Appendix 1. Algorithms

The appendix contains mathematical pseudo code that describes algorithms referenced in the main text. This is a mix of formal mathematical notation, and notation common to many computer programming languages (for example, “+“ is addition, “*” is multiplication, “**” is exponentiation).

Algorithm 1: Pseudocode to Generate Type 1 DALEC in Spherical Coordinates

Input:

$\mathbf{r}_{\text{min}} \equiv \{\rho_{\text{min}}, \theta_{\text{min}}, \phi_{\text{min}}\} \in \mathbb{R}^3$,
$\mathbf{r}_{\text{max}} \equiv \{\rho_{\text{max}}, \theta_{\text{max}}, \phi_{\text{max}}\} \in \mathbb{R}^3$,
$J_\phi \in \mathbb{R}^1$,
$N \in \mathbb{N}^1$

Output:

$\mathbf{r}_{\text{out}} \equiv \{\boldsymbol{\rho}, \boldsymbol{\theta}, \boldsymbol{\phi}\} \in \mathbb{R}^{(2N+2)\times 3}$,
$\mathbf{J}_{\text{out}} \equiv \{\mathbf{J}_\rho, \mathbf{J}_\theta, \mathbf{J}_\phi\} \in \mathbb{R}^{(2N+2)\times 3}$,
$\mathbf{l}_{\text{out}} \equiv \{\mathbf{l}_\perp, \mathbf{l}_\parallel\} \in \mathbb{R}^{(2N+2)\times 2}$

Assumes:

- '**a**[0]' is the 1st element of array **a**;
- '**a**[1..3]' are elements 2 through 4;
- '$\{b, \ldots, c\}$' is a list of regularly spaced values between b and c;
- mathematical operators are applied element wise to arrays and lists;
- angles are in radians.

\# allocate temporary working arrays

$\mathbf{q} \in \mathbb{R}^{2N+2}$
$\mathbf{p} \in \mathbb{R}^{2N+2}$
$\boldsymbol{\alpha} \in \mathbb{R}^{2N+2}$
$\boldsymbol{\beta} \in \mathbb{R}^{2N+2}$
$\boldsymbol{\gamma} \in \mathbb{R}^{2N+2}$
$\boldsymbol{\mu} \in \mathbb{R}^{2N+2}$
$\mathbf{I}_q \in \mathbb{R}^{2N+2}$
$\mathbf{I}_p \in \mathbb{R}^{2N+2}$
$\mathbf{I}_\phi \in \mathbb{R}^{2N+2}$

\# allocate output arrays

$\boldsymbol{\rho} \in \mathbb{R}^{2N+2}$
$\boldsymbol{\theta} \in \mathbb{R}^{2N+2}$
$\boldsymbol{\phi} \in \mathbb{R}^{2N+2}$
$\mathbf{J}_{\boldsymbol{\rho}} \in \mathbb{R}^{2N+2}$
$\mathbf{J}_{\boldsymbol{\theta}} \in \mathbb{R}^{2N+2}$
$\mathbf{J}_\phi \in \mathbb{R}^{2N+2}$
$\mathbf{l}_\perp \in \mathbb{R}^{2N+2}$
$\mathbf{l}_\parallel \in \mathbb{R}^{2N+2}$

```
# locate loop elements in magnetic
# dipole coordinates

# ionospheric segment
q[0] ← cos((θ_max + θ_min) / 2) / ρ_min**2
p[0] ← ρ_min / sin((θ_max + θ_min) / 2)**2
ϕ[0] ← (ϕ_max + ϕ_min) / 2

# equatorial segment
q[1*N+1] ← 0
p[1*N+1] ← (
        ρ_min / sin((θ_max + θ_min) / 2)**2
)
ϕ[1*N+1] ← (ϕ_max + ϕ_min) / 2

# eastern field aligned current
# (centers of N evenly spaced segments)
q[0*N+1..1*N] ← (
        {q[0], …, q[0] / N} / 2 +
        { q[0]*(N−1) / N, …, 0} / 2
)
p[0*N+1..1*N] ← p[0]
ϕ[0*N+1..1*N] ← ϕ_max

# western field aligned current
# (centers of N evenly spaced segments)
q[1*N+2..2*N+1] ← (
        {0, …, q[0]*(N−1) / N} / 2 +
        { q[0] / N, …, q[0]} / 2
)
p[1*N+2..2*N+1] ← p[0]
ϕ[1*N +2..2*N+1] ← ϕ_min

# convert dipole to spherical coordinates
α ← 256/27 * q**2 * p**4
β ← (1 + sqrt(1 + α))**(2/3)
γ ← α**(1/3)
μ ← (
        ((β**2 + β * γ + γ**2) / β)**(3/2) / 2
)
ρ ← (
        4 * μ * p / (1 + μ) / (1 + sqrt(2 * μ – 1))
)
θ ← arcsin(sqrt(ρ / p))
```

calculate lengths perpendicular and
parallel to current elements

36. $dp \leftarrow ($
$\quad\quad \rho_{\min} / \sin(\theta_{\min})**2 -$
$\quad\quad \rho_{\min} / \sin(\theta_{\max})**2$
$)$

37. $d\phi \leftarrow \phi_{\max} - \phi_{\min}$

ionospheric segment

38. $\mathbf{l}_{\perp}[0] \leftarrow \rho_{\min} * (\theta_{\max} - \theta_{\min})$

39. $\mathbf{l}_{\parallel}[0] \leftarrow \rho_{\min} * \sin((\theta_{\max} + \theta_{\min}) / 2) * d\phi$

equatorial segment

40. $\mathbf{l}_{\perp}[1*N+1] \leftarrow ($
$\quad\quad dp * \sin(\boldsymbol{\theta}[1*N+1])**3 /$
$\quad\quad \mathrm{sqrt}(1 + 3 * \cos(\boldsymbol{\theta}[1*N+1])**2)$
$)$

41. $\mathbf{l}_{\parallel}[1*N+1] \leftarrow ($
$\quad\quad \boldsymbol{\rho}[1*N+1] * \sin(\boldsymbol{\theta}[1*N+1]) * d\phi$
$)$

eastern field aligned current

42. $\mathbf{l}_{\perp}[0*N+1..1*N] \leftarrow ($
$\quad\quad dp * \sin(\boldsymbol{\theta}[0*N+1..1*N])**3 /$
$\quad\quad \mathrm{sqrt}(1 + 3 * \cos(\boldsymbol{\theta}[0*N+1..1*N])**2)$
$)$

43. $dqE \leftarrow \mathbf{q}[0*N+1] - \mathbf{q}[0*N+2]$

44. $\mathbf{l}_{\parallel}[0*N+1..1*N] \leftarrow ($
$\quad\quad dqE * \boldsymbol{\rho}[0*N+1..1*N]**3 /$
$\quad\quad \mathrm{sqrt}(1 + 3 * \cos(\boldsymbol{\theta}[0*N+1..1*N])**2)$
$)$

western field aligned current

45. $\mathbf{l}_{\perp}[1*N+2..2*N+1] \leftarrow ($
$\quad\quad dp * \sin(\boldsymbol{\theta}[1*N+2..2*N+1])**3 /$
$\quad\quad \mathrm{sqrt}(1 + 3 * \cos(\boldsymbol{\theta}[1*N+2..2*N+1])**2)$
$)$

46. $dqW \leftarrow \mathbf{q}[1*N+3] - \mathbf{q}[1*N+2]$

47. $\mathbf{l}_{\parallel}[1*N+2..2*N+1] \leftarrow ($
$\quad\quad dqW * \boldsymbol{\rho}[1*N+2..2*N+1]**3 /$
$\quad\quad \mathrm{sqrt}(1 + 3 * \cos(\boldsymbol{\theta}[1*N+2..2*N+1])**2)$
$)$

generate discrete current vectors in
magnetic dipole coordinates

ionospheric current density to current
$I_\phi \leftarrow J_\phi * \rho_{\min} * d\phi$

ionospheric segment
$\mathbf{I}_q[0] \leftarrow 0$
$\mathbf{I}_p[0] \leftarrow 0$
$\mathbf{I}_\phi[0] \leftarrow I_\phi$

equatorial segment
$\mathbf{I}_q[1*N+1] \leftarrow 0$
$\mathbf{I}_p[1*N+1] \leftarrow 0$
$\mathbf{I}_\phi[1*N+1] \leftarrow -I_\phi$

eastern field aligned current
$\mathbf{I}_q[0*N+1..1*N] \leftarrow -I_\phi$
$\mathbf{I}_p[0*N+1..1*N] \leftarrow 0$
$\mathbf{I}_\phi[0*N+1..1*N] \leftarrow 0$

western field aligned current
$\mathbf{I}_q[1*N+2..2*N+1] \leftarrow I_\phi$
$\mathbf{I}_p[1*N+2..2*N+1] \leftarrow 0$
$\mathbf{I}_\phi[1*N+2..2*N+1] \leftarrow 0$

convert magnetic dipole current vectors
to spherical coordinate current densities
$\mathbf{J}_\rho \leftarrow ($
$(-2 * \cos(\boldsymbol{\theta}) * \mathbf{I}_q + \sin(\boldsymbol{\theta}) * \mathbf{I}_p) /$
$\mathrm{sqrt}(1 + 3 * \cos(\boldsymbol{\theta})**2) / \mathbf{l}_\perp$
$)$
$\mathbf{J}_\theta \leftarrow ($
$(-\sin(\boldsymbol{\theta}) * \mathbf{I}_q - 2 * \cos(\boldsymbol{\theta}) * \mathbf{I}_p) /$
$\mathrm{sqrt}(1 + 3*\cos(\boldsymbol{\theta})**2) / \mathbf{l}_\perp$
$)$
$\mathbf{J}_\phi \leftarrow \mathbf{I}_\phi / \mathbf{l}_\perp$

organize into output lists
$\mathbf{r}_{\text{out}} \leftarrow \{\boldsymbol{\rho}, \boldsymbol{\theta}, \boldsymbol{\phi}\}$
$\mathbf{J}_{\text{out}} \leftarrow \{\mathbf{J}_\rho, \mathbf{J}_\theta, \mathbf{J}_\phi\}$
$\mathbf{l}_{\text{out}} \leftarrow \{\mathbf{l}_\perp, \mathbf{l}_\parallel\}$

Algorithm 2: Pseudocode to Generate Type 2 DALEC in Spherical Coordinates

Input:

$\mathbf{r}_{\min} \equiv \{\rho_{\min}, \theta_{\min}, \phi_{\min}\} \in \mathbb{R}^3$,
$\mathbf{r}_{\max} \equiv \{\rho_{\max}, \theta_{\max}, \phi_{\max}\} \in \mathbb{R}^3$,
$J_\theta \in \mathbb{R}^1$,
$N \in \mathbb{N}^1$

Output:

$\mathbf{r}_{\text{out}} \equiv \{\boldsymbol{\rho}, \boldsymbol{\theta}, \boldsymbol{\phi}\} \in \mathbb{R}^{(2N+2)\times 3}$,
$\mathbf{J}_{\text{out}} \equiv \{\mathbf{J}_\rho, \mathbf{J}_\theta, \mathbf{J}_\phi\} \in \mathbb{R}^{(2N+2)\times 3}$,
$\mathbf{l}_{\text{out}} \equiv \{\mathbf{l}_\perp, \mathbf{l}_\parallel\} \in \mathbb{R}^{(2N+2)\times 2}$

Assumes:

- '**a**[0]' is the 1st element of array **a**;
- '**a**[1..3]' are elements 2 through 4;
- '$\{b, \ldots, c\}$' is a list of regularly spaced values between b and c;
- mathematical operators are applied element wise to arrays and lists;
- angles are in radians.

```
# allocate temporary working arrays
```
$\mathbf{q} \in \mathbb{R}^{2N+2}$
$\mathbf{p} \in \mathbb{R}^{2N+2}$
$\boldsymbol{\alpha} \in \mathbb{R}^{2N+2}$
$\boldsymbol{\beta} \in \mathbb{R}^{2N+2}$
$\boldsymbol{\gamma} \in \mathbb{R}^{2N+2}$
$\boldsymbol{\mu} \in \mathbb{R}^{2N+2}$
$\mathbf{I}_q \in \mathbb{R}^{2N+2}$
$\mathbf{I}_p \in \mathbb{R}^{2N+2}$
$\mathbf{I}_\phi \in \mathbb{R}^{2N+2}$

```
# allocate output arrays
```
$\boldsymbol{\rho} \in \mathbb{R}^{2N+2}$
$\boldsymbol{\theta} \in \mathbb{R}^{2N+2}$
$\boldsymbol{\phi} \in \mathbb{R}^{2N+2}$
$\mathbf{J}_{\boldsymbol{\rho}} \in \mathbb{R}^{2N+2}$
$\mathbf{J}_{\boldsymbol{\theta}} \in \mathbb{R}^{2N+2}$
$\mathbf{J}_\phi \in \mathbb{R}^{2N+2}$
$\mathbf{l}_\perp \in \mathbb{R}^{2N+2}$
$\mathbf{l}_\parallel \in \mathbb{R}^{2N+2}$

```
# locate loop elements in magnetic
# dipole coordinates

# ionospheric segment
q[0] ← cos((θ_max + θ_min) / 2) / ρ_min**2
p[0] ← ρ_min / sin((θ_max + θ_min) / 2)**2
ϕ[0] ← (ϕ_max + ϕ_min) / 2

# equatorial segment
q[1*N+1] ← 0
p[1*N+1] ← (
        ρ_min / sin((θ_max + θ_min) / 2)**2
)
ϕ[1*N+1] ← (ϕ_max + ϕ_min) / 2

# southern field aligned current
# (centers of N evenly spaced segments)
qS ← cos(θ_max) / ρ_min**2
q[0*N+1..1*N] ← (
        {qS, …, qS / N} /2 +
        {qS * (N−1) / N, …, 0} / 2
)
p[0*N+1..1*N] ← ρ_min / sin(θ_max)**2
ϕ[0*N+1..1*N] ← ϕ[0]

# northern field aligned current
# (centers of N evenly spaced segments)
qN ← cos(θ_min) / ρ_min**2
q[1*N+2..2*N+1] ← (
        {0, …, qN * (N−1) / N} / 2 +
        {qN / N, …, qN} / 2
)
p[1*N+2..2*N+1] ← ρ_min / sin(θ_min)**2
ϕ[1*N +2..2*N+1] ← ϕ[0]


# convert dipole to spherical coordinates
α ← 256/27 * q**2 * p**4
β ← (1 + sqrt(1 + α))**(2/3)
γ ← α**(1/3)
μ ← (
        ((β**2 + β * γ + γ**2) / β)**(3/2) / 2
)
ρ ← (
        4 * μ * p / (1 + μ) / (1 + sqrt(2 * μ – 1))
)
θ ← arcsin(sqrt(ρ / p))
```

calculate lengths perpendicular and
parallel to current elements

38. $dp \leftarrow ($
ρ_{min} / $\sin(\theta_{\mathrm{min}})$**2 –
ρ_{min} / $\sin(\theta_{\mathrm{max}})$**2
)

39. $d\phi \leftarrow \phi_{\mathrm{max}} - \phi_{\mathrm{min}}$

ionospheric segment

40. $\mathbf{l}_{\perp}[0] \leftarrow \rho_{\mathrm{min}} * \sin((\theta_{\mathrm{max}} + \theta_{\mathrm{min}}) / 2) * d\phi$

41. $\mathbf{l}_{\parallel}[0] \leftarrow \rho_{\mathrm{min}} * (\theta_{\mathrm{max}} - \theta_{\mathrm{min}})$

equatorial segment

42. $\mathbf{l}_{\perp}[1*N+1] \leftarrow ($
$\boldsymbol{\rho}[1*N+1] * \sin(\boldsymbol{\theta}[1*N+1]) * d\phi$
)

43. $\mathbf{l}_{\parallel}[1*N+1] \leftarrow ($
$dp * \sin(\boldsymbol{\theta}[1*N+1])$**3 /
sqrt(1 + 3 * $\cos(\boldsymbol{\theta}[1*N+1])$**2)
)

southern field aligned current

44. $\mathbf{l}_{\perp}[0*N+1:1*N] \leftarrow ($
$\boldsymbol{\rho}[0*N+1:1*N] * d\phi *$
$\sin(\boldsymbol{\theta}[0*N+1:1*N])$
)

45. $dqS \leftarrow \mathbf{q}[0*N+1] - \mathbf{q}[0*N+2]$

46. $\mathbf{l}_{\parallel}[0*N+1:1*N] \leftarrow ($
$dqS * \boldsymbol{\rho}[0*N+1:1*N]$**3 /
sqrt(1 + 3 * $\cos(\boldsymbol{\theta}[0*N+1:1*N])$**2)
)

northern field aligned current

47. $\mathbf{l}_{\perp}[1*N+2:2*N+1] \leftarrow ($
$\boldsymbol{\rho}[1*N+2:2*N+1] * d\phi *$
$\sin(\boldsymbol{\theta}[1*N+2:2*N+1])$
)

48. $dqN \leftarrow \mathbf{q}[1*N+3] - \mathbf{q}[1*N+2]$

49. $\mathbf{l}_{\parallel}[1*N+2:2*N+1] \leftarrow ($
$dqN * \boldsymbol{\rho}[1*N+2:2*N+1]$**3 /
sqrt(1. + 3 * $\cos(\boldsymbol{\theta}[1*N+2:2*N+1])$**2)
)

generate discrete current vectors in
magnetic dipole coordinates

ionospheric current density to current
$I_\theta \leftarrow J_\theta * \rho_{\min} * \sin((\theta_{\max} + \theta_{\min}) / 2) * d\phi$

ionospheric segment
$\mathbf{I}_q[0] \leftarrow ($
$\quad -\sin((\theta_{\max} + \theta_{\min}) / 2) * I_\theta /$
$\quad \mathrm{sqrt}(1 + 3 * \cos((\theta_{\max} + \theta_{\min}) / 2)**2)$
$)$
$\mathbf{I}_p[0] \leftarrow ($
$\quad -2 * \cos((\theta_{\max} + \theta_{\min}) / 2) * I_\theta /$
$\quad \mathrm{sqrt}(1 + 3 * \cos((\theta_{\max} + \theta_{\min}) / 2)**2)$
$)$
$\mathbf{I}_\phi[0] \leftarrow 0$

equatorial segment
$\mathbf{I}_q[1*N{+}1] \leftarrow 0$
$\mathbf{I}_p[1*N{+}1] \leftarrow I_\theta$
$\mathbf{I}_\phi[1*N{+}1] \leftarrow 0$

southern field aligned current
$\mathbf{I}_q[0*N{+}1{:}1*N] \leftarrow -I_\theta$
$\mathbf{I}_{\boldsymbol{p}}[0*N{+}1{:}1*N] \leftarrow 0$
$\mathbf{I}_\phi[0*N{+}1{:}1*N] \leftarrow 0$

northern field aligned current
$\mathbf{I}_q[1*N{+}2{:}2*N{+}1] \leftarrow I_\theta$
$\mathbf{I}_p[1*N{+}2{:}2*N{+}1] \leftarrow 0$
$\mathbf{I}_\phi[1*N{+}2{:}2*N{+}1] \leftarrow 0$

convert magnetic dipole current vectors
to spherical coordinate current densities
$\mathbf{J}_\rho \leftarrow ($
$\quad (-2 * \cos(\boldsymbol{\theta}) * \mathbf{I}_q + \sin(\boldsymbol{\theta}) * \mathbf{I}_p) /$
$\quad \mathrm{sqrt}(1 + 3 * \cos(\boldsymbol{\theta})**2) / \mathbf{l}_\perp$
$)$
$\mathbf{J}_\theta \leftarrow ($
$\quad (-\sin(\boldsymbol{\theta}) * \mathbf{I}_q - 2 * \cos(\boldsymbol{\theta}) * \mathbf{I}_p) /$
$\quad \mathrm{sqrt}(1 + 3*\cos(\boldsymbol{\theta})**2) / \mathbf{l}_\perp$
$)$
$\mathbf{J}_\phi \leftarrow \mathbf{I}_\phi / \mathbf{l}_\perp$

organize into output lists
$\mathbf{r}_{out} \leftarrow \{\boldsymbol{\rho}, \boldsymbol{\theta}, \boldsymbol{\phi}\}$
$\mathbf{J}_{out} \leftarrow \{\mathbf{J}_\rho, \mathbf{J}_\theta, \mathbf{J}_\phi\}$
$\mathbf{l}_{out} \leftarrow \{\mathbf{l}_\perp, \mathbf{l}_\parallel\}$

Algorithm 3: Pseudocode for Biot-Savart Integration in Cartesian Coordinates

Input:

$\mathbf{r}' \equiv \{\mathbf{X}_J, \mathbf{Y}_J, \mathbf{Z}_J\} \in \mathbb{R}^{N\times3}$,
$\mathbf{J} \equiv \{\mathbf{J}_X, \mathbf{J}_Y, \mathbf{J}_Z\} \in \mathbb{R}^{N\times3}$,
$\mathbf{dV} \in \mathbb{R}^{N}$,
$\mathbf{r} \equiv \{X_B, Y_B, Z_B\} \in \mathbb{R}^3$

Output:

$\mathbf{B} \equiv \{B_X, B_Y, B_Z\} \in \mathbb{R}^3$

Assumes:

- '**a**[0]' is the 1st element of array **a**;
- '**a**[1..3]' are elements 2 through 4;
- '$\{b, \ldots, c\}$' is a list of regularly spaced values between b and c;
- units are meters, amperes, and teslas.

```
    # sum cross products of current densities
    # and (r-r') location vector differences
1.  FOR i in {0, …, N−1} DO
2.      dX ← X_B − X_J[i]
3.      dY ← Y_B − Y_J[i]
4.      dZ ← Z_B − Z_J[i]
5.      dR ← (
          sqrt(dX * dX + dY * dY + dZ * dZ)
        )
6.      dR^3 ← dR * dR * dR
7.      B_X ← B_X + (
          (J_Y[i] * dZ − J_Z[i] * dY) / dR^3 *
          dV[i] * 10^−7
        )
8.      B_Y ← B_Y + (
          (J_Z[i] * dX − J_X[i] * dZ) / dR^3 *
          dV[i] * 10^−7
        )
9.      B_Z ← B_Z + (
          (J_X[i] * dY − J_Y[i] * dX) / dR^3 *
          dV[i] * 10^−7
        )
10. ENDFOR
11. B ← {B_X, B_Y, B_Z}
```