The Concept of Entropy in Landscape Evolution

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By LUNA B. LEOPOLD *and* WALTER B. LANGBEIN

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THE CONCEPT OF ENTROPY IN LANDSCAPE EVOLUTION

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ABSTRACT

The concept of entropy is expressed in terms of probability of various states. Entropy treats of the distribution of energy. The principle is introduced that the most probable condition exists when energy in a river system is as uniformly distributed as may be permitted by physical constraints. From these general considerations equations for the longitudinal profiles of rivers are derived that are mathematically comparable to those observed in the field. The most probable river profiles approach the condition in which the downstream rate of production of entropy per unit mass is constant.

Hydraulic equations are insufficient to determine the velocity, depths, and slopes of rivers that are themselves authors of their own hydraulic geometries. A solution becomes possible by introducing the concept that the distribution of energy tends toward the most probable. This solution leads to a theoretical definition of the hydraulic geometry of river channels that agrees closely with field observations.

The most probable state for certain physical systems can also be illustrated by random-walk models. Average longitudinal profiles and drainage networks were so derived and these have the properties implied by the theory. The drainage networks derived from random walks have some of the principal properties demonstrated by the Horton analysis; specifically, the logarithms of stream length and stream numbers are proportional to stream order.

GENERAL STATEMENT

In the fluvial portion of the hydrologic cycle a particle of water falls as precipitation on an uplifted land mass. Its movement to the sea gradually molds a path that will be taken by succeeding particles of water. In its movement it will do some minute portion of the grand task of reducing the land mass in average elevation by carrying ultimately to the ocean in solution and as transported sediment molecules or particles of the continental materials.

The paths taken by the various droplets that make their way, through time, from higher to lower elevations represent the drainage network of the land surface, the patterns of the channels, and the longitudinal profiles of the waterways. As these paths are being carved by erosion and solution, the features of the landscape are expressed in the topography of the constantly changing surface.

The paths possible for water and its load have a large variety. There are obviously certain constraints which can be identified. Because gravity is the main force moving the materials, and works vertically downward, each particle of water and its associated load can move downward in the fluvial portion of the hydrologic cycle. The horizontal distance of movement is governed ultimately by the relation between the uplifted land mass and the ultimate base level—the ocean. Within these two general constraints, however, a large variety of paths is still possible.

Geomorphology is concerned with these paths, with the forms assumed during the process of landscape evolution, and with the principles governing the development of paths and forms.

W. M. Davis stated (1909) that the interpretation of land forms necessitates consideration of the processes acting, the effects of lithology and geologic structure, and the stage in the total evolution of the landscape. In attempts to define governing principles, geomorphologists have expressed various aspects of process and structure in mathematical form by equations of more or less generality. Available equations have not been found useful, however, as general expressions of spatial and time relations.

The most general equations which must be satisfied are, as in all the physical sciences, the equation of continuity which states that matter is not lost, and the equation of conservation which states that energy can neither be created nor destroyed.

These laws are so obviously general that they characterize each element, or reach, in any fluvial system. Further, the equations must characterize each unit of any path and at each instant in time. These two equations, then—however necessary—are insufficient to explain the paths of particles or the relation between one part of a path to another. Therefore, they can alone tell us nothing about the surface form of the landscape. Nor can they treat completely of the progressive development or change of form with time.

The present paper is an attempt to apply another general law of physics to the subject for the purpose of obtaining some additional insight into energy distributions and their relation to changes of land forms in space and time.

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ASPECTS OF ENTROPY

The thesis of the present paper is that the distribution of energy in a river system tends toward the most probable state. This principle, somewhat analogous to that implied in the second law of thermodynamics in relation to thermal energy, governs ultimately the paths of movement in the fluvial process and the spatial relations between different parts of the system at any one time or stage. Further, it is suggested that this general principle also tends to govern the sequence of development of these paths from one stage in geomorphic history to the succeeding one.

The development of the landscape involves not only the total available energy, but its distribution as well, a factor that may appropriately be described as entropy, adapting that term from the comparable concept in thermodynamics.

The first law of thermodynamics is merely a restatement of the principle of conservation of energy when heat is included in the energy forms. The second law of thermodynamics stated in simplest terms is that there is an increase in entropy in every natural process providing all the system taking part in the process is considered. The increase in entropy is a measure of the decrease in availability of the energy in the sense that a certain amount of energy is no longer available for conversion to mechanical work.

The second law expressed in thermal terms is not obviously related to geomorphic systems in which mechanical rather than heat energy is of principal concern. Its applicability will be explained in the discussion that follows.

With the understanding that it is necessary to define the system to which the second law of thermodynamics is to be applied, the essential idea to be adapted is that the entropy of a system is a function of the distribution or availability of energy within the system, and not a function of the total energy within the system. Thus, entropy has come to concern order and disorder. Information theory utilizes this aspect of the entropy concept.

The order-disorder aspect of entropy may be demonstrated by looking at your desk top. Ours is in disorder. Because of this we need offer no explanation of where things are. It is equally probable that a given piece of paper is anywhere. If we should take the trouble to place things in order, it would be necessary to label the various piles, and the amount of labelling (i.e. information) would increase with the degree of classification. One can tolerate a certain amount of disorder on his desk, but he must continually correct this by effort expended in search. This effort is putting negative entropy into the desk system; but the work done in this effort itself involves a general increase in entropy of the environment.

The degree of order or disorder in a system may be described in terms of the probability or improbability of the observed state. This aspect might best be stated first as a simple example: In a room filled with air the individual molecules are moving at random and, because of the effects of this movement and the collision of molecules, the gas tends to become more or less uniformly distributed throughout the room. Owing to the fact that these same collisions occur at random, it is physically possible that the random motion of particles might create a situation in which all particles of the gas are for a moment concentrated in one small volume in a corner of the room. The improbability that this would happen by chance is very great indeed. The molecules of the gas tend to become distributed in a random or disorderly way. The chance that given degrees of order prevail may be described in the form of a probability statement.

It is quite possible, of course, to pump the gas in the room into a small volume, but work would have to be expended on the system in order to accomplish this result. The overall entropy including that of the external pump would thence be increased.

The distribution of the energy may be stated in terms of the probability of the given distribution occurring relative to alternative distributions possible. As expressed by Brillouin (Bell, 1956, p. 159)—

> Entropy and probability are practically synonymous for the physicist who understands the second principle of thermodynamics as a natural tendency from improbable to more probable structures.

With the statistical conception of entropy in mind, the possible application to geomorphic systems becomes recognizable. The distribution of energy in a geomorphic system is one way of expressing the relative elevation of particles of water and of sediment which gradually will, in the process of landscape evolution, move downhill toward base level. The longitudinal profile of the river, for example, is a statement of the spatial distribution of streambed materials with regard to their elevation and, thus, with regard to their potential energy.

In thermodynamics heat energy is referred to absolute temperature as a base. The absolute temperature defines an absolute limit or a base datum, the situation in which molecular motion becomes zero. It is, then, the base level or the datum against which the energy content of a thermal system can be measured.

Systems in geomorphology also have a base datum with regard to the distribution of energy. This base datum is a datum of elevation, in most cases represented by mean sea level. The longitudinal profile of the river may be described as the distribution of the potential energy of both particles of water and particles of sediment, as they are traversing the fluvial path from higher elevation toward base level. The energy distribution may be defined in terms of the probability of the occurrence of that particular distribution. This again will be developed first for the thermodynamic case.

If, in a closed thermodynamic system at absolute temperature T, E is the thermal energy per unit mass of a substance having a specific thermal energy C (energy change per unit of temperature change), then a change dE in a unit mass is equal to

$$dE = CdT. \tag{1}$$

In this situation T may be thought of as being a measure of the adverse probability p that the energy exists in the given state above absolute zero. T is not equal to p because T is defined in arbitrary units, where p is an absolute number constrained between limits of zero and 1, but T is a function of p.

Because thermodynamic entropy is defined as

$$\phi = \int \frac{dE}{T} \tag{2}$$

then, per unit of mass,

$$\phi = C \int \frac{dT}{T}$$

$$= C' \int \frac{dp}{p}$$

$$\phi = C' \log_e p + \text{constant} \tag{3}$$

where C' is the specific heat energy in appropriate units.

Thus, entropy in the abstract sense may be defined as the logarithm of a probability and may express, for example, the ratio of the probability of a given physical state to the probability of all other possible alternative states. Such a statistical definition of entropy has been used in physical chemistry, information theory, and elsewhere (Bell, 1956; Lewis and Randall, 1961).

An example of possible alternatives would be the following: If consecutive single draws are made from a deck of playing cards, replacing the drawn card and reshuffling after each operation, the probabilities of drawing an ace, a face card, or a numbered card would differ. The alternative states (aces, face cards, numbered cards) have individual and different probabilities of turning up in consecutive draws. The entropy of the system is defined not in terms of the results of any single experiment but in terms of probabilities among the alternatives and is of the form $\phi = c(\log p_A + \log p_F + \log p_N)$ where p_A is the probability of drawing an ace, p_F the probability of drawing a face card, and p_N the probability of drawing a numbered card, and c is a constant to convert to appropriate units.

To generalize, then, if a system included various alternative states 1, 2, 3, . . . n, the individual probabilities of which occurring in various examples are $p_1, p_2, p_3 \ldots p_n$, the entropy of the system is defined as the sum of the logarithms of these probabilities

$$\phi = c \Sigma \log p. \tag{4}$$

The probability of a given state, p_1, represents the fractional chance as compared with unity that any example or sample will be in the state designated "1" among the n alternatives.

To explain in another way, if two states, statistically independent, of probabilities respectively, p_1 and p_2, and corresponding entropies, ϕ_1 and ϕ_2 from an assembly of states are combined, then the probability of the combination is $p_1 p_2$ since the probabilities are multiplicative. On the other hand, entropy being an extensive property is additive. This relation between entropy and probability is satisfied by a logarithmic relation as before, thus,

$$\phi \propto \log p + \text{constant}. \tag{5}$$

In terms of a gas in an isolated system, the Boltzman relation between entropy and probability is usually stated as

$$\phi = \frac{R}{N} \log p + \text{constant} \tag{5a}$$

where R is the gas constant, N is Avogadro's number, and p represents, as before, the probability of being in a given state. Because $\log p$ is a negative number, the sign of ϕ depends on the difference between $R/N \log p$ and the constant, representing the base level for measuring entropy.

Since p is a positive number less than 1 equation 5a can be written

$$p \propto e^{-\phi N/R}$$

or since

$$\phi_i = E_i/T$$

$$p \propto e^{-E_i/kT} \tag{5b}$$

where k is Boltzman's constant R/N. In other words, the probability of a given energy E_i of one state i is proportional to the negative exponential of its ratio to the total energy kT of all possible states.

It is now possible to specify the *most probable* distribution of energy in a system. If the system is composed of possible states, 1, 2, 3, . . . n, the individual probabilities of which are $p_1, p_2, p_3, \ldots p_n$, the entropy of the system is maximum when the sum

$$\Sigma \log p \text{ is a maximum.} \tag{6}$$

This may be written as

$$(\log p_1 + \log p_2 + \log p_3 + \ldots + \log p_n) = \text{a maximum.} \tag{7}$$

Because the states 1, 2, 3, . . . n represent various alternative states, each represented in the whole system, the sum of the probabilities is unity, or

$$p_1 + p_2 + p_3 + \ldots + p_n = 1.0. \tag{8}$$

We are now concerned with the values of the individual probabilities that would make the sum of their logarithms a maximum. It can be shown that the sum $\Sigma \log p$ is a maximum when $p_1 = p_2 = p_3 = \ldots p_n$. In other words, the most probable condition among alternatives occurs when the individual probabilities of the various alternatives are equal.

Whether a system can attain this distribution depends on the constraints on the distribution of energy. However, in this connection it must be remarked that although the quantity $[\log p_1 + \log p_2 + \log p_3 + \ldots + \log p_n]$ is a maximum when all probabilities are equal, this maximum is not a sensitive one. That is, there can be some considerable irregularity between the several values of the probabilities p, and yet the sum of their logarithms may not diverge greatly from the maximum value obtained when all probabilities are equal. For example, if there are 100 possible states and the probability of each state is 0.01, the value of $\Sigma \log_e p = -460$. If, on the other hand, 50 states have a probability of 0.015 and the remaining 50 a probability of 0.005, the value of $\Sigma \log_e p = -475$, not greatly less than that for exactly equal partition of probabilities. The geomorphic significance of this mathematical fact seems to be that one might expect considerable deviations from the theoretic "most probable" state even in the absence of marked physical constraints. Complementary to the flat nature of the peak of the curve of the sum of the logarithms, the side limbs are very steep, indicating that relatively short intervals of time are needed to reach a state approaching the maximum probable, although the rate of adjustment to the theoretic most probable state thereafter may be quite slow if ever achieved.

OPEN SYSTEMS

The classical treatment of entropy in thermodynamics deals with closed systems in which entropy continuously increases to a maximum stationary level at equilibrium. In closed systems there is no loss or addition of energy. Geomorphic processes operate, on the other hand, in open systems in which energy is being added in some places while in other places energy is being degraded to heat and is thus lost insofar as further mechanical work is concerned. A river system is an example of an open system. Let the system be defined as the water and the debris in the river channel. As water flows down the channel it gives up potential energy which is converted first to kinetic energy of the flowing water and, in the process of flow, is dissipated into heat along the channel margins.

Precipitation brings increments of energy into the system because water enters at some elevation or with some potential energy. Heat losses by convection, conduction, and radiation represent energy losses from the system. Yet, the channel may be considered in dynamic equilibrium.

The steady state possible in an open system differs from the stationary state of static equilibrium of closed systems. We shall therefore equate the term steady state with dynamic equilibrium in geomorphology as defined by Hack (1960). For the general case of an open system the statement of continuity of entropy is as follows (Denbigh, 1951, p. 40):

> rate of increase of entropy in system + rate of outflow of entropy = rate of internal generation of entropy.

In an open system in dynamic equilibrium, the rate of increase of entropy in the system is zero (Prigogine, 1955, p. 82; Denbigh, 1951, p. 86). The continuity equation above then takes the form that the rate of outflow of entropy equals the rate of internal generation of entropy.

In a river system in dynamic equilibrium, therefore, between any two points along its length, the rate of outflow of entropy is represented by the dissipation of energy as heat. This equals the rate of generation of

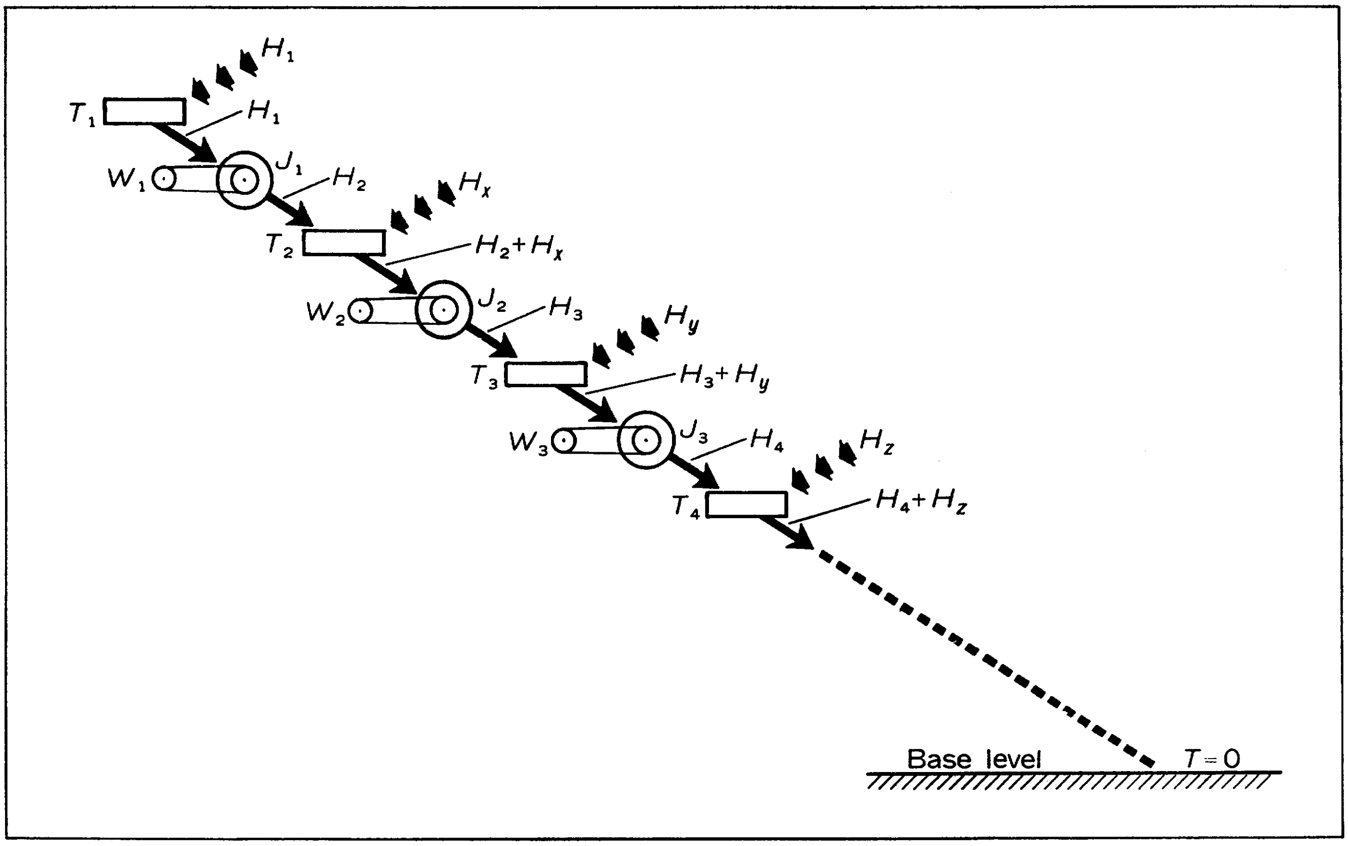

FIGURE 1.—Thermodynamic engine model of steady state.

entropy which is represented by the energy gradient toward base level. At this point it is desirable to have an equation for the generation of entropy in a river system. For this purpose we shall again consider a thermodynamic model depicted on figure 1. This model consists of a series of perfect engines, J_1, J_2, etc., operating between heat sources and sinks designated by their absolute temperatures, T_1, T_2, etc. The work produced by each engine in unit of time Δt is W_1, W_2, etc., which is delivered to a Prony brake which dissipates the work as heat. The working substance is a flow of a fluid from temperature source T_1 at a rate q_1 carrying a quantity of heat H_1 in a unit of time. The amount of work (in heat units) that is done by engine J_1 equals, in unit of time Δt,

$$\frac{W_1}{\Delta_t}=\frac{H_1}{\Delta_t}\frac{T_1-T_2}{T_1}=q_1(T_1-T_2). \tag{9}$$

It will be noted that the work done by the engine is equal to the heat transferred multiplied by the Carnot efficiency factor. Further, in thermal units, work done in unit time equals flow rate times temperature difference, so the equation is dimensionally consistent. Now the heat delivered to sink T_2 is

$$H_2=[H_1-W_1]=H_1\frac{T_2}{T_1}=q_1T_2\Delta t. \tag{10}$$

At temperature level of T_2 let us introduce an additional flow of fluid at rate q_x carrying a quantity of heat q_xT_2. The total heat therefore carried to engine J_2 is H_2+H_x and the work done in unit time by engine J_2 is

$$W_2=(H_2+H_x)\frac{T_2-T_3}{T_2}=(q_1+q_x)(T_2-T_3)\Delta t.$$

The heat delivered by J_2 to T_3 is

$$H_3=H_2+H_x-W_2=(H_2+H_x)\frac{T_3}{T_2}.$$

Hence, the heat delivered to each sink decreases in proportion to the ratio of the absolute temperatures of sink and source.

A unit of work done by engine J_1 and dissipated by the Prony brake, when divided by the absolute temperature, represents a generation of entropy or a unit export of entropy from the system. This work per unit of time is $\frac{W_1}{\Delta t}$, accomplished at temperature T_1, the quotient of which represents the change of entropy per unit of time or

$$\frac{W_1}{\Delta t}\cdot\frac{1}{T}=\left(\frac{\Delta\phi}{\Delta t}\right)_1$$

and from equation 9

$$\frac{W_1}{\Delta t T_1}=q_1\frac{T_1-T_2}{T_1}$$

and

$$\left(\frac{\Delta\phi}{\Delta t}\right)_1=q_1\frac{T_1-T_2}{T_1}.$$

Similarly at engine J_2

$$\left(\frac{\Delta\phi}{\Delta t}\right)_2=\frac{W_2}{\Delta t T_2}=(q_1+q_x)\frac{T_2-T_3}{T_2}. \tag{11}$$

Thus, in each engine the rate of production of entropy is proportional to the rate of flow of fluid, the temperature difference between source and sink, and inversely proportional to the absolute temperature.

Hence, in more general terms, if in time dt the fluid flows a distance dx, which represents the distance apart of the adjacent engines, operating between a corresponding temperature difference dT

$$\frac{d\phi}{dt}=q\frac{dT}{dx}\frac{1}{T} \tag{12}$$

or in terms of entropy per unit of flow

$$\frac{d\phi/dt}{q}=\frac{dT}{dx}\frac{1}{T}. \tag{13}$$

Thus the right-hand side of equation 13 may be interpreted as describing the most probable distribution of energy in the thermodynamic model. This distribution is such that energy is in inverse proportion to temperature above base level $T=0$. Equation 13 is a thermal statement having no direct relationship to a river system. There are, nevertheless, several analogous points, which, if supported by field evidence, can lead to further inferences about river systems.

The analogy to the river system is that each engine represents a reach, Δx, of the river which receives at its upper end a flow of water at total head H_1, and delivers at its lower end the same flow at H_2. The difference in head is degenerated into heat which, like the Prony brake, exerts no influence on the transport of energy or mass in the river. Within the river reach, as within the engine, certain reversible reactions may take place, as for example transfers between kinetic and potential forms of energy as the flow expands or contracts. These, too, may be necessary adjuncts to the operation of the engine or the transport of water in a river, but being isentropic, have no effect on the equation of continuity of entropy.

If one interprets equation 13 as stating that the most probable distribution of energy in the engine model is in inverse proportion to temperature above base level $T=0$, then a corresponding statement for a river system is

$$\frac{d\phi'/dt}{Q}=\frac{dH/dx}{H} \tag{14}$$

where ϕ' is the entropy in the special sense of the river system, H is elevation or total energy content above base level, and Q is rate of river discharge. Again note that tributary entrance does not alter the relationship because equation 14 is written in terms of rate of entropy change per unit of flow rate.

According to Prigogine (1955, p. 84), in the evolution of the stationary state of an open system the rate of production of entropy per unit volume corresponds to a minimum compatible with the conditions imposed on the system. In the case of the engine model, this means that if a stationary state prevails, the work done by each engine (the source of the entropy production) is a minimum. Hence, a stable system corresponds to one of least work, a point to which we refer in the next section, after which we shall examine the distribution of the production of entropy in a river system.

PRINCIPLE OF LEAST WORK AND ENTROPY

There is therefore considerable logic to the adage that nature follows the principle of least work. The idea is well developed in the analysis of stresses of a class of framed structures where the principles of statics are insufficient to determine the division of stresses. These structures are, therefore, called "statically indeterminate," and illustrate how inferentially the principle of maximum entropy is used to complete the solution. Consider the simple truss on figure 2. The crossed

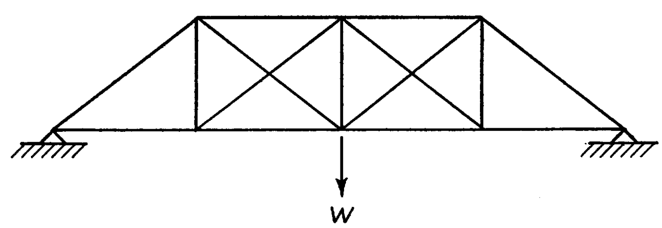

FIGURE 2.—Example of a simple statically indeterminate truss.

braces make this a common form of statically indeterminate truss. The principles of statics give 10 equations, but there are 11 unknowns. The civil engineer provides the necessary additional equation by introducing the assumption, first applied by Castigliano in 1879, that the stresses are distributed so that the total strain energy in the several members is a minimum. The strain energies are defined as $F^2L/2EA$ where F is the total stress on each member, L is its length and A its cross-sectional area, and E is the modulus of elasticity of the material. This is called the principle of least work.

In structural engineering no attention has been paid to the fact that the distribution of stresses so determined is also the most probable.

We shall demonstrate the equivalence of the principle of least work with that of maximum probability by application of the principles just discussed. As in the river profile, an articulated system such as a truss has an entropy equal to

$$\phi=\Sigma \log p+\text{constant}.$$

The most probable distribution exists when ϕ is a maximum. Let the ratios E_{n_1}/E, E_{n_2}/E, E_{n_3}/E . . ., etc., represent the proportional division of the strain energy among the several members, where E is the total strain energy in the system—the product of the weight W times deflection. (Note the analogy to the discussion leading to equation 5 and immediately following equation 5b.)

However, the joint probability that a particular combination of strain energies exists among the several members is

$$p \propto e^{-\frac{E_{n_1}}{E}} e^{-\frac{E_{n_2}}{E}} e^{-\frac{E_{n_3}}{E}} \ldots \text{etc.} \tag{15}$$

Where there are no physical limitations, this joint probability would be a maximum when all probabilities are equal; but in this case, conditions of statics set constraints on the values of the strain energies so that we can state only that the most probable combination exists when $E_{n_1}+E_{n_2}+E_{n_3}+$ . . . is a minimum. This is exactly equivalent to the statement of least work; but the principle of least work must be recognized as only one species of a larger class or genus including all of the states of maximum probability.

In the structural example the laws of statics left only one degree of freedom to be met by the principle of maximum probability. No general solution could be made based entirely on one principle—that is, entirely on the basis of statics, or entirely on the basis of maximum probability. Similarly in geomorphic problems, the set of physical factors includes many variables such as the amount of water and sediment to be carried, the fluid friction, and the river transport capacity. The equations connecting these factors leave several degrees of freedom remaining. In other words, a river system is "hydraulically indeterminate." A river can adjust its depth, width, or velocity to a given slope in several ways so that it is necessary to establish the river profile and the hydraulic geometry on the basis of maximum probability.

The general implication for rivers may then be stated as follows: The principle of least work is one of several ways in which the condition of maximum probability may be satisfied. The river channel has the possibility of internal adjustment among hydraulic variables to meet the requirement for maximum probability, and these adjustments tend also to achieve minimization of work. In systems other than rivers wherein the adjustment is actually to a condition of least work, maximum probability is achieved. The geomorphologist's intuitive inference that river equilibrium is a condition of least work (see, for example, Rubey, 1952, p. 135) is not complete. As will be shown in the section on hydraulic geometry, other factors restrain the system from attaining a state of least work.

LONGITUDINAL PROFILE OF RIVERS

A concept developed in a preceding section states that in an open system the distribution of energy tends toward the most probable. In the open system represented by the river, the unit under consideration is a unit of length along the river channel. The concept applied to the river system yields the result that the most probable distribution of energy exists when the rate of gain of entropy in each interval of length along the river is equal.

To develop the reasoning leading to that conclusion, refer back to equation 14,

$$\frac{d\phi'}{dt}\frac{1}{Q}=\frac{dH}{dx}\frac{1}{H}. \tag{14}$$

The rate of production of entropy in a river system is $\frac{d\phi'}{dt}$, and per unit of mass volume rate (discharge) is $\frac{d\phi'}{dt}\cdot\frac{1}{Q}$. This quotient is inversely proportional to the energy content above base level, H, and directly proportional to the loss of potential per unit of distance, $\frac{dH}{dx}$.

The right side of equation 14 is a statement of the proportional distribution of energy relative to base level, $H=H_0$. Then it also can be considered a statement of the probability of a given distribution of energy between $H=H_0$ and the maximum value of H in the system in question.

The probability of a particular combination of values of energy content of unit distances along the course of the open system of the river would be similar in form to equation 15, or

$$p \propto e^{-\frac{dH_1}{dx}\frac{1}{H}} e^{-\frac{dH_2}{dx}\frac{1}{H}} \ldots e^{-\frac{dH_n}{dx}\frac{1}{H}}$$

where $\frac{dH_1}{dx}$, $\frac{dH_2}{dx}$, etc., represent the loss of head in successive units of length along the river length.

This joint probability would be maximum when all the probabilities are equal. Because in each unit length, dx, the loss of head, dH, would have a given probability, p

$$\frac{dH_1}{dx}\frac{1}{H}\propto p_1$$

$$\frac{dH_2}{dx}\frac{1}{H}\propto p_2,$$

etc., and the joint probability is greatest when $p_1=p_2=p_3$ . . . p_n, then $\frac{dH_1}{dxH}=\frac{dH_2}{dxH}=\frac{dH_3}{dxH}$ or $\frac{dH}{dxH}$ is equal in each unit of length along the river and thus is constant. Also, then, the time rate of change of entropy per unit of flow rate is constant. Because

$$\frac{dH}{dx}\frac{1}{H}=\text{constant}, \tag{16}$$

integrating

$$H=ae^{-bx}+C \tag{17}$$

where a and b are coefficients and C represents the base level.

To summarize, the most probable sequence of energy losses in successive units of river length corresponds to a uniform increase of entropy in each unit length. When this specification is fulfilled without constraint on length, the longitudinal profile tends to become exponential in form, a result in agreement with many actual river profiles.

However, it may not be clear how the rate of loss of energy at any point is necessarily related to elevation above base level. We refer back to our statement that entropy must represent a ratio between a given state to the number of alternate states. Note that dh/dx is the gradient of energy or river slope, S, and then

$$\frac{d\phi/dt}{Q}=\frac{S}{H}=\frac{dH}{Hdx}\propto\log p. \tag{18}$$

The quantity $S=\frac{dH}{dx}$ is the rate of energy loss and represents the given state. Then the quantity H must represent the range of choice of the rate of loss of energy at any level. The probability p then represents the proportion of the total elevation lost as the river approaches base level. This we assert to be implicit in the statement that the higher the landscape above base level, the greater becomes the distribution of possible slopes. This range must decrease as one approaches base level; in the same way the possibility of orderliness among molecules of a gas becomes less and less as temperature is lowered and becomes zero at the base level of zero absolute temperature.

As demonstrated with the engine model example, the most probable state is one in which the rate of energy expenditure per unit mass of fluid is proportional to the height above base level. Because the statement applies per unit mass of fluid, it will apply throughout a river composed of a branched network of channels of increasing size downstream which collect and convey water and sediment. It is only necessary to postulate that water and sediment once entrained remain in transit; i.e., there are no losses in transit and there is no constraint on length.

DEMONSTRATIONS BY USE OF A RANDOM-WALK MODEL

The equations state that a profile of energy distribution in the open system represented by a river channel should be exponential in form and that this profile corresponds to the most probable one under the conditions stated in the previous section—that is, no constraint on length. This is demonstrated by a random-walk model constructed in the following way: Consider a point on the surface of a landscape above base level as shown in figure 3. The flow moves in

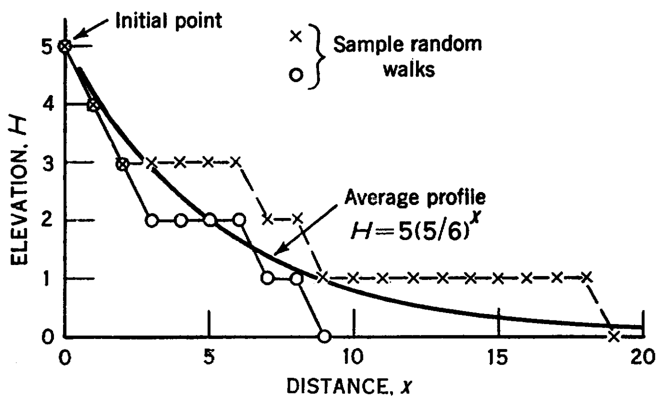

FIGURE 3.—Sample random walks used in the generation of an average stream profile.

unit steps to the right with two choices at each step. There is the probability p that it will move downward one unit of elevation or the probability q that it will continue at the same level. Since there are only these two alternatives, $p+q$ must equal unity. The condition that the rate of energy expenditure remains proportional to height above base level is included in the model by setting the probability p of a downward step proportional to the height above base level, decreasing to zero when a random walk reaches base level.

In this model, figure 3, an initial point was set at an elevation of 5 units above base level. Since there are 6 levels (0, 1, 2, 3, 4, and 5), the probability of a downward step is $H/6$. Thus, the probability of a downward step decreases with elevation. Hence, random walks were constructed as follows: A pack consisting of 5 white cards and 1 black card was prepared. The number of cards represents the total number of levels, and the number of white cards in the pack represents the elevation above base level, in this case initially 5. The cards were shuffled and a card selected at random and then replaced by a black card. If a black card is selected and replaced by a black, there is no change in elevation. If a white card is replaced by a black card, then elevation decreases by one unit. This process is continued and a record made of the number of white cards in the pack at each step. The probability of choosing a white card (equivalent to a downward step) decreases as the proportion of white cards in the pack decreases. When all white cards have been replaced by black cards, the random walk has reached base level.

Table 1 lists the frequency of random walks at different elevations and distances constructed by this model, and figure 3 shows the results of two of these random walks. The graph of the equation defining the mean position of all possible walks generated by the above model is also plotted on figure 3. The equation of this line is

$$H=5\left(\frac{5}{1+5}\right)^{x}. \tag{19}$$

Referring back, equation 18 states that

$$\frac{dH}{dx}\frac{1}{H}\propto \log p.$$

Integrating

$$H\propto H_0(p)^x \tag{20}$$

H_0 is height of an initial point above base level. The equation is that of a longitudinal profile expressed in terms of a statistical parameter.

Equation 20 is then a more general form of equation 19. The probability p in equation 20 corresponds to the fraction $\frac{5}{1+5}$, or more generally, $H/1+H$. In each unit of distance, x, the elevation declines by the ratio $H/1+H$. In other words, p in equation 20 equals the probability that the rate of loss of energy in a unit length of river is a fixed proportion of the total head above base level. This probability is a constant.

The profiles of many major streams do indeed follow an exponential law as is well known (Schoklitsch, 1933). There is no point in attempting to explore or compare actual profiles with the exponential law because, as we shall show, the most probable profile under various constraints differs from the simple exponential one. Analyzing the possible agreement of specific examples with the theory becomes a problem of identifying the several influences upon the actual profiles—which is not the objective of this theoretical paper.

TABLE 1.—*Frequencies of random walks having given elevation at various distances from origin, in percent, subject to condition that probability of a downward step equals* H/6

Elevation (H)	Distance										
	0	1	2	3	4	5	6	7	8	9	10
5	100	17	3	-----	-----	-----	-----	-----	-----	-----	-----
4	-----	83	42	17	6	2	-----	-----	-----	-----	-----
3	-----	-----	55	56	39	24	14	7	3	2	1
2	-----	-----	-----	27	46	50	45	37	29	20	14
1	-----	-----	-----	-----	9	23	36	45	50	52	50
0	-----	-----	-----	-----	-----	1	5	11	18	26	35
Average elevation	5.0	4.17	3.48	2.90	2.42	2.02	1.68	1.40	1.17	0.98	0.81

EFFECT OF CONSTRAINT ON STREAM LENGTH

The exponential form of stream profile is the most probable when there are no constraints on channel length; the profile approaches the base level asymptotically. If, as in mountainous headwater streams, the constraint on stream length greatly exceeds the constraint exerted by base level, then the probability of a vertical step decreases downstreamward with respect to distance rather than with respect to elevation as before. The resulting profile is therefore the converse of the preceding case. In other words, height and length are exchanged in their relationship; thus

$$x=L(p)^H \tag{21}$$

where $p=L/1+L$, L is total stream length, and x is distance downstream from the headwaters. Equation 21 also defines an exponential profile, but with respect to distance rather than with respect to elevation as in the previous case. In other words, the elevations vary with the logarithms of distance. In this case, H is the height above the elevation where x equals L. The profile is not asymptotic to this level.

Equation 21 can be written in the form $H\propto \log_p x/L$. Profiles of the form defined by this equation were found by Hack (1957, p. 70 and following) in many headwater streams in Virginia and Maryland.

The effect of a constraint on length upon the development of a profile may be illustrated by a random-walk model. In this case, one limits the random walks to those paths that reach base level at or before reaching a given limiting distance. For example, according to the frequencies listed in table 1, 35 percent of all random walks developed in the model used to define figure 3 reached base level at or before distance 10. The frequency distribution of these walks is listed in table 2, and the average elevation as calculated from these frequencies is plotted on figure 4. For comparison there is also plotted on figure 4, the profile for the random

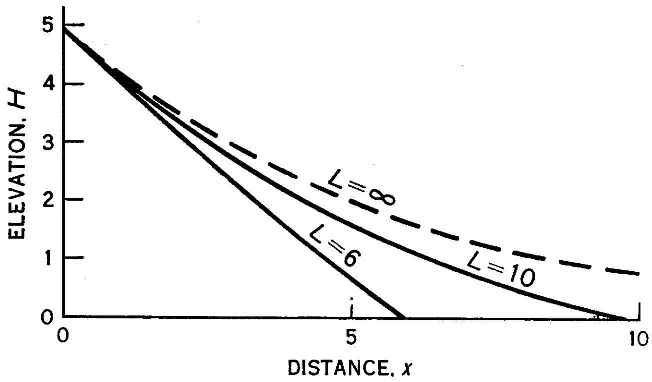

FIGURE 4.—Random-walk profiles subject to constraints of stream length.

walks that reached base level at or before distance 6, and the profile for all random walks (that is, without restriction as to length) as shown in figure 3. The concave profile is evident on all, but none of the restricted profiles is tangent to base level and the angle at which the profile intersects base level increases with increasing constraint on stream length.

This result leads us to judge that constraints of length tend to diminish the influence of base level on the profile, and so to impair the proportional distribution of head loss in a river system. Other factors are operative, of course, and their influence must also be investigated. Some of these other factors are hydraulic. Their interaction is also illuminated through application of the probability principle, as will be shown in a subsequent section where a theoretical derivation of the hydraulic geometry of rivers is presented.

TABLE 2.—*Frequencies of random walks contained in table 1 that reach base level at or before distance 10 (figures shown are percent)*

Elevation	Distance										
	0	1	2	3	4	5	6	7	8	9	10
5	35	4.7	0.6	-----	-----	-----	-----	-----	-----	-----	-----
4	-----	30.3	12	3.5	0.7	-----	-----	-----	-----	-----	-----
3	-----	-----	22.4	18	9.3	3.8	1.2	0.2	-----	-----	-----
2	-----	-----	-----	13.5	19	16.4	10.3	5	1.6	-----	-----
1	-----	-----	-----	-----	6.0	13.8	18.5	18.8	15.4	8.7	-----
0	-----	-----	-----	-----	-----	1	5	11	18	26.3	35
Average elevation	5.0	4.13	3.37	2.72	2.15	1.66	1.22	0.85	0.53	0.25	0

EFFECT OF CONSTRAINT OF TEMPORAL BASE LEVELS

Only the sea is a perfect sink—an absolute base level—in the geomorphic sense. There are other base levels, but all these must, in the scope of time and place, be considered temporary or imperfect. Base levels may be as temporary as a bar of gravel deposited in a channel by a flood from a tributary; base levels may be as imperfect as a riffle which transmits downstream only part of the energy of the flowing water and absorbs the remainder, and which are drowned out at high river stages. The valley floor is an example of an imperfect base level relative to streams draining to it from the valley sides. This is especially true in an arid region where both water and sediment are lost in the valley alluvium in small floods, but carry through the valley in large floods.

One can set up a random-walk model of imperfect or temporal base levels, by interposing an imperfect absorbing medium in the random walks. As Feller (1950, p. 279) explains, random walks may be subject to absorbing barriers or reflecting barriers. All random walks terminate at an absorbing barrier. The latter have a geomorphic significance. In the random-walk model of the river profile, the sea becomes a perfect absorbing barrier. However, the effects of a temporal base level as described may be modeled by a partial absorbing barrier, which "absorbs" a given percentage of the random walks and permits the remainder to pass through unaffected.

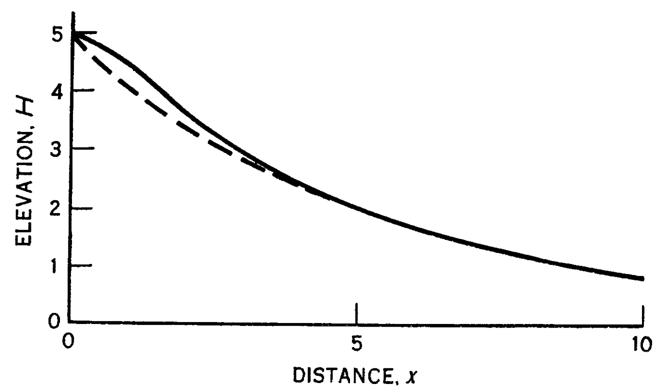

FIGURE 5.—Effect of an absorbent medium in a stream profile.

Figure 5 shows the results of such a calculation using the same random-walk model as used in the development of figure 3. The absorbing medium was placed between elevations 4 and 5 and it was assumed to absorb 75 percent of the random walks; that is, the frequencies at elevation 4 and below, as listed in table 1, were reduced by 75 percent. The data from which figure 5 was derived are presented in table 3. Average elevations at the various distances were computed accordingly. The resulting profile plotted on Cartesian coordinates, figure 5, is convex in its upper part leading into an exponential curve. This profile appears comparable to a typical hillside profile or profile of certain headwater draws or rills. This suggests that a tributary stream to which a hill slope or draw is graded is a partial or imperfect base level relative to the hill slope or draw.

The absorption factor does not mean that 75 percent of the water and sediment is absorbed—we refer, rather, to the transmission of the influence of lower base levels. The complement of the absorption factor describes the transmission of the influence of base level upon higher elevations.

TABLE 3.—Frequencies of random walks contained in table 1 after introduction of an absorbing layer (75 percent of random walks absorbed between elevation levels 4 and 5; figures listed are percent)

	Distance										
Elevation	0	1	2	3	4	5	6	7	8	9	10
5	100	17	3								
4		21	10	4	1						
3			14	14	10	6	3	2			
2				7	11	12	11	9	7	5	3
1					2	6	9	11	12	13	13
0							1	3	4	7	9
Average elevation	5.0	4.45	3.60	2.90	2.42	2.02	1.68	1.40	1.17	0.98	0.81

PROBABILITY AND THE PROFILE OF THE GRADED RIVER

The probability nature of the profile also carries certain implications about deviations from the mean or most probable profile. Departures from the exponential form represent increasing degrees of improbability. Variations in the profile due to changes in rock type or even of stream length will persist for a short or long period of time, depending on the scale of the constraint. However, local departures from a general rule of a proportional decrease in energy expenditure, when considered in the time scale of physiographic processes, are ephemeral elements. This may be true even when the irregularities are as conspicuous as lakes, waterfalls, or torrents. This conclusion merely affirms a view long held in geomorphology.

A contribution made by the entropy concept is that the "equilibrium profile" of the graded river is the profile of maximum probability and the one in which entropy is equally distributed, constituting a kind of isentropic curve.

CONSTRAINTS—SUMMARY

It may be visualized that the study of the landscape is the study of constraints imposed by geologic structure, lithology, and history. To identify those constraints is the purpose of much of geologic inquiry; to evaluate the effects of constraints upon the landscape is an object of quantitative geomorphology. The way in which some constraints affect the river profile can be evaluated if one considers the profile to approximate its maximum probable condition under a given set of constraints. Those treated in this paper are summarized below.

1. The absence of all constraints leads to no solution—an obvious but not a trivial conclusion.
2. The longitudinal profile of a stream system subject only to the constraint of base level is exponential with respect to elevations above base level.
3. The profile of a stream subject only to the constraint of length is exponential with respect to stream length (a logarithmic function with respect to elevation).
4. The constraint of length added to that of base level decreases the concavity and diminishes the influence of base level.
5. Introduction of the constraint of a partial base level above that of the sea adds a measure of convexity in the profile.

HYDRAULIC GEOMETRY

The use of a probability statement to complete the solution of a set of hydraulic equations can be illustrated through evaluation of the hydraulic geometry of river systems. In attempting to explain the morphology of rivers that are authors of their own geometries, to introduce additional hydraulic relationships adds more unknowns. Usually their number exceeds the number of equations and a direct solution becomes impossible. For example, Leopold and Maddock (1953, p. 50) outline a set of eight conditions that must be met to evaluate the hydraulic geometry of a river system. There are not as many as eight known interrelationships between these variables. In the following simplified example, evaluation of only five unknowns (velocity, depth, width, slope and friction) is sought. However, only three equations connecting these variables are available. The remaining conditions will be satisfied by a statement of maximum probable distribution of energy in the system—the basic concept implied in the minimum production of entropy in an open system.

Leopold and Maddock (1953) describe and evaluate from field data the hydraulic geometry of river channels by a set of relations as follows:

$$v \propto Q^m$$
$$D \propto Q^f$$
$$w \propto Q^b$$
$$s \propto Q^z$$
$$n \propto Q^y$$

where v is the mean velocity, D is the mean depth, w is the surface width, and s is the energy slope, and n is the friction factor at a cross section along a river channel where the mean discharge is Q. It is desired to evaluate the exponents in a downstream direction as discharge of uniform frequency increases.

Some of the principles we have described can provide estimates of the magnitude of the exponents of the above relationships. The exponents m, f, b, z, and y describe the *variability* in velocity, depth, width, slope, and friction along a river channel, but do not uniquely determine the magnitudes of these properties.

The first condition is that specified by the equation of continuity $Q=vDw$, which requires that

$$m+f+b=1.0.$$

A second condition is introduced by the Manning equation for hydraulic friction. This requires that velocity, depth, energy slope, and friction factor have the following relationship:

$$v\propto D^{\frac{2}{3}}s^{\frac{1}{2}}/n.$$

Since

$$Q^{m}\propto Q^{\frac{2}{3}f}Q^{\frac{1}{2}z}/Q^{y}.$$

Hence

$$m=\frac{2}{3}f+\frac{1}{2}z-y.$$

The third condition introduced is that the transport of sediment per unit mass of water is uniform. This is also a reasonable assumption and is a close approximation to the conclusion reached by Leopold and Maddock from analysis of suspended sediment transport data (1953, p. 26). Although many formulas for sediment transport have been devised, most can be expressed in terms of stream power as suggested by Bagnold (1960). Power is an important factor in the formulation of the hydraulic geometry of river channels. As explained by Bagnold, the stream power at flows sufficiently great to be effective in shaping the river channel is directly related to the transport of sediment, whose movement is responsible for the channel morphology.

Laursen (1958, p. 198), commenting on the general unsatisfactory nature of our knowledge of transport phenomena, lists several typical equations for the transport of sediment, based on flume experiments, that have been reported in the literature. The average relation shows sediment transport in excess of the point of incipient motion to vary about as $(vDs)^{1.5}$ where vDs is the stream power per unit area. In terms of sediment per unit discharge, that is the concentration, C, the several equations average out as $C\propto n$ $(vD)^{0.5}s^{1.5}$, a result that is consistent with the conclusion reached by Bagnold (1960). There is in addition to be considered the effect of sediment size. Examination of several equations indicates that sediment transport varies inversely as about the 0.8 power of the particle size. There have been several attempts to relate particle size to the friction factor n; at this point we shall use the Strickler relation that the value of n varies as the $\frac{1}{6}$ power of the particle size. It is realized full well that both the sediment transport and the friction factor are influenced by many other factors such as bed form and the cohesiveness, sorting, and texture of the material. These are the kinds of influences, themselves effects of the river, that prevent a straightforward solution of river morphology. In order to limit the number of variables only the effect of particle size on transport will be considered, as this factor varies systematically along a river from headwater to mouth. Thus, sediment transport used in this study reads $C\propto(vD)^{0.5}s^{1.5}/n^{4}$.

The sediment transport per unit discharge in the river system will be recognized as a hydrologic factor that is independent of the hydraulic geometry of a river in dynamic equilibrium. Consequently sediment concentration may be considered constant, and therefore $0.5m+0.5f+1.5z-4y=0$.

Thus, there are three equations: continuity, hydraulic friction, and sediment transport. There are five unknowns. The two remaining equations will be derived from a consideration of the most probable distribution of energy and total energy in the river system.

The probability of a given distribution of energy is the product of the exponential functions of the ratio of the given units to the total, as in equation 15. The ratios of the units of energy E_1, E_2, etc., representing the energy in successive reaches along the river sufficiently long to be statistically independent, to the total energy E in the whole length, are $\frac{E_1}{E}, \frac{E_2}{E}, \ldots \frac{E_n}{E}$. The product of the exponentials of these is the probability of the particular distribution of energy or, as was seen in equation 15,

$$p\propto e^{-\frac{E_1}{E}}e^{-\frac{E_2}{E}}e^{-\frac{E_3}{E}}\ldots e^{-\frac{E_n}{E}}$$

where n remains finite.

As previously, the most probable condition is when this joint probability, p, is a maximum and this exists when $E_1=E_2=E_3\ldots=E_n$. Thus energy tends to be equal in each unit length of channel.

Equable distribution of energy corresponds to a tendency toward uniformity of the hydraulic properties along a river system. Consider the internal energy distribution. Uniform distribution of internal energy per unit mass is reached as the velocity and depth tend toward uniformity in the river system. Uniformity of velocity and depth requires that the exponents m and f be as near zero as possible, and therefore according to the Manning equation $\frac{1}{2}z-y$ should approach zero.

Since the energy is largely expended at the bed, equable distribution of energy also requires that stream power per unit of bed area tend toward uniformity, that is, that Qs/w tend toward uniformity. This condition is met as the sum of $m+f+z$ tends toward zero. An opposite condition is indicated by Prigogine's rule of minimization of entropy production that characterizes a stable open system and which, therefore, leads to the tendency that the total rate of work, $\Sigma Qs\Delta Q$ in the system as a whole be a minimum. Because $s \propto Q^z$, then $\Sigma Q^{1+z}\Delta Q \rightarrow$ a minimum. For a given drainage basin this condition is met as z takes on increasingly large negative values. However, there is a physical limit on the value of z, because for any drainage basin the average slope $\Sigma s\Delta Q/\Sigma\Delta Q$ must remain finite. This condition is met only for values of z greater than -1, and therefore z must approach -1 or $1+z$ approaches zero.

To summarize, we have introduced three statements on the energy distribution—

$$\frac{1}{2}z-y\rightarrow 0$$

$$m+f+z\rightarrow 0$$

$$(1+z)\rightarrow 0.$$

The condition of minimum total work tends to make the profile concave; whereas the condition of uniform distribution of internal energy tends to straighten the profile. Hence, we seek the most probable state.

The most probable combination is the one in which the product of the probabilities of deviations from expected values is a maximum. It is unnecessary to evaluate the probability function, provided one can assume normality, as we can then state directly that the product of the separate probabilities is a maximum when their variances are equal.

$$\left(\frac{F_1}{\sigma_{F_1}}\right)^2=\left(\frac{F_2}{\sigma_{F_2}}\right)^2=\left(\frac{F_3}{\sigma_{F_3}}\right)^2=\text{etc.}$$

where F_1, F_2, F_3 represent the several functions stated above. Therefore,

$$\left(\frac{F_1}{\sigma_{F_1}}+\frac{F_2}{\sigma_{F_2}}\right)\left(\frac{F_1}{\sigma_{F_1}}-\frac{F_2}{\sigma_{F_2}}\right)=0$$

for which there are two possible solutions:

$$\frac{F_1}{\sigma_{F_1}}+\frac{F_2}{\sigma_{F_2}}=0$$

or

$$\frac{F_1}{\sigma_{F_1}}-\frac{F_2}{\sigma_{F_2}}=0$$

of which only one leads to a possible result.

The absolute values of the standard deviations need not be known, as we can infer their relative values. For example, letting $F_1=\frac{1}{2}z-y$, the standard deviation of F_1 is

$$\sqrt{(\sigma_{z/2})^2+\sigma_y^2}$$

and

$$F_2=m+f+z;\quad \sigma_{F_2}=\sqrt{\sigma_m^2+\sigma_f^2+\sigma_z^2}$$

$$F_3=(1+z);\quad \sigma_{F_3}=\sigma_z.$$

The standard deviations σ_m, σ_f, σ_z, and σ_y represent the variability of the several factors as may occur along a river system. In this solution, they are taken to be proportional to their respective values. Since these values are not known initially, the problem must be solved by iteration. Fortunately, the solution is not sensitive to the values of the several standard deviations, so the solution converges rapidly. No knowledge of their values is available for the first trial solution so all values are considered equal. Thus:

$$\left(\frac{m+f+z}{\sqrt{3}}\right)^2=\left(\frac{\frac{1}{2}z-y}{\sqrt{1.5}}\right)^2$$

$$\left(\frac{\frac{1}{2}z-y}{\sqrt{1.5}}\right)^2=\left(\frac{(1+z)}{1}\right)^2$$

$$\left(\frac{m+f+z}{\sqrt{3}}\right)^2=\left(\frac{(1+z)}{1}\right)^2$$

The results of the first solution are then used to derive first estimates of the standard deviations, and a second computation carried forward. A third calculation is made to confirm the results of the second trial solution.

These three equalities lead to two independent equations—

$$y=-\frac{1}{2}(m+f)$$
$$z=-0.53+0.93y$$

which, together with the three hydraulic conditions

$$m+f+b=1.0$$
$$m=\frac{2}{3}m+\frac{1}{2}z-y$$
$$\frac{1}{2}m=\frac{1}{2}f+1.5z-4y=0,$$

lead to a solution for m, f, b, z, and y. The final values, derived without reference to field data, are compared below with others obtained previously from analysis of field data on actual rivers.

Values of exponents of the hydraulic geometry, in downstream direction

	Theoretical values, this paper	*Average values from data on rivers in midwestern United States*
Velocity, m =	0.09	[1] 0.10
Depth, f =	.36	[1] .40
Width, b =	.55	[1] .50
Slope, z =	−.74	[2] −.49
Friction, y =	−.22	

[1] Leopold and Maddock, 1953, p. 16.

[2] Leopold, 1953, p. 619, from practically same river data used by Leopold and Maddock and therefore considered comparable.

Because velocity is such a large factor in energy expenditure, the value of m, the exponent of velocity, is close to zero, making this characteristic the most conservative.

The value of z (the measure of the variability in slope) is least satisfactorily defined from field data, as it is affected by variation in the friction factor. Thus, the value of z as reported above is $-0.53+0.93y$. Hence the value of z is -0.53 where grain size is constant. The value of z was found by Leopold (1953, p. 619) to be -0.49 for midwestern rivers. Henderson (1961) shows that for a stable channel of uniform grain size the slope varies as the -0.46 power of the discharge. However, when a large variety of river data were averaged, including data on ephemeral channels, Leopold and Miller (1956, p. 25, 26) obtained a value of z averaging near -0.95. This latter average is probably less comparable with the Leopold-Maddock data than the average value Leopold (1953) compiled from nearly the same list of midwestern rivers. Values of z in excess of 0.50 may be attributed to the common tendency for the friction factor to decrease downstream (increasing discharge) as particle size decreases. In canals, for example, where y is positive, the value of z is less than 0.50.

This study of hydraulic geometry develops a method of solution through introduction of probability statements that we consider to be sound in principle. Though the results obtained by this theoretical derivation of exponents in the hydraulic geometry do agree quite well with field data, we are far from satisfied with some of the physical relationships. For example, there is uncertainty whether the transport equations represent physical relationships or are in fact formulas for sediment transport under regime conditions. Several alternative assumptions were tried and the resulting exponents also agreed quite well with field data. But the present state of knowledge about some of these interactions among the hydraulic and sediment factors allows various reasonable but empiric assumptions that, nevertheless, yield results of the same order of magnitude.

THE DRAINAGE NETWORK

As has been shown in the example of the longitudinal profile, the probability that a random walk will fall in certain positions within the given constraints can be ascertained. There is also a mean or most probable position for a random walk within those constraints. This statement suggests the possibility that a particular set of constraints might be specified that would describe the physical situation in which drainage channels would develop and meet, eventuating in the drainage network.

Let us postulate that precipitation falling on a uniformly sloping plain develops an incipient set of rills near the watershed divide and that they are oriented generally downhill. As the rills deepen with time, crossgrading begins owing to overflow of the shallow incipient rills in the manner postulated by Horton (1945, p. 337). The direction that the crossgrading takes place and the micropiracy of incipient rills is, as Horton implies, a matter of chance until the rills deepen sufficiently to become master rills.

This randomness in the first stages of crossgrading might be approximated in the following conceptual model which is amenable to mathematical description. Consider a series of initial points on a line and equidistant from one another at spacing a. Assume random walks originating at each of these points. In each unit of time each random walk proceeds away from the initial line a unit distance. But let us specify that each walk may move forward, left or right at any angle, but may not move backward. The accumulation of moves will produce in time sufficient cumulative departures from the orthogonal to the direction of the original line of points that some pairs of paths might meet. After such a junction only one walk proceeds forward in like manner and would behave similarly, just as when two stream tributaries join the single stream proceeds onward.

The physical situation described is analogous to the statistical model called the "gambler's ruin" (Feller, 1950, chap. 14). This model treats the probability that in a certain number of consecutive plays, a gambler playing against the "house" will lose all of his capital to the house. In the model, the capital assets of the gambler and the house are specified. If the gambler has a lucky run he accumulates capital at the expense of the house. If the reverse takes place, he gradually may lose his capital completely.

The statistical statement of the probable duration of the game (number of plays) is given by $D=Z(A-Z)$ provided the probabilities of a win and loss are equal, and where Z is the capital of the gambler, and A is the total capital in the game (house plus gambler). The size of Z and A are measured in units of the size of equal wagers.

The analogy to the physical situation of the postulated developing drainage network may now be specified quantitatively. It is desired to compute the distance two random walks will proceed before joining. In this model each walk has the same probability that it will accumulate a deviation of some given distance from the mean or orthogonal path.

This would be analogous to the condition in the game where the house and gambler begin with the same capital, that is, $Z=\frac{1}{2}A$. If the size of each wager is equal, then $D=\frac{1}{4}A^2$.

The duration of the game under the conditions stated is proportional to the square of the size of the initial capital. However, this describes the duration of a game with fixed boundaries. In the stream case, contiguous streams may join either on the right or left, so that the distance travelled before joining would not necessarily increase as the square of the separation distance at the last junction. On the other hand, the distance travelled would increase at least as the first power of the separation distance. Hence, there are two limits

$$D \propto A^2$$

or

$$D \propto A.$$

In the stream case, the quantity A is equivalent to the average distance between streams, and we may assume that on the average the streams are evenly spaced (equivalent to the statement $Z=\frac{1}{2}A$).

Consider the geometry of a simple and regular network of joining streams. Let us follow the definition introduced by Horton (1945) for stream order. The smallest unbranched tributary is called first order, the stream receiving as tributaries only streams of first order is called second order. A stream receiving as a tributary a second-order stream is called third order, and so on.

The geometry of joining streams may be described by the relation $A=a2^{R-1}$ where A is the mean distance between streams of order R, and a is the mean distance between the first-order streams. Hence, we can state that the average stream length from one junction to the next varies with the mean interfluve distance, in the form

$$L \propto 4^R$$
$$L \propto 2^R$$

depending on whether the length varies as the square or the first power of the distance between streams of a given order. We deduce therefore that the logarithms of stream length vary linearly as the order number, and that the mean ratio between the lengths of streams differing in order by 1 will be between 2 and 4. This result derived from considerations of entropy (most probable state) is in agreement with the findings of Horton's (1945) analysis of field data on river systems.

The mathematical relationship derived may perhaps best be visualized by actual trial, constructing random walks on cross-section paper using a table of random numbers.

We constructed a stream network from the following elemental structure: Beginning with a series of points 1 inch apart on a line, a random walk was constructed, limited, however, only to motion right, left, or ahead. Backward or uphill steps were excluded. The several random walks when extended far enough made junctions. The resulting walk proceeding onward again joined and so on. A portion of the work sheet on which the random walks were plotted is shown as figure 6.

To test how this constructed network compared with nature, we have made what is called a Horton analysis. This consists of plotting the logarithms of the average length of streams of various orders against the order number. Horton (1945, p. 298) found the length of streams of a given order increased in constant ratio with increasing order number. Figure 7 presents the results obtained from the random walks plotted, and shows that stream lengths increased by a ratio of about 2.8 between consecutive order numbers. This result lies within the range of 2 and 4 as suggested previously. Horton obtained, from field data he studied, ratios between 2.5 and 3.7. Ephemeral channels near Santa Fe, N. Mex., were found to exhibit a ratio of about 2.1 (Leopold and Miller, 1956, p. 18, fig. 13). Thus the random-walk results give results in general quantitative agreement with field data.

The constraints used in construction of the random walks in figure 6 may seem too limited. Another type of construction is as follows: Using a sheet of rectangular cross-section graph paper, each square is presumed to represent a unit area. Each square is to be drained, but the drainage channel from each square has equal chance of leading off in any of the four cardinal directions, subject only to the condition that, having made a choice, flow in the reverse direction is not possible. Under these conditions, it is possible for one or more streams to flow into a unit area, but only one can flow out. By the random selection of these directions under the conditions specified, a stream network was generated as shown on figure 8 which has striking similarities to natural drainage nets. Divides were developed, and the streams joined so as to create rivers of increasing size. One can say that the random pattern represents a most probable network in a structurally and lithologically homogeneous region. The Horton analysis of the network pictured on figure 8 is presented as figure 9.

Figure 10 shows the relation between stream length and drainage area. The slope of the line indicates that for this random-walk model length increases as the 0.64 power of the drainage area. Simple geometric relations would suggest that stream lengths vary as the square root of the drainage area. This model accounts for the higher values that Langbein (1947, p. 135) and Hack (1957, p. 66–67) found for natural streams.

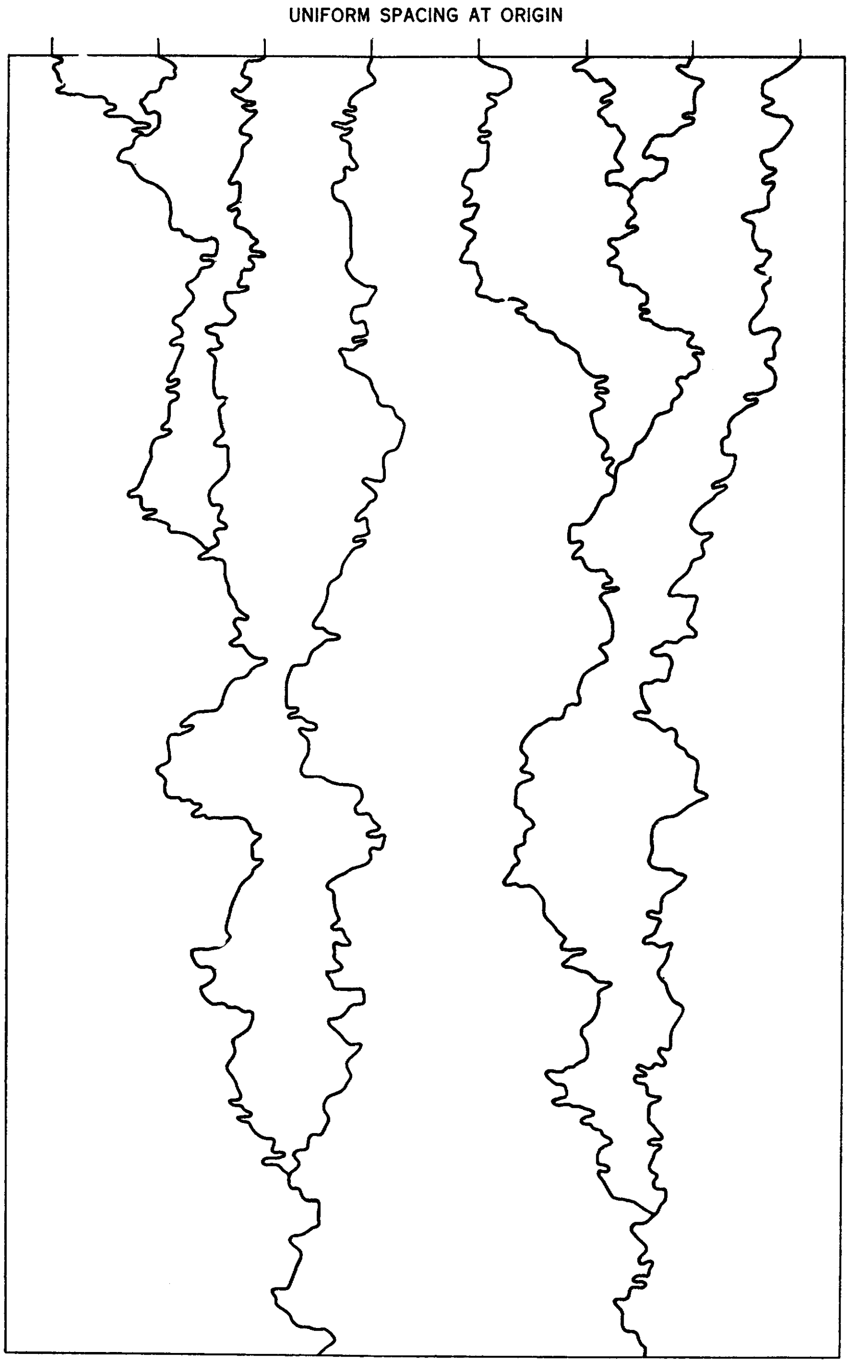

FIGURE 6.—Portion of random-walk model of stream network.

Although in this example each elemental area is a square, we note that the diagram might have been drawn on an elastic sheet. If this sheet were stretched in various ways to alter the geometric figures, none of the relations between numbers and lengths with orders, shown on figure 9, would have been changed.

Horton also showed that in natural stream networks, the logarithms of the number of streams of various orders is proportional to the order number. More specifically, the number of streams of a given order decreased in constant ratio with increasing order number. This ratio he called the "bifurication ratio." Figure 9 shows the relation between numbers of streams of the orders 1 to 5 that were developed by the streams on figure 8. When compared with similar diagrams from data on actual rivers, given by Horton in his original treatment of the subject, one may note that in each case there is a logarithmic relationship between

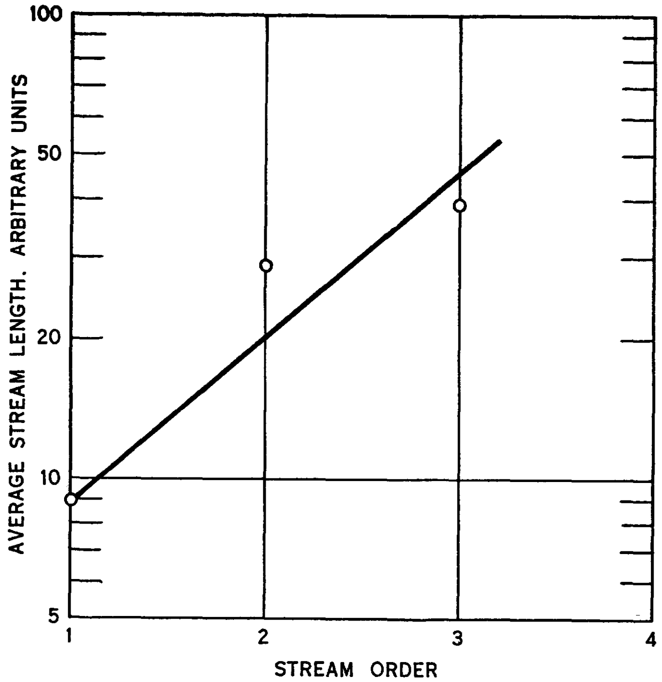

FIGURE 7.—Relation of length of streams to stream order.

orders and numbers of streams, and moreover, the bifurication ratios are quite comparable.

The essential point of these demonstrations is that the logarithmic increase, both of stream length and number of streams with order number as found in natural stream networks, accords with the geometrical properties and the probabilities involved. The logarithmic relationship is thus one of optimum probability. Optimum probability, in this sense, represents maximum entropy.

DISCUSSION AND GEOMORPHIC IMPLICATIONS

In statistical mechanics the probability aspect of entropy has been demonstrated. The word entropy has, therefore, been used in the development of systems other than thermodynamic ones, specifically, in information theory and in biology. Its use is continued here.

With this understanding of the term, the present paper shows that several geomorphic forms appear to be explained in a general way as conditions of most probable distribution of energy, the basic concept in the term "entropy."

It is perhaps understandable that features such as stream profiles which occur in nature in large numbers should display, on the average, conditions that might be expected from probability considerations because of the large population from which samples may be drawn. The difficulty in accepting this proposition is that there is not one but many populations owing to the variety of local geologic and lithologic combinations that occur. Further, the geomorphic forms seen in the field often are influenced by previous conditions. Stream channels, in particular, show in many ways the effects of previous climates, of orogeny, and of structural or stratigraphic relations that existed in the past. In some instances the present streams reflect the effects of sequences of beds which have been eradicated by erosion during the geologic past.

In a sense, however, these conditions that presently control or have controlled in the past the development of geomorphic features now observed need not be viewed as preventing the application of a concept of maximum probability. Rather, the importance of these controls strengthens the usefulness and generality of the entropy concept. In the example presented here we have attempted to show that the differences between patterns derived from averaging random walks result from the constraints or controls imposed upon the system. We have, in effect, outlined the mathematical nature of a few of the controls which exist in the field. The terms in the genetic classification of streams reflect the operation of constraints. Terms like "consequent" and "subsequent" are qualitative statements concerning constraints imposed on streams which, in the absence of such constraints, would have a different drainage net and longitudinal profile.

In a sense, then, much of geomorphology has been the study of the very same constraints that we have attempted to express in a mathematical model.

The present paper is put forward as a theoretical one. It is not the purpose of a theoretical paper to compare in detail the variety of field situations with the derived theoretical relations. Rather, it is hoped merely to provide the basis for some broad generalizations about the physical principles operating in the field situations.

On the other hand, the random-walk models used here are simple demonstrations of how probability considerations enter into the problem. They are intended to exemplify how the basic equations can be tested experimentally. Thus the present paper should not be considered to deal with random walks. We hope it is concerned with the distribution of energy in real landscape problems. The random-walk models exemplify the form of the equations, but the equations describe the distribution of energy in real landscapes, simplified though the described landscape may be.

One of the interesting characteristics of the experiments with random walks is the relatively small number of trials that need be carried out to obtain average results that closely approximate the final result from a large number of trials. This suggests, though it does not prove, that random processes operating in the landscape within certain constraints develop rather quickly

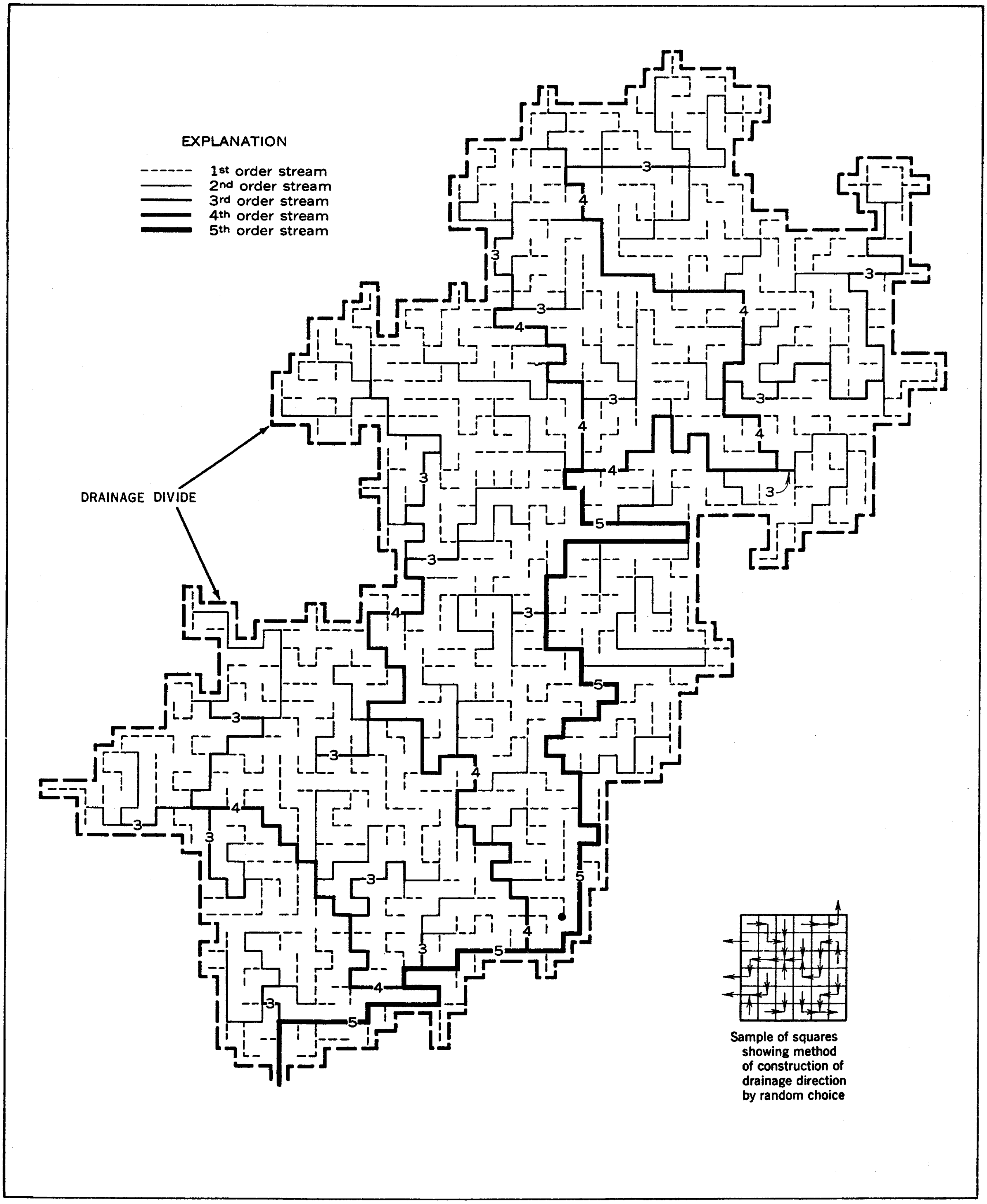

FIGURE 8.—Development of a random-walk drainage basin network.

the characteristics that obtain after a much longer period of time.

Further, the relative insensitivity of the results to the lack of exactly equal probabilities among alternatives suggests that the approach to the most probable condition is at first very rapid. In the mathematical models the final elimination of minor deviations from the condition of maximum entropy requires a very large number of samples from the population of possible alternatives. In terms of the field development of forms, it seems logical that this may be equated to a long time-period required to eliminate minor variations from the theoretic most probable state.

These observed characteristics of the averages of samples used in the mathematical models make it difficult to believe that in field conditions, time periods measured in geologic epochs could elapse before fluvial systems approach the condition of quasi-equilibrium. This line of reasoning quite fails to lend support to the Davisian concept that the stage of geomorphic youth is characterized by disequilibrium whereas the stage of maturity is characterized by the achievement through time of an equilibrium state. Rather, the reasoning seems to support the concept recently restated by Hack (1960) that no important time period is necessary to achieve a quasi-equilibrium state.

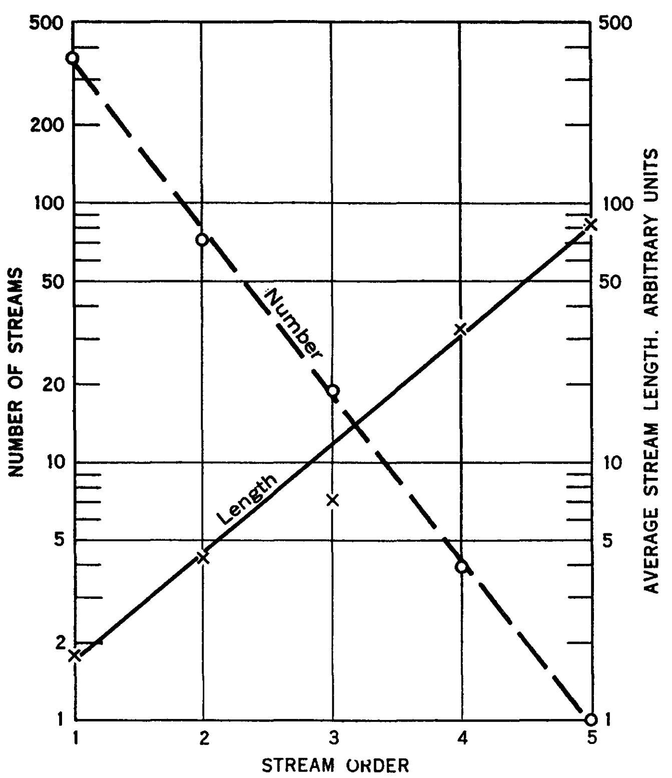

FIGURE 9.—Relation of number and average lengths of streams to stream order for model in figure 8.

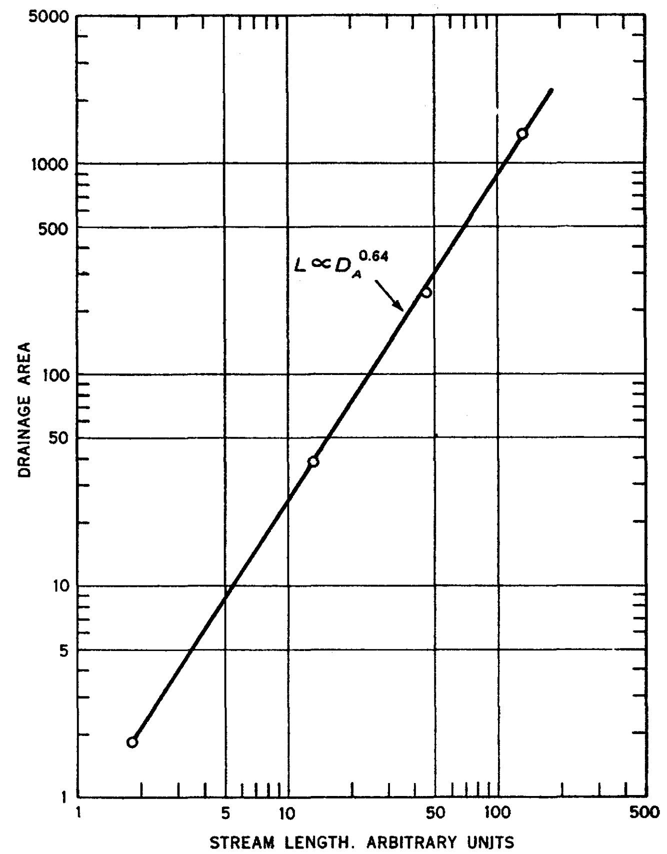

FIGURE 10.—Relation between stream length and drainage area for random-walk model.

The differing results obtained from varying the constraints imposed in the mathematical models lead us to following view as a working hypothesis: Landscape evolution is an evolution in the nature of constraints in time, maintaining meanwhile and through time essentially a dynamic equilibrium or quasi-equilibrium.

This conclusion also is in agreement with the view of Hack (1960), though arrived at by a quite different line of reasoning.

Whether or not the particular inferences stated in the present paper are sustained, we believe that the concept of entropy and the most probable state provides a basic mathematical conception which does deal with relations of time and space. Its elaboration may provide a tool by which the various philosophic premises still characterizing geomorphology may be subjected to critical test.

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