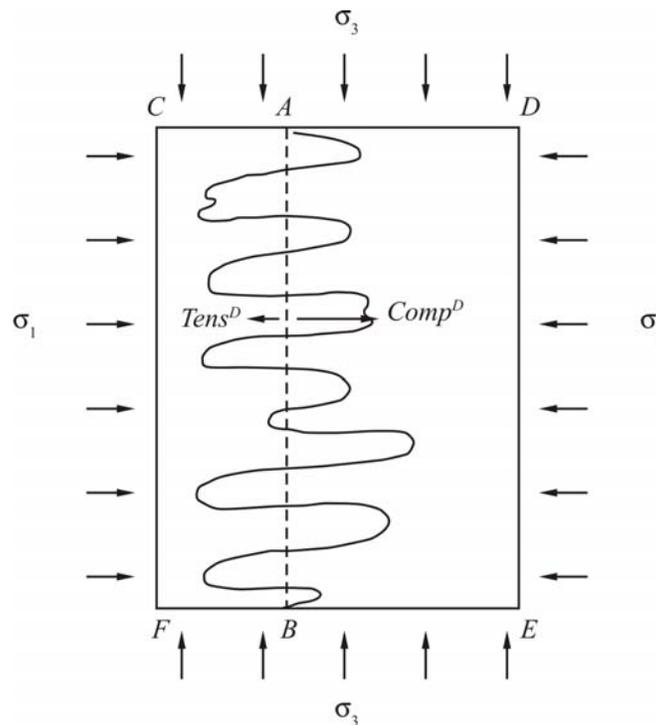


Figure 18 Schematic distribution of localized tension and compression forces along a vertical section of a laterally unconfined specimen loaded vertically in compression

In summary, then, axial cracking parallel to the direction of major compression is likely to occur in most rocks. The extend of such cracking in macroscopically uniform compressive loading system is likely to be limited, and is rapidly attenuated in the presence of confining stress. Axial cracking will not produce unstable collapse in uniform compressive stress fields. Some other mechanism must be responsible for collapse in compression. Axial shortening due to shear displacement along planes inclined to the major compression is an obvious possibility, assuming shear displacement is possible.

Cook (1965) has considered the problem of unstable propagation of a shear crack in a uniform compressive stress field. The procedure parallels that used by Griffith in the analysis of tensile rupture. It is implicitly assumed that the shear stress developed at the tip of the crack is sufficient to exceed the cohesive strength σ_m by shearing.

Starr (1928) showed that the increase in strain energy (ΔU) in a plate due to the introduction of a crack that undergoes shear displacement in the applied plan (shear) stress field is given by

$$\Delta U = \frac{\pi \tau^2 c^2}{E} \quad (40)$$

where τ is the shear stress parallel to the frictionless crack,
 c is the half-length of the crack, and
 E is the modulus of the plate material.

It is assumed in (40) that there is no discontinuity of normal stress across the shear crack.

Cook (1965) modified (40) to take account of friction along the crack in compressive shearing, replacing τ in (40) by τ_{eff} , the effective shear stress, given by

$$\tau_{eff} = \tau - \mu \sigma_n \quad (41)$$

where σ_n is the compressive normal stress, and
 μ is the coefficient of friction between crack surfaces.

Equation (40) then becomes

$$\Delta U = \frac{\pi (\tau - \mu \sigma_n)^2 c^2}{E} \quad (42)$$

Proceeding as with the original Griffith criterion, we have

$$\Delta S = 4c\gamma_s \quad (43)$$

where γ_s is the “Work of Fracture in Shear”.

Unstable shear fracture will occur when

$$\frac{\partial}{\partial c} \left(4c\gamma_s - \frac{\pi(\tau - \mu\sigma_n)^2 c^2}{E} \right) \leq 0 \quad (44)a$$

That is, instability develops when

$$(\tau - \mu\sigma_n) = \sqrt{\frac{2E\gamma_s}{\pi c}} \quad (44)b$$

The criterion of (44b) will be reached first at the crack orientation for which, in a given applied compression field, $(\tau - \mu\sigma_n)$ is a maximum. This orientation is shown to be given by the expression

$$\psi = \frac{1}{2} \tan^{-1} \frac{1}{\mu} \quad (45)$$

where ψ is the angle between the crack axis and the direction of major compression, as in Figure 16. τ and σ_n can, of course, be expressed in terms of σ_1 and σ_3 , and ψ (or, equivalently, μ). The shear criterion (44b) again is equivalent to the Coulomb criterion, differing from the McClintock-Walsh criterion and the original Coulomb criterion only in the value of the constant — i.e., the right-hand side term in (44b).

The value of γ_s , the Work of Fracture in Shear, probably is dependent on several factors and is unlikely to be a constant. Thus, in unconfined compression, axial splitting, by developing ‘ligaments’ in the rock, will tend to reduce the resistance to shear. With the addition of confining pressure and elimination of splitting, the work of fracture in shear may increase rapidly. At still higher confining pressures, some rocks may develop some form of plastic deformation at the crack tip, which may reduce the work of fracture. Variation of γ_s in this way could explain the ‘less than linear’ increase in compressive ‘strength’ with increase in confinement, particularly at high values of confining pressure. Cook (1965) has expressed the shear failure criterion of (44b) in terms of a stress-strain curve in compression in essentially the same way as did Berry for the Griffith Tensile Rupture Criterion. The same form of fracture locus is obtained.

Complete load-deformation curves have been obtained experimentally for compression tests on rock specimens. In general, the curves tend to have the characteristic appearance shown in Figure 19. Examination of sectioned specimens removed from tests at various stages during the complete load-deformation event indicates that the sudden drop in load-bearing capability exhibited in all tests up to quite high confining pressure is associated with microscopic shear fracture along essentially a single planar zone. The major shear is preceded in

- (a) unconfined tests, by axial splitting, slabbing of edges, grain rotation, and minor shears distributed through the rock; and

- (b) confined tests, mainly by shears distributed throughout the specimen. Axial cracks and slabbing are generally absent.

Detailed interpretation of the compressive load-deformation curve is difficult, particularly as the peak load is approached. The rock tends to dilate appreciably at loads above 50% of the peak, indicating inelastic deformations. (The dilation is accompanied by micro-seismic activity; see Fig. 19.)⁶ The dilation usually is restricted in the vicinity of the loading platens, so that the applied load must become non-uniformly distributed to an increasing extent.

It appears that no simple quantitative meaning can be assigned to the peak load in compression, which is often referred to as “the compressive strength”.

Disintegration in compression tests proceeds in a very heterogeneous way, often being concentrated along a central shear zone located between more or less intact parts of the specimen at each end, adjoining the loading platens. Reduction of the data to average ‘stress-strain curves’ is very misleading and should be avoided. It is preferable to record the load-deformation response, noting the specimen dimensions, and the mode of disintegration.

12.0 GENERALIZATION OF ENERGY INSTABILITY CONCEPT OF FRACTURE AND COLLAPSE

Although details of the Griffith approach to fracture may be of uncertain validity, the general view of fracture (and eventual collapse) as the result of energy instability in the system can be a very fruitful one. It seems that Griffith's high-stress condition can always be achieved in brittle rocks and so can be taken for granted. The deformation behavior of any system can be examined in terms of the Energy Supply (or Release) Rate and the Energy Demand Rate at all stages. When supply exceeds demand, the system becomes unstable, either temporarily or permanently. In the latter case, the system will collapse.

The advent of high-speed digital computers, and the associated development of approximate numerical methods of analysis in mechanics, facilitates application of the energy instability approach to many situations for which it previously had been impractical. This is particularly true of situations involving complicated geometry.

Hardy (1971), for example, has analyzed the so-called Brazilian test, an indirect tension-test used on rocks, as an energy instability problem. The same approach has been used on other indirect tension tests, such as the beam test and the ring test. It is well known that computation of the ‘tensile strength’ on the basis of the peak stress generated in the various tests results in a wide variation (up to 600%) of strength values. This suggests that the calculated peak stress developed prior to fracture (usually computed on the basis of linear elasticity) is not a meaningful quantity. Test results

⁶ The onset of microseismic activity corresponds, in principle, to the start of the (stable) crack growth predicted by the Griffith Criterion for the ‘strength’ of brittle materials in uniaxial compression, as discussed in connection with Figure 16. If we assume that the actual strength is 16~20 times greater than the uniaxial tensile strength, and that the Griffith uniaxial ‘strength’ in compression is 8 times the tensile strength, then microseismic activity should start at about 40%~50% of the peak load in uni-axial compression. This compares well with observations on specimens in laboratory tests.

correlated based on an energy instability analysis provide more consistent values for the rock 'strength'. The adoption of Fracture Toughness in preference to tensile strength as a relevant engineering measure in metals technology is a development along these lines, stemming from Irwin's (1958) adaptation of Griffith's theory of rupture.

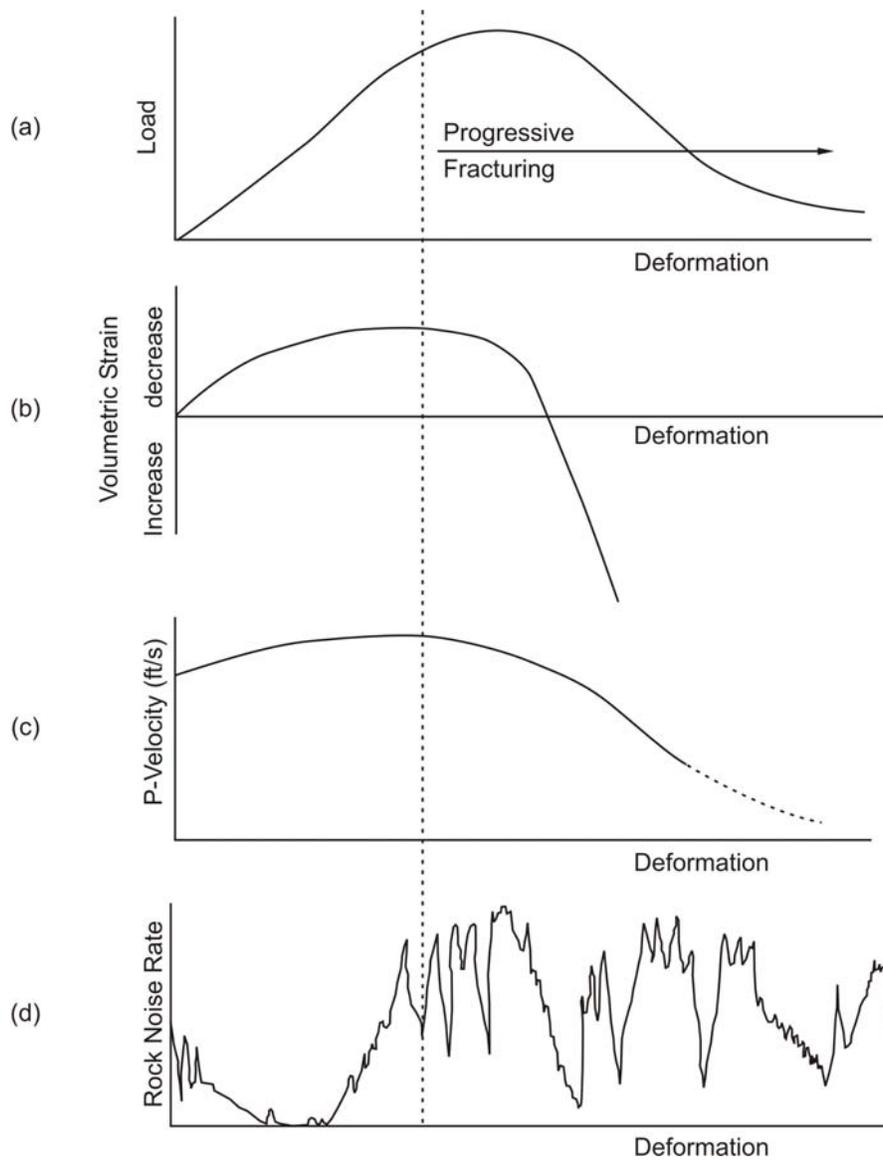


Figure 19 Typical behavior of a cylindrical specimen of rock compressed to disintegration in a controlled laboratory test: (a) complete load-deformation curve; (b) volumetric strain-deformation curve; (c) seismic velocity (transverse to axis of loading) — deformation curve; (d) seismic noise rate-deformation curve (All curves relate to tests on the same specimen.)

The energy-instability approach is not limited to microscopic events. It can be used, for example, in the analysis of slabbing around tunnels (Fairhurst and Cook 1966), tunnel stability (Daemen and Fairhurst 1971), crater formation in blasting (Porter and Fairhurst 1971) and, indeed, any situation involving rupture.

13.0 SIZE-STRENGTH RELATIONSHIPS FOR ROCK

As noted earlier, relationships (19), and (40), of the general form

$$S^2 d = \text{constant} \quad (46)$$

where S is the rupture stress, or ‘strength’ in compression, and

d is a parameter with dimensions of length that characterizes the critical flaw size

are a consequence of the type of elastic instability assumed to govern structural collapse (or rupture). The ‘energy supply’ is released from a volume proportional to d^2 (per unit thickness).

Thus, if the size of the critical flaw increased at a rate directly proportional to the size (linear dimension L) of the test ‘specimen’, the strength (S) should be related to the size, for geometrically similar specimens and applied load distributions, as follows:

$$S^2 L = \text{constant} \quad (47)$$

This relationship appears to be approximately confirmed from test data on coal (Holland 1964; Salamon and Munro 1967) — i.e., the strength S decreases as $L^{-0.5}$. Coal is an extensively cracked brittle material, and it is reasonable to presume that coal specimens and coal mine pillars will contain flaws just as large as the pillar size will allow.

If, however, the critical flaw size is a constant (i.e., does not increase with specimen size), then it may be expected that, provided the specimen is large enough to accommodate a reasonably large number of critical flaws, the strength will be independent of size. Limited tests on the bending strength of granite beams, ranging from 1 ft. long x 1 inch thick to 30 ft. long x 12 inches thick, indicate a virtually constant strength for all sizes. Laboratory compression tests on homogeneous limestones and marbles from 1/2 inch to 6 inch in diameter also tend to indicate a size-independent strength. In such rocks, it appears that the ‘flaw size’ is determined by the maximum grain size.

If tests are conducted on a scale such that the region critically stressed is limited to a size approaching that of the ‘flaws’ themselves, then the general concept of a flaw located within a relatively large (constant modulus) energy supply zone is no longer valid. In an extreme case, the critically stressed region may lie entirely within a (‘hard’) single grain; in a repeat test, it may lie, although with lower probability, entirely adjacent to a weak grain boundary. On this small scale, test results are likely to be quite erratic, usually exhibiting markedly increased strengths. The scale is then too small, and, as in the case of single atom-pairs, notions (and formulae) from classical continuum mechanics are simply not applicable.

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APPENDIX

1. Change in Strain Energy of a Plate Due to Introduction of a Crack

A simple derivation of the change in strain energy of the plate with and without the crack has been presented by Irwin (reported as Appendix 1 of the paper by Roberts and Wells(1954). Using Westgaard's solution for the displacement around a crack in a biaxially loaded plate in plane stress, he shows that the displacement 'v' of the crack edges from the major crack axis ($y=0$) due to a stress σ applied on the outer boundary of the infinite plate is given by the expression:

$$v = \frac{2\sigma\sqrt{c^2 - x^2}}{E} \quad (A1)$$

where $+c \geq x \geq -c$.

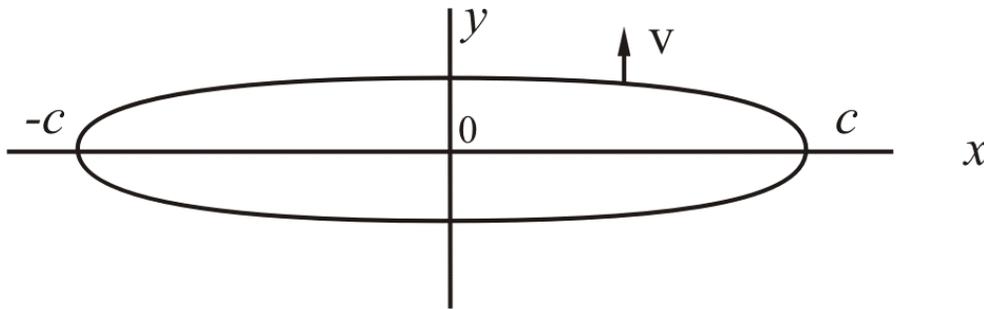


Figure A1 *Opening of a Line Crack Due to Constant Stress σ at Infinity*

To close the crack it would be necessary to apply a uniform tension normal to the crack surfaces. Since the plate is assumed to be linearly elastic, the crack will close uniformly with increase in the crack tension, until the latter reaches the stress $\tilde{\sigma}$. At this value the crack surfaces will meet and the crack, in effect, will no longer exist i.e. the plate is everywhere under homogeneous tension σ .

If the volume of the fully open crack is V per unit thickness, then the work (W) done in closing the crack will be

$$W = 1/2(V\sigma) \quad (A2)$$

Noting the symmetry of the crack edge displacement 'v' (Figure A1), we may write

$$W = 2\sigma^2 \int_0^c v \, dx \quad (\text{A3})$$

From A2, we have

$$W = \frac{2\sigma}{E} \int_0^c \sqrt{c^2 - x^2} \, dx = \frac{\pi c^2 \sigma^2}{E} \quad (\text{A4})$$

Thus, the difference between the strain energy of the plate with and without the crack is $(\pi c^2 \sigma^2) / E$.

This method of computation, based on the recognition that the energy change can be computed by considering displacements at the crack edges only (rather than throughout the plate), is very useful, particularly when finite plates are considered. Irwin's method allows easy numerical computation of energy changes.

2. Kinetic Energy of a Moving Crack

Consider the planar crack, length $2c$, (as in Figure A2) moving with the instantaneous velocity, v_0 , along the x-axis. An arbitrary element $dx dy$ has been displaced by an amount u , as shown. The kinetic energy of the element, mass ρ , is given by

$$\begin{aligned} & \frac{1}{2} \rho dx dy \left(\frac{du}{dt} \right)^2 \\ &= \frac{1}{2} \rho dx dy \left(\frac{dc}{dt} \right)^2 \left(\frac{du}{dc} \right)^2 \\ &= \frac{1}{2} \rho v^2 \left(\frac{du}{dc} \right)^2 dx dy \end{aligned} \quad (\text{A5})$$

The total kinetic energy (KE) of the (infinite) plate is then

$$K.E = \frac{1}{2} \rho v^2 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left(\frac{du}{dc} \right)^2 dx dy \quad (\text{A6})$$

Mott then reasons “The value of u near the surface of the crack is of order σ_c / E and, thus, the integral is, on dimensional grounds, a multiple of $c^2 (\sigma / E)^2$; the numerical factor could easily be calculated.” Hence, the kinetic energy is

$$K.E = \frac{1}{2} k \rho c^2 v^2 \left(\frac{\sigma}{E} \right)^2 \quad (A7)$$

where k is the numerical factor. Its value has been computed by Roberts and Wells (1954) ($k \cong 45$ for a material with Poisson's ratio $\nu = 0.25$).

Roberts and Wells (1954) have determined that, for a material for which Poisson's ratio $\nu = 0.25$

$$\sqrt{\frac{2\pi}{k}} \cong 0.38.$$

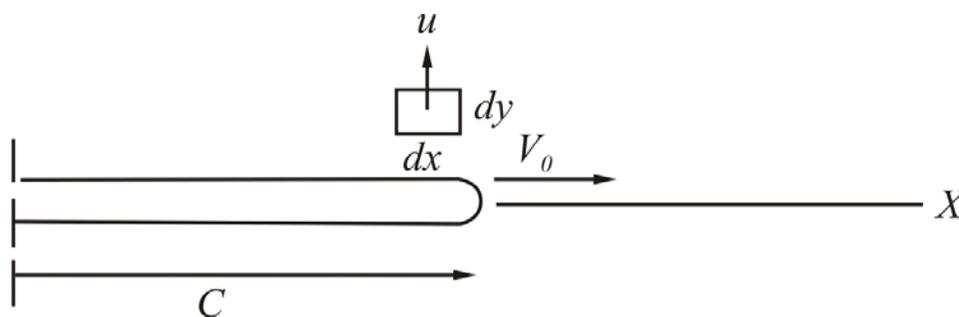


Figure A2 Planar crack, length $2c$, moving at velocity V_0