# Statistics of a Runoff-Precipitation Relation

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*By* NICHOLAS C. MATALAS

STATISTICAL STUDIES IN HYDROLOGY

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# STATISTICS OF A RUNOFF-PRECIPITATION RELATION

By Nicholas C. Matalas


### ABSTRACT

This report presents the results of an analysis of the influence of the water-retardation characteristics of a river basin on runoff distribution. The runoff was assumed to be generated by a moving average of the effective precipitation, where the extent of the moving average is assumed to be equal to the carryover—a function of the water-retardation characteristics of the river basin.

The probability distribution of the runoff is a function of the extent of the carryover period. Even though the characteristics of the effective precipitation may be the same for all river basins, the probability distribution of the runoff is not the same, because water-retardation characteristics vary from one river basin to another.

Owing to the carryover period, runoff is nonrandomly distributed in time. The serial correlation coefficients that are used to measure the nonrandomness of runoff are functions of the coefficients of the moving-average model, if it is assumed that effective precipitation is randomly distributed in time.

The moving-average model and the theoretical results derived from it are supported by experimental results obtained by analyzing several long-term runoff records.


## INTRODUCTION

### OBJECT OF INVESTIGATION

The purpose of this investigation was to study the influence of the water-retardation characteristics of a river basin on the distribution of runoff. This investigation is based on a simple hydrologic model, where runoff is assumed to be generated by a moving average of the effective precipitation. The extent of the moving average is assumed to be equal to the carryover period. It is also assumed that runoff and effective precipitation correspond to a time interval such that the effective precipitation can be considered as randomly distributed in time.

Theoretically, the runoff during a given interval of time, such as day, month, and year, is a function of all climatic factors, present and past, since the beginning of time. The dominant climatic factor is the effective precipitation, which is defined as the total precipitation less all losses. The effective precipitation prior to a given time interval is referred to as antecedent effective precipitation.

As the contribution of effective precipitation to the runoff converges to zero very rapidly with an increase in antecedent time, the effective precipitation for only a finite period of antecedency affects the runoff. The finite period of antecedency, which is defined as the carryover period, is a function of the water-retardation characteristics of the river basin and the distribution of the effective precipitation with respect to time.

### HYDROLOGIC MODEL

If $Q_j$ denotes the runoff during the $j^{th}$ time interval (where $j=1, 2, \ldots, N$, with $N$ the total number of time intervals) and if $p_{j-i}$ denotes the effective precipitation during the $(j-i)^{th}$ time interval (where $i=0, 1, 2, \ldots, m$ is the time interval antecedent to $j$, and $m$ is the extent of the carryover), the relationship between runoff and effective precipitation is expressed as

$$Q_j=b_o p_j+b_1 p_{j-1}+ \ldots +b_m p_{j-m}. \qquad (1)$$

Equation (1) expresses the runoff as a moving average of extent $m$ of the effective precipitation. The weights of the moving average, $b_o, b_1, \ldots, b_m$ are subject to the linear constraint

$$\sum_{i=0}^{m} b_i=1, \qquad (2)$$

since for large values of $N$ the mean runoff is equal to the mean effective precipitation. If the time interval is much greater than the concentration time of floods on the river basin, the weights of the moving average decrease monotonically with an increase in $i$. That is,

$$b_o>b_1>b_2> \ldots >b_m. \qquad (3)$$

Since the values of effective precipitation are positive and since antecedent effective precipitation contributes to the runoff, the weights of the moving average are positive. That is,

$$b_i>0; i=0, 1, 2, \ldots, m. \qquad (4)$$

The runoff-effective precipitation relationship given by equation (1) is the form of equation adopted by Folse (1929). Other forms can be considered, such as those involving cross-product terms. However, owing to the limited knowledge of the true runoff-effective precipitation relationship, a restriction is imposed on considering complex models. A complex model is warranted only when it can be demonstrated to yield results in closer agreement with the observed facts than the results obtained by simpler models.

### STATISTICAL PARAMETERS CHARACTERIZING RUNOFF

The statistical parameters that describe the probability distribution of runoff are as follows: (1) $\overline{Q}$, the mean runoff; (2) $[\mu_2(Q)]^{1/2}$, the standard deviation; (3) $\beta_1(Q)$, the coefficient of skewness; and (4) $\beta_2(Q)$, the coefficient of kurtosis. These parameters suffice to describe adequately the frequency distributions that are applied to hydrologic investigations of runoff.

If the runoff during any given time interval is independent of the runoff during any other time interval, runoff is said to be distributed randomly in time. If, however, the contrary is true, runoff is said to be distributed nonrandomly in time. The serial correlation, $R_k$, between any two runoff events of interval $k$ apart, gives a measure of the degree and extent of the nonrandomness of the runoff values.

### GENERAL THEORY

#### MOMENTS OF RUNOFF

Basically, the moments are a set of parameters of a distribution that measure its properties and in certain cases specify the probability distribution. Summing equation (1) over all values of $j$, dividing by $N$, and using equation (2), then

$$\overline{Q}=\overline{p}, \tag{5}$$

whereby the mean value of runoff is equal to the mean value of effective precipitation. Equation (5) is valid if $N$ is very large and $m$ is much smaller than $N$. Since the $z^{th}$ central moment of the distribution of a variable $x$ around the mean, $\overline{x}$, of the distribution is defined as

$$\mu_z(x)=\frac{1}{N}\sum_{1}^{N}(x_j-\overline{x})^z, \tag{6}$$

it follow that the $z^{th}$ central moment of runoff is

$$\mu_z(Q)=\frac{1}{N}\sum_{1}^{N}(Q_j-\overline{Q})^z. \tag{7}$$

Substituting equation (1) for $Q_j$ and equation (5) for $\overline{Q}$, equation (7) becomes

$$\mu_z=\frac{1}{N}\sum_{j=1}^{N}\sum_{i=1}^{M}b_i(p_{j-i}-p)^z. \tag{8}$$

Since the probability distributions that have been found to fit the observed values of runoff are of such forms for which all moments exist, it is sufficient for this investigation to specify only the first four central moments. These central moments are as follows:

$$\mu_1(Q)=0 \tag{9}$$

$$\mu_2(Q)=\mu_2(p)\sum_{i=0}^{m}b_1^2 \tag{10}$$

$$\mu_3(Q)=\mu_i(p)\sum_{i=0}^{m}b_i^3 \tag{11}$$

$$\mu_4(Q)=\mu_4(p)\sum_{i=0}^{m}b_i^4+3[\mu_2(p)]^2\left[\left(\sum_{i=0}^{m}b_i^2\right)^2-\sum_{i=0}^{m}b_i^4\right] \tag{12}$$

The subscripts of the $\mu$'s refer to the order of the central moments, so that $\mu_z(Q)$ and $\mu_z(p)$ are the $z^{th}$ central moments of the runoff and effective precipitation, respectively. The central moments given by equations (9) through (12) are valid under the assumption that the effective precipitation is randomly distributed in time.

It follows from equation (2) and inequality (4) that

$$\sum_{i=0}^{m}b_i^d<1; d>1 \tag{13}$$

It is therefore apparent that the second and third central moments of runoff, equations (10) and (11), respectively, are less than the corresponding central moments of effective precipitation. It does not necessarily follow that the fourth central moment of runoff is less than the fourth central moment of effective precipitation. It can be proved that if the square of the second central moment of effective precipitation is equal to or less than the fourth central moment of effective precipitation, then the fourth central moment of runoff is less than the fourth central moment of effective precipitation. On the other hand, if the square of the second central moment of effective precipitation is greater than the fourth central moment of effective precipitation, then depending upon the difference between these two central moments and upon the carryover period and the function of the $b_i$'s, the fourth central moment of runoff can be greater than the fourth central moment of effective precipitation.

#### PROBABILITY DISTRIBUTION OF RUNOFF

It is assumed that the probability distributions of runoff and effective precipitation are such that all moments exist and that the two probability distributions can be defined by their respective means, standard deviations, coefficients of skewness, and coefficients of kurtosis. Each of these parameters is a function of one or more of the first four central moments.

The mean value is defined as the first moment of the distribution around the origin. Equation (5) shows that the mean value of runoff is equal to the mean value of effective precipitation. Therefore, the carryover period, $m$, which integrates the water-retardation characteristics, does not influence the mean value. The equality of the two means is ensured by the linear constraint given by equation (2).

The variance is defined as the second central moment and is given by equation (10). Equation (13) indicates that the variance of runoff is less than that of effective precipitation. The water-retardation characteristics cause the values of runoff to have less dispersion about the mean than do the values of effective precipitation. Thus the water-retardation characteristics act to smooth out the irregularities of effective precipitation.

For probability distributions of the form just specified, relationships exist between the coefficients of skewness and kurtosis. These relationships can be expressed mathematically for many of the probability distributions used in hydrology. The coefficients of skewness and kurtosis, which are dimensionless ratios, are defined, respectively, as

$$\beta_1=\mu_3^2/\mu_2^3 \tag{14}$$

$$\beta_2=\mu_4/\mu_2^2. \tag{15}$$

By using equations (10) and (11), the coefficient of skewness of runoff becomes

$$\beta_1(Q)=[\mu_3(p)]^2\left(\sum_{i=0}^{m} b_i^3\right)^2\Big/[\mu_2(p)]^3\left(\sum_{i=0}^{m} b_i^2\right)^3 \tag{16}$$

or

$$\beta_1(Q)=\beta_1(p)\left(\sum_{i=0}^{m} b_i^3\right)^2\Big/\left(\sum_{i=0}^{m} b_i^2\right)^3, \tag{17}$$

wherein

$$\left(\sum_{i=0}^{m} b_i^2\right)^3>\left(\sum_{i=0}^{m} b_i^3\right)^2. \tag{18}$$

If $m=0$, there is no carryover, in which case the river basin possesses no water-retardation characteristics with respect to the time interval taken for the runoff and the effective precipitation. As the carryover period, $m$, increases, then both sides of inequality (18) approach zero as $m$ approaches infinity. Because the left-hand side of inequality (18) is the denominator in equation (17) and because it approaches zero slower that the right-hand side of inequality (18), which is the numerator is equation (17), the coefficient of skewness of the runoff approaches zero as $m$ tends to infinity. Because a coefficient of skewness equal to zero is associated with a symmetrical probability distribution, the probability distribution of the runoff, due to the water-retardation characteristics, is more symmetrical than the probability distribution of the effective precipitation, provided $\beta_1(p)>0$. If $\beta_1(p)=0$, then $\beta_1(Q)=0$ regardless of $m$.

By using equations (10) and (12) in equation (15), the coefficient of kurtosis of the runoff becomes

$$\beta_2(Q)=\frac{\mu_4(p)\sum_{i=0}^{m} b_i^4+3[\mu_2(p)]^2\left[\left(\sum_{i=0}^{m} b_i^2\right)^2-\sum_{i=0}^{m} b_i^4\right]}{[M_2(p)]^2\left(\sum_{i=0}^{m} b_i^2\right)^2} \tag{19}$$

or

$$\beta_2(Q)=\beta_2(p)\frac{\sum_{i=0}^{m} b_i^4}{\left(\sum_{i=0}^{m} b_i^2\right)^2}+3\left[1-\frac{\sum_{i=0}^{m} b_i^4}{\left(\sum_{i=0}^{m} b_i^2\right)^2}\right], \tag{20}$$

wherein

$$\sum_{i=0}^{m} b_i^4<\left(\sum_{i=0}^{m} b_i^2\right)^2. \tag{21}$$

As $m$ approaches infinity, both sides of inequality (21) approach zero with the left-hand member approaching zero faster than the right-hand member. Therefore, as $m$ approaches infinity, the coefficient of kurtosis of the runoff approaches 3. By inequalities (18) and (21), it is seen that in the limit, for $m$ equal to infinity, the probability distribution of the runoff has coefficients of skewness and kurtosis equal to 0 and 3, respectively; whereby, the runoff is normally distributed. This may also be shown by the central limit theorem (Cramer, 1954).

If $m=0$, in which case there is no carryover, the coefficient of kurtosis of the runoff is equal to that of the effective precipitation. And if the coefficient of kurtosis of the effective precipitation is equal to 3, it is seen by equation (20) that the coefficient of kurtosis of the runoff is 3 regardless of the carryover period, $m$.

As $m$ tends to infinity, the coefficients of skewness and kurtosis approach their limiting values 0 and 3, respectively. However, the coefficient of skewness approaches 0 faster than the coefficient of kurtosis tends to 3. This condition can be shown by proving that

$$\frac{\left(\sum_{i=0}^{m} b_i^3\right)^2}{\left(\sum_{i=0}^{m} b_i^2\right)^3}<\frac{\sum_{i=0}^{m} b_i^4}{\left(\sum_{i=0}^{m} b_i^2\right)^2} \tag{22}$$

or that

$$\left(\sum_{i=0}^{m} b_i^3\right)^2<\sum_{i=0}^{m} b_i^4\sum_{i=1}^{m} b_i^2. \tag{23}$$

By expanding both sides of inequality (23), then

$$\sum_{i=0}^{m} b_i^6+2\sum_{s=0}^{m-1}\sum_{i=0}^{m-s} b_i^3 b_{i+s}^3<\sum_{i=0}^{m} b_i^6+\sum_{s=0}^{m-1}\sum_{i=0}^{m-s} b_i^2 b_{i+s}^2(b_i^2+b_{i+s}^2). \tag{24}$$

Since the first terms on both sides of inequality (24) are identical, it is necessary to consider only the second terms on both sides of the inequality. For any given value of $i$ and $s$, the second terms give

$$2b_i b_{i+s}<b_i^2+b_{i+s}^2 \tag{25}$$

or

$$0<(b_i-b_{i+s})^2. \tag{26}$$

Since $b_i>b_{i+s}$, which follows from inequality (3), the term on the right hand side of inequality (26) must be a positive number greater than zero. Thus inequalities (22) and (23) are proved.

The extent of the carryover, $m$, is not a constant, because water-retardation characteristics vary from one river basin to another. The probability distribution of the runoff is a function of the carryover, $m$, provided that the coefficients of skewness and kurtosis of the effective precipitation are not equal to 0 and 3, respectively. Therefore, the probability distribution of the runoff is not necessarily the same from one river basin to another.

If, however, the coefficients of skewness and kurtosis of the effective precipitation are equal to 0 and 3, respectively, the probability distribution of the runoff is characterized by the fact that the coefficients of skewness and kurtosis of the runoff are also 0 and 3, respectively. Thus the water-retardation characteristics do not influence the form of the runoff probability distribution when it is the same as that of the effective precipitation. However, owing to the carryover, the variance of the runoff is less than that of the effective precipitation.

## DISTRIBUTION OF RUNOFF IN TIME

If there is carryover, the runoff during any given time interval is dependent upon the runoff during previous time intervals; therefore, the runoff is nonrandomly distributed in time. Theoretically, this dependency is a function of all previous runoff since the beginning of time. For practical purposes, however, the carryover period is considered as finite, so that the runoff during a given time interval is dependent only upon the runoff during a finite number of antecedent time intervals.

A measure of the nonrandomness is the serial correlation coefficient of order $k$. This measure is defined as

$$R_k=\frac{\sum_{j=i}^{N-k}(Q_j-\overline{Q})(Q_{j+k}-\overline{Q})}{\sum_{j=1}^{N-k}(Q_j-\overline{Q})^2}, \tag{27}$$

which is the correlation between values of runoff $k$ time intervals apart. By using equations (1) and (5), equation (27) becomes

$$R_k=\frac{\sum_{j=1}^{N-k}\sum_{i=0}^{m} b_i(p_{j+1-i}-\overline{p})\sum_{i=0}^{m} b_{i+k}(p_{j+k+1-i}\overline{p})}{\sum_{i=1}^{N-k}\sum_{i=0}^{m} b_i(p_{j+1-i}-p)^2} \tag{28}$$

or

$$R_k=\frac{\sum_{i=0}^{m-k} b_i b_{i+k}}{\sum_{i=0}^{m} b_i^2}, \tag{29}$$

under the assumption that the effective precipitation is distributed randomly in time. Equation (29) holds for $k\leq m$ where $R_0=1$. However, for $k\geq(m+1)$, then $R_k=0$.

If there is no carryover, $m=0$, $R_k$ is equal to zero for all values of $k\geq 1$, since only the coefficient $b_1$ has a value greater than zero. For $m>0$, each value of $b_o$ and $b_{i+k}$ is greater than zero for all values of $k\leq m$ so the $R_k$ is positive. And because all values of $b_{i+k}$ are zero for $k\geq(m_{+}1)$, $R_k$ is zero for all values of $k\geq(m+1)$. That $R_k$ is a monotonic function, decreasing from $R_0=1$ to $R_k>(m+1)=0$, can be proved as follows. Since the denominator of equation (29) is a constant for each value of $k$, it suffices only to prove that the numerator of equation (29) for a given value of $m$ decreases with an increase of $k$. If the numerator of equation (29) is expanded, the number of terms in the expansion equals $(m+1-k)$. Hence as $k$ increases from zero to its maximum value, $m$, the number of terms decreases from $(m+1)$ to 1. Since $b_i>b_{i+k}$ for $1\leq k\leq m$, the product $b_i b_{i+k}$ decreases with an increase of $k$. It therefore follows that the numerator of equation (29) decreases monotonically as $k$ increases.

For any given function of the $b_i$'s, $R_k$ increases as $m$ increases and tends to unity as $m$ tends to infinity. If the function of the $b_i$'s is different for two river basins and if the carryover period, $m$, is the same for both basins, then all possible values of $R_k$ for the runoff from one river basin are not necessarily either less or greater than the corresponding values of $R_k$ for the runoff from the other river basin.

By equation (29), the nonrandomness of the runoff is a function of the water-retardation characteristics of the river basin. For any given function of the coefficients, $b_i$'s, $R_k$ increases with an increase of $m$. Equation (16) shows that the coefficient of skewness of the runoff decreases as $m$ increases. Since the coefficient of skewness of the effective precipitation, $\beta_1(p)$, is independent of $m$ and of the coefficients, $b_i$'s, then $R_k$, the nonrandomness of the runoff, is independent of $\beta_1(Q)$, the coefficient of skewness of the runoff.

The values of $R_k$ can be determined directly from the observed values of runoff. Hence, if the distribution of the $b_i$'s is known, it is possible to determine the carryover period, $m$, by solving equation (29) for $m$. For example, if the $b_i$'s vary linearly with $i$, then

$$b_i=\frac{2}{(m+1)}\left[1-\frac{(i+1)}{(m+2)}\right]. \qquad (30)$$

Using equation (30), equation (29) becomes

$$R_k=1-\frac{k}{(m+1)}\left[1+\frac{(m+1)^2-k^2}{(2m+3)(m+2)}\right]. \qquad (31)$$

Letting $k=1$ and solving for $m$, it appears that

$$m=\frac{3R_1}{2(1-R_1)}. \qquad (32)$$

## STUDY OF PRECIPITATION-RUNOFF DATA

### DETERMINATION OF EFFECTIVE PRECIPITATION

The time interval selected for the runoff and effective precipitation was a year. By determining the stored water at the end of the water year, $W_e$, and the stored water at the beginning of the water year, $W_b$, then the difference

$$\Delta_j=W_{ej}-W_{bj} \qquad (33)$$

represents the stored water in the river basin during the water year. If $Q_j$ denotes the total runoff during the water year, then

$$P_j=Q_j+\Delta_j \qquad (34)$$

represents the total effective precipitation during the water year.

By using equations (33) and (34), it is possible to estimate the effective annual precipitation, and this was done for several gaging stations. These stations are listed in table 1.

TABLE 1.—*Streamflow records*

[Data furnished by Dr. V. M. Yevdjevich (Colorado State University) from his current study "The fluctuation of annual river flows"]

| Stream | Location | Drainage area (sq mi) | Period of record | $N$ Number of years of record |
|---|---|---|---|---|
| St Lawrence | Ogdensburg, N.Y. | 295,200 | 1860–1957 | 97 |
| Göta | Sjötorp-Vänersborg, Sweden | 18,076 | 1807–1957 | 150 |
| Nemunas | Smalininkai, Lithuania | 30,900 | 1811–1943 | 132 |
| Danube | Orshava, Rumania | 216,300 | 1837–1957 | 120 |



### CORRELOGRAMS OF RUNOFF AND EFFECTIVE PRECIPITATION

In figures 1 through 4, the correlograms for the annual runoff and effective precipitation are given for the streams listed in table 1. The correlograms formed by the solid lines apply to the runoff and the correlograms formed by the alternate long-short dashed lines apply to the effective precipitation. For each correlogram, approximate 90- and 95-percent confidence limits are given. These confidence limits are based on Anderson's (1942) test of significance of serial correlation coefficients.

Figure 1 shows that for the St. Lawrence River runoff, the serial correlation coefficients for $k$ equal to 1 through 9 are significant at the 95-percent level. For $k>9$, the serial correlation coefficients fluctuate within the confidence bands. With respect to the effective precipitation, shown by the alternate long-short dashed lines, the values of $R_k$ fluctuate within the confidence bands.

The first serial correlation coefficients for the runoff for the Göta River (fig. 2) and Nemunas River (fig. 3) are significant at the 95-percent level. The values of $R_k$ for $k\geq 2$, for both of these rivers, fluctuate within the confidence bands and therefore can be considered as not significant. For the effective precipitation corresponding to both of these rivers, the values of $R_k$ fluctuate within the confidence limits.

With respect to the Danube River (fig. 4) the values of $R_k$ for both the runoff and the effective precipitation fluctuate within the 95-percent confidence bands.

The correlograms for effective precipitation indicate that effective precipitation is randomly distributed in time. Runoff may be either randomly or nonrandomly distributed in time. The correlograms for runoff indicate that $R_k$ decreases with an increase in $k$ and that beyond a certain value of $k$, the values of $R_k$ can be considered as not significant. Therefore, the correlograms give reasonable support to the moving average as the generating scheme of runoff.

### STATISTICAL PARAMETERS OF RUNOFF AND EFFECTIVE PRECIPITATION

The above investigations of the correlograms gave support to the hydrologic model, equation (1), and its underlying assumptions. A mathematical treatment of equation (1) indicated the following conditions: (1) That the mean runoff is equal to the mean effective precipitation; (2) that the variance of the runoff is less than the variance of the effective precipitation; (3) that the skewness of the runoff is less than the skewness of the effective precipitation; and (4) that the kurtosis of the runoff is not necessarily less than the kurtosis of the effective precipitation. Condition 3 applies to the case for which the effective precipitation

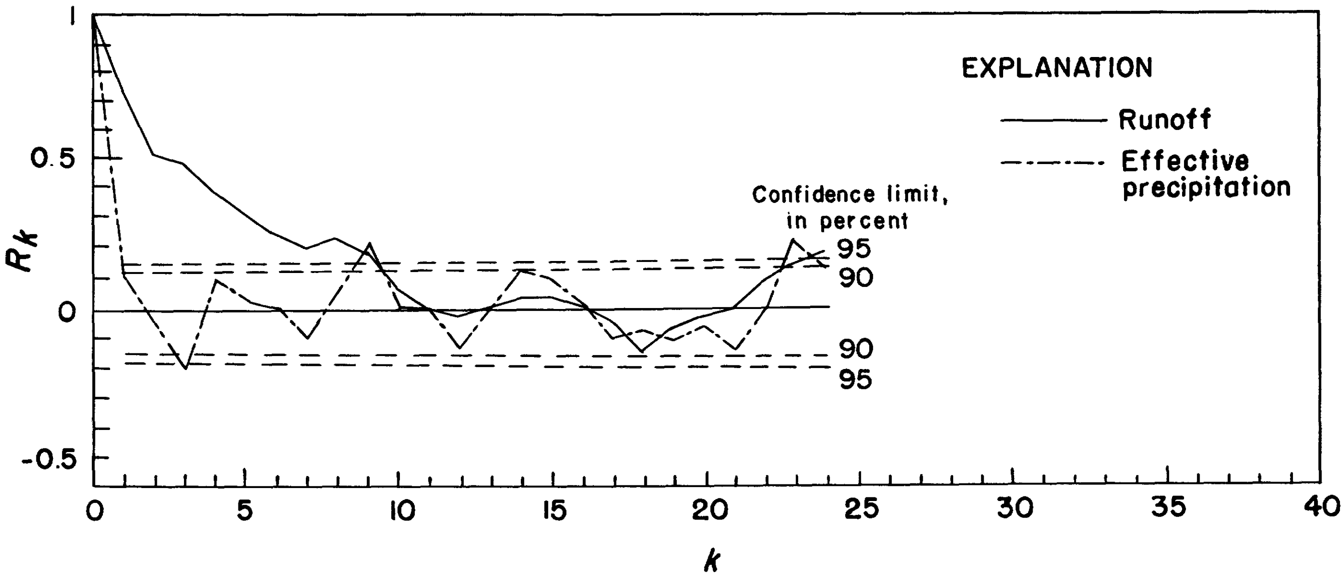


FIGURE 1.—Correlograms of runoff and effective precipitation for the St. Lawrence River at Ogdensburg, N.Y.

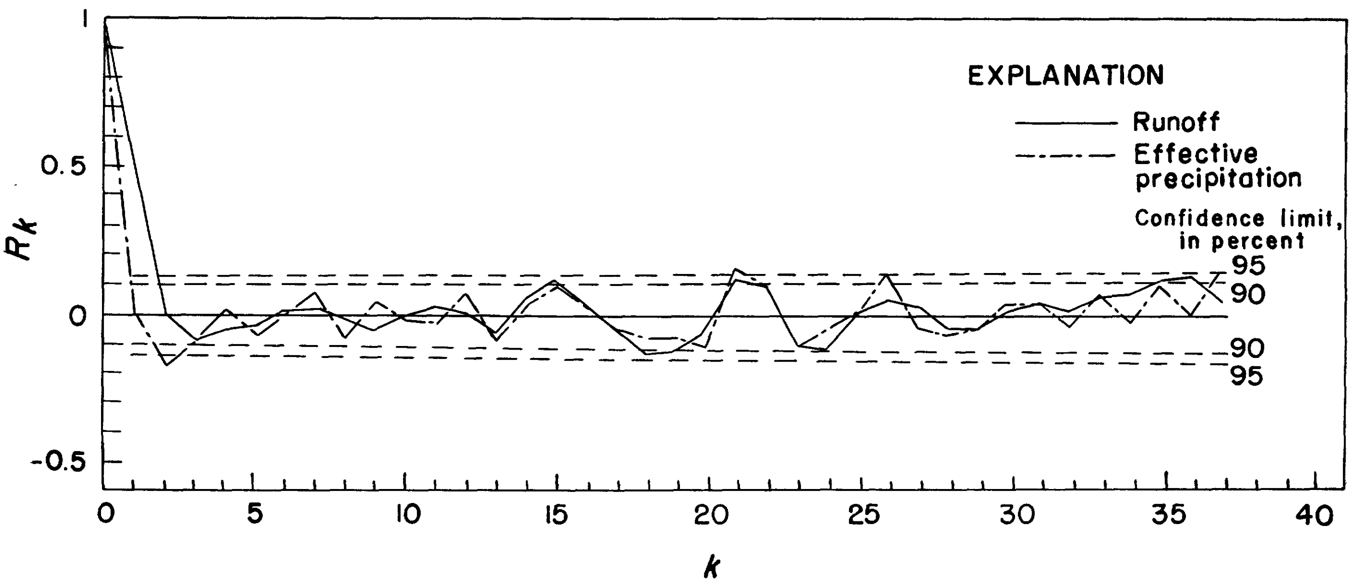


FIGURE 2.—Correlograms of runoff and effective precipitation for the Göta River at Sjötorp-Vänersborg, Sweden.

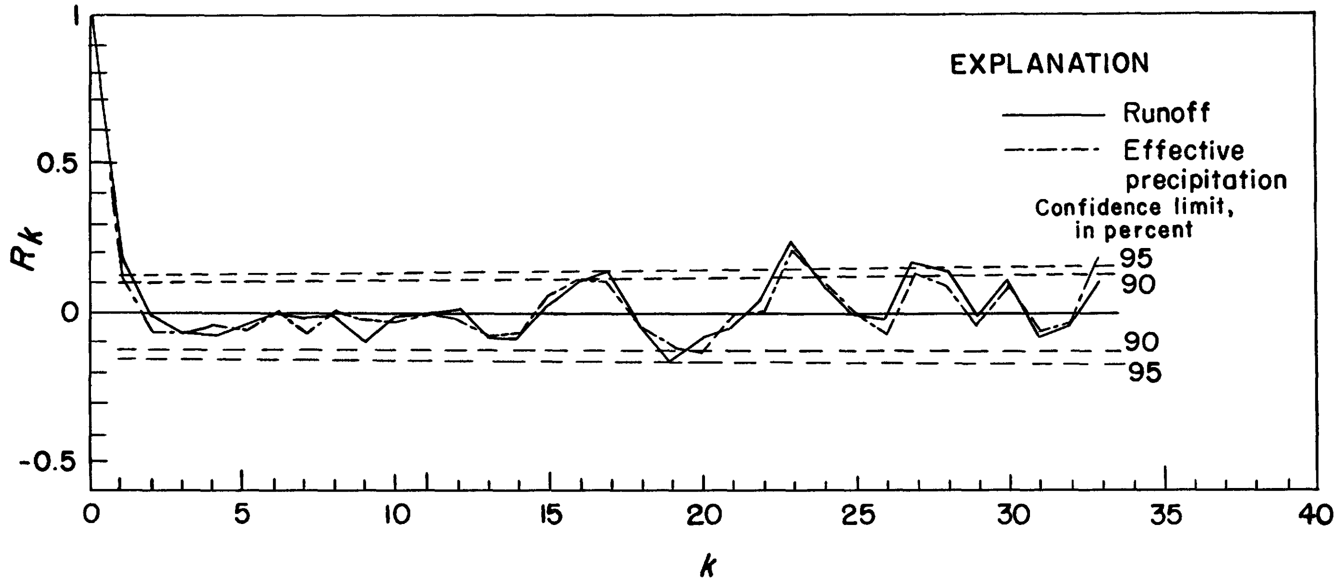


FIGURE 3.—Correlograms of runoff and effective precipitation for the Nemunas River at Smalininkai, Lithuania.

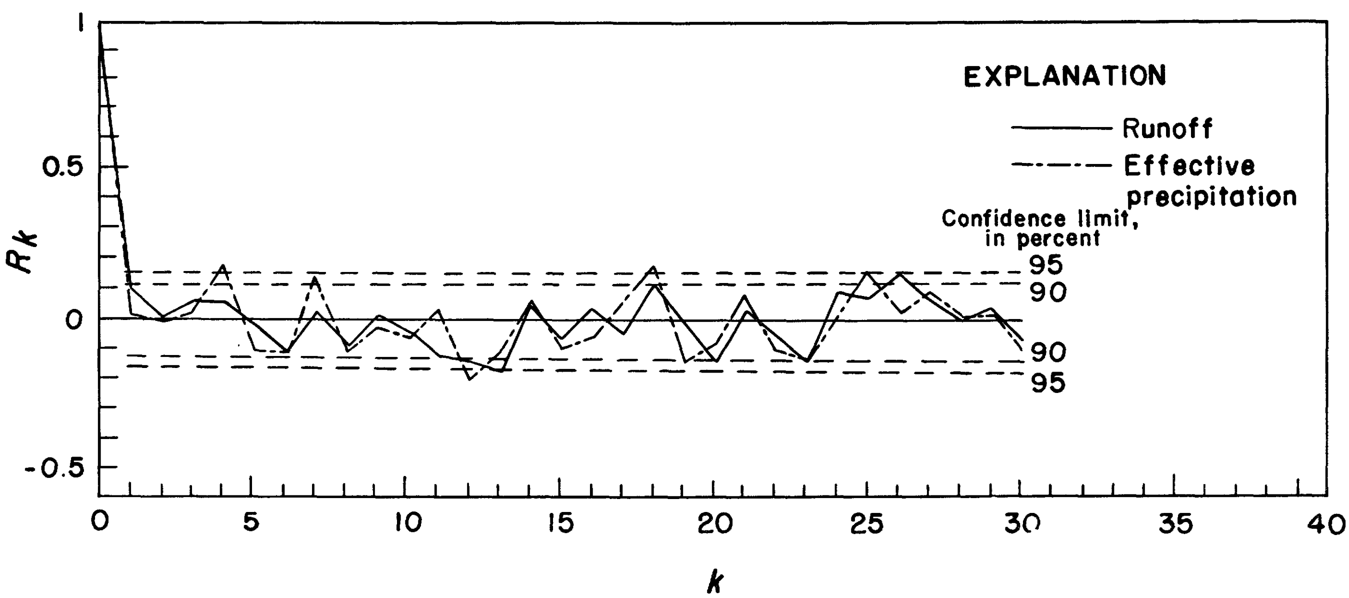


FIGURE 4.—Correlograms of runoff and effective precipitation for the Danube River at Orshava, Rumania.

does not follow a symmetrical probability distribution. Condition 4 applies to the case for which the kurtosis of the effective precipitation is not equal to 3.

In table 2 values are given for the mean, the standard deviation, the coefficient of skewness, and the first order serial correlation coefficient for runoff and effective precipitation in the St. Lawrence, Göta, Nemunas, and Danube River basins. The coefficients of kurtosis were not considered, because it would have been necessary to define the function of the $b_i$'s in order to carry out the comparison between runoff and effective precipitation.

Table 2 shows that the standard deviation of the effective precipitation is greater than that of the runoff. For the St. Lawerence and Göta Rivers, the standard deviation of the effective precipitation is nearly twice as large as that of the runoff; however, the extent of the carryover for these two rivers is different as shown by their correlograms in figure 1. This difference can be attributed to the fact that the function of the $b_i$'s is different for the two streams.

According to equation (17), the coefficient of skewness of runoff is less than that of effective precipitation provided $\beta_1(p) \neq 0$. Table 2 shows that this is the case with respect to the Göta and Nemunas Rivers. The St. Lawrence and Danube Rivers show the contrary. However, in the case of the St. Lawrence River the skewnesses are nearly zero so that the difference between the skewness of runoff and the skewness of effective precipitation probably is not significant.

With respect to the first order serial correlation coefficients, it appears that $R_1$ for runoff is greater than $R_1$ for effective precipitation with respect to each of the four streams.

TABLE 2.—*Statistical characteristics of runoff and effective precipitation*

| River | Hydrologic variable [1] | $\mu_1$ Mean (cfs) | $\mu_2^{1/2}$ Standard deviation | $\beta_1$ | $R_1$ |
|---|---|---|---|---|---|
| St. Lawrence | $Q$ | 240,820 | 20,950 | 0.080 | 0.705 |
| | $P$ | 240,820 | 47,440 | .020 | .090 |
| Göta | $Q$ | 35,300 | 6,420 | .003 | .463 |
| | $P$ | 35,300 | 10,870 | .154 | .009 |
| Nemunas | $Q$ | 19,253 | 3,410 | .216 | .181 |
| | $P$ | 19,253 | 3,700 | .368 | .119 |
| Danube | $Q$ | 189,500 | 36,380 | .729 | .090 |
| | $P$ | 189,500 | 40,550 | .511 | .001 |

[1] $Q$ is runoff; $P$ is effective precipitation.

## SUMMARY

### HYDROLOGIC MODEL AND ITS CHARACTERISTICS

Owing to the water-retardation characteristics of a river basin, the runoff during a given time interval is a function of the effective precipitation during the given time interval and during all previous time intervals. From a practical point of view, however, the runoff is considered to be a function of the effective precipitation during a finite number of antecedent time intervals. It was assumed that the runoff was generated by a moving average of the effective precipitation, when the extent of the moving average was equal to a finite number of antecedent time intervals. This finite number of antecedent time intervals was defined as the carryover period, which is a function of the water-retardation characteristics of the river basin.

The coefficients defining the moving average, denoted by $b_i$'s, were assumed to satisfy the following conditions: (1) the sum of the coefficients is equal to unity; (2) the magnitudes of the coefficients decrease monotonically with an increase in the antecedent time; and (3) the values of the $b_i$'s are greater than zero.

### PROBABILITY DISTRIBUTION OF RUNOFF

On the basis of the assumed model of runoff and effective precipitation, the mean value of runoff is equal to the mean value of effective precipitation. The second and third central moments of runoff are less than those of effective precipitation. However, the fourth central moment of runoff is not necessarily less than that of effective precipitation.

Investigation of the skewness and kurtosis of runoff and effective precipitation showed that if the effective precipitation were normally distributed, then the runoff was normally distributed regardless of the carryover period. If, however, the effective precipitation followed a skewed probability distribution, then the runoff followed a skewed probability distribution, which was different and less skewed than that of the effective precipitation, provided that there was carryover. For a carryover period equal to infinity, runoff is normally distributed regardless of the probability distribution of effective precipitation, provided, of course, that the probability distribution of effective precipitation is such that all moments exist. Therefore, the greater the carryover period, the more the probability distribution of runoff departs from that of effective precipitation, provided that effective precipitation follows a skewed probability distribution.

The carryover period is a function of the water-retardation characteristics of a river basin. Even though the characteristics of the effective precipitation may be the same for each river basin, the probability distribution of the runoff is not the same, because water-retardation characteristics vary from one river basin to another.

### TIME DISTRIBUTION OF RUNOFF

Owing to the water-retardation characteristics, the runoff during a given time interval is dependent upon the runoff during previous time intervals. If the carryover period is denoted by $m$, then the runoff during a given time interval is dependent upon the runoff during $(m-1)$ previous time intervals. Thus the water-retardation characteristics generate a nonrandom distribution of the runoff, and the nonrandomness is measured by the serial correlation coefficients.

The serial correlation coefficients of the runoff are dependent only upon the function of the coefficients of the moving average, $b_i$'s, and upon the carryover period, $m$, under the assumption that the effective precipitation is randomly distributed in time. Thus by specifying the function of the $b_i$'s, it is possible to determine $m$, as the serial correlation coefficients can be determined from the observed values of the runoff.

The carryover period, $m$, can be estimated without specifying the function of the $b_i$'s by means of the correlogram, provided that the number of time intervals, $N$, is large. Theoretically, the correlogram for a moving-average process vanishes for all orders of serial correlation equal to and greater than a certain value of $k$. If this value of $k$ is denoted by $k_0$, then the carryover period, $m$, is equal to $(k_o+1)$. With actual data, however, zero values of serial correlation are not likely to be obtained, owing to sampling errors. In practice, $k_0$ may be taken as that value of $k$ beyond which all values of serial correlation coefficients fluctuate within a confidence band of nonsignificance.

## RELATION BETWEEN SKEWNESS AND NONRANDOMNESS OF RUNOFF

The coefficient of skewness of the runoff, $\beta_1(Q)$, is dependent upon the function of the coefficients, $b_i$'s, of the moving average, the carryover period, $m$, and the coefficient of skewness of the effective precipitation, $\beta_1(P)$. However, the serial correlation coefficients, $R_k$, of the runoff are dependent only upon the function of the coefficients of the moving average and the carryover period and are independent of the skewness of the effective precipitation.

For a given value of skewness of the effective precipitation, the skewness of the runoff decreases and the nonrandomness of the runoff increases with an increase in the carryover period. However, since the skewness of the effective precipitation is independent of the function of the coefficients of the moving average and the carryover period, the serial correlation coefficients of the runoff can assume any value within the range 0 to 1 independently of the values of the coefficients of skewness of the runoff.

## RESULTS

In order to test the theoretical results, annual values of runoff and estimated values of effective precipitation for four long-term records were considered. The correlograms of runoff and effective precipitation show that the moving average provides a satisfactory model of the generating scheme of runoff. The second central moments of the runoff were found to be less than those of the effective precipitation. However, the coefficient of skewness of the runoff was not always found to be less than the coefficient of skewness of the effective precipitation.

These investigations show that the data provide reasonable support for the theoretical results. These results are, however, applicable to annual events only. If time intervals other than a year are considered, the theoretical results will be applicable only if the effective precipitation can be assumed to be distributed randomly in time and if no cyclic effects are present.

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