

River Meanders— Theory of Minimum Variance

GEOLOGICAL SURVEY PROFESSIONAL PAPER 422-H



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By WALTER B. LANGBEIN and LUNA B. LEOPOLD

PHYSIOGRAPHIC AND HYDRAULIC STUDIES OF RIVERS

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The geometry of a meander is that of a random walk whose most frequent form minimizes the sum of the squares of the changes in direction in each unit length. Changes in direction closely approximate a sine function of channel distance. Depth, velocity, and slope are adjusted so as to decrease the variance of shear and the friction factor in a meander over that in an otherwise comparable straight reach of the same river



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SYMBOLS

a	ratio of variances (table 3)
c	a coefficient (equation 1)
k	sinuosity, ratio of path distance to downvalley distance
R	radius of curvature of a bend
r	coefficient of correlation (table 3)
s	unit distance along path
x	a factor or variable
p	probability of a particular direction (equation 1)
σ^2	variance of a particular factor, x
D	water depth
S	energy slope
M	total path distance in a single wavelength
N	number of measurements
v	velocity
α	a constant of integration (equation 2)
γ	unit weight of water
λ	wavelength
π	pi, ratio of circumference to radius
ρ	radius of curvature of path
σ	standard deviation
ϕ	angle that path at a given point makes with mean downpath direction
ω	maximum angle a path makes with mean downpath direction

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ABSTRACT

Meanders are the result of erosion-deposition processes tending toward the most stable form in which the variability of certain essential properties is minimized. This minimization involves the adjustment of the planimetric geometry and the hydraulic factors of depth, velocity, and local slope.

The planimetric geometry of a meander is that of a random walk whose most frequent form minimizes the sum of the squares of the changes in direction in each successive unit length. The direction angles are then sine functions of channel distance. This yields a meander shape typically present in meandering rivers and has the characteristic that the ratio of meander length to average radius of curvature in the bend is 4.7.

Depth, velocity, and slope are shown by field observations to be adjusted so as to decrease the variance of shear and the friction factor in a meander curve over that in an otherwise comparable straight reach of the same river.

Since theory and observation indicate meanders achieve the minimum variance postulated, it follows that for channels in which alternating pools and riffles occur, meandering is the most probable form of channel geometry and thus is more stable geometry than a straight or nonmeandering alignment.

INTRODUCTION

So ubiquitous are curves in rivers and so common are smooth and regular meander forms, they have attracted the interest of investigators from many disciplines. The widespread geographic distribution of such forms and their occurrence in various settings had hardly been appreciated before air travel became common. Moreover, air travel has emphasized how commonly the form of valleys—not merely the river channels within them—assumes a regular sinuosity which is comparable to meanders in river channels. Also, investigations of the physical characteristics of glaciers and oceans as well as landforms led to the recognition that analogous forms occur in melt-water channels developed on glaciers and even in the currents of the Gulf Stream.

The striking similarity in physical form of curves in these various settings is the result of certain geometric proportions apparently common to all, that is, a nearly constant ratio of radius of curvature to meander length and of radius of curvature to channel width (Leopold

and Wolman, 1960, p. 774). This leads to visual similarity regardless of scale. When one looks at a stream on a planimetric map without first glancing at map scale, it is not immediately obvious whether it is a large river or a small stream.

Explanation of river meanders have been varied. It has been suggested that meanders are caused by such processes as—

- regular erosion and deposition;
- secondary circulation in the cross-sectional plane;
- seiche effect analogous to lake seiches.

Attempts, however, to utilize these theories to calculate the forms of meanders have failed.

Although various phenomena—including some of those mentioned above, such as cross circulation—are intimately involved in the deviation of rivers from a straight course, the development of meanders is patently related to the superposition of many diverse effects. Although each of these individual effects is in itself completely deterministic, so many of them occur that they cannot be followed in detail. As postulated in this paper, such effects can be treated as if they were stochastic—that is, as if they occurred in a random fashion.

This paper examines the consequences of this postulate in relation to (1) the planimetric geometry of meanders, and (2) the variations in such hydraulic properties as depth, velocity, and slope in meanders as contrasted with straight reaches.

The second problem required new data. These were obtained during the snowmelt season of high discharge in the years 1959–64 by the junior author with the invaluable and untiring assistance of William W. Emmett and Robert M. Myrick. Leon W. Wiard was with us for some of the work. To each of them, the authors are very grateful for their important contribution, measured in part by the internal consistency of the various field data, and the satisfactory closures of surveys made under difficult conditions.

MEANDER GEOMETRY

Meandering in rivers can be considered in two contexts. The first involves the whole profile from any headwater tributary downstream through the main trunk—that is, the longitudinal profile of the river system. The second context includes a meandering reach of a river in which the channel in its lateral migrations may take various planimetric forms or paths between two points in the valley.

The river network as a whole is an open system tending toward a steady state and within which several hydraulically related factors are mutually interacting and adjusting—specifically, velocity, depth, width, hydraulic resistance, and slope. These adjust to accommodate the discharge and load contributed by the drainage basin. The adjustment takes place not only by erosion and deposition but also by variation in bed forms which affect hydraulic resistance and thus local competence to carry debris. Previous application of the theory of minimum variance to the whole river system shows that downstream change of slope (long profile) is intimately related to downstream changes in other hydraulic variables. Because discharge increases in the downstream direction, minimum variance of power expenditure per unit length leads toward greater concavity in the longitudinal profile. However, greater concavity is opposed by the tendency for uniform power expenditures on each unit of bed area.

In the context of the whole river system, a meandering segment, often but not always concentrated in downstream rather than upstream portions of the system, tends to provide greater concavity by lengthening the downstream portion of the profile. By increasing the concavity of the profile, the product of discharge and slope, or power per unit length becomes more uniform along a stream that increases in flow downstream. Thus the meander decreases the variance of power per unit length. Though the occurrence of meanders affects the stream length and thus the river profile, channel slope in the context of the whole river is one of the adjustable and dependent parameters, being determined by mutual interaction with the other dependent factors.

In the second context of any reach or segment of river length, then, average slope is given, and local changes of channel plan must maintain this average slope. Between any two points on the valley floor, however, a variety of paths are possible, any of which would maintain the same slope and thus the same length. A thesis of the present paper is that the meander form occupies a most probable among all possible paths. This path is defined by a random-walk model.

PATH OF GREATEST PROBABILITY BETWEEN TWO POINTS¹

The model is a simple one; a river has a finite probability p to deviate by an angle $d\phi$ from its previous direction in progressing an elemental distance ds along its path. The probability distribution as a function of deviation angle may be assumed to be normal (Gaussian) which would be described by

$$dp = c \exp \frac{1}{2} \left(\frac{d\phi/ds}{\sigma} \right)^2 \quad (1)$$

where σ is the standard deviation and where c is defined by the condition that $\int dp = 1.0$.

The actual meander path then corresponds to the most probable river path proceeding between two points A and B , if the direction of flow at the point A and the length of the path between A and B is fixed and the probability of a change in direction per unit river length is given by the probability distribution above. This formulation of the problem of river meanders is identical to that of a class of random-walk problems which have been studied by Von Schelling (1951, 1964). The Von Schelling solution demonstrated that the arc length s is defined by the following elliptic integral:

$$s = \frac{1}{\sigma} \int \frac{d\phi}{\sqrt{2(\alpha - \cos \phi)}} \quad (2)$$

where ϕ is the direction angle measured from the line AB and α is a constant of integration. It is convenient to set

$$\alpha = \cos \omega \quad (3)$$

in which ω becomes the maximum angle the path makes from the origin with the mean direction. Curves for $\omega = 40^\circ, 90^\circ, 110^\circ$, which all show patterns characteristically seen in river meanders, are given in figure 1 (after Von Schelling, 1964, p. 8). Von Schelling (1951) showed that a general condition for the most frequent path for a continuous curve of given length between the two points, A and B , is

$$\sum \frac{\Delta s}{\rho^2} = \text{a minimum}, \quad (4)$$

where Δs is a unit distance along the path and ρ is the radius of curvature of the path in that unit distance. But since

$$\rho = \left(\frac{\Delta s}{\Delta \phi} \right) \quad (5)$$

where $\Delta \phi$ is the angle by which direction is changed in distance Δs , therefore,

$$\sum \frac{(\Delta \phi)^2}{\Delta s} = \text{a minimum}. \quad (6)$$

¹ Acknowledgments are due A. E. Scheldegger for his assistance in clarifying the mathematical relationships of this section.

Since the sum of all the directional changes is zero, or

$$\sum \Delta\phi = 0. \quad (7)$$

equation 6 is the most probable condition in which the variance (mean square of deviations in direction) is minimum.

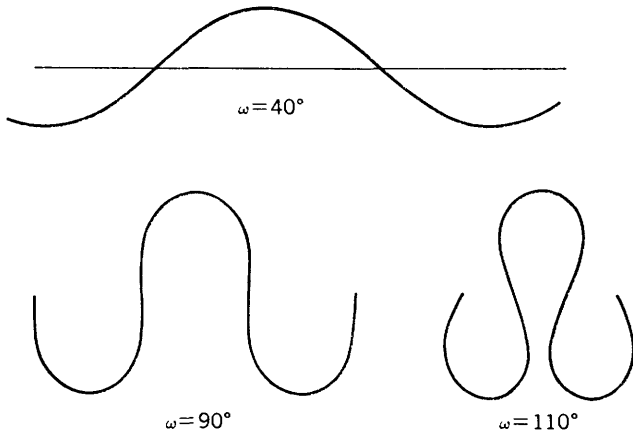


FIGURE 1.—Examples of most frequent random-walk paths of given lengths between two specified points on a plane. Adapted from H. von Schelling.

THE SINE-GENERATED CURVE

For graphing meanders it is easier to make use of the approximation that

$$\frac{d\phi}{ds} = \sigma \sqrt{2(1 - \cos \omega)} \left[1 - \left(\frac{\phi}{\omega} \right)^2 \right] \quad (8a)$$

is a close approximation for

$$\frac{d\phi}{ds} = \sigma \sqrt{2(\cos \phi - \cos \omega)} \quad (8b)$$

where angles ω and ϕ are deviations from the central axis with the downstream direction as zero. With this simplification

$$\phi = \omega \sin \frac{\sigma \sqrt{2(1 - \cos \omega)}}{\omega} s \quad (9)$$

or

$$\phi = \omega \sin \frac{s}{M} 2\pi \quad (10)$$

where M is the total path distance along a meander, and where ω is the maximum angle the path makes from the mean downvalley direction. The maximum possible value of ω is 2.2 radians, for at this angle meander limbs intersect. When ω equals 2.2 radians, the pattern is a closed figure "8."

Comparison of equations 9 and 10 indicates that

$$M = \frac{2\pi}{\sigma} \frac{\omega}{\sqrt{2(1 - \cos \omega)}} \quad (11)$$

all angles being in radians, as before.

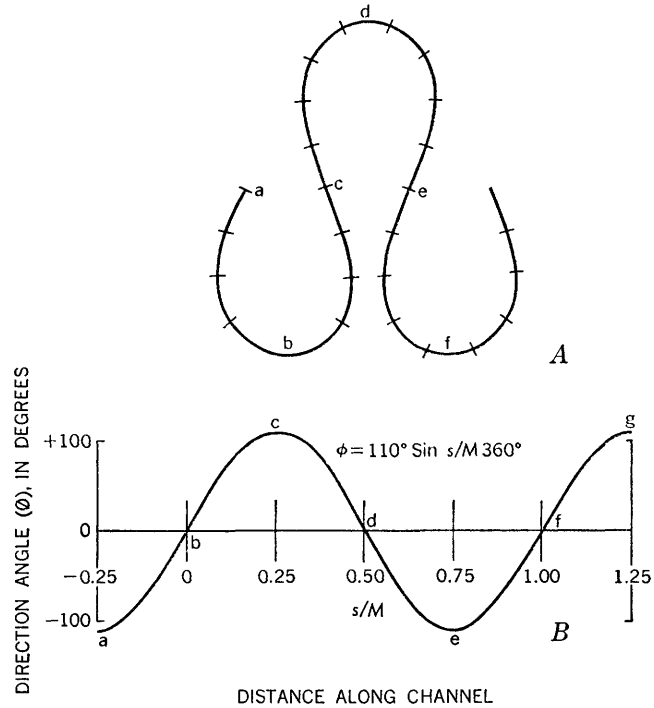


FIGURE 2.—Theoretical meander in plan view (A) and a plot of its direction angle ϕ as a function of distance along the channel path (B).

Inasmuch as the ratio $\omega/\sqrt{2(1 - \cos \omega)}$ is nearly constant ($=1.05$), over the range of possible values of ω , meander path lengths, M , are inversely proportional to the standard deviation σ ; thus, $M = 6.6/\sigma$.

Equation 10 defines a curve in which the direction ϕ is a sine function of the distance s along the meander. Figure 2 shows a theoretical meander developed from a sine function. The meander thus is not a sine curve, but will be referred to as a "sine-generated curve."

Thus meander loops are generated by the sine curve defined by equation 10, and the amount of horseshoe looping depends on the value of ω . This means that at relative distance $\frac{s}{M}$ equal to $\frac{1}{2}$ and 1, ϕ has a value of zero or the channel is locally directed in the mean direction. At distance $\frac{s}{M}$ equal to $\frac{1}{4}$ and $\frac{3}{4}$ the value of ϕ has its largest value ω , as can be seen in figure 2. The graph on figure 2A has been constructed for a value of $\omega = 110^\circ$, and corresponds almost exactly to the 110° curve of figure 1 calculated by the exact equations of Von Schelling.

It will be noted that the plan view of the channel (fig. 2A) is not sinusoidal; only the channel direction changes as sinusoidal function of distance (fig. 2B). The meander itself is more rounded and has a relatively uniform radius of curvature in the bend portion. This can be noted in the fact that a sine curve has quite a straight segment as it crosses the x -axis.

The tangent to the sine function in figure 2*B* at any point is $\Delta\phi/\Delta s$ which is the reciprocal of the local radius of curvature of the meander. The sine curve is nearly straight as it crosses the zero axis. Therefore, the radius of curvature is nearly constant in a meander bend over two portions covering fully a third of the length of each loop.

There are, of course, plenty of irregular meanders but those which are fairly uniform display a close accordance to a true sine curve when channel direction is plotted against distance along the channel, as will be shown.

ANALYSIS OF SOME FIELD EXAMPLES

Some field examples presented in figure 3 will now be discussed briefly with a view to demonstrating that among a variety of meander shapes the sine-generated curve fits the actual shape quite well and better than alternatives.

Figure 3*A*, upper part, shows the famous Greenville bends at Greenville, Miss., before the artificial cutoffs changed the pattern. The crosses in the lower diagram represent values of channel direction, ϕ —relative to a chosen zero azimuth, which for convenience is the mean downstream direction—plotted against channel distance. The full curve in the lower diagram is a sine curve chosen in wavelength and amplitude to approximate the river data. A sine curve that was fitted to the crosses was used to generate a plot of channel direction against distance which has been superimposed on the map of the river as the dashed line in the upper diagram. It can be seen that there is reasonable agreement.

The same technique has been used in the other examples in figure 3 chosen to cover a range in shape from oxbows to flat sinuous form, and including a variety of sizes. Figure 3*B* shows a stream that not only is not as oxbowed as the Greenville bends but also represents a river of much smaller size, the Mississippi River being about three-quarters of a mile wide and Blackrock Creek about 50 feet wide.

Figure 3*C* shows the famous Paw Paw, W. Va., bends of the Potomac River, similar to and in the same area as the great meanders of the tributary Shenandoah River. The curves in these rivers are characterized by being exceptionally elongated in that the amplitude is unusually large for the wavelength and both are large for the river width, characteristics believed to be influenced by elongation along the direction of a fracture system in the rock (Hack and Young, 1959). Despite these peculiarities the sine-generated curve fits the river well.

Figures 3*D* and 3*E* are among the best known incised meanders in the western United States, where the Colorado River and some of its tributaries are a thousand or more feet below the lowest of the surrounding benches, the cliffs rising at very steep angles up from the river's very edge. Even these provide a plot of direction versus length which is closely approximated by a sine curve.

By far the most symmetrical and uniform meander reach we have ever seen in the field is the Popo Agie River near Hudson, Wyo., for which flow data will be discussed later. The relation of channel direction to distance is compared with a sine curve in the lower part of figure 3*F*.

The laboratory experiments conducted by Friedkin (1945) provide an example of near-ideal meanders. The meander shown in figure 4 is one of those that developed greatest sinuosity. The channel shown on figure 4*A* was developed in Mississippi River sand on a "valley" slope of 0.006, by rates of flow that varied from 0.05 to 0.24 cfs over an 18-hour period on a schedule that simulated the fluctuations of discharge on the Mississippi River. The sand bed was homogeneous and the meander mapped by Friedkin is more regular than those usually observed in natural streams.

The direction angles of the thalweg were measured at intervals of 2.5 feet. These direction angles are plotted on the graph, figure 4*B*. Since each meander had a thalweg length of 50 feet, the data for the two meanders are shown on the same graph. Where the two bends had the same direction only one point is plotted. The data correspond closely to the sine curve shown.

COMPARISON OF VARIANCES OF DIFFERENT MEANDER CURVES

As a close approximation to the theoretical minimum, when a meander is such that the direction, ϕ , in a given unit length Δs is a sine function of the distance along the curve, then the sum of squares of changes in direction from the mean direction is less than for any other common curve. In figure 5, four curves are presented. One consists of two portions of circles which have been joined. Another is a pure sine curve. The third is a parabola. A fourth is sine-generated in that, as has been explained, the change of direction bears a sinusoidal relation to distance along the curve. All four have the same wavelength and sinuosity—the ratio of curve length to straight line distance. However, the sums of squares of the changes in direction as measured in degrees over 10 equally spaced lengths along the curves differ greatly as follows:

Curve	$\Sigma(\Delta\phi)^2$
Parabola.....	5210
Sine curve.....	5200
Circular curve.....	4840
Sine-generated curve.....	3940

The "sine-generated" curve has the least sum of squares. The theoretical minimum curve is identical within practical limits of drawing.

MEANDER LENGTH, SINUOSITY, AND BEND RADIUS

The planimetric geometry of river meanders has been defined as follows

$$\phi = \omega \sin \frac{s}{M} 2\pi \quad (10)$$

where ϕ equals the direction at location s , ω is the maximum angle the meander takes relative to the general direction of the river, and M is the channel length of a meander. The values of ω and M can be defined further as follows:

The angle ω is a unique function of the sinuosity and is independent of the meander length. It ranges from zero for a straight line path of zero sinuosity to a maximum of 125° for "gooseneck" meanders at point of incipient crossing. The sinuosity, k , equals the average of the values of $\cos \phi$ over the range from $\phi=0$ to $\phi=\omega$. Thus a relationship can be defined between k and ω . An approximate algebraic expression ² is

$$\omega \text{ (radians)} = 2.2 \sqrt{\frac{k-1}{k}}$$

or

$$\omega = 125^\circ \sqrt{\frac{k-1}{k}}$$

Sinuosity as measured by parameter k , as explained, is thought to be a consequence of the profile development which is only secondarily influenced by reaction from the meander development.

In the random-walk model, the standard deviation of changes in direction per unit of distance is σ which

² Relation of ω to sinuosity, k .

$$k = \frac{M}{\Sigma \cos \phi \Delta s}$$

where

$$\phi = \omega \sin \frac{s}{M} 2\pi.$$

With an assumed value of ω , values of ϕ at 24 equally spaced intervals of s/M were computed. The reciprocal of the average value of $\cos \phi$ equals the sinuosity, k .

Values of ω and k so computed were as follows:

ω (radians)	k	ω (radians)	k
0.5.....	1.06	1.5.....	1.96
1.0.....	1.30	1.75.....	2.67

The values of k approximate the following

$$k = 4.84 / (1.84 - \omega^2)$$

or

$$\omega = 2.2 \sqrt{\frac{k-1}{k}}$$

has the dimension of L^{-1} or the reciprocal of length. As previously shown, σ is inversely proportional to meander length.

Bend radius is related to wave length and is virtually independent of sinuosity. Defined as before, as the average over the $\frac{1}{2}$ of channel length for which ϕ is nearly linearly related to channel distance, bend radius R is

$$R = \frac{\frac{1}{2}M}{\Delta\phi}.$$

Since ϕ ranges from $+0.5\omega$ to -0.5ω over this near-linear range, $\Delta\phi = \omega$. Substituting for ω its algebraic equivalent ³ in terms of sinuosity,

$$\omega = 2.2 \sqrt{\frac{k-1}{k}}$$

and since $M = k\lambda$, bend radius equals

$$R = \frac{\lambda}{13} \frac{k^{3/2}}{\sqrt{k-1}}.$$

Some typical values for bend radius in terms of sinuosity are

k	Bend radius (R)	Ratio $\frac{\text{wavelength}}{\text{bend radius}}$
1.25.....	0.215 λ	4.6
1.5.....	.20 λ	5.0
2.0.....	.22 λ	4.5
2.5.....	.24 λ	4.2

These values agree very closely with those found by measurement of actual meanders. Leopold and Wolman (1960) measured the meander characteristics of 50 samples, including rivers from various parts of the United States, that range in width from a few feet to a mile. For this group, in which sinuosities were dominantly between 1.1 and 2.0, the average ratio of meander length to radius of curvature was 4.7, which is equivalent to .213 λ . The agreement with values listed above from the theory is satisfactory.

The curve defined by $\phi = \omega \sin \frac{s}{M} 360^\circ$ is believed to underlie the stable form of meanders. That actual meanders are often irregular is well known, but as observed above, those meanders that are regular in geometry conform to this equation. Deviations (or "noise"), it is surmised, are due to two principal causes: (1) shifts from unstable to a stable form caused by random actions and varying flow, and (2) non-homogeneities such as rock outcrops, differences in alluvium, or even trees.

The irregularities are more to be observed in "free" (that is, "living") meanders than in incised meanders. During incision, irregularities apparently tend to be

³ See footnote ², this page.

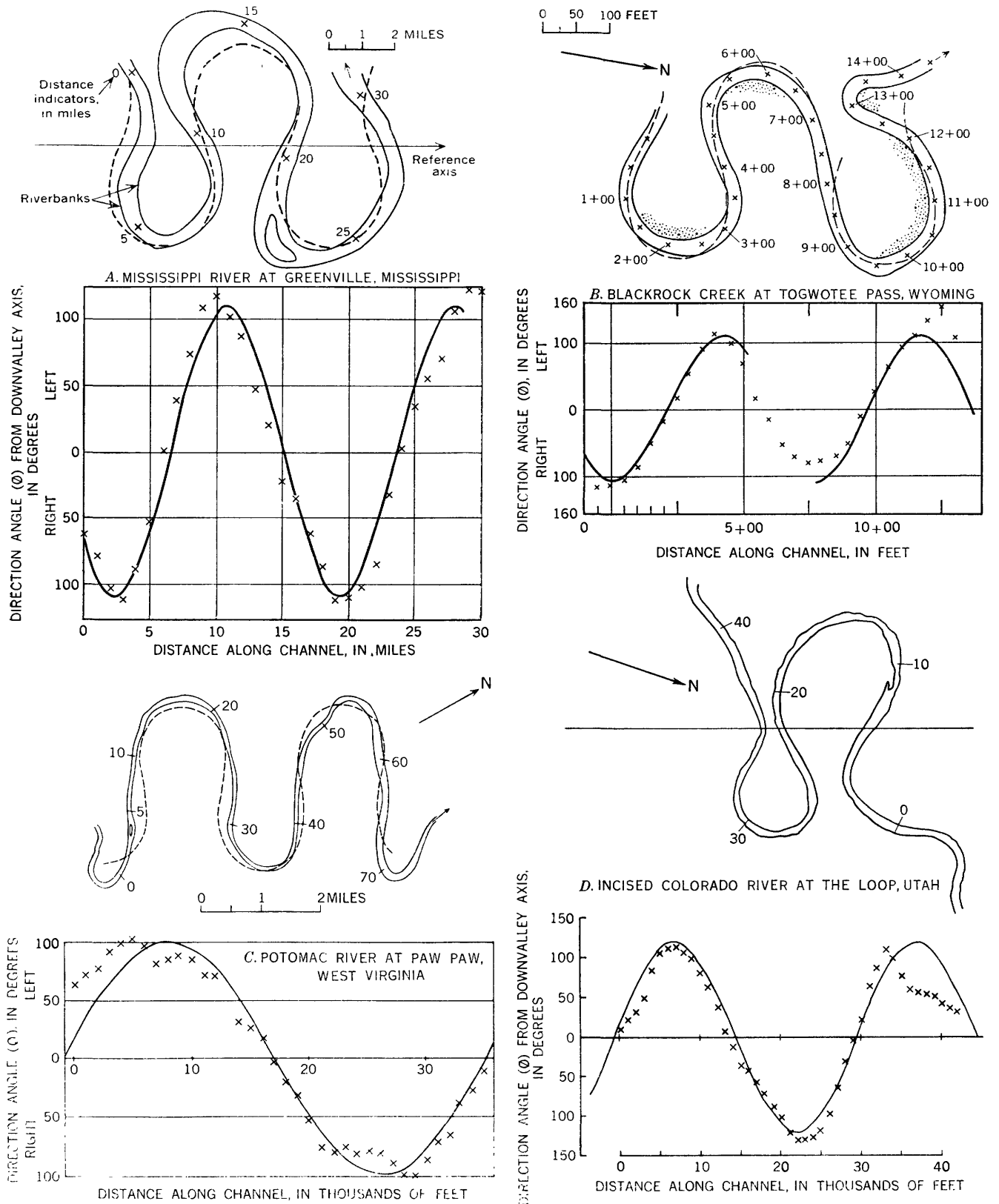
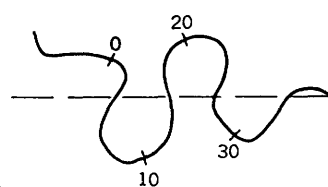
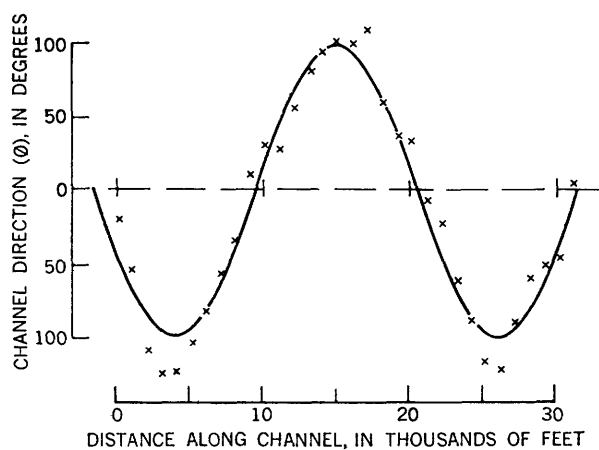
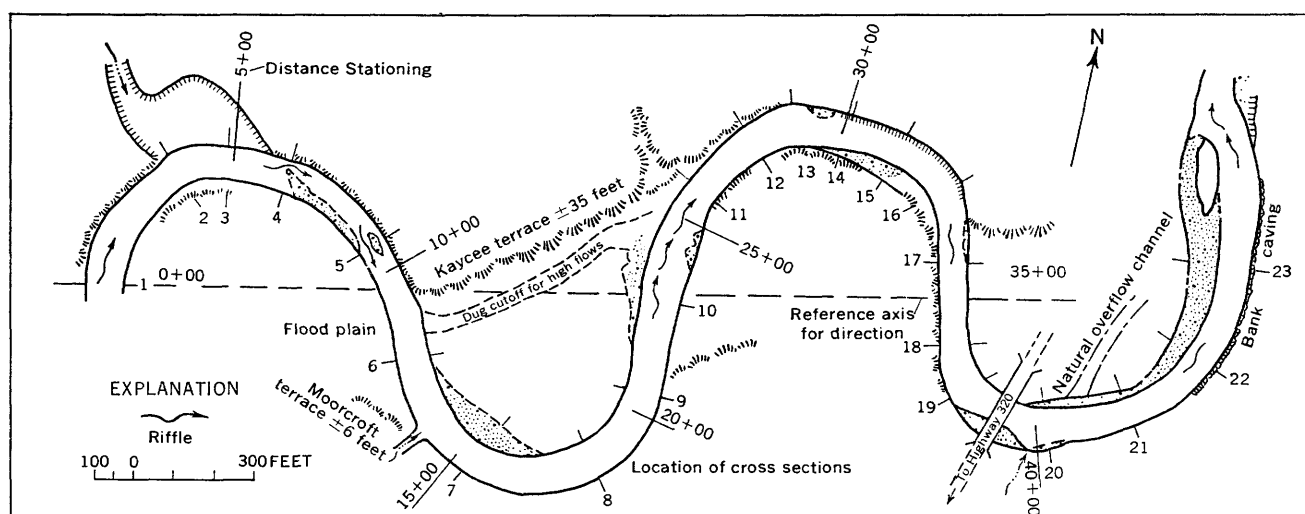


FIGURE 3.—MAPS OF RIVER CHANNELS AND

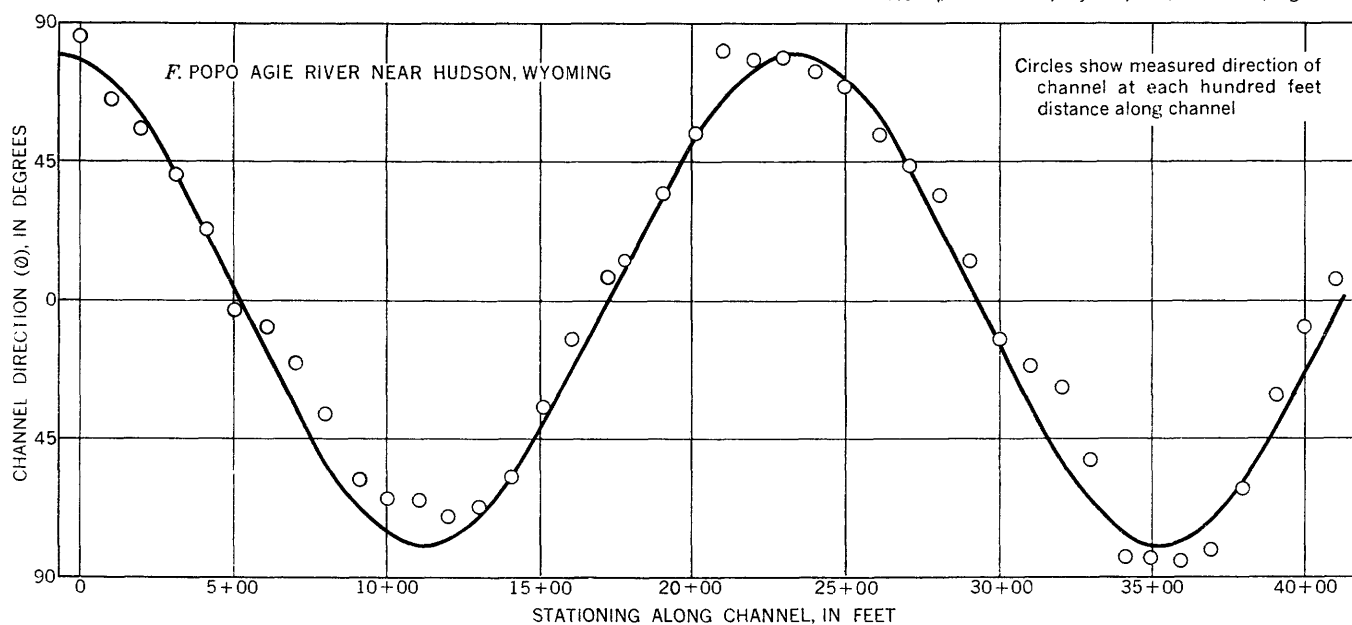
A, Upper, Map of channel compared to sine-generated curve (dashed); lower, plot of actual channel direction against distance (crosses) and a sine curve (full line).
 B, Upper, Map of channel compared to sine-generated curve (dashed); lower, plot of actual channel direction against distance (crosses) and a sine curve (full line).
 C, Upper, Map of channel compared to sine-generated curve (dashed); lower, plot of actual channel direction against distance (crosses) and a sine curve (full line).
 D, Upper, Planimetric map of river; lower, plot of actual channel direction against distance (crosses) and a sine curve (full line).



E. INCISED SAN JUAN RIVER
AT THE GOOSENECKS, UTAH



From planimetric map by Leopold and Wolman, August 1953



PLOTS OF CHANNEL DIRECTION AGAINST DISTANCE

E. Right, Planimetric map of river; left, plot of actual channel direction against distance (crosses) and a sine curve (full line). *F.* Upper, Planimetric map of river; lower, plot of actual channel direction against distance (crosses) and a sine curve (full line).

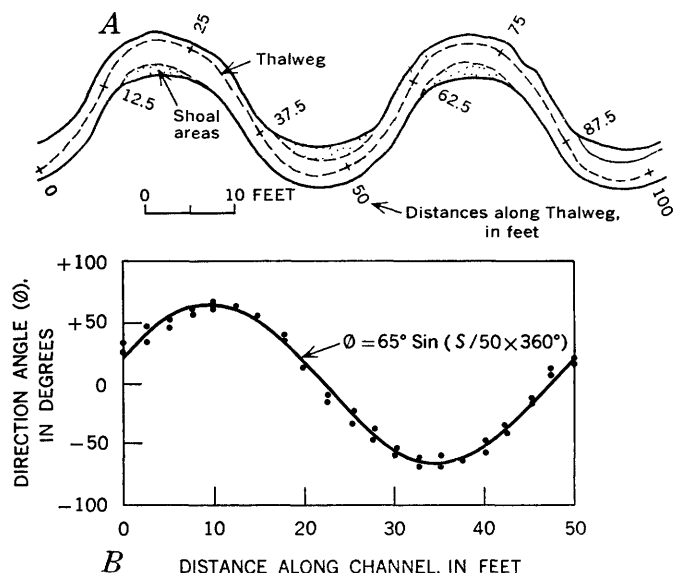


FIGURE 4.—Map of laboratory meander and plot of thalweg direction against distance. A, Plan view; B, Comparison of direction angles with sine curve.

averaged out and only the regular form is preserved, and where these intrench homogeneous rocks, the result is often a beautiful series of meanders as in the San Juan River or the Colorado River.

MEANDERS COMPARED WITH STRAIGHT REACHES

Meanders are common features of rivers, whereas straight reaches of any length are rare. It may be inferred, therefore, that straightness is a temporary state. As the usual and more stable form, according to the thesis of this report, the meander should be characterized by lower variances of the hydraulic factors, a property shown by Maddock (Written commun., 1965) to prevail in self-formed channels.

The review of the characteristics of meanders and the theories which had been advanced to explain them showed that quantitative data were nearly nonexistent on many aspects of the hydraulics of flow during those high stages or discharges at which channel adjustment actively takes place.

Because there are certain similarities and certain differences between straight segments or reaches of river and meandering reaches, it also seemed logical to devote special attention to comparison of straight and curved segments during channel-altering flow. An initial attempt to obtain such data was made during 1954–58 on a small river in Maryland, near enough to home and office to be reached on short notice during the storm flows of winter. Though something was learned, it became clear that rivers which rise and fall in stage quickly do not allow a two-man team with modest equipment to make the desired observations.

Attention was then devoted to the rivers which drain

into the Missouri and Colorado River systems from the Wind River Range in central Wyoming, where good weather for field work generally exists at the base of the mountains during the week or two in late spring or early summer when snowmelt runoff reaches its peak. Some of the smaller streams presented the difficulty that the area of study lay 15 to 20 miles from the farthest watershed divide—a distance sufficient to delay the peak discharge by 9 to 12 hours. Maximum snowmelt in midafternoon caused a diurnal peak flow which reached the study area shortly after midnight and meant surveying and stream gaging by flashlight and lantern illumination of level rod and notebook.

The plan of field work involved the selection of one or more meanders of relatively uniform shape which occurred in the same vicinity with a straight reach of river at least long enough to include two riffles separated by a pool section. No tributaries would enter between the curved and straight reaches so that at any time discharge in the two would be identical. It was assumed and checked visually, that the bed and bank materials were identical in the two reaches as well. Later bed sampling showed that there were some differences in bed-material size which had not been apparent to the eye.

A curve introduces an additional form of energy dissipation not present in a comparable straight reach, an energy loss due to change of flow direction (Leopold and others, 1960). This additional loss is concentrated in the zone of greatest curvature which occurs midway between the riffle bars. The additional loss is at a location where, in a straight reach, energy dissipation is smaller than the average for the whole reach.

The introduction of river bends, then, tends to equalize the energy dissipation through each unit length, but does so at the expense of greater contrasts in bed elevation. Moreover, curvature introduces a certain additional organization into the distribution of channel-bed features. The field measurements were devised to measure and examine the variations in these quantities.

Though more will be said about individual study reaches when the data are discussed, a salient feature was apparent which had not previously been noticed when the first set of such field data was assembled; this feature can be seen in figure 6. Where the profiles of bed and water surface are plotted on the same graph for comparison, it is apparent that at moderately high stage—from three-fourths bankfull to bankfull—the larger-scale undulation of the riverbed caused by pool and shallow has been drowned out, as the streamgager would say, or no longer causes an undulation in the water-surface profile of the meander. In contrast, at the same stage in the straight reach the flat and steep alternations caused by pool and riffle are still discernible.

On the other hand, the profile revealed that the undulation of the bed along the stream length is of larger magnitude in the curved than in the straight reach.

Considering the features exemplified in figure 6, one could reason as follows. In a straight reach of channel the dunes, bars, pools, and riffles form more or less independently of the channel pattern owing to grain interaction. To the occurrence of pool and riffle the channel must adjust. The riffle causes a zone of greater-than-average steepness which is also a zone of greater-than-average energy expenditure. The pool, on the other hand, being relatively free of large gravel on the bed and of larger depth relative to bed roughness elements, offers less resistance to flow and there the energy expenditure is less than average. The result is a stepped water-surface profile—flat over pool and steep over riffle.

Once pools and riffles form with their consequent variations in depth, width remaining relatively uniform, then slopes must vary. If slopes vary so that bed shear remains constant, then slope would vary inversely with depth. However, this result would require that the friction factor vary directly as the square of the depth. If, on the other hand, the friction factor remained uniform and the slope varied accordingly, then bed shear would vary inversely as the square of the depth. Uniformity in bed shear and the friction factor is incompatible, and the slope adjusts so as to accommodate both about equably and minimize their total variance.

FIELD MEASUREMENTS

The general program of field measurements has been described. The main difficulty experienced was that the combination of conditions desired occurs rarely. A straight reach in proximity to a regular meander

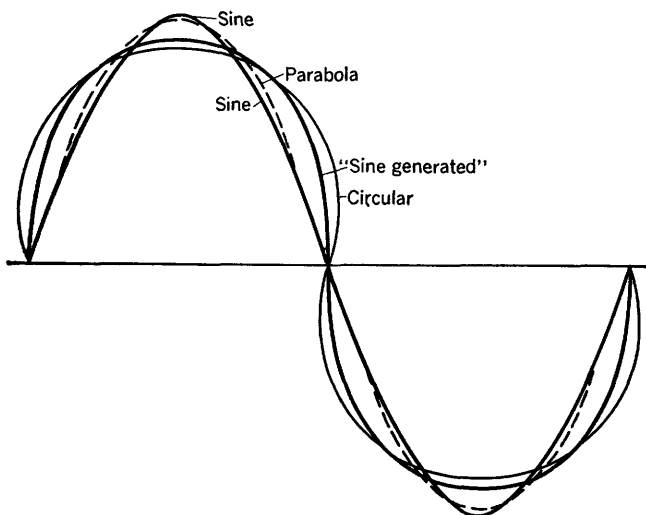


FIGURE 5.—Comparison of four symmetrical sinuous curves having equal wave-length and sinuosity.

curve does not have a high probability of occurrence, and to find such a combination, airplane photographic traverses at about 1,000 feet above the ground were flown for some distance along the east front of the Wind River Range.⁴ Even after reaches were chosen from the photographs, ground inspection resulted in discard of significant proportions of possible sites.

When a satisfactory site was located, a continuous water-stage recorder was established; benchmarks and staff gage were installed and the curved and straight reaches mapped by plane table. Cross sections were staked at such a spacing that about seven would be included in a length equal to one pool-and-riffle sequence.

Water-surface profiles were run by leveling at one or more stages of flow. Distances between water-surface shots were equal for a given stream; 10-foot distances were used on small streams having widths of 10 to 20 feet and 25-foot spacing used for streams 50 to 100 feet wide. Usually two rods, one on each streambank, accompanied the instrument. Shots were taken opposite one another along the two banks. Water-surface elevations were read to 0.01 foot on larger rivers, and to 0.001 foot on small ones. For the small rivers an attachment was used on a surveying rod which in construction resembled a standard point gage used in laboratory hydraulic practice for measurement of water-surface elevation.

Velocity measurements by current meter were usually made at various points across the stream at each cross section or at alternate ones. In the small streams these current-meter measurements were made by wading, those in the larger rivers from a canoe. For canoe measurements a tagline had to be stretched across the river at the staked cross sections, an operation fraught with some difficulty where brush lined the banks and when the velocity was high at near-bankfull stage.

Figures 6, 7, and 8 show examples of the planimetric maps and bed and water-surface profiles. The sinusoidal change of channel direction along the stream length for figure 7 was presented in figure 3.

REDUCTION OF DATA

Mean depths and mean velocities were calculated for each cross section. The average slopes of the water surface between the cross sections were also computed from the longitudinal profile. To reduce these quantities to nondimensional form, they were expressed in ratio to the respective means over each reach. The variances of these ratios were computed by the usual formula

⁴ The authors acknowledge the assistance of Herbert E. Skibitzke and David E. Jones in the photo reconnaissance.

$$\sigma_x^2 = \frac{\sum X^2}{N} - \left(\frac{\sum X}{N} \right)^2 \quad (11)$$

where σ_x^2 is the variance of the quantity X and N is number of measurements in each reach.

As shown by Maddock (1965), the behavior of many movable-bed streams can be explained by a tendency toward least variation in bed shear and in the friction factor. Accordingly, the variances of the bed shear

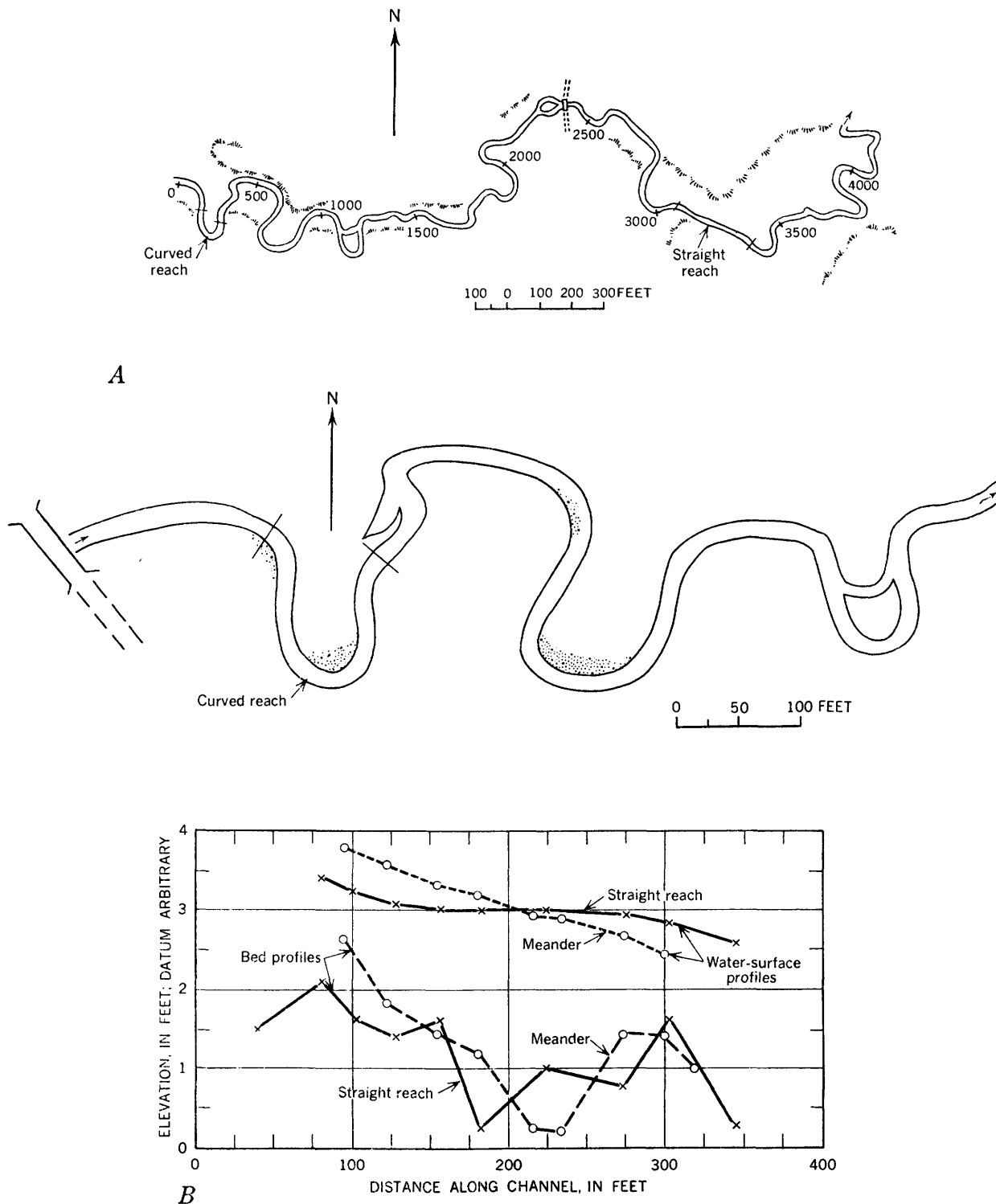


FIGURE 6.—Plan and profiles of straight and curved reaches, Baldwin Creek near Lander, Wyo. *A*, Planimetric map showing location of reaches; *B*, Profiles of centerline of bed and average of water-surface elevations on the two sides of the stream; curved reach is dashed line and straight reach is full line.

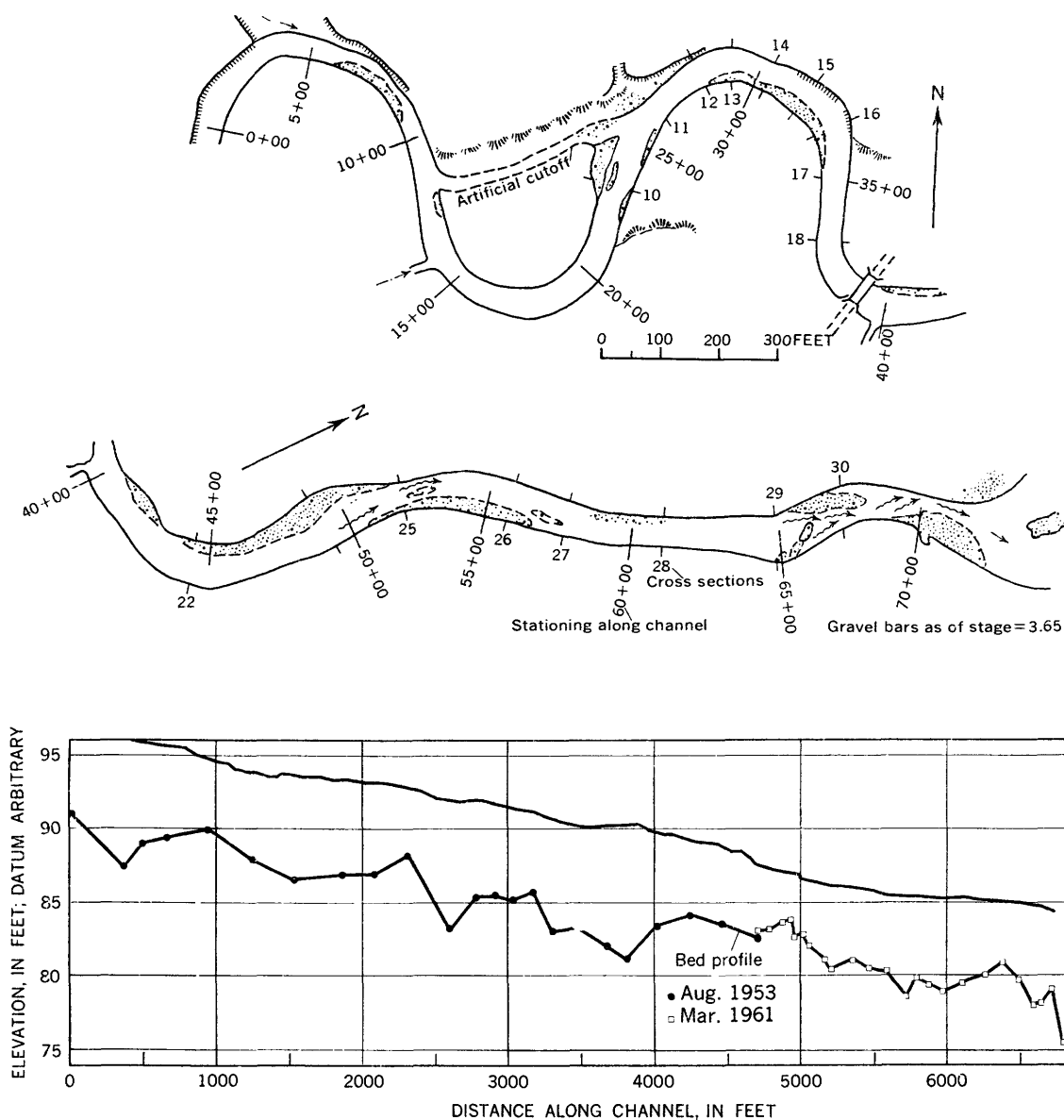


FIGURE 7.—Plan and profile of curved and straight reaches of Popo Agie River, $\frac{1}{2}$ mile below Hudson, Wyo.

and of the friction factor—again considered as ratios to their respective means—were also computed. Bed shear is equal to the product γDS , where γ is the unit weight of water, D is the depth, and S is the slope of the energy profile. The unit weight of water is a constant that may be neglected in this analysis of variances. In this study water-surface slopes will be used in lieu of slopes of the energy profile. This involves an assumption that the velocity head is small relative to the accuracy of measurement of water surface. In any case, where the velocity-head corrections were applied, they were either small or illogical, and so were neglected.

Similarly, the variance of the quantity DS/v^2 , which is proportional to the Darcy-Weisbach friction factor, was computed.

Width is not included in the analysis because it is relatively uniform, and has no characteristic that distinguishes it between curved and straight reaches.

Table 1 shows a sample computation for Pole Creek, and table 2 lists a summary for the five streams surveyed.

COMPARISON OF RESULTS

The results are listed in table 2. It will be noticed that the variance of slope in each reach is larger than that of depth or velocity. The contrast, however, is less in the meandering than in the straight reach; this reduction may be noted by the figures in the column headed a , which is the ratio of the variance of slope to the sum of those of depth and velocity. Among the

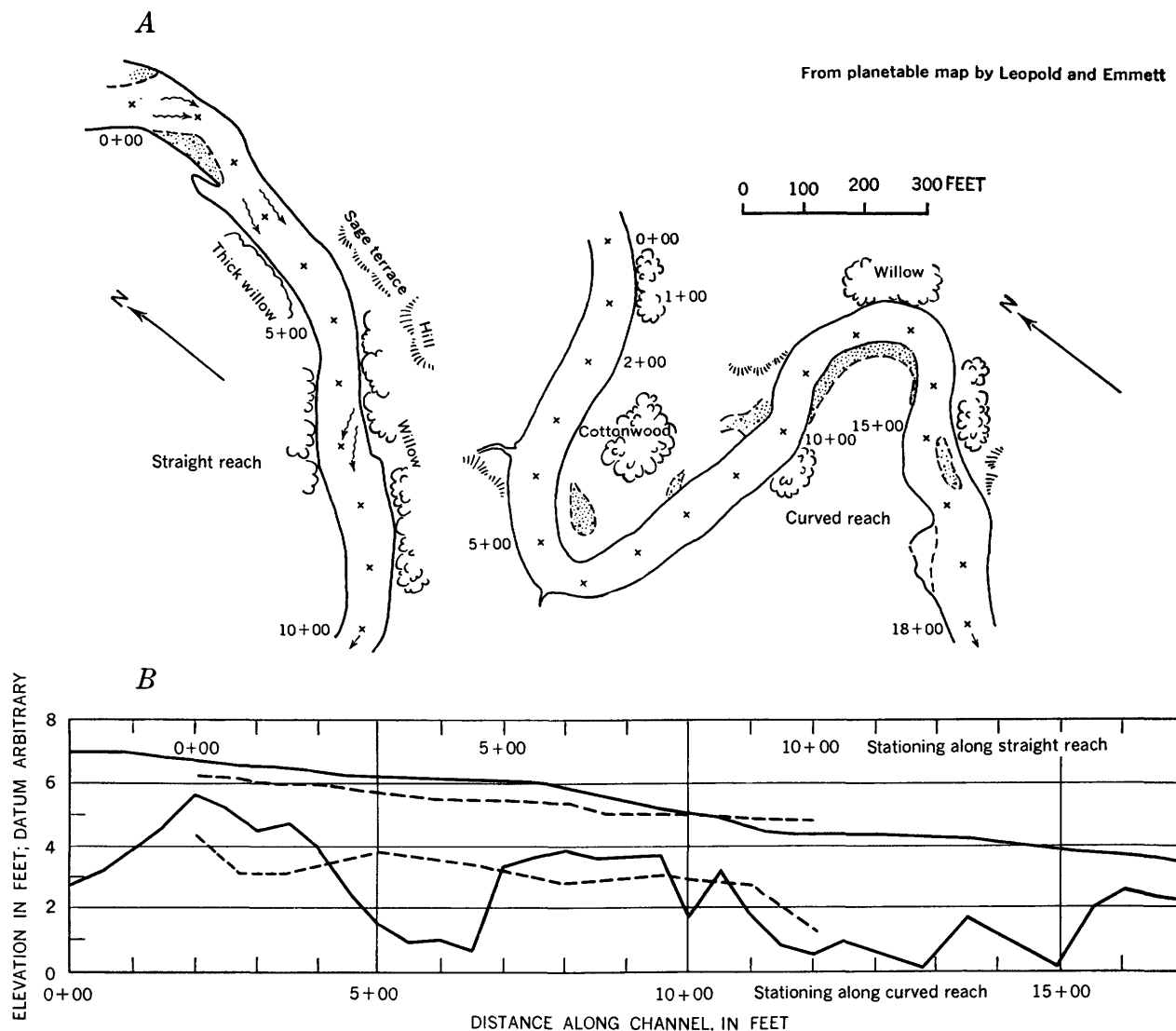


FIGURE 8.—Plan and profile of straight and curved reaches on Pole Creek at Clark's Ranch near Pinedale, Wyo. *A*, Planimetric map showing location of reaches; *B*, Profiles of centerline of bed and average of water-surface elevations on the two sides of the stream; curved reach is dashed line and straight reach is a full line.

five rivers studied, this ratio is uniformly less in the meandering reaches than in the straight reaches.

The data listed in table 2 also show that the variance in bed shear and in the friction factor are uniformly lower in the curved than in the straight reach. The data on table 2 suggest that the decrease in the variance of bed shear and the friction factor is related to the sinuosity. As shown on figure 9, the greater the sinuosity the less the average of these two variances.

When one considers, for example, that the variances of depth and velocity are greater in the curved reach, and even that the variance of slope may be greater (as for example Pole Creek), the fact that the product DS and the ratio DS/V^2 have consistently lower variances must reflect a higher correlation between these variables in the meandered than in the straight reach.

This improved correlation is shown in the following tabulation (table 3) of the respective coefficients of determination (= square of the coefficient of correlation.)

Figure 10 shows graphical plots for the Pole Creek data listed in table 2. The graph shows that in the meanders, depths, velocities, and slopes are more systematically organized than in straight reaches.

To summarize, the detailed comparisons of meandering and straight reaches confirm the hypothesis that in a meander curve the hydraulic parameters are so adjusted that greater uniformity (less variability) is established among them. If a river channel is considered in a steady state, then the form achieved should be such as to avoid concentrating variability in one aspect at the expense of another. Particularly a steady-state form minimizes the variations in forces on the boundary—

TABLE 1.—Field data for Pole Creek at Clark's Ranch near Pinedale, Wyo., June 1964

[Discharge=350 cfs, ¾ bankfull]

Station (feet)	Depth		Velocity (v)		Mean depth (D) in reach (ratio to mean)	Mean velocity (v) in reach (ratio to mean)	Slope (S) in reach		DS (ratio to mean)	DS/v ² (ratio to mean)
	Feet	Ratio to mean	Feet per sec.	Ratio to mean			Feet per 1,000 ft.	Ratio to mean		
Straight reach										
0-----	1. 52	0. 79	3. 67	1. 30						
150-----	2. 26	1. 17	3. 26	1. 12	0. 98	1. 21	1. 67	1. 20	1. 18	0. 81
300-----	1. 92	1. 00	2. 44	. 89	1. 08	1. 01	2. 14	1. 55	1. 67	1. 64
450-----	2. 01	1. 05	2. 44	. 87	1. 02	. 88	1. 14	. 83	. 85	1. 10
600-----	1. 79	. 93	3. 52	1. 25	. 99	1. 06	. 80	. 58	. 58	. 52
750-----	1. 69	. 88	2. 12	. 76	. 90	1. 01	2. 24	1. 62	1. 46	1. 43
900-----	2. 29	1. 19	2. 26	. 80	1. 04	. 78	. 30	. 22	. 23	. 38
Mean-----	1. 92		2. 81				1. 38			
Variance-----		. 019		. 042				. 26	. 25	. 21
Curved reach										
0-----	2. 68	1. 23	2. 24	0. 82						
150-----	1. 96	. 90	2. 45	. 90	1. 06	0. 88	1. 03	0. 50	0. 53	0. 69
300-----	1. 44	. 66	3. 46	1. 27	. 78	1. 08	2. 20	1. 07	. 84	. 72
450-----	2. 37	1. 09	2. 45	. 90	. 88	1. 08	2. 00	. 98	. 86	. 74
600-----	3. 77	1. 74	1. 63	. 60	1. 42	. 75	. 70	. 34	. 48	. 85
750-----	1. 95	. 90	2. 62	. 96	1. 32	. 89	1. 14	. 56	. 74	1. 22
900-----	1. 25	. 58	4. 10	1. 50	. 74	1. 23	3. 94	1. 92	1. 42	. 94
1050-----	1. 25	. 58	4. 02	1. 48	. 58	1. 49	3. 97	1. 94	1. 13	. 51
1200-----	2. 34	1. 08	2. 20	. 81	. 83	1. 15	3. 17	1. 55	1. 29	. 98
1350-----	2. 67	1. 23	1. 98	. 73	1. 16	. 77	. 27	. 13	. 15	. 25
Mean-----	2. 17		2. 72				2. 05			
Variance-----		. 067		. 088				. 41	. 15	. 07

that is, the steady-state form tends toward more uniform distribution of both bed stress and the friction factor.

INTERPRETIVE DISCUSSION

After a review of data and theories concerning river meanders Leopold and Wolman (1960) concluded that meander geometry is "related in some unknown manner to a more general mechanical principle" (p. 774). The status of knowledge suggested that the basic reason for meandering is related to energy expenditure, and they concluded (p. 788) as follows. "Perhaps abrupt discontinuities in the rate of energy expenditure in a reach of channel are less compatible with conditions of equilibrium than is a more or less continuous or uniform rate of energy loss."

Subsequent work resulted in the postulate that the behavior of rivers is such as to minimize the variations in their several properties (Leopold and Langbein, 1963), and the present work shows how this postulate applies to river meanders as had been prognosticated.

Total variability cannot be zero. A reduction in the variability in one factor is usually accompanied by an increase in that of another. For example, in the meander, the sine-generated curve has greater variability in changes in direction than the circle. This greater variability in changes in direction is, however, such as to decrease the total angular change. The sine-generated curve, as an approximation to the theoretical curve, minimizes the total effect.

The meandering river has greater changes in bed contours than a straight reach of a river. However,

TABLE 2.—Sinuosity and variances for straight and curved reaches on five streams

Sinuosity: ratio of path distance to downvalley distance. Variances: σ_D^2 , depth; σ_v^2 , velocity; σ_s^2 , slope; σ_{Ds}^2 , bed shear; σ_f^2 , Darcy-Weisbach friction factor

$$a = \frac{\sigma_s^2}{\sigma_D^2 + \sigma_s^2}$$

Stream	Sinuosity (k)	σ_D^2	σ_v^2	σ_s^2	a	σ_{Ds}^2	σ_f^2
Straight reaches							
Pole Creek near Pinedale, Wyo.....	1.05	0.019	0.042	0.26	4.2	0.25	0.21
Wind River near Dubois, Wyo.....	1.12	.03	.017	.16	3.4	.14	.11
Baldwin Creek near Lander, Wyo.....	1.0	.065	.058	.94	7.8	.55	.40
Popo Agie River near Hudson, Wyo.....	1.02	.026	.017	.70	16.	.46	.30
Mill Creek near Ethete, Wyo.....	1.02	.053	-----	.28	2.6	.32	-----
Curved reaches							
Pole Creek near Pinedale, Wyo.....	2.1	0.067	0.088	0.41	2.6	0.15	0.07
Wind River near DuBois, Wyo.....	2.9	.074	.032	.13	1.2	.06	.073
Baldwin Creek near Lander, Wyo.....	2.0	.048	.017	.20	3.1	.15	.14
Popo Agie River near Hudson, Wyo.....	1.65	.044	.029	.38	5.2	.28	.12
Mill Creek near Ethete, Wyo.....	1.65	.145	-----	.64	2.2	.21	-----

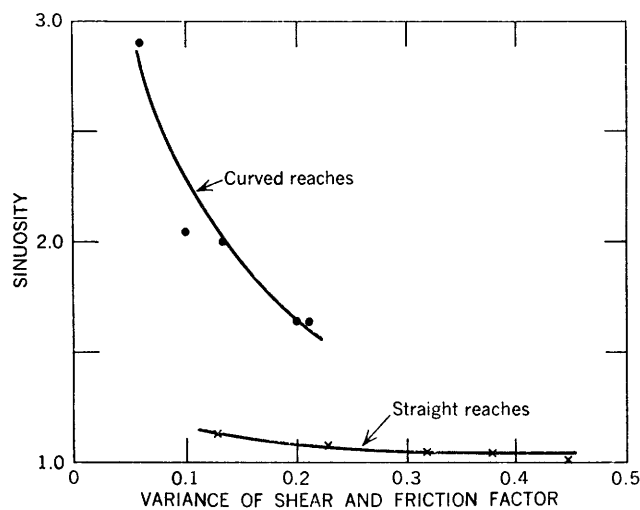
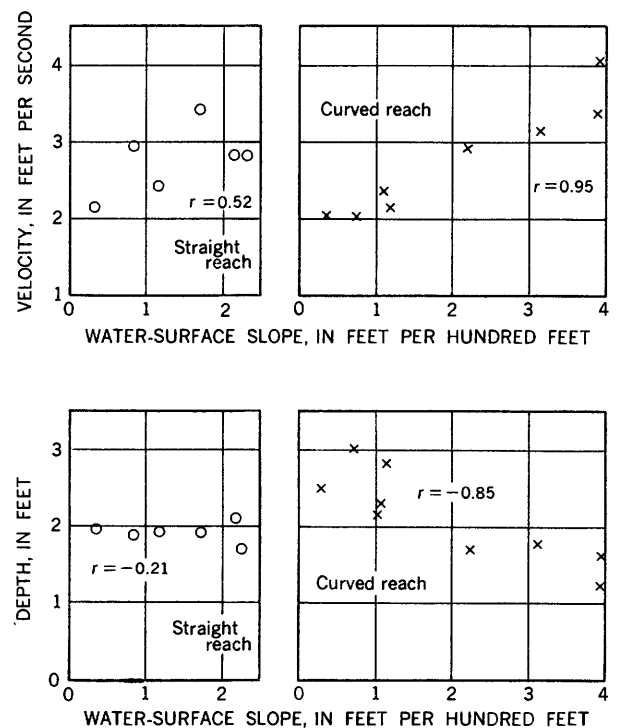


FIGURE 9.—Relation of meander sinuosity to average of variances of friction factor and shear.

TABLE 3.—Comparison of coefficients of determination

[r_{Ds}^2 , coefficient of linear determination between depth and slope; r_{vs}^2 , coefficient of linear determination between velocity and slope]

	Straight reach		Curved reach	
	r_{Ds}^2	r_{vs}^2	r_{Ds}^2	r_{vs}^2
Pole Creek.....	0.044	0.27	0.72	0.90
Wind River.....	.072	.062	.55	.42
Baldwin Creek.....	.56	.28	.42	.73
Popo Agie River.....	.39	.49	.58	.62
Mill Creek.....	0	-----	.88	-----

FIGURE 10.—Relations between velocity, depth, and slope, in straight and curved reaches, Pole Creek, near Pinedale, Wyo. r =correlation coefficient.

these changes in depth produce changes in the slope of the water surface and so reduce the variability in bed shear and in the friction factor, as well as to lessen the contrast between the variances of depth and slope.

These considerations lead to the inference that the meandering pattern is more stable than a straight reach in streams otherwise comparable. The meanders themselves shift continuously; the meandering behavior is stable through time.

This discussion concerns the ideal case of uniform lithology. Nature is never so uniform and there are changes in rock hardness, structural controls, and other heterogeneities related to earth history. Yet, despite these natural inhomogeneities, the theoretical forms show clearly.

This does not imply that a change from meandering to straight course does not occur in nature. The inference is that such reversals are likely to be less common than the maintenance of a meandering pattern. The adjustment toward this stable pattern is, as in other geomorphic processes, made by the mechanical effects of erosion and deposition. The theory of minimum variance adjustment describes the net river behavior, not the processes.

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