Mohr's Theory of Strength and Prandtl's Compressed Cell in Relation to Vertical Tectonics

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Mohr's Theory of Strength and Prandtl's Compressed Cell in Relation to Vertical Tectonics

By S. P. KANIZAY

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Plastic mechanics of geologic deformation

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MOHR'S THEORY OF STRENGTH AND PRANDTL'S COMPRESSED CELL IN RELATION TO VERTICAL TECTONICS

By S. P. Kanizay

ABSTRACT

Mohr's theory of strength, given in functional form, $\tau = f(\sigma)$, where τ is the shear stress and σ is the normal stress on the surface of failure, is particularized graphically such that failure occurs along planes oriented at right angles to the largest principal stress when failure is under tension, and along complementary orthogonal slip planes, whose orientation ranges from 30° to 45° with the largest principal stress in failure under compression. It is proposed that a correlation exists between mean stress, as measured along the σ axis of a modified Mohr yield envelope, and depth in the earth's crust such that there is a functional relationship in which mean stress increases with depth. The type of deformation varies with mean stress; as mean stress increases, deformation is characterized by less brittle and more plastic behavior.

The types of deformations discussed here are assumed to be associated with rising geologic masses. Under this assumption a structural unit exhibiting characteristics of brittle deformation is presumed to have undergone deformation while nearer the surface than a unit that exhibits characteristics of plastic deformation. A further extension of this assumption is that structures of the brittle type are younger than structures of the plastic type, assuming that the rates of upward movement are the same for each unit.

Prandtl's solution for a compressed cell is used as an example of perfectly plastic deformation in which the angle of internal friction is zero, and Hartmann's solution is used for the case where there is internal friction. Hence, plastic solutions exist under Mohr's theory which hold throughout the compressive domain of the yield envelope.

Prandtl's solution is used to define active and passive cells of deformation which together with wedge variations may simulate structural units, particularly under the condition of vertical tectonics.

INTRODUCTION

Many mountainous masses are much longer than they are wide. Moreover, the rocks in the exposed central parts are elevated above their stratigraphic position in adjacent basins. This linear or striplike topographic outline together with differential uplift suggests the possibility that useful ideas may arise from a comparison of such geologic structures with theoretically derived analyses of deformed ideal bodies with simple outline.

A framework under which this comparison can be made is Mohr's (1928) theory of strength. That is, if it can be shown that Mohr's theory is applicable to geologic deformations in general, then solutions to specific deformations under the theory should also be applicable.

With this in mind, the writer has examined the existing solutions for deformation within rectilinear plastic strips and wedges, as presented by Prandtl (1924), Nádai (1950), and Varnes (1962). Inasmuch as these analyses were made for the condition of plane strain, they are here applied to two-dimensional vertical sections taken centrally through long mountain masses in order that deformations parallel to the long horizontal axis of the deformed units may be neglected.

The whole basis for the concepts presented rests on the mathematical theory of plasticity which is here only briefly discussed. Comprehensive treatment of the subject is given in several texts including Nádai (1950), Hill (1950), and Prager and Hodge (1951). Varnes (1962) gives a definitive discussion of the subject and points his development toward geologic application. It is suggested that the Varnes' report be read as a companion to this report. Odé (1960) brings out the core of the theory in its application to faulting. The essence of the theory lies in allowing permanent deformations. Certain differential equations involving velocity are found to be hyperbolic in form. Being hyperbolic, these equations thus permit real discontinuities in velocity (slip lines). In geology they are the trajectories along which there may be real faults.

Application of plastic theory to geology can be approached from either of two ways. The postulated stress field may be given as a boundary condition and from which a solution may be obtained that defines the slip-line field. This approach requires that the stresses which give rise to a particular deformation are known. Before the stresses can be ascertained in geologic structures, one must have a rather thorough knowledge of what the actual structure looks like. The stresses can then be assumed and a problem solved for the slip-line field. It should be apparent that this approach can give useful templates or typical solutions for comparison with actual or implied geologic structures.

A second approach is through the solution of a boundary-value problem where some part of the slip-line field is given. Then by use of the plastic theory the rest of the slip-line field and the appropriate boundary stresses may be derived and the theoretical concept completed. This latter approach was employed by Varnes (1962) in applying plastic theory to the Silverton, Colo., area.

In the present report a variation of the first approach will be employed. That is, the limits of the plastic domain are assumed, linear mountain elements with boundary lines that are parallel or converge like wedges; then the internal mechanics of the structure may be determined according to plastic theory.

MOHR'S THEORY OF STRENGTH

The Mohr theory is virtually an empirical theory of yield which accounts for the behavior of permanently deformed materials. As portrayed on a Mohr stress diagram the theory assumes a functional relation between mean stress and maximum shear stress on the plane of failure. In this report the particular shape of the yield envelope is assumed to be more or less parabolic in form as shown in figure 1.

Two equations are noteworthy in defining Mohr's theory,

$$\tau = f(\sigma) \tag{1}$$

$$\frac{\sigma_1 - \sigma_3}{2} = c_1 + c_2\left(\frac{\sigma_1 + \sigma_3}{2}\right). \tag{2}$$

Equation 1 says that the shear stress on the plane of failure is a function of the normal stress across that plane. Equation 2 says that the maximum shear stress or the diameter of the largest stress circle tangent to the yield envelope is a function of the mean or hydrostatic stress component of the total stress.

Nádai (1950, p. 218), after having discussed Mohr's theory of strength, said in a footnote,

In summing up we may state that the Mohr theory predicts many of the principal facts known from strength tests for a large variety of materials embracing the perfectly ductile metals and the solids behaving in a brittle manner, including the observations on the orientation of the surfaces of slip in plastic

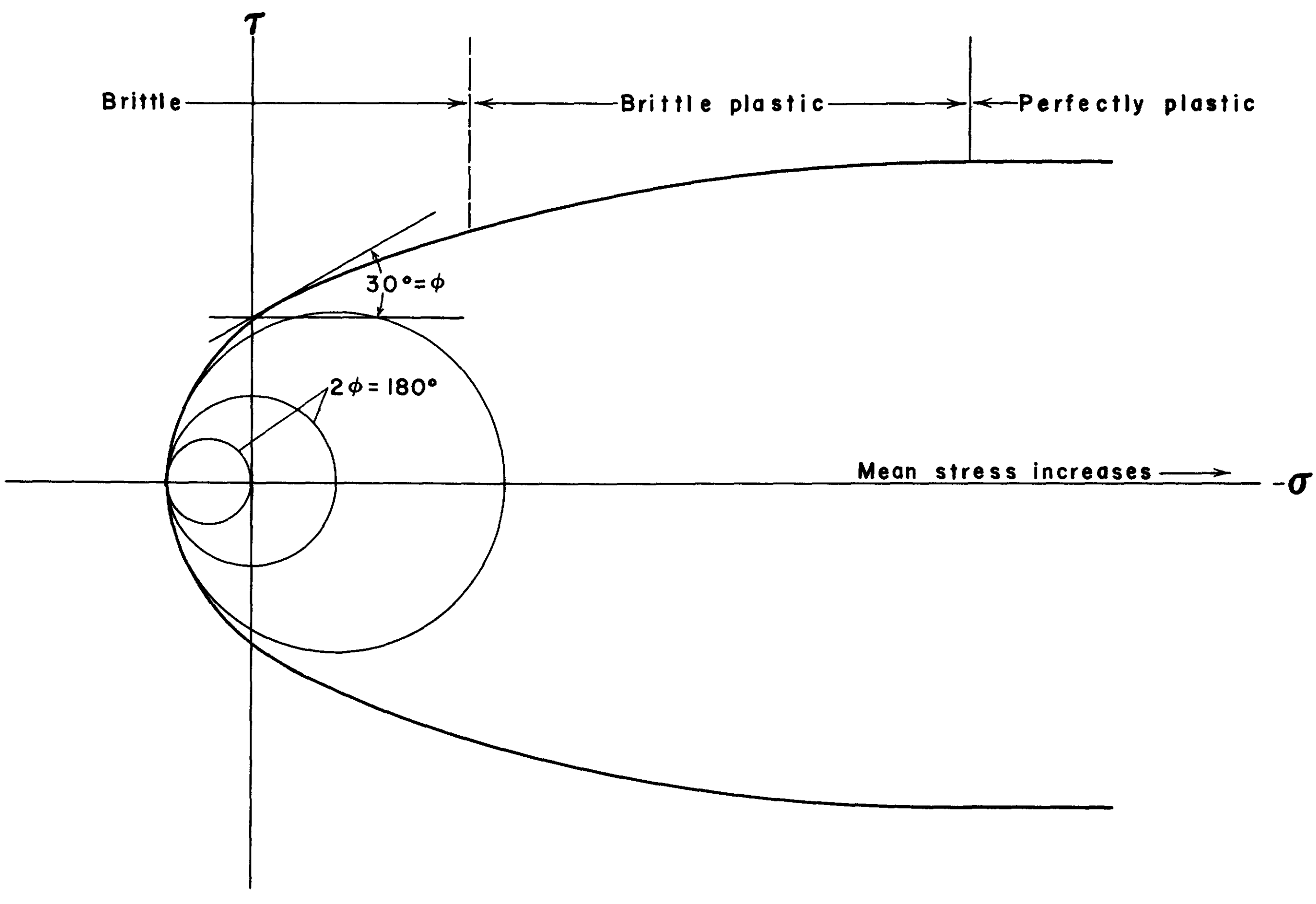

FIGURE 1.—Postulated form of yield envelope.

materials and, as we shall see below, of the shear and cleavage fractures. In the region of large mean pressures when the envelope tends to become parallel to the σ axis, it predicts what might be called "pure" shear fractures ($\phi=45°$), in the transition zone the ordinary obliquely inclined shear fracture ($0<\phi<45°$) and near the apex of the envelope the cleavage fracture ($\phi=0$) in the direction normal to the maximum tensile stress.

The assumed parabolic envelope specified herein follows Mohr's theory basically, but it is a modification to the extent that certain additional empirical factors also are considered. There are two principal factors. First, failure under tension is to be specified as occurring on a single plane (not complementary families), and second, the angle of internal friction where the yield envelope crosses the shear stress axis is taken to be 30°. The first factor is credited to Leon (*in* Nádai, 1950, p. 221), the second factor is taken as an average figure. The exact value of the angle of internal friction at the shear stress axis is not so important as the convergence toward the σ axis from a position of parallel lines to an orientation, on the tension side, in which the envelope is normal to the σ axis. Thus 30° represents simply a reasonable figure.

Within the parallel line part of the envelope the material deforms under a constant maximum shearing stress, the prescribed slip lines are inclined at 45° to the largest principal stress and occur in complementary orthogonal sets. This is true regardless of the value of the mean stress, implying that once the perfectly plastic state is reached any increase in mean stress does not affect the geometry of the slip-line field.

At the other end of the envelope, near the origin, the behavior of the material shall be specified as brittle. This means that failure takes place on discrete, macroscopic, slip lines. Where the tensile stress is equal or greater than the compressive stress the failure is termed "cleavage fracture" and the slip line is oriented at 90° to the largest principal stress.

The brittle and perfectly plastic zones grade into one another through an intermediate zone. If this zone were narrowed to a line there would be two zones, one brittle and one plastic. Heard (1960, p. 193) investigated the effects of temperature, confining pressure and interstitial fluid pressure on such a transition and noted that under conditions comparable to his experiments, normal faulting of dry limestone could exist to a depth of 15 kilometers and reverse faulting to a depth of 3.5 kilometers. Interstitial fluid pressure would serve to increase these depths.

The zone between the perfectly plastic behavior and brittle behavior is characterized by internal friction. Whereas in the parallel-line zone the influence of the mean stress was ineffectual; in the brittle plastic zone there is a continuous change in the angle of internal friction with a change in the mean stress. The degree of nonorthogonality between the slip lines also changes value in this interval.

From the brittle zone to the perfectly plastic zone the configuration of the envelope requires that the behavior of the material change with mean stress. That yield is a function of mean stress is apparent from this inspection but should not imply the exclusion of other independent variables which contribute to the control of deformation, that is, time and temperature.

Perfectly plastic behavior is assumed to be a simultaneous movement along countless slip lines. Brittle behavior would modify this definition to the extent that certain lines would be favored for movement.

It should be emphasized that Mohr's theory is not explicit, that is, the functional relationship $\tau=f(\sigma)$, which considers observed deformation phenomena, is not specified. Generally yield criteria which are more specific are also more restricted in their application owing to mathematical considerations.

Later in this report the solutions to two problems of plastic deformation in plane strain will be cited; one of these uses the parallel-line part of Mohr's envelope; that is, Prandtl's problem of the compressed cell using von Mises' yield criteria (τ_{max}=constant), and the other uses the convergent-line part of the envelope, that is, Hartmann's solution using a Coulomb criterion. While both criteria can be considered as special cases of Mohr's theory (Nádai, 1950, p. 219), they should not be considered less noteworthy than Mohr's theory. They are mathematical rather than empirical.

MEAN STRESS AND DEPTH

Mean stress is defined as the average normal stress at a point; it is also called the hydrostatic component. In terms of principal stresses,

$$\text{Mean stress}=\frac{\sigma_1+\sigma_3}{2}=\sigma_2 \text{ (plane strain).} \qquad (3)$$

It corresponds to confining pressure of the triaxial test. Total stress at a point in a solid body consists of the mean stress plus the deviator stress. In contrast, the stress in a fluid body has no deviatoric component. In terms of principal stresses,

$$\text{Total stress}=\text{mean stress}+\text{deviator stress}$$

$$\sigma_1=\frac{(\sigma_1+\sigma_3)}{2}+\sigma_1'$$

$$\sigma_3=\frac{(\sigma_1+\sigma_3)}{2}+\sigma_3' \qquad (4)$$

Thus the deviator is the actual deforming agent or component while the mean stress acts as a sort of conditioner for the deformation.

In recognition of the importance of this single parameter, mean stress, a search is made for some corresponding element in the earth's crust in order that Mohr's envelope might be more directly correlated with an element of the earth's crust. Depth is such an element. As a simple linear measure depth by itself does not tell the whole story. In a general sense its magnitude can be taken as a rough measure of increase not only in mean stress but also in other variables that strongly control earth deformation; namely, temperature. Handin and Hager (1958, p. 2892) have shown experimentally that an increase in temperature tended to weaken rocks and enhance their ductility. The net effect of increased temperature would increase the possibility of perfectly plastic behavior.

Thus the mean stress axis of the Mohr envelope can be placed side by side with a section of the earth's crust in order to give a graphic representation of the correlation, as in figure 2.

Although a quantitative relation between mean stress and depth cannot be stated, the general correlation may be useful in discussing certain postulates about deformation in the earth's crust, as follows:

1. According to the position of the coordinate (σ,τ) origin relative to the earth's surface it is seen that the condition of no tension in the crust is specified. This statement needs modification to the extent that tension perhaps may exist to shallow depths in special cases.
2. Near the surface the response of material to deformation will be of the brittle type whereas at depth it will be plastic.
3. Failure will occur along lines of maximum shear in complementary families. Near the surface these lines will be nonorthogonal, whereas in the perfectly plastic zone the angles between them will be 90°. The absolute orientation in space cannot be specified unless the orientation of the principal stresses is known. At the surface, the acute angle between members of a family of slip lines is about 60°.
4. The material is specified to have a movement pattern such that, in a continuity of deformation, movement is generally in an upward direction from a high mean stress to a lower mean stress.
5. The geometry and mode of deformation should reflect the relative age of deformation in a cycle where the latest mode of deformation should be of the most brittle type.

Support for these remarks is now developed.

PRANDTL'S COMPRESSED CELL

Under the concept of linear mountain belts with differential uplift we recognize the nature of the deformation and seek to explain the mechanism whereby it developed. The internal mechanism of deformation is presumed to be a shear mechanism. The linear aspect of the mountain belts indicates that a two-dimensional theory, that is, plane strain, will give useful results. Since the direction of movement of material is upward, the stresses that give rise to such a movement should be associated with a mean-stress compo-

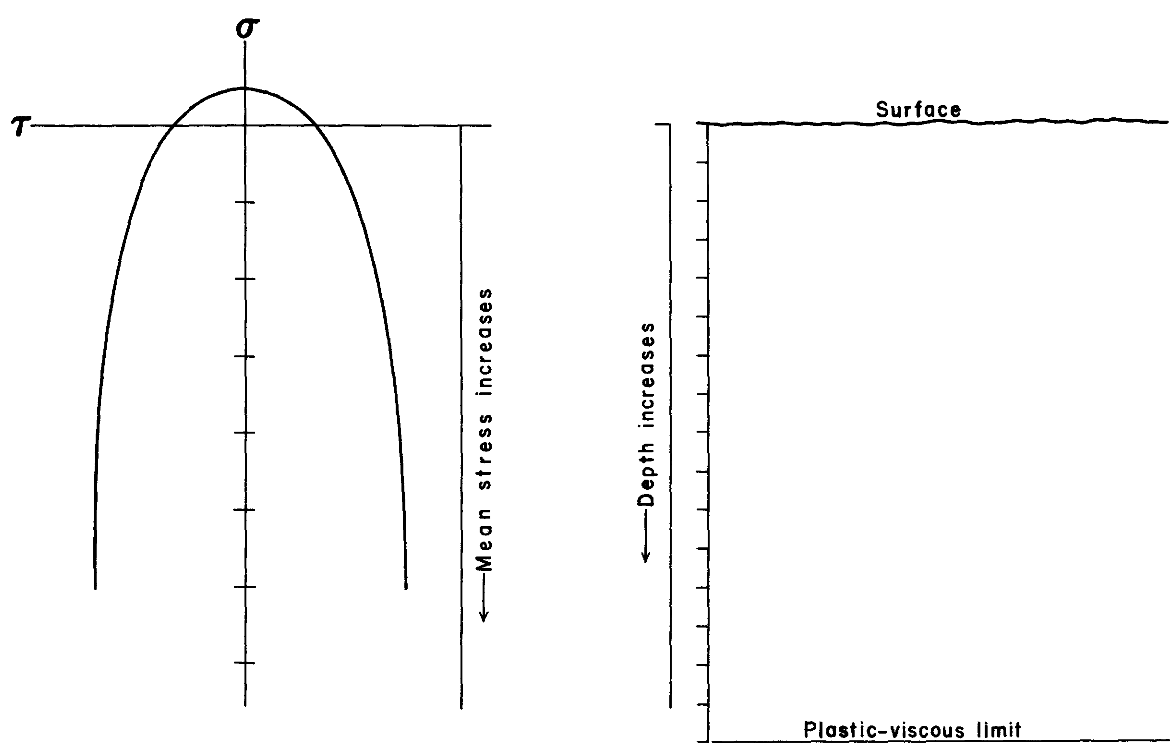

FIGURE 2.—Mean stress correlated with depth.

nent that decreases upward. In turn this implies that boundary stresses should decrease upward. The nature of the physical boundaries of the mountain masses are necessarily conjectural; nevertheless a variation in geometrical shape ranging from parallel walled boundaries to wedges with sides convergent upward or downward again should provide useful approximations. The concept of having straight-line walls as boundaries may not meet with agreement on the part of researchers who assert that there are no such sharp boundaries in nature but rather a gradual decrease of deformation intensity. This lack of agreement is recognized and cannot be discounted other than to note that solutions that involve gradual transition from plastic domains into elastic or rigid domains cannot be obtained from plastic theory at the present time. The approach is to treat the deformation within a unit as all of one type—permanent. By selecting boundaries somewhat wider apart than what might be dictated by field evidence, some allowance can be made for the lack of a transitional change. Similarly the selection of a wedge geometry instead of a parallel-shaped cell can account for lack of symmetry or variation of deformation if referred to rectangular coordinates.

Prandtl (1924) solved for the case which serves as a basic guide to deformation as applied to vertical tectonics here, namely the compressed cell.

According to his theory a mass is confined between two approaching, parallel, perfectly rough plates such that the shear stress along the plates will be at a maximum constant value. The material is permitted to extrude in one direction only, $+x$, and the plates squeezing the material remain parallel to one another and at right angles to the y axis as the deformation ensues. Figure 3 illustrates the problem and its solution in the form of the derived slip lines which are cycloids, as well as the distribution of the normal stresses on the boundaries.

Prandtl's solution was for a semi-infinite cell (x direction goes to infinity) in which case the slip lines would be orthogonal families of cycloids.

Prandtl's (1924) exposition is less detailed regarding the compressed cell than that of some workers who have enlarged on this work. For the purpose of detailing the solution, reference might be made to previously mentioned authors including Nâdai, Varnes, and Geiringer. Nâdai (1950, p. 533) gives the development somewhat as follows.

Taking the equilibrium conditions in plane strain

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}=0,\ \frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{xy}}{\partial x}=0 \tag{5}$$

and cross differentiating one with respect to x and the other with respect to y and then subtracting, the equation is

$$\frac{\partial^2(\sigma_x-\sigma_y)}{\partial x \partial y}=\frac{\partial^2(\tau_{xy})}{\partial x^2}-\frac{\partial^2(\tau_{xy})}{\partial y^2} \tag{6}$$

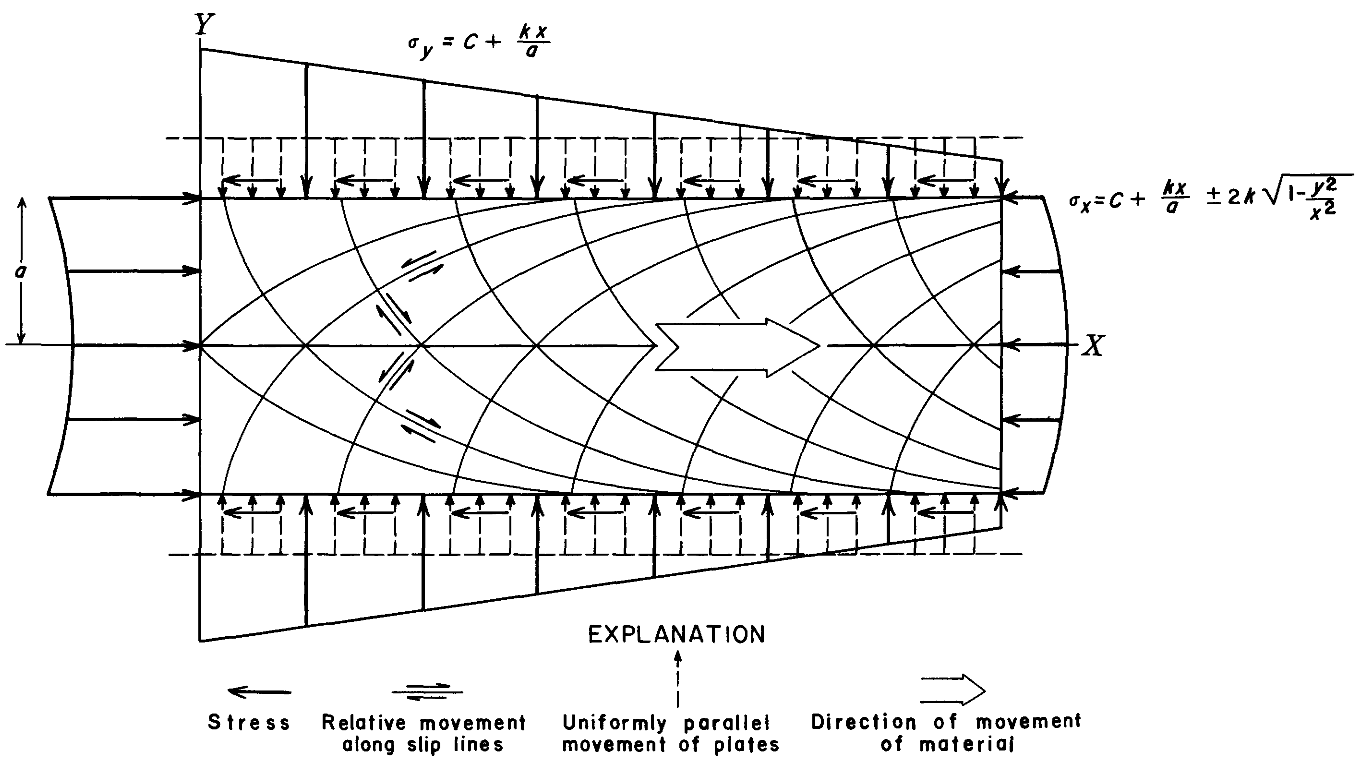

FIGURE 3.—General case of Prandtl's compressed cell.

Substituting from the von Mises' yield condition

$$\frac{(\sigma_x-\sigma_y{}^2)}{4}+\tau_{xy}^2=k^2 \tag{7}$$

$$(\tau_{\max}=\text{constant})$$

the partial differential equation 8 is obtained.

$$\frac{\partial^2\tau_{xy}}{\partial x^2}-\frac{\partial^2\tau_{xy}}{\partial y^2}=\pm 2\,\frac{\partial^2\sqrt{k^2-\tau_{xy}^2}}{\partial x\partial y} \tag{8}$$

Under the assumption that the shearing stress is a function of the y direction only, $\tau_{xy}=f(y)$, it is seen that equation 8 reduces to $\frac{\partial^2\tau_{xy}}{\partial y^2}=0$, for which a solution is $\tau_{xy}=cy$. At the boundaries (plates) it is seen that $\tau_{xy}=k$ since y will be a constant, and this will be the maximum shearing stress. Thus the contacts between the two plates and the material will be slip lines. From the equilibrium conditions (eq 5) the expressions for the coordinate stresses can now be obtained

$$\sigma_x=+\frac{kx}{a}+f_1(y),\quad \sigma_y=f_2(x) \tag{9}$$

where $2a$ is the distance between the plates. Moreover, from the yield condition the functions $f_1(y)$ and $f_2(x)$ are found to be

$$f_1(y)=c+2k\sqrt{1-\left(\frac{y}{a}\right)^2},\ f_2(x)=c+\frac{kx}{a} \tag{10}$$

Substitution of these functions into the expressions for coordinate stress the final expressions for stresses are obtained

$$\sigma_x=c+\frac{kx}{a}\pm 2k\sqrt{1-\left(\frac{y}{a}\right)^2},\ \sigma_y=c+\frac{kx}{a},\ \tau_{xy}=-\frac{ky}{a} \tag{11}$$

The distribution of these stresses is shown in figure 3. Note that σ_y is a linear function of the x direction only.

From equations which specify the relation between principal and coordinates stresses we have

$$\sigma_x=\sigma+k\sin 2\beta$$

$$\sigma_y=\sigma-k\sin 2\beta$$

$$\tau_{xy}=-k\cos 2\beta \tag{12}$$

where $\sigma=\frac{\sigma_1+\sigma_3}{2}$, and where β is the inclination between the maximum shear stress trajectory and the x axis. The relation between the coordinates and the slip lines is given as

$$\frac{dy}{dx}=\tan\beta,\ \frac{dy}{dx}=\tan\left(\frac{\pi}{2}+\beta\right) \tag{13}$$

Suitable substitution and integration leads to expressions for the slip lines given in parametric form as

$$x=a[2\beta+\sin 2\beta]+\text{const.},\quad y=a\cos 2\beta, \tag{14}$$

for the first slip-line family, and

$$x=a[2\beta-\sin 2\beta]+\text{const.},\quad y=a\cos 2\beta \tag{15}$$

for its orthogonal complement. These are also shown in figure 3 with the relative direction of slip required indicated along the lines.

Hartmann (1928, p. 495) gives a similar development for a material having internal friction. In his solution the slip lines are cycloids whose degree of nonorthogonality to one another depends on the coefficient of internal friction.

Prandtl used a von Mises' yield condition for his solution, whereas Hartmann used a Coulomb yield condition for his solution. Hartmann thus permitted failure on planes having less than the maximum shear stress. Since both yield conditions may be considered as special cases under the Mohr theory it follows that there may exist a solution to a compressed cell in which the derived slip lines range continuously from that of a Prandtl solution through a Hartmann-type solution with varying angles of internal friction.

Nádai (1950, p. 535) has shown that Prandtl's solution (hence Hartmann's) may be modified to give the so-called active and passive cases (fig. 4). Essentially, the slip-line patterns are rotated 180° from active case to passive case, but the material extrudes in the same direction as before. Prandtl's original solution is termed "the passive case" where the mass being extruded is said to passively receive the pressure being exerted by the plates. In the passive case, the slip lines are concave toward the direction of movement of material. In the active case a push on the material from one end in the direction of movement causes the mass to actively press on the plates and move them outward and at the same time extrude the material forward. Here the slip lines are convex toward the direction of movement of the material.

In the situation cited above, one dimension (x) is infinite. What happens with finite dimensions? The guiding geometry of a basic cell is as illustrated in figure 5.

The material being deformed is characterized by a central rigid kernel, bounded by the complementary first slip lines and the plates. At the ends of the plates, A and A', where the material is extruded, another rigid plug is bounded by a slip-line pattern that consists of radii and arcs of circles. The boundary between rigid material already extruded and the continuously deforming material still in the cell consists of

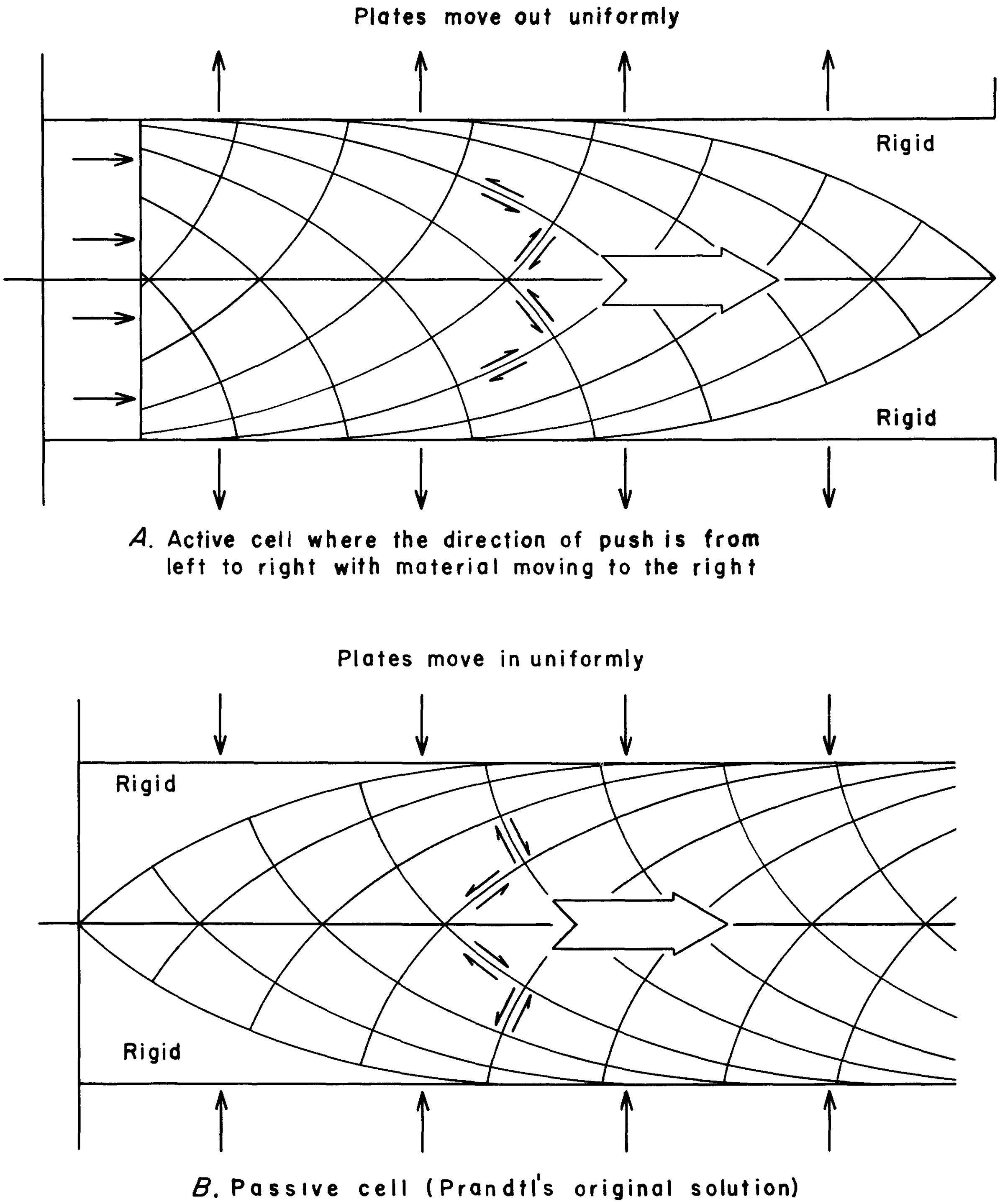

FIGURE 4.—Active and passive cells. *A*, Active cell, where the direction of push is from left to right with material moving to the right. *B*, Passive cell (Prandtl's original solution).

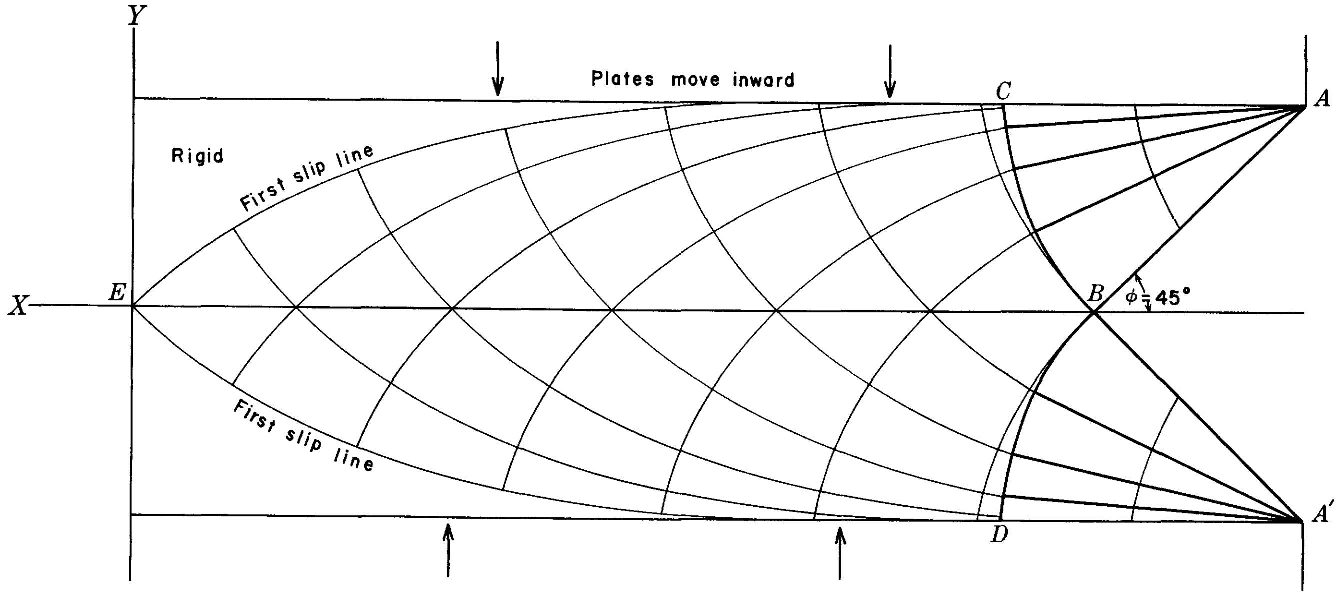

FIGURE 5.—Slip-line field in a finite compressed cell.

straight slip lines inclined at 45° to the x axis. In figure 5 these are shown by lines AB and $A'B$. If the material is not perfectly plastic the 45° angle decreases as the angle of internal friction increases. To further construct the slip-line pattern at the extruding end of the cell an arc of a circle is drawn from B to C with radius AB and the center at A. A similar construction is made in the lower half for the complementary family. Material in the areas ABC and $A'DB$ will have radii and arcs of circles as slip lines. Further inside the cell the slip lines will be cycloids bounded by the first slip lines or the plates.

The critical dimension of a cell occurs when the length of the cell approaches the width of the cell, a 1:1 dimensional limit (Prandtl, 1924, p. 51). A cell that approaches this limit is shown in figure 6. There is

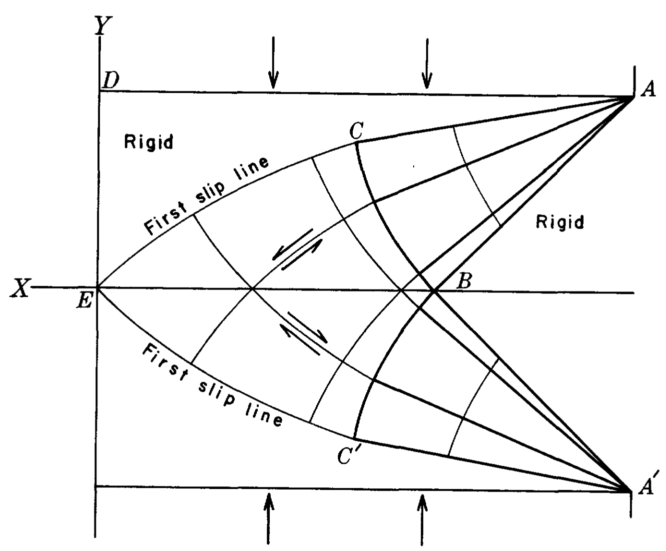

FIGURE 6.—Prandtl's cell approaching 1:1 dimensional limit.

virtually no change in the construction from the one outlined above since the extruding end of the cell is characterized by radii and arcs of circles, and the inner part by cycloids. However, the first slip line does not complete its trajectory to the plate; instead it is intersected by the circular arc at the end of the plate.

If the ratio of the length to width of the cell under compression is less than 1:1 the material behaves as if it were being punched rather than compressed between two plates (Prandtl, 1924, p. 51).

So far the discussion has been concerned with the passive case of deformation. The active case can be considered as the reverse. That is, if one had been viewing motion pictures of the passive case of deformation and now reversed the direction of film movement one would see what happens in the active case of deformation. The configuration of rigid areas for both the active and the passive cases is the same even though the slip lines are reversed relative to the direction of movement. Hence, for outward movement in the passive case the end of the cell is characterized by a rigid area (fig. 5), whereas in the active case the end of the cell is characterized by two rigid areas, having cycloids as one of the sides.

If the cells are rotated 90° so that the direction of movement is either up or down, it is seen that there are two possible positions of the individual cells. If the movement is in an upward direction, the slip lines of the passive case must be concave upwards, and in the active case slip lines must be convex upwards. The reverse orientation holds for downward movements (fig. 7).

The body force of gravity did not enter into Prandtl's solution, because this function acts normal to the xy

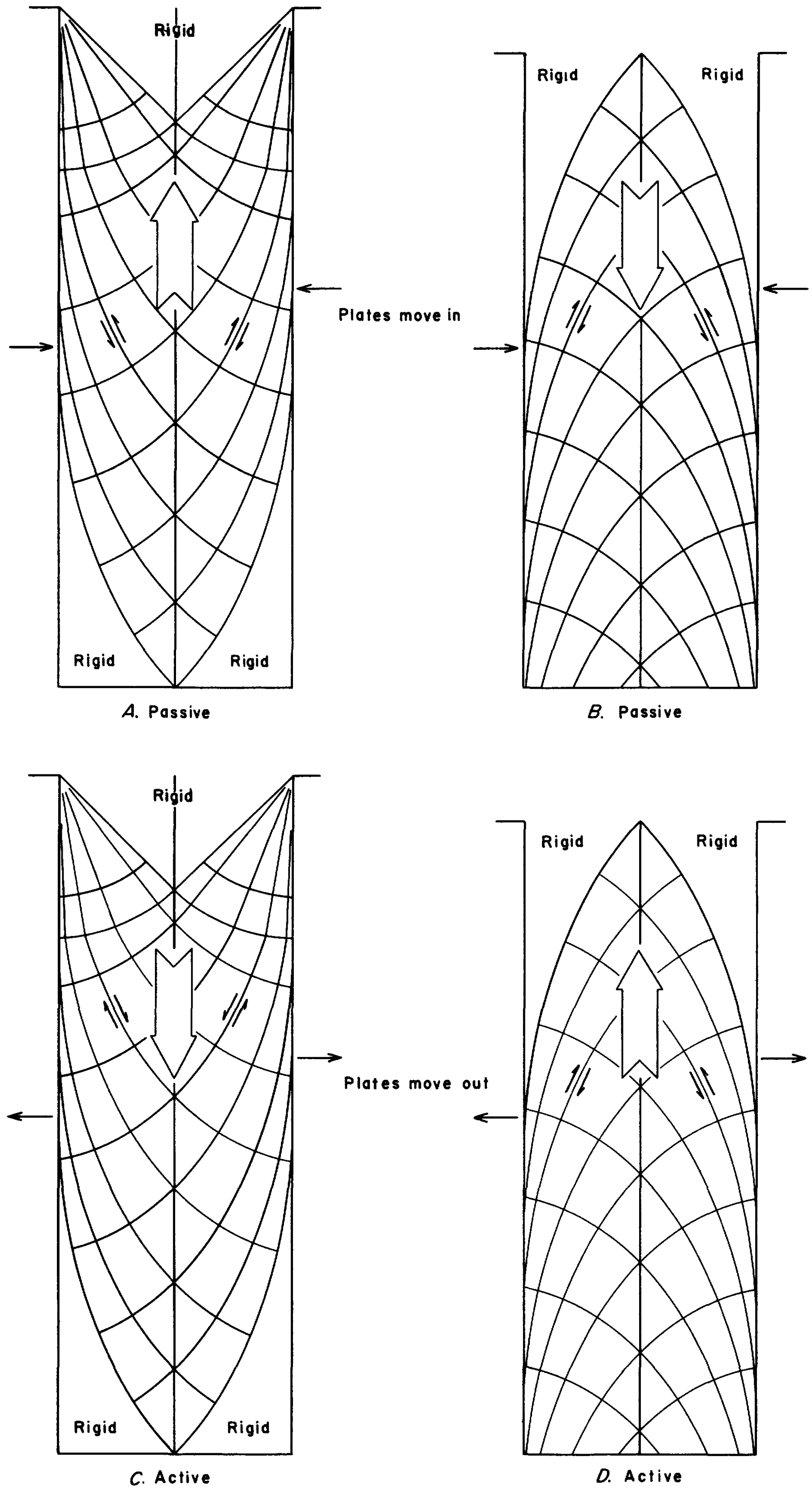

FIGURE 7.—Variations of Prandtl's cells oriented vertically.

plane and does not vary with either of the coordinates in this plane. Since the cell is rotated into the vertical plane, for application to vertical tectonics, it can be seen that the body force of gravity must be considered as a variable function of the vertical coordinate; the gravitational constant is effectively a constant to a depth of 2,400 kilometers (Bullen, 1947, p. 222). However, when cross derivatives of the equilibrium equations are taken, we would have the partial derivative of the gravitational body force with respect to a horizontal coordinate direction with which it does not vary; hence the derivative would vanish. Thus, the presence of a gravitational body force does not affect the slip-line configuration of a Prandtl cell in an upright position.

Attention is called to the linear increase of lateral compressive stresses required by Prandtl's solution, as this sort of stress distribution may well exist in the earth. The assumption that there is equilibrium postulates then that the lateral stresses must increase with depth to compensate for the increase in weight of a column of rock at depth. By addition of a uniform lateral compressive stress, the principal stress difference could be increased to the point of yield. The addition of a uniform, or even of a linearly distributed lateral compression, to the original lateral stress—which is assumed to increase downward linearly—still results in a linear distribution of normal lateral stress as required by the solution of Prandtl's cell. This is one of the principal reasons why the solution seems appropriate to the tectonics of long mountain ranges.

Nádai (1950, p. 543) has shown that the solution for the compressed cell of Prandtl is related to the solution of a compressed wedge where the only difference in the boundary conditions is in the angle between the compressing plates. For Prandtl's problem it is 0°, for Nádai's wedge the angle may range from 0° to 180°.

Nádai's solution for the wedge assumed movement of material out of the constricted end of the wedge. Varnes (1962) gives a similar solution for a compressed wedge where the material moves out of the wedge toward the open end. Thus, both of these wedge solutions are for passive cases. It can be shown that there are both active and passive solutions for a wedge just as there were for Prandtl's rectangular cell. The slip lines for the wedge solutions are exponential curves rather than the cycloids. However, the geometry of Nádai's active case is the same as Varnes' passive case, but the movement of material is in the opposite direction. Similarly, the Varnes active case has the same slip-line geometry as Nádai's passive case.

The four variations for the wedge solutions are shown schematically in figure 8.

Hence, for application to geologic structure, particularly in consideration of vertical tectonics, some geometric patterns range from those with parallel boundaries to those with boundaries inclined to one another at large angles. Furthermore, these boundaries may move toward one another as in the passive case or away from one another as in the active case giving rise to thrust or normal movement along slip lines, respectively.

STREAMLINES AND FOLDING

What will be the shape of a straight reference line after deformation? It can be shown that a straight line oriented parallel to the direction of transport will remain virtually straight although displaced, but a line originally oriented at right angles to the direction of transport will have an elliptic distribution for some time after deformation starts. This can be illustrated by two types of evidence. First it has been shown by Nádai (1950, p. 537), experimentally, that the distribution will be an elliptic one. Secondly, from a streamline pattern derived by Geiringer and Freudenthal (1958, p. 430), it is apparent that the shape of a straight line after deformation will be elliptic (see also Hill, 1950, p. 234). Figure 9 shows the pattern of streamlines in relation to the slip lines as derived by Geiringer and Freudenthal for Prandtl's cell. The relative instantaneous direction of slip along the slip lines follows as illustrated. Thus the particles closer to the X axis will have the greatest component of velocity upward parallel to the X axis, whereas those particles near the boundaries will have relatively less of a component in this direction.

Under the assumption that all particles of the mass have the same velocity, those particles of a straight line (AB) near the X axis will have moved upward farther than the marginal particles; hence the elliptic distribution. The assumption of the same constant velocity for all particles in the deforming mass is termed "steady flow." In the solution to Prandtl's compressed cell it is assumed that only scale changes occur so that the pattern of streamlines and slip lines will remain geometrically the same as in a previous state. This is termed "the pseudosteady state." If the geometry of the slip lines and streamlines were to change from one instant to the next the material would be under a condition of unsteady flow. Such a condition probably would tend to make the material flow faster in the middle than at the boundaries and consequently increase the arching of the elliptic distribution as well as to thicken the ellipse in the middle and thin it near the boundaries.

Figure 10 illustrates two successive stages in a case of steady flow for a Prandtl cell where only scale change

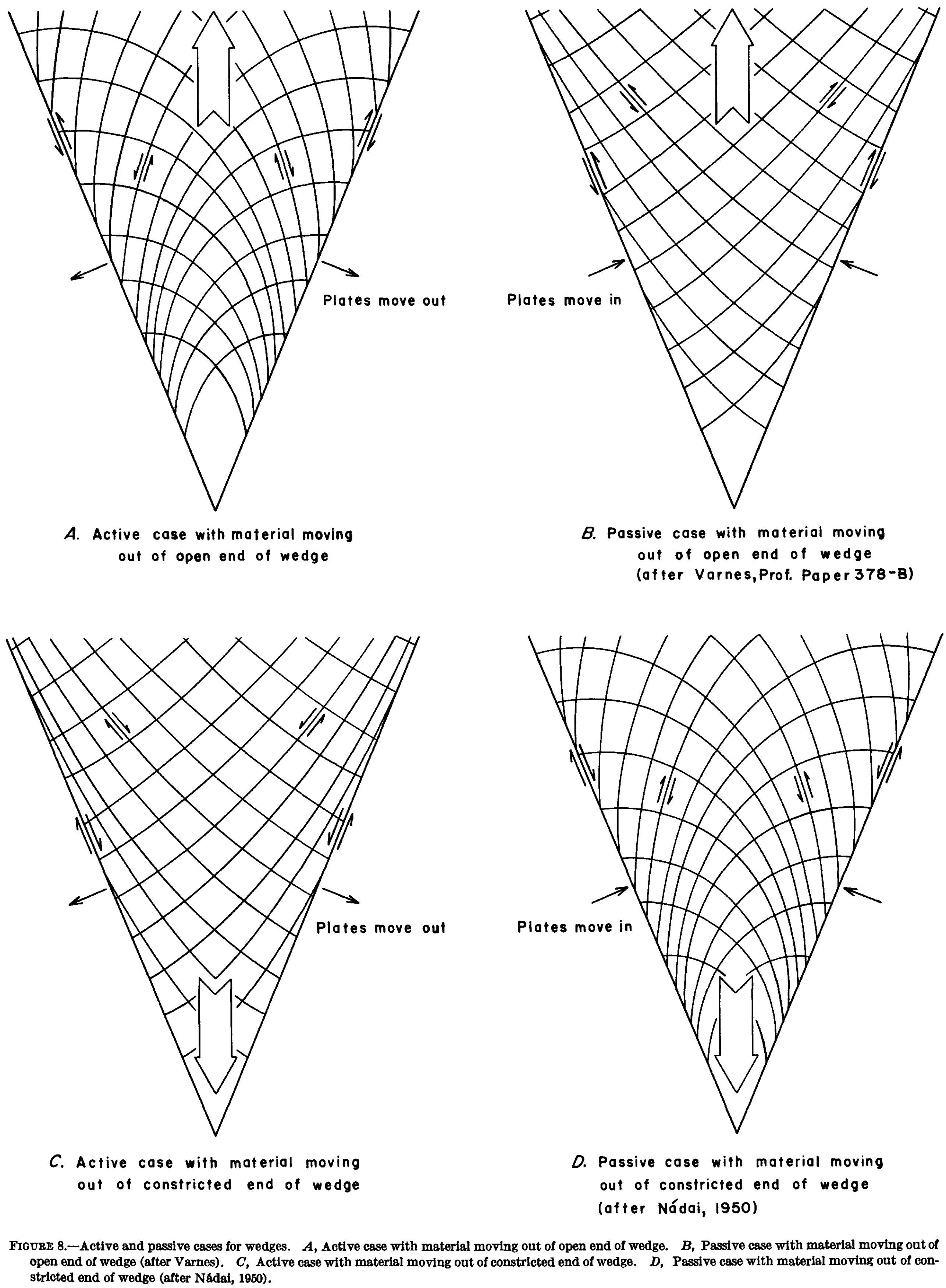

FIGURE 8.—Active and passive cases for wedges. *A*, Active case with material moving out of open end of wedge. *B*, Passive case with material moving out of open end of wedge (after Varnes). *C*, Active case with material moving out of constricted end of wedge. *D*, Passive case with material moving out of constricted end of wedge (after Nâdai, 1950).

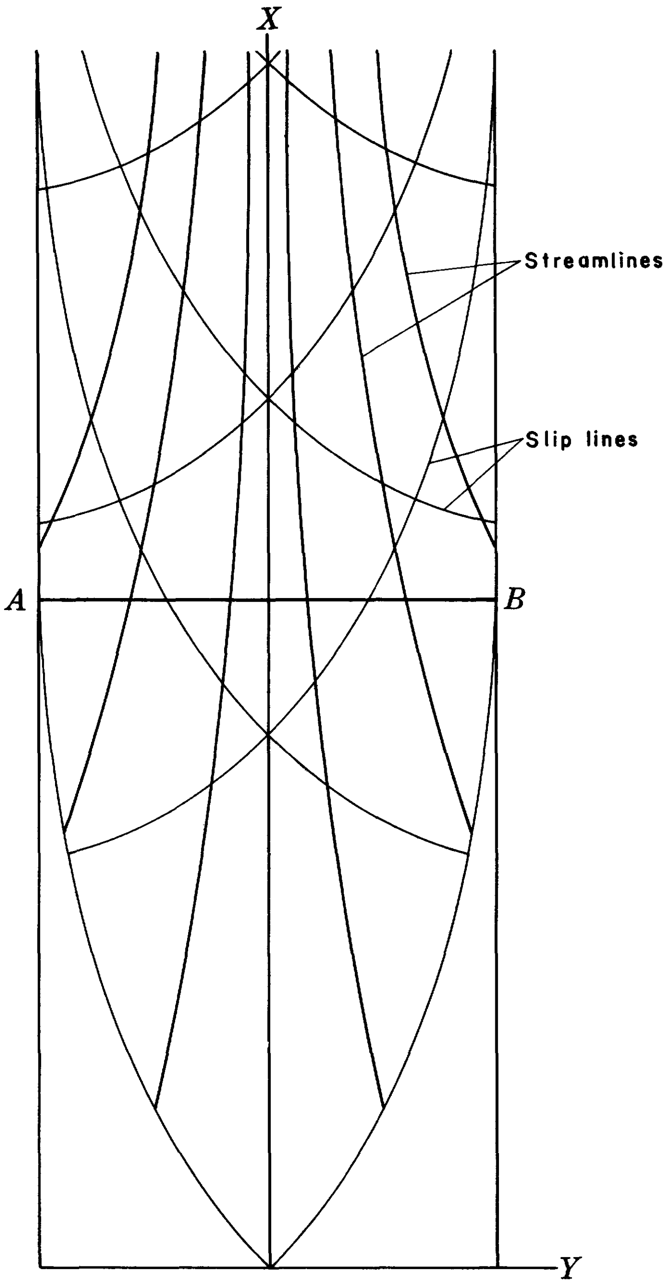

FIGURE 9.—Relation of streamlines to slip lines (passive case). Showing instantaneous particle movement and relative slip particles on a normal (AB) to axis of symmetry (after Geiringer and Freudenthal, 1958, p. 430).

has taken place. Similar flow phenomena will hold for the cases of wedges.

In contrast to the passive case as mentioned previously, the active case shows an opposite direction of movement. The form of an original straight band normal to the direction of movement will be elliptic; however, this time it will be thinned. Hence, from the condition of no volume change, both the active and the passive cases are marked by an arching, but the difference is that the passive case exhibits a thickening of the line, whereas the active case exhibits a thinning of the line.

SLIP LINES AND FAULTING

The relation of slip lines to faults is somewhat more tenuous than the relation of slip lines to folds. In folds the deformation is uniform and symmetrically distributed. Faulting on the other hand may be considered as deformation along one or a few of the many possible slip lines, hence, an asymmetrically distributed deformation. Ultimately the distinction to be made between faults and folds is one of why deformation occurs along a single line rather than along infinitely many. Such a question will not be resolved in this report. The writer believes that an asymmetry of deformation does not preclude a single slip line being analogous to a fault.

An entire cell may be considered as a fault zone as in figure 11, or one or more of the slip lines within the cell can be taken as a fault or fault zone as in figure 12.

The former case leads to a discussion of faults where the material outside the cell boundaries undergoes movement different from that specified in Prandtl's problem and where the material outside the boundaries constitutes an integral part of the structure. This case will not be discussed here.

The case where the entire deformation is within the cell boundaries, and where faulting is manifested along a restricted set of slip lines consisting of one or more slip lines, is discussed briefly in the following quotation by Nâdai (1950, p. 550).

> Incidentally, it may be noted that among the systems of displacement satisfying eq. (37–57) states of strain may occur in which only one layer of slip will form in the direction of one of the two characteristics m=const. or n=const. This is the case, for example, if one portion of the body adjoining the edge is fixed in space, while the other part remaining rigid beyond the slip layer is allowed to move as a rigid body according to the motion prescribed through the shear in the layer. Thus, we see that slip along a single family of characteristics is also a possible case of distortion and that the requirement of the Mohr theory claiming the formation of two symmetrical systems of slip is not a necessary attribute in their laws of formation. * * * It may be worth noting that, in the case of more general plastic states of stress when the slip layers are curved, the requirement of having certain regions displaced as rigid bodies between the wedge shaped plastic layers may not so easily be satisfied as for straight characteristics.

Some remarks are needed to justify the correlation between theoretically derived slip lines and faults in the geologic sense. A requirement of theory is that there be a continuity of material across slip lines even though the slip lines represent velocity discontinuities. In the geologic analogy faults generally exhibit this continuity of material even though particles have been displaced along the fault. In other words voids are

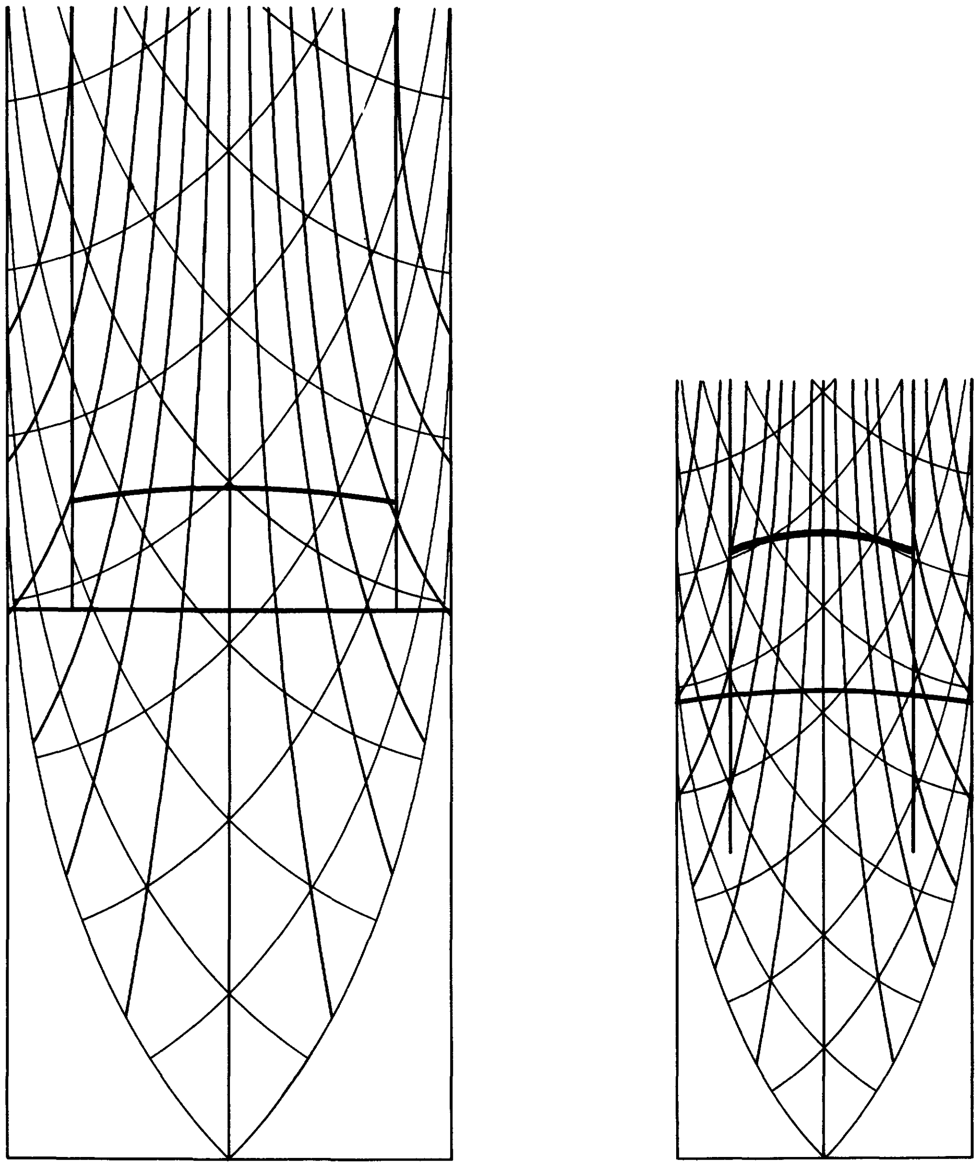

FIGURE 10.—Deformation of an originally straight line (heavy) showing the thickening and arching as boundaries move successively inwards, under constant velocity throughout, and only scale changes in slip-line and streamline pattern (passive case).

not permitted in the theoretical case nor are large continuous voids generally observed in real faults. This analogy does not ignore the occurrence of small-scale irregularities giving rise to permeability variations, brecciated zones and similar phenomena that occur along faults or fault zones. Such irregularities do not alter the genesis of faults under plastic theory.

Displacements along faults generated under plastic theory have two aspects.

1. If the slip lines are virtually straight, then both rigid body translations and rotations may occur; hence there need be no internal deformation of a mass contained within preexisting slip lines.
2. If there is a curvature to the preexisting slip lines such that translations and rotations are restricted through the assumption of no volume change, then new internal deformations must take place if there is to be yield without "holes." Hence new boundary conditions must be satisfied with the old slip lines that constitute part of the new boundary conditions. The old slip lines thus will control subsequent faulting although they have their genesis in an earlier deformation.

RELATION TO VERTICAL TECTONICS

A direct application of the foregoing theory to a specific area will not be made; rather the principles involved in such an application will be briefly reviewed. For the most part the terms "vertical tectonics" and "lateral tectonics" have been contrasted with one another in the sense that "vertical" meant movement as a result of a primary upward or downward force system

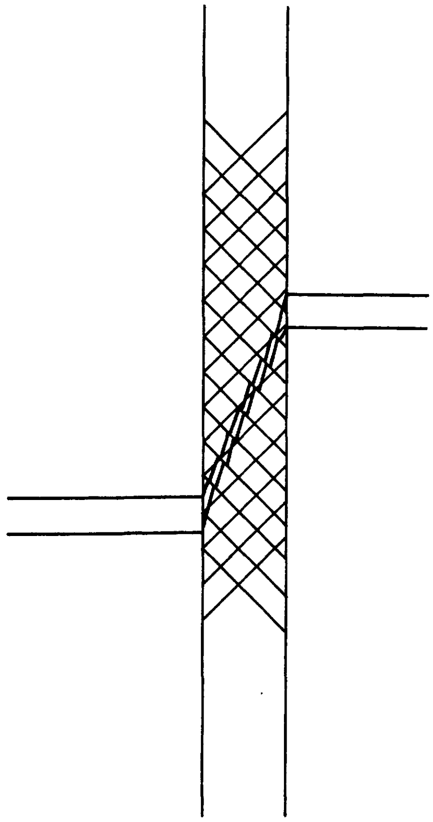

FIGURE 11.—An entire cell as a fault zone.

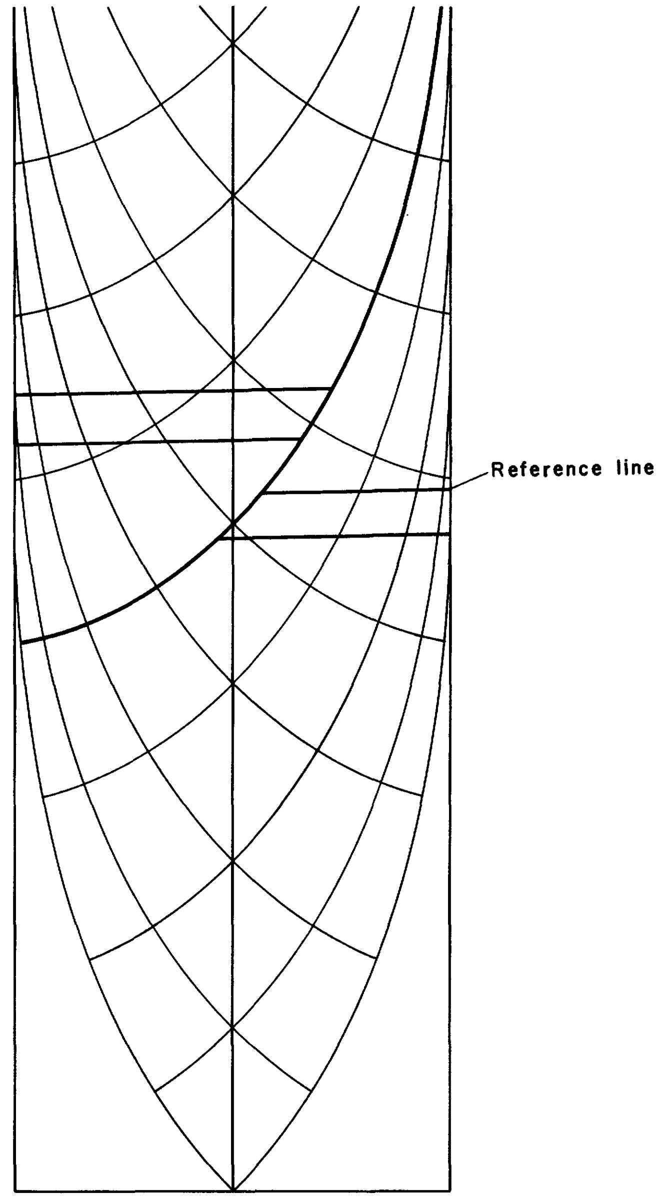

FIGURE 12.—A slip line as a fault.

whereas "lateral" implied movement as a result of tangential forces. In this report, implication concerning the direction of forces is omitted since vertical movement can result either from an upward push or a lateral squeeze. Hence, the term "vertical tectonics" is used only in the sense of the predominantly upward direction of movement of deformation.

Essentially two ideas are developed. The first is that Mohr's theory of strength implies zones of deformation in which the perfectly plastic behavior increases with depth. At this point nothing is specified about how the slip lines are oriented with respect to the earth's surface. The second idea is that plastic solutions for rectangles and wedges may furnish useful approximations to deformation in the earth's crust.

Figure 13 shows a diagrammatic composite of several successive deformations characterized by mountain building and erosional cycles. The diagram shows from left to right three successive episodes of mountain building and deformation, each followed by a period of quiescence and erosion. The internal deformation is assumed to take place on genetically related slip lines throughout, but the slip lines in the upper parts of the diagrams are not mutually at right angles, whereas those at the bottom are. The only distinction in the type of deformation in the three different zones concerns how deformation is accomplished along slip lines in a given zone. Hence, in zone A the deformation is assumed to be entirely along a restricted few of the possible slip lines permitted by theory; thus the movement would occur by faulting without significant deformation in the area between the slip lines. Zone B can be characterized by two types of movement, faulting along restricted sets of slip lines and flow deformation in the area between the faults. In zone C deformation is along extremely closely spaced slip lines (of which only a few are drawn) in such a manner that continuous displacement produces folding rather than rupturing.

These three zones can be arranged one on top of the other as shown in figure 13. The zone nearest the sur-

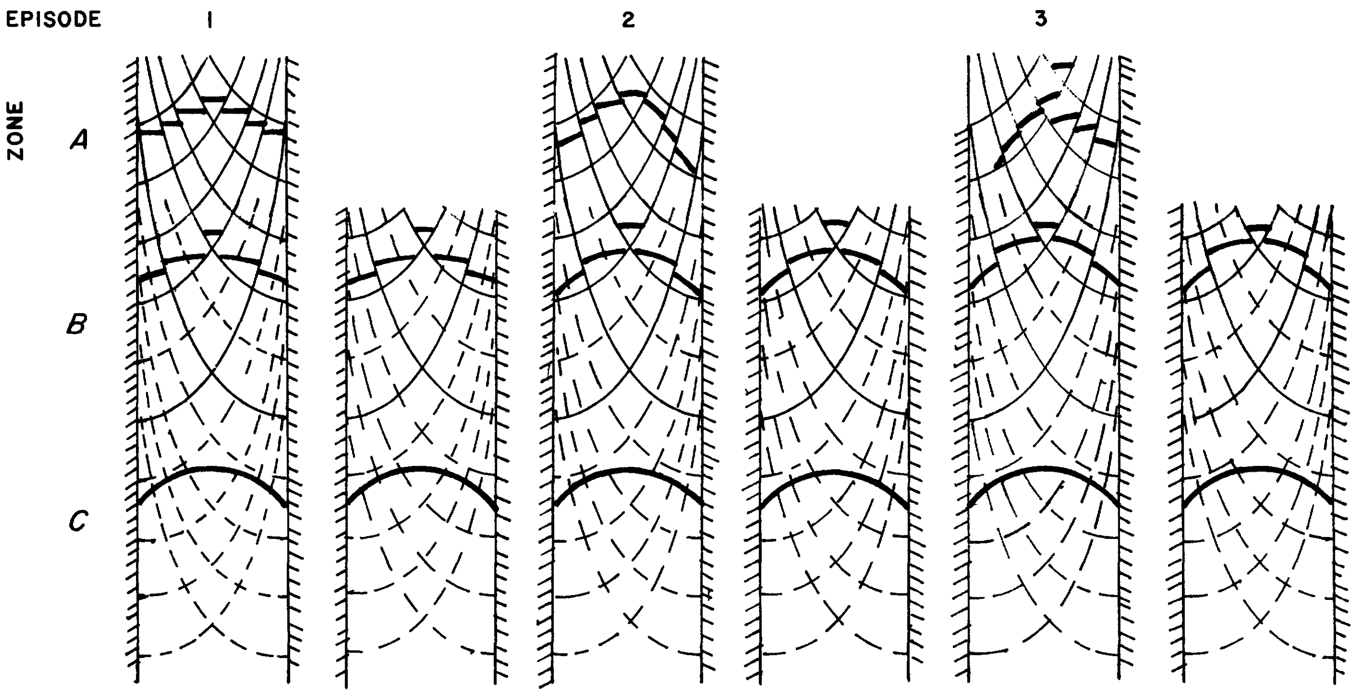

FIGURE 13.—Diagrammatic representation of superimposed zones of deformation according to a particularized yield condition under Mohr's theory of strength (passive Prandtl cells).

face will be denoted by the letter *A*, the zone next deeper by *B* and the deepest by *C*. The number to which the letter is appended will refer to the particular episode or orogeny during which the deformation takes place. Hence each combination of number and letter designates a time and place of deformation under a succession of similar stress systems.

At the end of the first episode an observer on the surface would see evidence of 1*A*, and would have noticed the formation of a mountain range. A period of quiescence and erosion of the mountain range would follow the first episode during which part or all of 1*A* would be removed. A deeper zone, 1*B*, might be exposed by the erosion. Another subsequent episode of orogenic activity would again be characterized by the three zones of deformation. The observer on the surface would see, as in the first episode, the formation of zone 2*A*. He would note that the geometry would be superimposed on that of 1*B*. Repetition of the process would afford the observer with superimposed patterns of deformation.

At the close of the mountain building stage of episode 3 the observer would note a compound structure at the surface consisting of parts of 3*A*, 2*B*, and 1*C*. If the material had been originally homogeneous before deformation of 1*C*, there would be little likelihood that it would remain homogeneous to the end of episode 3.

An element of analysis frequently overlooked is the one that ties together parts of the same genesis. If the age relations are not simple and straightforward, care must be taken to distinguish one genetically related set of structures from another. The slip-line trajectories will be concave upward when the movement along the slip lines is of a thrust type, and convex upward when the movement along the slip lines is of the normal type. Under these conditions, a thrust fault should decrease in dip as it is traced to greater depth; a normal fault should steepen. Furthermore, as one observes structure along a horizontal plane the dips of faults should steepen progressively as the boundaries are approached from the center.

For the passive case with upward movement of material, it might well be expected that the slip-line geometry at the surface would always consist of radii and arcs of circles as in figure 10, provided that deformation is continuous. If deformation ceases and erosion strips the upper end of the cell, exposing parts formed deeper within the cell, a fault pattern will be seen, which is not expected to be at the end of the cell. Isostatic uplift could then provide additional material for removal by erosion so that ultimately the roots of mountains would be exposed.

High positive gravity anomalies thus would suggest areas of possible compound structure inasmuch as such anomalies can be interpreted to mean uplifted roots of mountains.

Exactly what physical manifestations the boundaries of cells will have in the geologic analogy is not clear. From the purely theoretical point of view there does not need to be a plate or similar feature, merely an end to the solution. The boundary may be a vertical fault predating the deformation taking place within the boundaries. On the other hand the boundary might appear completely arbitrary as an intermediate line separating an anticline from a syncline.

Thus far geologic structure has been related to prime orogenic forces. The mechanics of the removal or erosion of mountains in the evolution toward base level is a secondary effect of the prime orogenic forces. That is, prime orogenesis is related to constructional aspects of mountains whereas the secondary effects are related to destructional features.

Hence, those phenomena commonly studied in the science of soil mechanics—particularly the effects on slope stability, such as pore water pressure, fabric of rock or soil, vegetation, and ground water gradients—can be presumed to have an important effect even where the scale approaches that of the mountain-building type. Gravitational sliding phenomena are probably the type related to soil mechanics rather than the type associated with the prime orogenic forces.

Thus with two separate genetic types of deformation possible at the surface care must be used not to apply the plastic theory as here proposed to encompass that of the gravitational sliding type. Accordingly it is suggested that the test of the theory should come in the investigation of older structure or at least those structures where the "extruded" parts have been removed by erosion.

Plastic theory can be applied also to the analysis of destructional features, though such features are not considered in this report.

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