

# Extending Darcy's Concept of Ground-Water Motion

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GEOLOGICAL SURVEY PROFESSIONAL PAPER 411-F

*Prepared in cooperation with the  
U.S. Atomic Energy Commission,  
Division of Isotope Development and  
Division of Reactor Development*



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By H. E. SKIBITZKE

FLUID MOVEMENT IN EARTH MATERIALS

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## PREFACE

The ground-water hydrologist seeks to identify and use the physical parameters controlling ground-water flow. Scientists are seldom required to define a more heterogeneous or complex region than that involved in an underground flow system. In this respect, the hydrologist's problem is different from that of, say, the electrical engineer or the physicist, who normally analyzes regions constructed with homogeneous characteristics and definable geometries. The heterogeneity of the flow region involved in ground-water studies is so great that it is difficult to find physical characteristics adequate to describe the region. In effect, then, the region must be described by more complex parameters than are generally used in other scientific fields.

Two such parameters are particularly significant. The first is the factor of heterogeneity, which somehow must be identified. But the nature of the heterogeneous region can hardly be described through reference to the individual geometric discontinuities. Such a description would require an endless compendium of individual descriptions, a device so obviously impractical that it renders the region not amenable to description by measurement of any of the characteristics visible or accessible from the surface of the region. The second region embraces the flow relations at any given point and these relations often reflect an angular discrepancy between potential gradient and direction of flow. Thus, tensor description is required because any scalar factor is inadequate.

The measurement of the two factors—heterogeneity and flow relations—has been one of the principal difficulties in accomplishing an engineering analysis of ground-water flow. Laboratory experiments and study have illuminated some of the basic principles involved in the measurement of these factors. The effects of heterogeneity have been studied by hydraulic experiments on two types of laboratory models, one type constructed of heterogeneous materials and the other of homogeneous materials. A comparison of the experimental results for flow through the two model types shows that for heterogeneous porous material a dissolved constituent is dispersed widely; conversely, for material that is simply homogeneous a dissolved constituent is not dispersed in the moving fluid. This paper discusses results of experiments on the former type material. The paper was prepared and should be read as a companion to USGS Professional Paper 411-G, by Akio Ogata, in which experiments on homogeneous materials are discussed.

Because the effects of heterogeneity cannot be expressed quantitatively, experimental procedures must entail the use of a more suitable concept. The observation of the flow is generally accomplished by following a tracer element in its progress downgradient. However, as tracer elements flow downstream they are dispersed to a degree depending on the conditions met in the porous medium—assuming the flow and fluid properties are unchanged. If it were possible to delineate a given series of streamlines, as they would be defined by Darcy's law, the need for describing the flow might be circumvented provided some measure of heterogeneity could be established. Generally, however, establishment of such measure is impossible because of physical limitations in experimental techniques.

Because it is physically impossible to trace out any given streamline, the description of the dispersive mechanism may be of importance. Dispersion in porous media has been studied to some extent, and a general description of the spreading of the flowlines can be predicted for isotropic media. The paper by Ogata describes, for such a medium, an experiment on dispersion transverse to the direction of flow. Good correlation was obtained between analytical and experimental results. The distribution of concentration about the centerline thread of flow was almost symmetrical.

The tracer elements illustrate the history of events as the fluid progresses downstream. This seems to imply that if the dispersive effects can be separated the flow system can be com-



pletely defined within the region. Although the tracer elements show the historical progress of each fluid element and even if it were possible to describe geometrically any flowline, the process of integrating this description throughout a region is extremely formidable—more likely impossible; hence, there is the need to establish a new outlook in describing the flow system, as discussed in some detail in this paper.

Further research is required to determine the utility of a macroscopic concept of the flow system to replace the prevailing microscopic concept. The use of the microscopic analysis has led to the conclusion that the Navier-Stokes equation is valid; however, the integration of the point function throughout the region becomes impossible. In other words, each streamline needs to be described as a separate entity. Further, the microscopic concept has led to a statistical analysis of flow employing an ordered porous body and a random flow system, which is in direct contradiction to the conclusion that the Navier-Stokes equation is valid. In essence, then, the microscopic analysis has produced a method of describing a point, whereas the description of a large-scale aquifer is the goal of the hydrologist.

The implications of studies already accomplished are that further experiments should be tried in larger regions and that a few small-scale supporting laboratory experiments should be added. Because in many of the tests made to date only dissolved constituents were observed, some different and more suitable type of tracer must be used to reveal the components of the dispersion process. Radiotracers have proven especially well suited for this purpose in most laboratory experiments.

The analysis of dispersion processes will necessarily require the use of some tracer techniques, simply because dispersion is the product of the motion of the dissolved component. In a macroscopic sense, of the order involved in ground-water field studies, the dispersion of contaminant does not depend upon any measurable hydraulic factor. Therefore, part of the experimental technique must involve addition of dissolved components to the fluid. Because the dispersion is also dependent upon the location and geometry of the injection site, no naturally occurring tracer element will suffice. Furthermore, once a tracer is decided upon, it will, in general, need to be a radiotracer because the problem of remote sensing is involved in ground-water studies. It seems necessary, then, that techniques be developed for extending the laboratory use of radiotracers into the field.

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## FLUID MOVEMENT IN EARTH MATERIALS

### EXTENDING DARCY'S CONCEPT OF GROUND-WATER MOTION

By H. E. SKIBITZKE

#### ABSTRACT

The tensor nature of permeability and its effect in a given ground-water flow regime have been acknowledged by various investigators. The effect on the spread of a given dissolved substance can be qualitatively discussed on the basis of the tensor characteristic of the permeability coefficient. The correlation between the dispersion coefficient and the tensor characteristic is one possible means of defining the flow regime within a given aquifer. Because of the large-scale changes in magnitude and direction of the tensor components of permeability, the concept of a mean line of dispersion may produce a more significant statement on the characteristics of flow than does the present concept of hydraulic potential and the streamline.

#### THE PERMEABILITY CONCEPT

The analysis of a region harboring ground-water flow requires the description of the permeability function in a quantitative sense. In studying most ground-water flow problems, the permeability of the porous earth material through which the flow occurs has been assumed a constant. At best the flow analysis may sometimes acknowledge the permeability as a scalar, variable in space. This practice is illustrated by studies where permeabilities are determined by discharging-well tests of an aquifer and then are plotted on a regional map and used to develop contours of permeability. This simple scalar relationship is one wherein the permeability is constant at a given point—that is, it does not change in the different directions. The permeability can thus be defined by a single number at that point. If another point in the region is considered, a new number may represent the permeability at that point, but again the permeability is the same in any direction. Thus the scalar coefficient—permeability—of the Darcy equation may vary in space, but it will be constant for all directions at any point.

Now suppose that the permeability differs in various directions at any point; that is, if the permeability were measured in the east-west direction, it might differ from that measured in the north-south direction, which in

turn might differ from the vertical direction. The manner in which the permeability may vary can no longer be described mathematically by a simple coefficient such as the scalar permeability previously used. An adequate description will necessarily lead to a tensor concept for the permeability relationship.

The permeability concept can be extended further by considering spatial relationships as well as directional characteristics. Assume first that the permeability can be described by a function that would account for variations in direction—that is, by a tensor. It may then be assumed that permeability can vary in any direction specified by the Cartesian coordinate system and, further, that these variations change as the position of measurement is moved laterally or vertically within the aquifer. Measurements made at various points within an aquifer will then define a function varying with position. If these changes are continuous—that is, vary smoothly in a mathematical sense—the change of permeability can be described as a continuous function. This limitation of continuity is unnecessary, however, and its removal leads to the final enlargement of the permeability concept into something that is a discontinuous function.

Imagine that the permeability of aquifer material is describable by a tensor relationship and that the tensor then can properly account for permeability variations in direction and magnitude. The tensor may vary not only in space but also in a discontinuous manner. Previously the permeability function was described as varying in space—both in the lateral and the vertical directions—but in a smooth manner, so that a continuous function accomplished its description. For discontinuous variation, permeability measurements at points adjacent to one another would reveal abrupt changes in the permeability tensor. This discontinuity condition leads to the most generalized concept of permeability necessary for describing the porous material constituting an aquifer. Thus an adequate

concept of the permeability function leads to the definition of a variable that changes with position in space as well as with direction at a point, the variations being discontinuous or continuous or both.

Often in aquifer materials the variation in permeability is continuous over a small region, at the boundary of which it abruptly changes in a discontinuous manner. An analysis of the permeability factor requires determination of the implications involved in its variations. First, Darcy's equation must be revised.

The Darcy equation has the general form

$$\vec{Q} = P \vec{\text{grad}} h. \quad (1)$$

To develop a more elaborate description than is generally used, the scalar permeability  $P$  must be replaced in equation 1 by the tensor permeability  $P_t$ , which represents a coefficient that would vary in space continuously or discontinuously—and generally both.

Some comprehension of the tensor relationship may be extracted from a definition given by Margenau and Murphy (1959, p. 161), a part of which is quoted as follows:

In many physical problems the notion of a vector is too restricted. For example, in an isotropic medium, stress  $\vec{S}$  and strain  $\vec{X}$  are related by the vector equation

$$\vec{S} = k \vec{X} \quad (2)$$

$\vec{X}$  and  $\vec{S}$  having the same direction. If the medium is not isotropic,  $\vec{S}$  and  $\vec{X}$  are not in general in the same direction; it is then necessary to replace the scalar  $k$  by a more general mathematical construct capable, when acting on the vector  $\vec{X}$ , of changing its direction as well as its magnitude. Such a construct is a "tensor." A similar generalization has to be made in the vector equations

$$\vec{P} = \epsilon \vec{E} \quad (3)$$

where  $\vec{P}$  and  $\vec{E}$  represent electric polarization and field strength, and

$$\vec{I} = \mu \vec{H} \quad (4)$$

where  $\vec{I}$  and  $\vec{H}$  represent intensity of magnetization and field strength; for anisotropic media, the susceptibilities  $\epsilon$  and  $\mu$  must be replaced by tensors.

Equally complex conditions confront the analysis of ground-water motion. If the permeability varies with direction, there is an angular difference between the hydraulic gradient and flow direction. Because of this difference, the vector descriptions require application of a tensor concept to the permeability coefficient. In the simplest situation the permeability function is a scalar, which in turn is the most degenerate form of tensor. More generally the tensor is a function that may be used to describe mathematically the relations between the hydraulic gradient and the direction and rate of ground-water flow, regardless of the directional characteristics of the rock permeability. This tensor

can vary both continuously and discontinuously in space.

The tensor relationship changes the continuity conditions for which flow equations can be specified. The Laplace equation, written with a simple scalar permeability varying as a continuous function, describes the continuity condition of the liquid in a moving state. In this form the Laplace equation appears different from the form usually given, and one might argue that it is no longer properly termed Laplace's equation. As it is commonly written, Laplace's equation has the form

$$\nabla^2 h = 0 \quad (5)$$

where  $h$  is the hydraulic head. When the scalar permeability  $P$  varies in space, the equation must necessarily contain  $P = P(x, y, z)$  as a variable. Thus equation 5 would be rewritten

$$\nabla \cdot P \vec{\nabla} h = 0. \quad (6)$$

Upon completing the indicated mathematical operations, equation 6 becomes

$$P \nabla^2 h + \vec{\nabla} h \cdot \vec{\nabla} P = 0. \quad (7)$$

That equation 7 is reducible to equation 5 is readily shown. For the simplest concept of permeability—that is, constant or unvarying with respect to space and direction—the permeability gradient  $\vec{\nabla} P$  is obviously zero, and the second term in equation 7 is therefore zero. The remaining term can be divided by the constant permeability  $P$ , and the simple Laplacian form of equation 5 is thus left. However, the simple Laplacian form is not applicable to the ground-water conditions commonly found in nature.

The continuity equation for ground-water motion becomes even more complex when the Laplace equation is written in terms of a tensor relationship. A formal derivation and a discussion of its significance are not included here. These will be covered in a separate chapter of this Professional Paper. Sufficient for the present is an indication of the complexity in the form of Laplace's differential equation when written to show, say, the distribution of velocity potentials  $V$  in terms of a completely general coordinate system and the fundamental metric tensor  $g_{ij}$ . In tensor notation

$$\text{the Laplacian relation } \nabla^2 V = 0 \text{ has the form} \quad (8)$$

$$g^{ij} V_{,ij} = 0$$

or

$$\frac{1}{\sqrt{g}} \frac{\partial (\sqrt{g} g^{ij} \partial V / \partial x^i)}{\partial x^j} = 0. \quad (9)$$

(See James and James, 1949, p. 206.)



Finding solutions for a relationship similar to equation 9 is much more difficult than for the simple Laplace equation. Actually, it would be virtually impossible to find any significant solutions for the tensor relationship. However, some flow features leading up to and involving the relationship might be briefly examined.

### THE GROUND-WATER FLOW REGIME

It is helpful first to have clearly in mind, as a reference base or point of departure, the ground-water flow regime as it is commonly visualized and depicted—permeability in every place and direction being constant for the porous medium. These simple conditions are presented as case 1 in figure 1. Observe that the flow-lines are coincident with the lines denoting hydraulic gradient, and the flow rate is labeled as inversely proportional to the distance between the equipotential contours.

Now consider the matter of scalar variations in permeability in space—that is, the simple situation in which the permeability is always a scalar but varies continuously in space. The pertinent conditions are described as case 2, shown in figure 2; the changing permeability produces a flow pattern distorted in that the flow rate is not proportional to the gradient vector  $grad\ h$ . The flow is in general parallel to the gradient vector, but the rate is not proportional to the hydraulic gradients indicated by the spacing of contours denoting equipotential surfaces. This set of conditions, case 2, represents the simplest conceivable departure from the common idealized conditions illustrated in case 1 (fig. 1).

A more general departure from the idealizations of case 1 is represented by the conditions shown as case 3 in figure 3. The permeability not only varies continu-

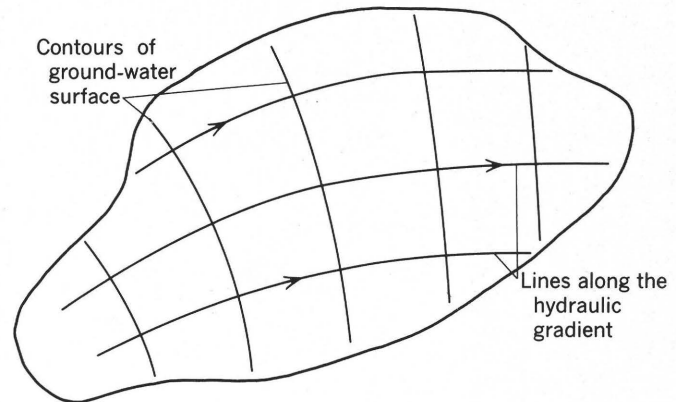


FIGURE 2.—Case 2:  $\nabla \cdot P \cdot grad\ h = 0$ . Diagram of the flow regime in a region in which permeability  $P$  is a scalar varying continuously in space. Flow rate is not inversely proportional to the distance between contours. Arrows indicate direction of flow.

ously with space but also varies with direction, and it has the characteristics of a tensor. Thus the ground water flows in directions differing from those indicated by the hydraulic gradients.

A corollary case deserving brief mention involves the discontinuous condition resulting from the abrupt change between the simple permeabilities of two adjacent parts of an aquifer. The permeability  $P$  is a scalar quantity in each part but abruptly changes at the interface between the parts. The flow regime in such an environment has been analyzed and discussed by a number of technical writers but is particularly well covered in the works of Muskat (1946) and Hubbert (1940). The sketches drawn in figure 4 are from Hubbert's discussion; they show how flowlines are abruptly refracted at the permeability discontinuity. Clearly illustrating the same refraction phenomena is the picture shown in figure 5, taken in the Phoenix research laboratory of the U.S. Geological Survey during

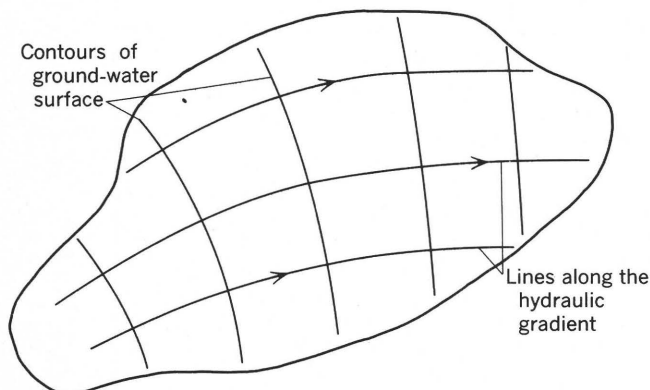


FIGURE 1.—Case 1:  $\nabla^2 h = 0$ . Diagram of the flow regime in a region in which permeability  $P$  is constant throughout. Lines along gradient are streamlines—that is, total flow  $Q$  is constant between streamlines. Flow rate is inversely proportional to the distance between contours. Arrows indicate direction of flow.

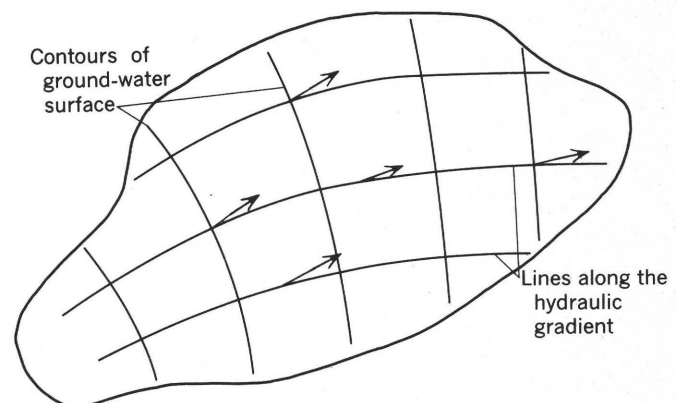


FIGURE 3.—Case 3:  $\nabla \cdot (P_t \cdot grad\ h) = 0$ . Diagram of the flow regime in a region in which permeability  $P_t$  is a nonscalar continuously distributed in space. Flow direction is not parallel to the gradient. Flow rate is not inversely proportional to the distance between contours. Arrows indicate direction of flow.

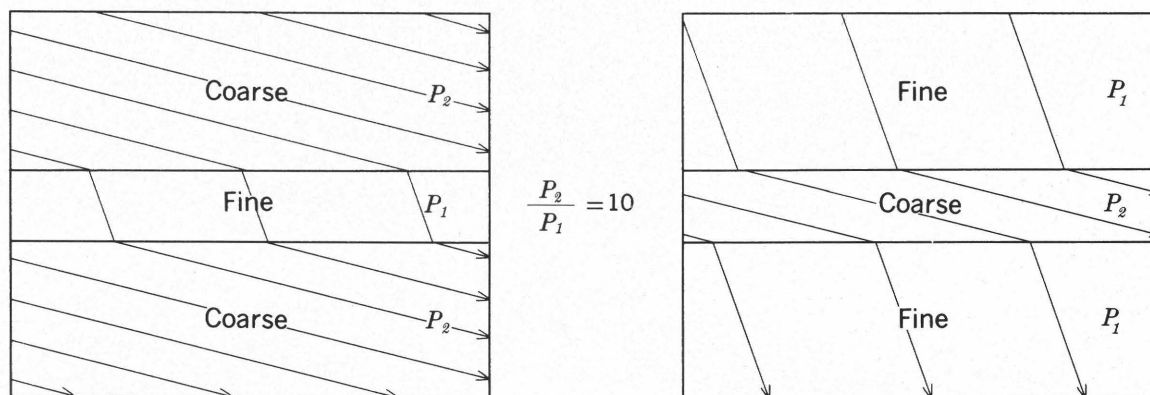


FIGURE 4.—Diagram showing theoretical flowline refraction across layers of coarse and fine sand. After Hubbert (1940).

a hydraulic experiment on an artificial-sandstone model. Uniform flow is from left to right and the refraction of the black-appearing dye stream is clearly evident as it passes through the tapered band of higher permeability sandstone in the center of the model.

To be examined as a final departure from the idealizations of case 1 are the completely general conditions presented as case 4 in figure 6. The permeability is now a tensor function and can vary with direction; it also can vary discontinuously in space. The groundwater flow is not only in directions differing from those indicated by the hydraulic gradients but appears to have several directions at each point.

The net effect upon the flow system of combining all the individual conditions and factors discussed in the preceding paragraphs is most difficult to illustrate. A possible approach might be to construct in model form a heterogeneous medium having all these factors inherent in the construction. The author has concluded, after many such experiments (see, for example, Skibitzke and Robinson, 1963), that the most general-

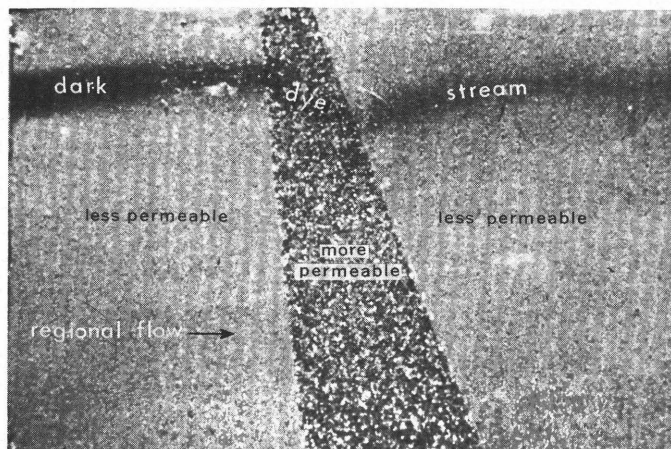


FIGURE 5.—Laboratory photograph showing refraction of flowlines in artificial-sandstone model.

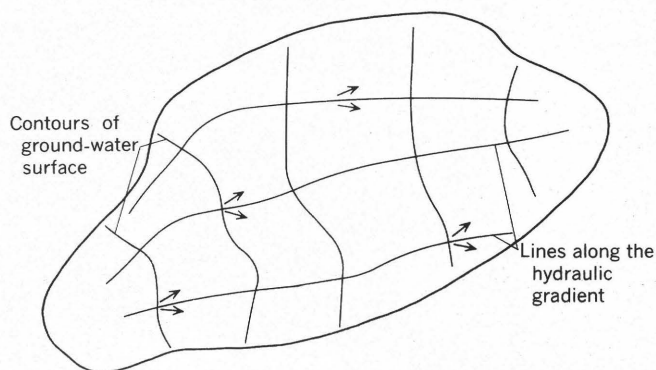


FIGURE 6.—Case 4:  $\nabla \cdot (P_t \text{ grad } h) = 0$ . Diagram of the flow regime in a region in which permeability  $P_t$  is a tensor function (either scalar or directional) and is discontinuous in space. Flow is dispersed, appearing to have several directions at a point. Flow is not inversely proportional to the distance between contours. Arrows indicate direction of flow.

ized state of the heterogeneous medium would lead to flowlines and related features as shown in figures 6 and 7. The details in figure 7 are explained in the following paragraphs. Observe, however, that if the path of a water particle is traced, starting from any point, the flowline begins by briefly indicating laminar motion, according to the classical thinking represented by Darcy's law. As distance from the starting point increases and the flow pattern becomes more widely dispersed, however, the flowline quickly departs from the classical concepts. The manner in which the flow disperses downstream from a reference point is described by Ogata (1964) in a companion chapter to the present report. Presented and discussed therein are the experimental data collected in a series of experiments designed specifically to show the effects of dispersion. The important conclusion drawn by Ogata is that the spread of a tracer or contaminant migrating from a deposit in a heterogeneous aquifer is completely the result of the heterogeneity, insofar as can be observed.



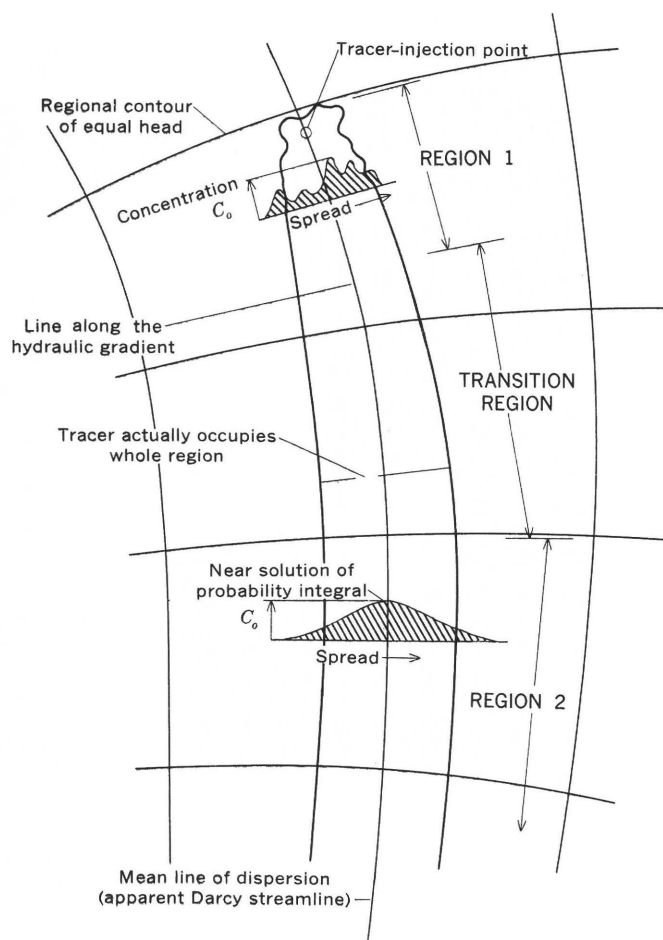


FIGURE 7.—Diagram showing dispersion of a tracer injected into a heterogeneous aquifer. Permeability  $P$  has scalar characteristics.

### DISPERSION PHENOMENA

The preceding conclusion leads to an important concept in describing the nature of flow in a ground-water reservoir. Two significant regions in the flow field can be visualized. In the first region the flow environment is so heterogeneous that the travel paths of a tracer or contaminant follow streamlines winding tortuously through the porous medium. Therefore, within this region, if a tracer-concentration profile were drawn for a section normal to the general direction of flow, it would be highly irregular in form (fig. 7). Further down the path of motion the effects of the intertwining of flowlines tend to blur out the apparent discontinuous features of tracer concentration. The region in which this blurring occurs can be regarded as merely a transition region (fig. 7). Ultimately, the flow reaches the second region to be visualized, which is at some distance downstream from the tracer or contaminant origin. In this second region (fig. 7), the porous medium appears to be more amenable to description

by dispersive processes in a mathematical sense. If a coordinate system moving with the liquid is postulated, the mathematical description would be the Fickian diffusion equation, having the form

$$\nabla^2 C = D \frac{\partial C}{\partial t} \quad (10)$$

In equation 10, however, the magnitude of  $D$  is many powers of ten greater than it is in the ordinary process of molecular diffusion.

Until region 2 is reached (fig. 7) and "complete" (in the mathematical sense) dispersion occurs, the discontinuities of flow virtually defy description. Their identification would require a complete knowledge of the permeability of the aquifer material in microscopic detail; in other words, the nature of about each cubic foot of the aquifer would have to be studied. Knowledge of the hydraulic gradient would also be required in the same detail. Obviously these data, required primarily in region 1 (fig. 7), could never be obtained. To summarize, therefore, it must be concluded that around the tracer or contaminant source and proceeding downstream from it there exists first a region of indeterminate flow where the tracer is mixed discontinuously. This kind of mixing is illustrated schematically by the upper inset cross section of figure 7. This first region grades gradually through a transition region, where the flow and dispersion discontinuities begin to smooth out, into a second region where the flow can be characterized as simple dispersed flow. The effects are again schematically illustrated in the lower inset cross section of figure 7. However, it may take many times the length of any normal aquifer to arrive at the second region. Conversely, there may well be some situations in which the second region is reached in a matter of a few feet. Even in the simplest and most uniform of aquifers there will still be some intertwining of flowlines within the pore spaces themselves. The flow-regime features sketched in figure 7 typify the nature of flow and dispersion in a heterogeneous aquifer and show that a more generalized description of the flow pattern is needed.

If the flow-regime idealizations based upon Darcy's law are compared with the features shown in figure 7, it may be concluded that the commonly used formula  $Q = P \text{ grad } h$  in reality describes only a generalized average direction of flow. The formula does not describe actual stream paths of tracer or contaminant elements, but describes instead what might be termed a mean path of dispersion. As a beginning step, therefore, in a more realistic analysis of flow through a heterogeneous aquifer, it would appear most desirable to promote this concept, or new definition, that the

central streamline, as it is identified classically by use of Darcy's equation, is more nearly the central line about which the dispersed tracer travels. That this is only a beginning step in a more realistic analysis can be inferred from the features shown in figure 6, characterizing the flow regime when the permeability is a tensor function. In this completely general situation the mean line of dispersion will depart from the lines representing the conventionally determined hydraulic gradient. Although the manner of departure is yet to be determined, it is schematically represented in figure 8.

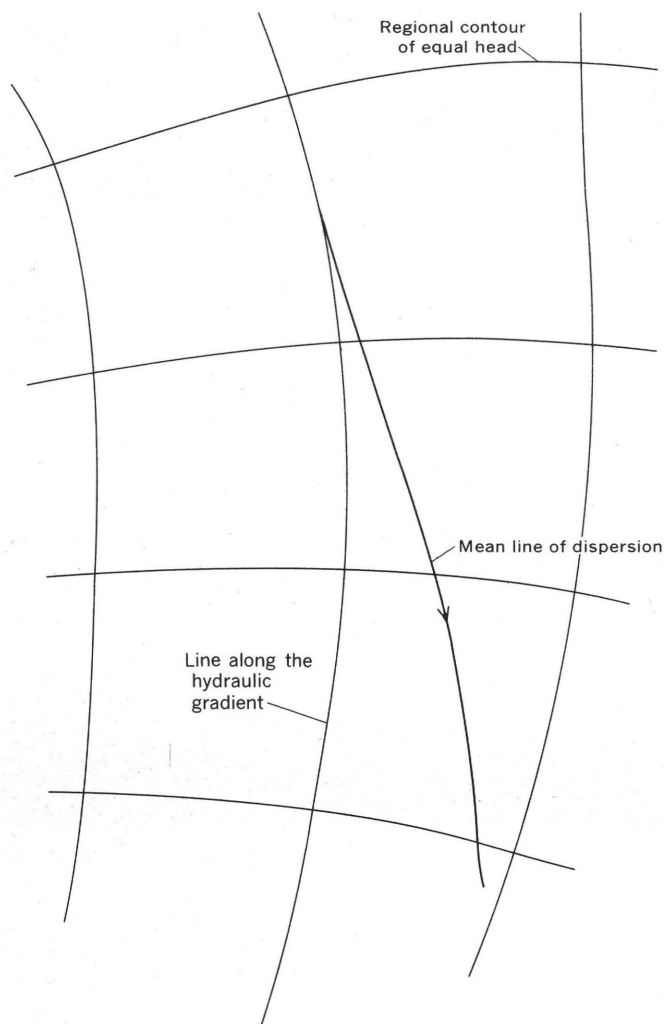


FIGURE 8.—Diagram showing the mean line of dispersion in a heterogeneous aquifer when permeability  $P_t$  has directional characteristics.

#### LIMITATIONS OF THE CALCULUS

One additional point merits brief discussion, and it concerns the nature of infinitesimals when working

with geologic bodies the size of regional aquifers. In mathematical analysis the infinitesimal generally must be considered as ever reducible to yet a smaller one without losing any of its inherent physical characteristics. If, for example, observation wells penetrate a regional aquifer on a  $\frac{1}{2}$ -mile-square grid pattern, it must be assumed in any mathematical analysis of data obtained from such wells that all aquifer elements smaller than the  $\frac{1}{2}$ -mile-square segments will be identical in nature. If these "infinitesimals" remained the same in character as the elemental areas approached zero, a fairly representative mathematical picture of the aquifer would be obtained. If on the other hand the successively smaller infinitesimal elements—considered within the  $\frac{1}{2}$ -mile-square segments—changed in their physical characteristics as the areas tended toward zero, the mathematical operations involving the calculus would not be valid. Commonly the heterogeneity of porous media is such that the permeability function will contain changes in its nature through all magnitudes of areal size that may be chosen throughout the regional aquifer. It is not unreasonable to expect, therefore, that flow patterns different from those described by simple solutions of Laplace's equation will be observed. In field practice the smallest infinitesimals that can be studied and described are those whose areas are delimited by the spacing of wells in the observation-well network. Thus any analytical conclusions dependent upon mathematical integration and differentiation processes will be invalid. For these reasons, employing the concept of the mean line of dispersion seems simpler and more promising than attempting to find the direct flow path. In heterogeneous aquifers the latter approach obviously requires a mode of description potentially involving an infinite number of paths.

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