

# Turbulence in Ground-Water Flow

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GEOLOGICAL SURVEY PROFESSIONAL PAPER 402-E



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By W. O. SMITH *and* A. NELSON SAYRE

INFILTRATION AND DRAINAGE IN UNIFORM SANDS

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## INFILTRATION AND DRAINAGE IN UNIFORM SANDS

### TURBULENCE IN GROUND-WATER FLOW

By W. O. SMITH and A. NELSON SAYRE

#### ABSTRACT

The problem of turbulent flow in ground water is analyzed by using pipes of circular section as an approximation of the flow channels of underground rocks. Pipe diameters ranging from 0.01 inch to 6 inches have been considered. Critical velocities have been calculated from the Reynolds equation. The hydraulic gradients necessary to produce these critical velocities have been calculated on the basis of laminar flow in pipes. The problem has been analyzed also for gravity flow in a channeled rock having a given angle of dip. A further approximation has been made for uniform sediments.

The results show that underground flow is turbulent in the larger pipes at relatively slight hydraulic gradients. They further indicate that turbulent flow does not occur in uniform sands finer than very coarse sand. Under natural conditions undisturbed by man, turbulent flow probably does not occur in material finer than medium to coarse gravel.

#### INTRODUCTION

The motion of fluids is governed by well-established laws of hydrodynamics. As in the motion of solids, inertial forces must exceed resistive forces if motion is to occur. In the motion of solids, however, the resistive forces are exerted outside the moving body because distortion is negligible; in the motion of liquids the resistive forces are exerted not only on the outside of the fluid, but also within the fluid itself. The motion may be either of two kinds—laminar or turbulent.

Laminar flow is usually slow and steady, one layer of particles sliding smoothly over the next and each layer advancing at a different rate of speed. In this type of motion, the resistance to flow is determined chiefly by the friction between succeeding layers of molecules. For a fluid of any given viscosity, velocity of flow and hydraulic head are directly related. Many investigators have shown that laminar flow occurs only when the inertial forces are slightly greater than the resistive forces due to shear.

Turbulent flow is usually more rapid than laminar flow; eddies are formed and, as a consequence, the resistance due to friction is greatly increased. The

relationship of velocity to flow is not linear, as flow varies with the square root of the loss in hydraulic head.

Reynolds (1883) demonstrated the nature of these two types of flow in a series of experiments using straight horizontal glass tubes of uniform diameter. He established that, for a liquid of given viscosity, there is a direct relationship between the velocity of flow and the diameter of the tube, that at low velocities the flow is laminar, but that when the velocity is increased beyond a certain point the flow becomes turbulent. It is generally accepted that the flow in large tubes or pipes is turbulent but that the flow in tubes of small diameter is laminar. However, Tolman (1937) pointed out that laminar flow is not necessarily associated only with tubes of small diameter. Whether flow is laminar or turbulent, in fact, depends upon the ratio between velocity of a liquid of any given viscosity and tube diameter.

In an attempt to apply this principle to the movement of ground water, we are faced with several difficulties. The pore spaces in granular materials are not straight tubes of uniform diameter, but rather are an assemblage of an infinite number of tubes of irregular diameter interconnected at many places. These materials represent the most common water-bearing deposits, in which the walls of the tubes are irregular, the openings small, and the flow velocity low. Thus the inertial forces are barely sufficient to overcome the resistive forces. However, in some coarse gravel, in cavernous limestone, in tubes in basalt, and in some fractured rocks, the shear resistance to fluid movement may be negligible in comparison to inertial forces. The relation between shear resistance and head is not linear; the flow is turbulent; and the velocity varies as the square root of the loss in head.

Because the diameter of the pore spaces in natural rocks is infinitely varied, it is not correct to assume that the flow of ground water is always laminar, even through granular materials, and that Darcy's law

always applies. Grain size has commonly been used to arrive at an approximation of the diameters of the pores in granular materials, and although the determination of the pore diameters is not as simple as the calculation of velocities and differences in head needed to provide precise determination of the limits of velocities of laminar flow in straight tubes, it may, by a series of approximations, provide a measure of the physical requirements needed to determine the upper limits of the applicability of Darcy's law.

Approximate values of the critical velocity and the hydraulic gradient required to produce turbulent flow may easily be obtained. An attempt will be made in this paper to analyze the problem on the basis of a simple approximation of the internal geometrical structure of a rock. Critical velocities will be calculated, and the critical values of the hydraulic gradient, in feet per mile, required to produce these critical velocities will be determined.

For the purpose of this discussion, the rock structure will be represented by a series of cylindrical pipes, each having a diameter of  $d_i$  and a circular cross section. Three types of flow are presented: first, horizontal flow under a hydraulic gradient  $G=(h_1-h_2)/l$ ; second, flow down an inclined bed under gravity, with no external hydraulic gradient; third, flow down an inclined bed under the combined effects of gravity and an external hydraulic gradient  $G=(h_1-h_2)/l$ .

#### CRITICAL VELOCITY OF HORIZONTAL FLOW UNDER A HYDRAULIC GRADIENT $G=(h_1-h_2)/l$

Figure 1A shows an assemblage of horizontal pipes approximating the channels through which water flows in the rock mass  $R$ . The nature of the flow in a single pipe is determined by the size  $d_i$  of the pipe and by the head producing the flow. Whether the flow is laminar or turbulent depends upon whether the actual velocity exceeds or is less than the critical velocity given by Reynolds' (1883) criteria. Gibson (1947, p. 49) stated this relation as

$$v_c=2,000(\mu/\rho)/d, \quad (1)$$

where  $v_c$  is the critical velocity (lower critical velocity), above which the flow may be turbulent and below which the flow will be laminar,  $\mu$  is the viscosity of the liquid and  $\rho$  is its density, and  $d$  is the pipe diameter.

Examples of typical vertical sections of the rocks referred to figure 1A, if a porosity of 25 percent is assumed, are approximately the same as those shown in figure 2. Figure 2A shows a single pipe in a 1-square-foot cross section of rock. Figure 2B shows nine pipes in a 1-square-foot cross section of rock. (The porosity in each rock is 25 percent.) The diam-

eter of the single pipe shown in figure 2A is 0.56 foot; that of each pipe shown in figure 2B is 0.14 ft. The total cross-sectional area of the nine pipes in figure 2B is equal to the total cross-sectional area of the single pipe shown in figure 2A.

The nature of the flow in a section of rock such as that shown in figure 2B is determined by the area of a single pipe rather than by the total area of all the pipes in the section. The pipe is a tube having rock walls for most of its course. The conductive paths approximate a bundle of parallel pipes. The flow through each pipe is determined by the nature of the pipe wall, its diameter, and the hydraulic head across its ends. Jointing and interconnection with other pipes may also affect the pattern of the flow.

The critical velocity will be determined from equation 1 with a value of  $d$  equal to the diameter of a single pipe. Equation 1, therefore, will give the critical velocity of the flow in the bundle of pipes constituting the permeability of the rock.

The flow in the rock is caused by the hydraulic gradient acting on the fluid in the rock interstices. In a bundle of rock tubes, a velocity that is less than the critical velocity  $v_c$  produces laminar flow. If the velocity produced by the gradient exceeds the critical velocity, the flow is turbulent. Thus, in the flow of water through the internal openings or pores of rocks, a critical hydraulic gradient exists for each rock, and it determines the critical velocity in the rock.

If  $p_1$  and  $p_2$  are the respective pressures across the ends of a horizontal pipe having a diameter of  $d$  and a length of  $l$  (Lamb, 1924; Prantl, 1952), the laminar flow will be

$$p_1-p_2=32\mu\bar{v}/d^2, \quad (2)$$

where  $\mu$  is the viscosity of the liquid flowing and  $\bar{v}$  is the mean velocity of flow in the pipe. If  $p_1$  and  $p_2$  are produced by heads of water  $h_1$  and  $h_2$ , respectively, then, as indicated in figure 3A, which shows in detail a single horizontal pipe of figure 1,

$$p_1-p_2=\rho g(h_1-h_2), \quad (3)$$

where  $\rho$  is the density of the liquid and  $g$  the acceleration of gravity. From equations 2 and 3 the hydraulic gradient is

$$G=(h_1-h_2)/l=32(\mu/\rho)\bar{v}/d^2g. \quad (4)$$

The flow becomes turbulent at the lower critical velocity  $v_c$  given by equation 1. At the critical hydraulic gradient  $G_c$ , obtained by substituting the value  $v_c$  for  $\bar{v}$  in equation 4, the flow is at the upper limit of laminar

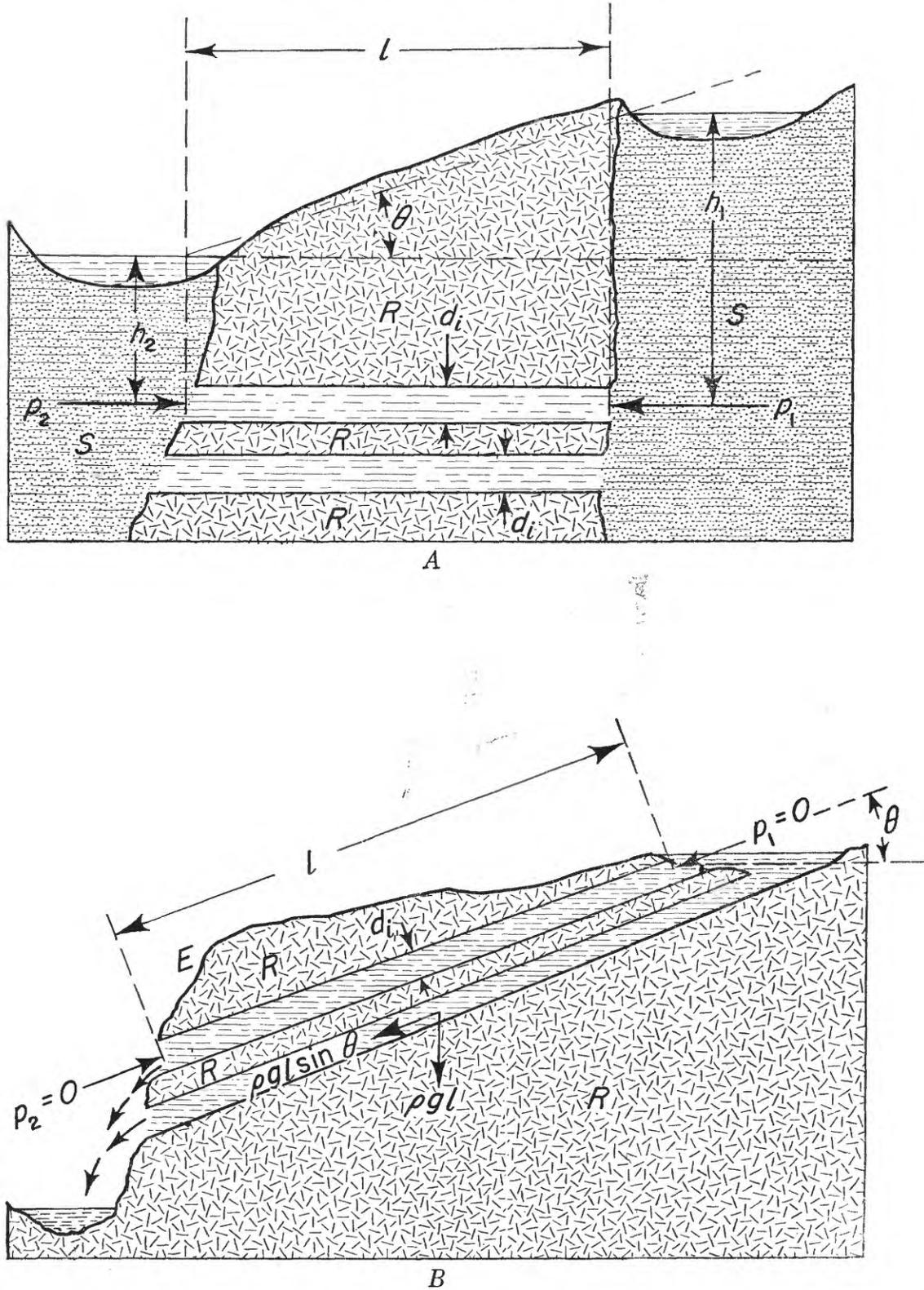
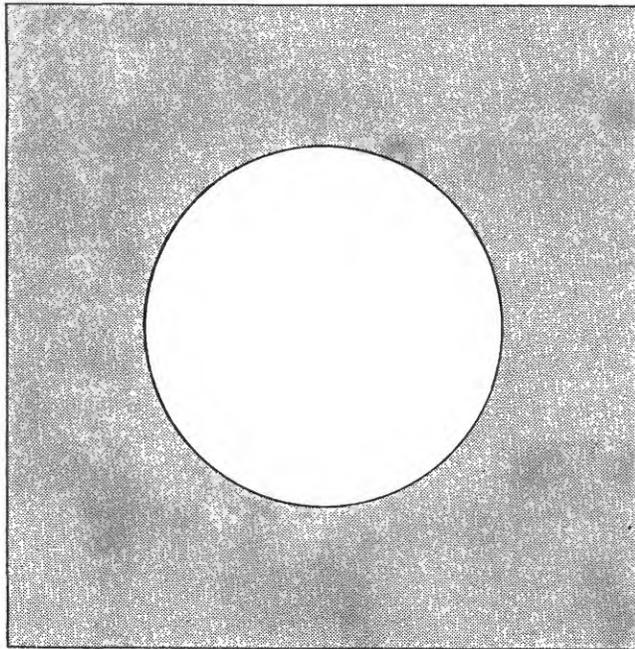
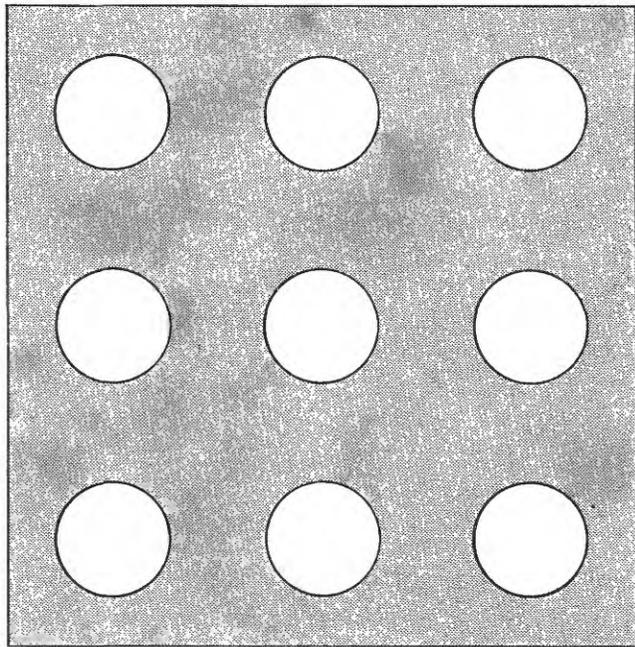


FIGURE 1.—A, An assemblage of horizontal pipes approximating the water-conducting channels of a rock mass,  $R$ .  $d_i$  is the diameter of a single pipe, and  $l$  is its length;  $h_1$  and  $h_2$  represent the elevations of the pipe  $d_i$  with respect to the free-water surfaces at the top of the saturated water-bearing sand or gravel,  $S$ . The channels  $d_i$  contain only water.  $\theta$  is the angle between  $h_1-h_2$  and  $l$ .  $p_1$  and  $p_2$  are the pressures at the entrance and exit, respectively, of  $d_i$ . B, Rock mass  $R$  channeled by an assemblage of pipes inclined at an angle,  $\theta$ . The pipes have a diameter  $d_i$  and length  $l$ . Gravity flow exists exclusively. The entrance pressure  $p_1=0$ ; the discharge is an orifice in an escarpment,  $E$ , and the exit pressure  $p_2=0$ .



A



B

FIGURE 2.—Typical pipe cross sections of the rock mass *S* of figure 1: *A*, single pipe in a 1-square-foot cross section of rock; *B*, nine pipes in a 1-square-foot cross section of rock. The total cross-sectional area of the nine pipes of *B* is equal to that of the single pipe shown in *A*. The porosity is 25 percent for each figure.

flow for this value of  $\bar{v}$ . Thus the critical hydraulic gradient from equations 1 and 4 is

$$G_c = (h_1 - h_2)/l_c = v32(\mu/\rho)/d^2g$$

and

$$[(h_1 - h_2)/l]_c = 64,000(\mu/\rho)^2/d^3g, \quad (5)$$

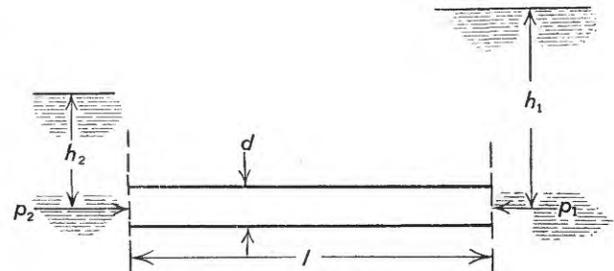
where  $h_1 - h_2$  and  $l$  are given in feet. If  $\theta$  is the angle between  $(h_1 - h_2)$  and  $l$ , then, referring to figure 1A (or fig. 3A),

$$\tan \theta = (h_1 - h_2)/l$$

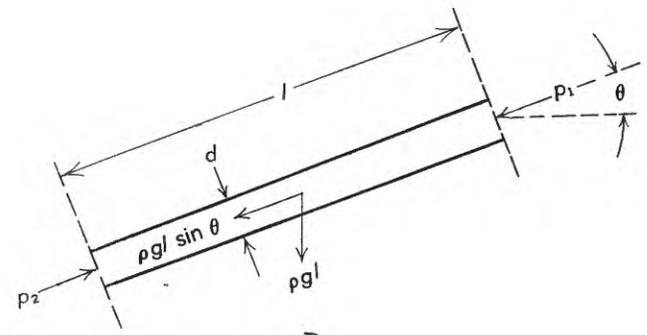
If the hydraulic gradient is in feet per mile, then

$$(h_1 - h_2)/L = 5,280(64,000)(\mu/\rho)^2/d^3g,$$

where  $L$  is given in miles and  $(h_1 - h_2)$  in feet.



A



B

FIGURE 3.—*A*, Single horizontal pipe of figure 1A. The entrance pressure  $p_1$  is produced by the head  $h_1$ ; the exit pressure  $p_2$  equals the head  $h_2$ .  $d$  is the diameter of the pipe of length  $l$ . *B*, Gravity flow in a tube of diameter  $d$  and length  $l$  inclined at an angle  $\theta$ .  $p_1$  is the entrance pressure and  $p_2$  the discharge pressure.

Table 1 shows values of critical hydraulic gradient as well as values of  $\theta$  for several pipe diameters ranging from 0.01 inch to 6 inches. Values of  $v_c$  in feet per mile are plotted (semilog) in figure 4 as a function of  $d$ , the pipe diameter. Values of  $G_c$  in feet per mile are plotted in figure 5 as a function of  $d$ ; a semilog plot is shown because of the extreme variation of  $G_c$  in the range of diameter values  $d$ .

**CRITICAL VELOCITY AS A FUNCTION OF GRAIN SIZE AND POROSITY**

If the rock consists of unconsolidated sediments and if the particles are of approximately uniform size the size of the grains constituting the rock mass may be estimated. The simplest packing is a spaced hex-

TABLE 1.—Critical velocities and critical hydraulic gradient for several tube diameters, and corresponding sediment size (uniform spheres, porosity 48 percent) for horizontal flow

Diameter of pipe (inches)	Critical velocity $v_c$ (ft per sec)	Critical hydraulic gradient ( $G_c$ )		$\theta$	Grain size (mm)	Class name (Lane, 1917)
		$G_c = \frac{h_1 - h_2}{l}$	$G_c$ in (ft per mile)			
6.0	0.482	$2.316 \times 10^{-6}$	0.012	0.5"	300.6	Small boulders.
3.0	.0968	$1.859 \times 10^{-6}$	.0982	6"	153.0	Large cobbles.
1.0	.2900	$5.016 \times 10^{-4}$	2.66	1'40.5"	50.8	Very coarse gravel.
0.50	.581	$4.009 \times 10^{-3}$	21.2	13'46"	25.5	Coarse gravel.
0.25	1.16	$3.192 \times 10^{-2}$	169.0	1°50'	12.5	Medium gravel.
0.10	2.90	$5.016 \times 10^{-1}$	2600	26°38'	5.1	Fine gravel.
0.050	5.81	4.009	21,000	75°59'	2.5	Very fine gravel.
0.025	11.62	$3.195 \times 10^1$	170,000	88°13'	1.25	Very coarse sand.
0.010	29.04	$5.016 \times 10^2$	2,600,000	89°53'	.51	Coarse sand.

As an example, let us suppose that the packing is open and that  $P$  is 48 percent. For this value of  $P$  we find

$$D = (\bar{A}/0.1872)^{1/2} \tag{7}$$

For the purposes of this approximation the values of  $\bar{A}$  will be taken as those of the pipe channels given in table 1. The value of  $D$  corresponding to each of these values of  $\bar{A}$  is shown in the next to last column of

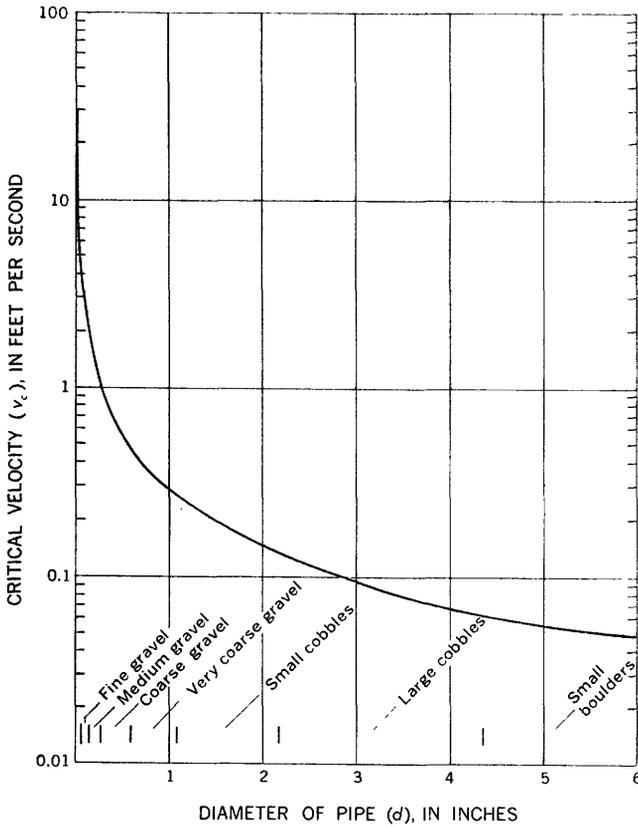


FIGURE 4.—Values of critical velocity ( $v_c$ ) plotted as a function of the pipe diameter  $d$ .

agonal array of uniform spheres of diameter  $D$ , with particle centers at a distance  $(2r+d)$  from each other, which is determined by the porosity,  $P$  (Smith and others, 1931, p. 24);  $r$  is the grain radius and  $(2r+d)$  is the distance between grain centers. The average area  $\bar{A}$  of a single channel through this packing is given by Smith (1932, p. 142) as

$$\bar{A} = 0.360 D^2 \left[ \left\{ \frac{0.9850}{(1-P)^{2/3}} - 1 \right\} \right], \tag{5a}$$

from which the particle diameter  $D$  is given by

$$D = \left[ \frac{\bar{A}/0.360}{\left\{ \frac{0.9850}{(1-P)^{2/3}} - 1 \right\}} \right]^{1/2} \tag{6}$$

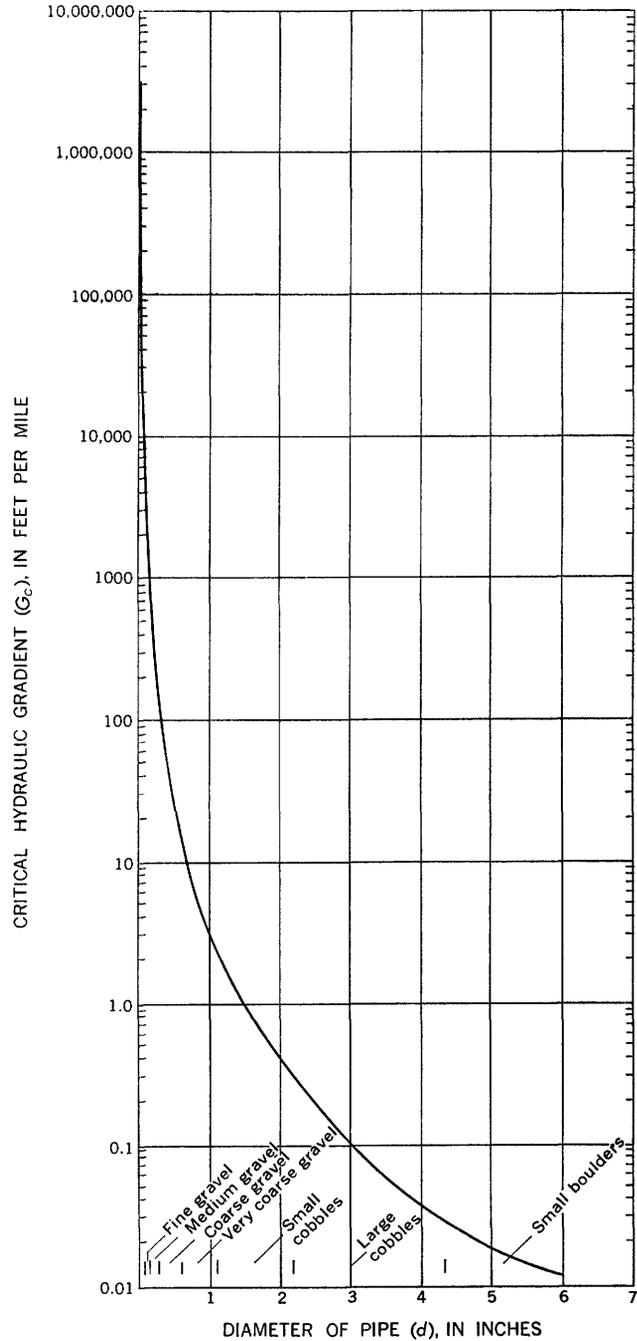


FIGURE 5.—Values of the critical hydraulic gradient ( $G_c$ ) plotted as a function of the pipe diameter  $d$ .

table 1, and the corresponding sediment class is shown in the last column of the table.

The sediment class (Lane, 1947) has been plotted at the bottoms of figures 4 and 5. Obviously, there is little likelihood of turbulent flow through sediments finer than coarse gravel in natural rock formations, unless the hydraulic gradient is exceptionally steep. Furthermore, the discussion seems to indicate that turbulent flow does not occur in sands unless the hydraulic gradient is very great, and then only in very coarse sands.

The general formula for the critical hydraulic gradient for horizontal flow in a uniform sand as approximated from equations 5 and 5a is, since  $A = \pi d^2/4$ ,

$$\left[ \frac{(h_1 - h_2)/l}{c} \right] = 64,000 \left( \frac{\mu}{\rho} \right)^2 / gD^3 \left[ \frac{1.440}{\pi} \{ [0.9850/(1-P)^{2/3}] - 1 \} \right]^{3/2}.$$

The hydraulic gradient in feet per mile can be calculated by multiplying the right member of this equation by 5,280.

The critical velocity  $v_c$  is derived from equations 1 and 5a:

$$v_c = 2,000 \left( \frac{\mu}{\rho} \right) / D \left[ \frac{1.440}{\pi} \{ [0.9850/(1-P)^{2/3}] - 1 \} \right]^{1/2}.$$

#### CRITICAL VELOCITY UNDER GRAVITY FLOW DOWN AN INCLINED ROCK BED

The flow just discussed is horizontal, as shown in figure 1A. For a tube of length  $l$  and diameter  $d$  inclined at an angle  $\theta$  to the horizontal, as shown in figure 3B, gravity must be considered and equation 2 becomes

$$(p_1 - p_2) + \rho g l \sin \theta = 32 \mu l v / d^2. \quad (8)$$

The term  $\rho g l \sin \theta$  is the component of the weight in the direction of the flow, and  $p_1$  and  $p_2$  are the pressures at the entrance and exit of the tube. Most rock structures in nature are covered by equation 8. Equation 8 shows that the general flow in tubes is produced by a combination of pressures, and such body forces as gravity acting on the liquid (Lamb, 1924). Pressures arise from the reaction of one part of the fluid to another (fig. 1A). For example, the flow in the horizontal tube is produced entirely by pressures. Figure 3B shows that the flow in the inclined tube is produced by the pressures that are acting on the fluid and by the pull of gravity on the mass of the liquid flowing down the pipe—specifically, the component of gravity in the direction of the tube.

A more practical example is shown in figure 1B, which shows a rock mass,  $R$ , channeled by an assemblage of pipes,  $d_i$ . The rock mass dips westward, the

angle of dip being  $\theta$ , and terminates in an escarpment,  $E$ , where the internal channels,  $d_i$ , discharge the water that is flowing through them. The flow to the channels originates in a shallow lake. The entrance pressure  $p_1$  is zero. The exit pressure  $p_2$  is likewise zero. Hence, gravity flow prevails. Equation 8 reduces to

$$\sin \theta = 32(\mu/\rho)\bar{v}/d^2g. \quad (9)$$

The critical angle  $\theta_c$  corresponding to the critical velocity  $v_c$  for a given value of  $d$  is obtained by substituting the value of  $v_c$  given by equation 1 for  $\bar{v}$  in equation 9, and

$$\sin \theta_c = 64,000(\mu/\rho)^2/d^3g. \quad (10)$$

The greatest value of  $\sin \theta_c$  is 1, in which  $\theta_c$  is  $90^\circ$  (vertical fall under gravity). The diameter of the channel for this situation is found from equation 10 to be

$$d^3 = 2,000(\mu/\rho)^2(32)/g. \quad (11)$$

For water at  $60^\circ\text{F}$ ,  $(\mu/\rho) = 1.21 \times 10^{-5}$  ft<sup>2</sup>/sec, and  $g = 32.2$  ft/sec<sup>2</sup>. If these values and equation 11 are used,  $d = 0.036$  inch, or 0.914 mm. The corresponding diameter  $D$  of a single particle in a uniform packing, again assumed to be spaced hexagonally for a porosity of 48 percent, is given by equation 7 with a value of  $\bar{A} = \pi(0.914/2)^2$ , and the limiting value of the grain diameter  $D$  given by equation 7 is

$$D = 1.87 \text{ mm}. \quad (12)$$

$D$  is the lower limit of grain size at which turbulent flow can occur in sands, through which the flow is vertical and where the component of gravity is at a maximum. On the basis of the Lane (1947) scale the lower limit corresponds to very coarse sand. If the approximation used to arrive at equation 12 is applicable, turbulent flow cannot occur through porous rocks unless the grain size is equal to or greater than very coarse sand because the flows are produced almost entirely by gravity. Furthermore, because Lane (1947) classifies very coarse sand as 1–2 mm in diameter and because the critical diameter  $d$  is 1.87 mm, the sediment must generally be larger than very coarse sand; therefore, when water flows into a well through very coarse sand, turbulent flow does not occur, even in the layer of sand adjacent to the well screen. When high gradients are created around pumping wells, turbulence can probably occur if the well is screened in gravel or has a gravel envelope.

Table 2 shows the values of  $d$ ,  $v_c$ ,  $\sin \theta_c$ ,  $\theta_c$ , and  $\tan \theta_c$  for several pipe diameters for gravity-flow computed from equation 9. The value of the equivalent hydraulic gradient is easily found from  $\tan \theta$ . The sediment class size and the grain-size diameter, based on the

Lan  (1947) scale and computed on the basis of equations 6 and 7, are also shown.

TABLE 2.—Critical velocities and corresponding critical angles of dip,  $\theta_c$ , for several tube sizes under gravity flow

[Critical hydraulic gradients ( $\tan \theta_c$ ), and corresponding sediment sizes are shown for a packing of uniform spheres (porosity, 48 percent). No turbulent gravity flows exist in tubes smaller than 0.036 in.]

Diameter of pipe (inches)	Critical velocities $v_c$ (ft per sec)	$\sin \theta_c$	$\theta_c$	$\tan \theta_c$	Grain size (mm)	Class name (Lane, 1947)
6.0	0.482	$2.316 \times 10^{-4}$	0.5''	$2.316 \times 10^{-6}$	300.6	Small boulders.
3.0	.0968	$1.859 \times 10^{-3}$	8''	$1.859 \times 10^{-5}$	153.0	Large cobbles.
1.0	.2900	$5.016 \times 10^{-4}$	1'40.5''	$5.016 \times 10^{-4}$	50.8	Very coarse gravel.
0.5	.581	.0042	14'10''	.0042	25.5	Coarse gravel.
0.25	1.16	.0308	1'48''	.0308	12.5	Medium gravel.
0.05	5.81	.506	30°23'50''	.5866	2.5	Very fine gravel.
0.036	8.07	1.000	90°	.0	1.87	Very coarse sand.

**CRITICAL VELOCITY UNDER ANY HEAD DOWN AN INCLINED ROCK BED**

A more general condition is shown in figure 6, which shows a rock mass,  $R$ , channeled by an assemblage of pipes,  $d_i$ , inclined at an angle  $\theta$ . There is a definite pressure on both the intake end and the discharge end

of each pipe. The entrance pressures are produced by the head in a water-bearing sand, which is completely filled with water to the land surface. Likewise, the discharge pressures arise from the head of water in a stream channel. If  $d$  is the diameter of the pipe in the rock mass  $R$ , and  $\bar{v}$  is the average velocity of water flowing through it, and if the entrance pressure  $p_1 = \rho g h_1$  and the exit pressure  $p_2 = \rho g h_2$ , then, using equation 8,

$$\rho g(h_1 - h_2) + \rho g l \sin \theta = 32 \mu \bar{v} / d^2 g,$$

where  $l$  is the length of the pipe,  $\mu$  is the coefficient of viscosity,

$$\{(h_1 - h_2) / l\} + \sin \theta = 32(\mu / \rho) \bar{v} / d^2 g,$$

and

$$\sin \theta = \{32(\mu / \rho) \bar{v} / d^2 g\} - \{(h_1 - h_2) / l\}. \tag{13}$$

For  $\bar{v} = v_c$  ( $v_c$  given by equation 1), the critical angle of dip  $\theta_c$  is given by

$$\sin \theta_c = \{64,000(\mu / \rho)^2 / d^3 g\} - \{(h_1 - h_2) / l\}. \tag{14}$$

The greatest value of  $\sin \theta_c$  is 1, in which  $\theta_c$  is 90° (vertical fall under gravity). The drawing shown in

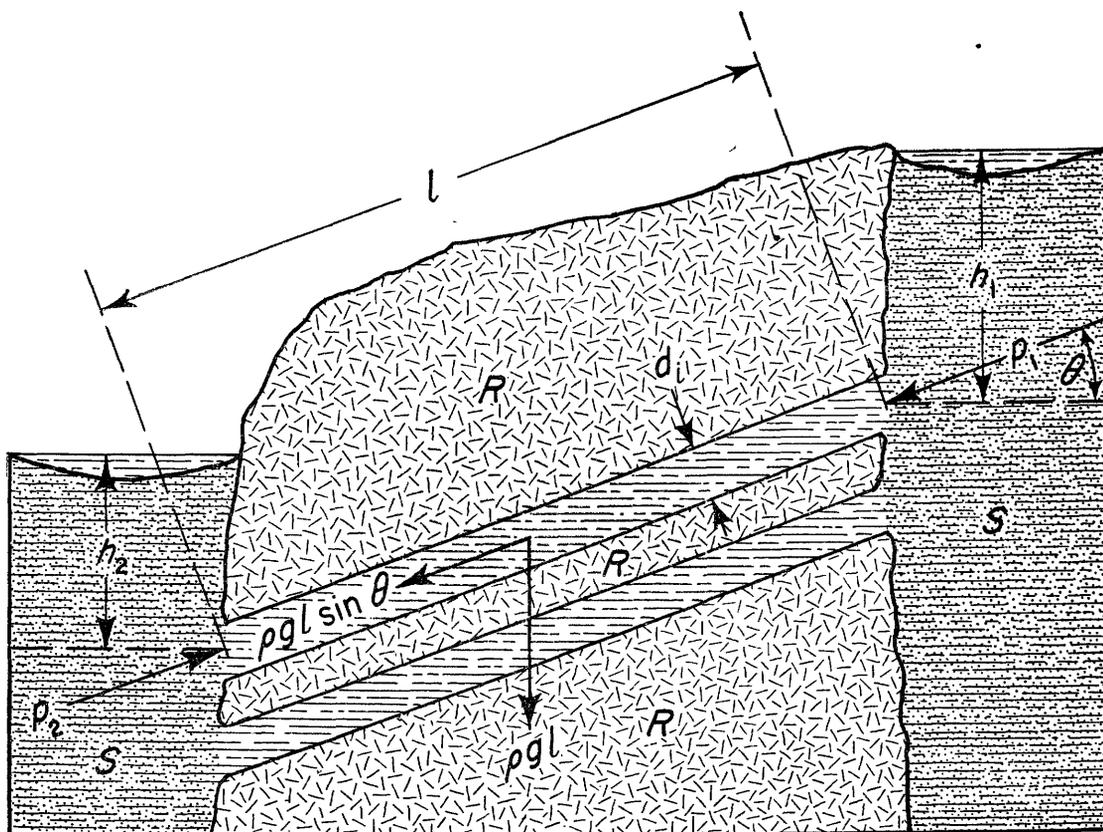


FIGURE 6.—Rock mass  $R$  channeled by pipes inclined at an angle,  $\theta$ . The intake pressure  $p_1$  is produced by the water head  $h_1$  in the sand bed  $S$ . The exit pressure  $p_2$  is produced by the water head  $h_2$  in a second sand bed  $S$ .  $d_i$  is the diameter of the pipe  $i$ , and  $l$  is its length.

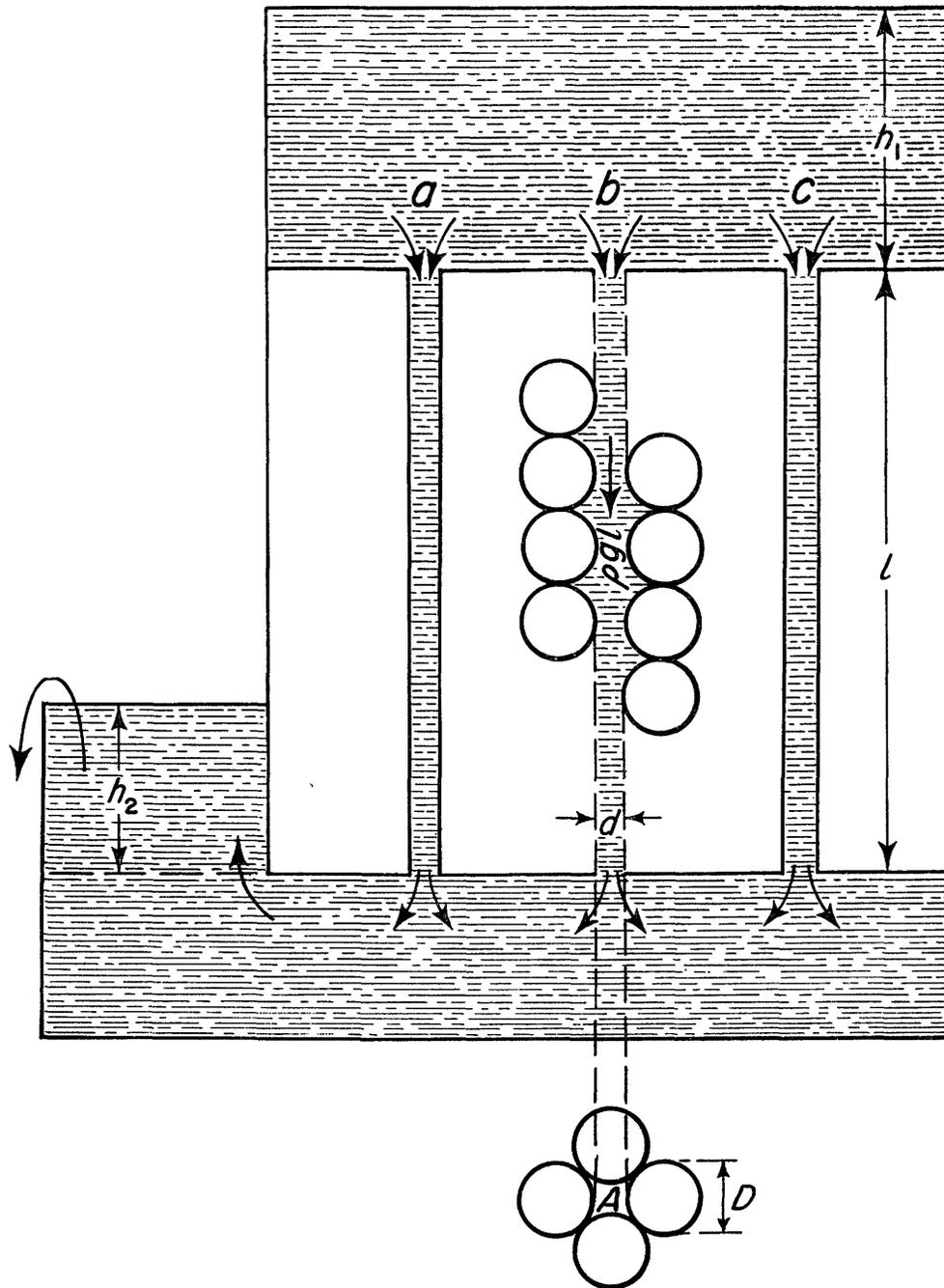


FIGURE 7.—Vertical flow under gravity in an assemblage of pipes of diameter  $d$  and length  $l$ . The center pipe  $b$  illustrates a gravel pore channel. Its wall is composed of gravel of diameter  $D$ . The small figure at the lower end shows the cross section of this channel;  $A$  is its area;  $h_1$  is the head of water at the entrance on top of the pipes;  $h_2$  is the head of water at the exit at the bottom.

figure 6 would have to be modified slightly, as shown by figure 7. The diameter  $d$  of the single channel where vertical fall occurs depends upon the entrance and discharge heads  $h_1$  and  $h_2$  and is found to be

$$d^3 = 64,000(\mu/\rho)^2/g \{ 1 + (h_1 - h_2)/l \}. \quad (15)$$

Let  $d_0$  be the value of the diameter  $d$  of the pipe inclined at an angle  $\theta$ , and let  $G = (h_1 - h_2)/l = 0$ ; that is,

no external pressure gradient exists across the pipe when turbulence first appears; and further let  $d_G$  be the value of  $d$ , when the same pipe is inclined at the same angle  $\theta$ , but under a definite hydraulic gradient  $g = (h_1 - h_2)/l$ , when turbulence first appears. Then, from equations 1 and 13,  $d_G$  may be expressed in terms of  $d_0$

$$d_G = d_0 / \left\{ 1 + \frac{(h_1 - h_2)/l}{\sin \theta} \right\}^{1/3}. \quad (16)$$

If a uniform sediment is considered and its conducting channels are considered as the pipes  $d_i$  shown in figure 6, then the diameter of the sediment particles may be found by using equation 6 for given values of  $d_i$ . By using equation 6 we find for a given porosity,  $P$ ,

$$D_\theta/D_G = (\bar{A}_\theta/\bar{A}_G)^{1/2} = d_\theta/d_G, \quad (17)$$

where  $D_\theta$  is the diameter of sediment grains in a bed that is inclined at an angle,  $\theta$ , when turbulence first appears, and that is not subjected to an external pressure gradient, and  $D_G$  is the corresponding grain diameter, when the bed, still inclined at an angle  $\theta$ , is subjected to an external hydraulic gradient  $G = (h_1 - h_2)/l$ . The values of the single channel areas  $A$  are given by  $\bar{A} = \pi d^2/4$ , where  $d$  is the equivalent tube diameter.  $A_\theta$  and  $A_G$  are the channel areas corresponding to grain diameters  $D_\theta$  and  $D_G$ , respectively.

For vertical fall (that is,  $\sin \theta$  is 1) and values of  $G$  equal to 100 feet per mile, 1,000 feet per mile, and 2,000 feet per mile (see fig. 7), equations 16 and 17 show that, for a given value of  $D_\theta$ ,  $D_G$  is obtained by dividing  $D_\theta$  by 1.006, 1.060, and 1.113, respectively. Thus, by using the value 1.87 mm for the given limiting grain diameter in vertical flow, which has no hydraulic gradient  $G$ , the corresponding limiting grain diameters for the three values of hydraulic gradient  $G$ , just given, are 1.86 mm, 1.76 mm, and 1.65 mm. These values are within the size classification "very coarse sand" (Lane, 1947). Thus, turbulent flow probably does not occur in sediments that are of smaller size than very coarse sand.

#### CONCLUDING REMARKS ON THE FOREGOING ANALYSIS

A detailed study of a given rock structure was not attempted in this paper. The effects of variation in pipe length  $l$  and pipe diameter  $d$  in a given structure will require more consideration. Jointing in structure, roughness of pipe wall, and similar problems require further investigation. The fact that the cross sections of rock channels deviate from a circular form requires additional study, especially as it relates to the critical velocity of a given rock channel. The theory developed in this discussion will probably give a fair approximation of the hydraulic gradients that are required to produce turbulent flow.

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