Techniques of Water-Resources Investigations of the United States Geological Survey

Chapter B3

TYPE CURVES FOR SELECTED PROBLEMS OF FLOW TO WELLS IN CONFINED AQUIFERS

By J. E. Reed

Book 3
APPLICATIONS OF HYDRAULICS
PREFACE

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TWI 3-B3. Type curves for selected problems of flow to wells in confined aquifers, by J. F. Reed.


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### SYMBOLS AND DIMENSIONS

[Numbers in parentheses indicate the solutions to which the definition applies. If no number appears, the symbol has only one definition in this report.]

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<th>Symbol</th>
<th>Dimension</th>
<th>Description</th>
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<tr>
<td>a</td>
<td>Dimensionless</td>
<td>( \sqrt{K/K_0} )</td>
</tr>
<tr>
<td>b</td>
<td>L</td>
<td>Aquifer thickness.</td>
</tr>
<tr>
<td>b'</td>
<td>L</td>
<td>Thickness of confining bed (4, 6, 7, 11); specifically the upper confining bed (5).</td>
</tr>
<tr>
<td>b&quot;</td>
<td>L</td>
<td>Thickness of lower confining bed.</td>
</tr>
<tr>
<td>d</td>
<td>L</td>
<td>Depth from top of aquifer to top of pumped well screen.</td>
</tr>
<tr>
<td>d'</td>
<td>L</td>
<td>Depth from top of aquifer to top of observation-well screen.</td>
</tr>
<tr>
<td>H</td>
<td>L</td>
<td>Change in water level in well.</td>
</tr>
<tr>
<td>H_a</td>
<td>L</td>
<td>Initial head increase in well.</td>
</tr>
<tr>
<td>h</td>
<td>L</td>
<td>Change in water level in aquifer.</td>
</tr>
<tr>
<td>K</td>
<td>LT^{-1}</td>
<td>Hydraulic conductivity of aquifer.</td>
</tr>
<tr>
<td>K_r</td>
<td>LT^{-1}</td>
<td>Hydraulic conductivity of the aquifer in the radial direction.</td>
</tr>
<tr>
<td>K_z</td>
<td>LT^{-1}</td>
<td>Hydraulic conductivity of the aquifer in the vertical direction.</td>
</tr>
<tr>
<td>K’</td>
<td>LT^{-1}</td>
<td>Hydraulic conductivity of confining bed (4, 6, 7); specifically the upper confining bed (5).</td>
</tr>
<tr>
<td>K’'</td>
<td>LT^{-1}</td>
<td>Hydraulic conductivity of lower confining bed.</td>
</tr>
<tr>
<td>I</td>
<td>L</td>
<td>Depth from top of aquifer to bottom of pumped well screen.</td>
</tr>
<tr>
<td>I’</td>
<td>L</td>
<td>Depth from top of aquifer to bottom of observation-well screen.</td>
</tr>
<tr>
<td>Q</td>
<td>L^3T^{-1}</td>
<td>Discharge rate.</td>
</tr>
<tr>
<td>Q(t)</td>
<td>L^2T^{-1}</td>
<td>Discharge rate.</td>
</tr>
<tr>
<td>r</td>
<td>L</td>
<td>Radial distance from center of pumping, flowing, or injecting well.</td>
</tr>
<tr>
<td>r_c</td>
<td>L</td>
<td>Radius of well casing or open hole in the interval where the water level changes.</td>
</tr>
<tr>
<td>r_w</td>
<td>L</td>
<td>Effective radius of well screen or open hole for pumping, flowing, or injecting well.</td>
</tr>
<tr>
<td>S</td>
<td>Dimensionless</td>
<td>Storage coefficient.</td>
</tr>
<tr>
<td>S_a</td>
<td>L^{-1}</td>
<td>Specific storage of aquifer.</td>
</tr>
<tr>
<td>S_b</td>
<td>L^{-1}</td>
<td>Specific storage of confining beds.</td>
</tr>
<tr>
<td>S_r</td>
<td>Dimensionless</td>
<td>Storage coefficient of upper confining bed.</td>
</tr>
<tr>
<td>S_b</td>
<td>Dimensionless</td>
<td>Storage coefficient of lower confining bed.</td>
</tr>
<tr>
<td>s</td>
<td>L</td>
<td>Drawdown in head (change in water level).</td>
</tr>
<tr>
<td>s_1</td>
<td>L</td>
<td>Drawdown in upper confining bed.</td>
</tr>
<tr>
<td>s_2</td>
<td>L</td>
<td>Drawdown in lower confining bed.</td>
</tr>
<tr>
<td>s_w</td>
<td>L</td>
<td>Constant drawdown in discharging well.</td>
</tr>
<tr>
<td>T</td>
<td>LT^{-1}</td>
<td>Transmissivity.</td>
</tr>
<tr>
<td>T_{x}, T_{y}, T_{y}</td>
<td>LT^{-1}</td>
<td>Components of the transmissivity tensor in any orthogonal x-, y-axis system.</td>
</tr>
<tr>
<td>T_{x}, T_{y}, T_{x}, T_{y}</td>
<td>LT^{-1}</td>
<td>Transmissivities along two principal axes, ( \epsilon ) and ( \eta ), such that ( T_{\epsilon \eta} = 0 ).</td>
</tr>
<tr>
<td>t</td>
<td>T</td>
<td>Time.</td>
</tr>
<tr>
<td>t'</td>
<td>Dimensionless</td>
<td>Variable of integration.</td>
</tr>
<tr>
<td>u</td>
<td>Dimensionless</td>
<td>( r^3S/4T(t, 2, 6) ); variable of integration (3, 7, 9).</td>
</tr>
<tr>
<td>v</td>
<td>Dimensionless</td>
<td>Variable of integration.</td>
</tr>
<tr>
<td>x</td>
<td>Dimensionless</td>
<td>Dummy variable (2, 5); variable of integration (3).</td>
</tr>
<tr>
<td>y, x</td>
<td>L</td>
<td>Distances from the pumped well for an arbitrary rectangular coordinate system (10).</td>
</tr>
<tr>
<td>y</td>
<td>Dimensionless</td>
<td>Variable of integration (1, 2, 4, 5, 6).</td>
</tr>
<tr>
<td>z</td>
<td>L</td>
<td>Depth from top of aquifer, also, specifically, the depth to bottom of a piezometer (2, 6); depth below top of upper confining bed (5).</td>
</tr>
<tr>
<td>z</td>
<td>Dimensionless</td>
<td>Dummy variable (10).</td>
</tr>
<tr>
<td>a</td>
<td>Dimensionless</td>
<td>( T(t)/Sr_w^\epsilon ).</td>
</tr>
<tr>
<td>\beta</td>
<td>Dimensionless</td>
<td>Variable of integration.</td>
</tr>
<tr>
<td>\Theta</td>
<td>Dimensionless</td>
<td>Angle between x axis and ( \epsilon ) axis.</td>
</tr>
<tr>
<td>\epsilon, \eta</td>
<td>L</td>
<td>Distances from pumped well in a coordinate system colinear with principal axes of transmissivity tensor.</td>
</tr>
<tr>
<td>\rho</td>
<td>Dimensionless</td>
<td>( r/r_w ).</td>
</tr>
<tr>
<td>\tau</td>
<td>Dimensionless</td>
<td>( T(t)/Sr_w^\epsilon ).</td>
</tr>
</tbody>
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Abstract

This report presents type curves and related material for 11 conditions of flow to wells in confined aquifers. These solutions, compiled from hydrologic literature, span an interval of time from Theis (1935) to Papadopoulos, Bredehoeft, and Cooper (1973). Solutions are presented for constant discharge, constant drawdown, and variable discharge for pumping wells that fully penetrate leaky and nonleaky aquifers. Solutions for wells that partially penetrate leaky and nonleaky aquifers are included. Also, solutions are included for the effect of finite well radius and the sudden injection of a volume of water for nonleaky aquifers. Each problem includes the partial differential equation, boundary and initial conditions, and solutions. Programs in FORTRAN for calculating additional function values are included for most of the solutions.

Introduction

The purpose of this report is to assemble, under one cover and in a standard format, the more commonly used type-curve solutions for confined ground-water flow toward a well in an infinite aquifer. Some of these solutions are only published in several different journals; some of these journals are not readily obtainable. Other solutions which are included in several references (for example, Ferris and others, 1962; Walton, 1962; Hantush, 1964a; Lohman, 1972) are included here for completeness.

The need for a compendium of type curves for aquifer-test analysis was recognized by Robert W. Stallman, who initiated the work on it. However, ill health and the press of other duties prevented him from personally carrying out his concept, but he never ceased to advocate the need for the compendium. Although it is reduced in scope from his original concept, this report should be recognized to be a result of Stallman's foresight and endeavors in the field of ground-water hydrology.

The type-curve method was devised by C. V. Theis (Wenzel, 1942, p. 88) to determine the two unknown parameters, $S$ and $T$, in the equations

$$s = \left(\frac{Q}{4\pi T}\right)W(u)$$

and

$$u = \frac{r^2S}{4Tt},$$

where $s$ is the drawdown in water level in response to the pumping rate $Q$ in an aquifer with transmissivity $T$ and storage coefficient $S$. The distance $r$ from the pumping well, and the elapsed time $t$ since pumping began, combine with $S$ and $T$ to define a dimensionless variable $u$ and corresponding dimensionless response $W(u)$. Briefly, the method consists of plotting a function curve or type curve, such as $(1/u,W(u))$ on logarithmic-scale graph paper, and plotting the time-drawdown $(t-s)$ data on a second sheet having the same scales. This is equivalent to expressing the preceding equations as

$$\log s = \log \left(\frac{Q}{4\pi T}\right) + \log W(u)$$

and

$$\log \frac{1}{u} = \log t + \log \frac{4T}{r^2S}.$$
Axes will be related by constant factors: \( s/W(u) = C_1 \) and \( t/(1/u) = C_2 \). The values of these two constants are

\[
C_1 = Q/(4\pi T)
\]

and

\[
C_2 = r^2 S/(4T).
\]

Thus, a common match point for the two curves may be chosen, and the four coordinate points—\( W(u), 1/u, s, \) and \( t \)—recorded for the common match point. \( T \) can be obtained from the equation \( T = QW(u)/(4\pi s) \), and then \( S \) can be solved from the equation \( S = 4Tut/r^2 \), where \( W(u), 1/u, s, \) and \( t \) are the match-point values.

It is apparent that the type curves, and data, can be plotted in several ways. That is, the function curve, using \( W(u) \) as an example, could be plotted as \( (u, W(u)) \) with corresponding data plots of \( (1/t, s) \) or \( (r^2/t, s) \); or could be plotted as \( (1/u, W(u)) \) with corresponding data plots of \( (t, s) \) or \( (dr^2, s) \). The type-curve method is covered more fully by Ferris, Knowles, Brown, and Stallman (1962, p. 94).

The type curves presented in this report are shown on two different plots. One plot has both logarithmic scales with 1.85 inches per log-cycle, such as K and E 467522. The other plot is arithmetic-logarithmic scale with the logarithmic scale 2 inches per log-cycle and the arithmetic scale with divisions at multiples of 0.1, 0.5, and 1.0 inches, such as K and E 466213.

Other methods exist for analysis of aquifer-test data. Among them are methods based on plots of data on semi-log paper, developed by...
Jacob (Ferris and others, 1962, p. 98) and by Hantush (1956, p. 703). These methods are useful, but they are beyond the scope of this report.

Aquifer tests deal with only one component of the natural flow system. The isolation of the effects of one stress upon the system is based upon the technique of superposition. This technique requires that the natural flow system can be approximated as a linear system, one in which total flow is the addition of the individual flow components resulting from distinct stresses.

The use of the principle of superposition is implied in most aquifer-test analyses. The term "superposition," as here applied, is derived from the theory of linear differential equations. If the partial-differential equation is linear (in the dependent variable and its derivatives), two or more solutions, each for a given set of boundary and initial conditions, can be summed algebraically to obtain a solution for the combined conditions. For instance, consider a situation (fig. 0.2) where a well has been pumping for some time at a constant rate \( Q_0 \), and the drawdown trend for that pumping rate has been established. Assume that the pumping rate increases by some amount \( \Delta Q \) at some time \( t_1 \). Then the drawdown for that step increase in rate will be the change in drawdown from that occurring due to the pumpage \( Q_0 \).

Programs, written in FORTRAN, for calculating additional function values are included for most of the solutions. Some of the type-curve solutions would require an unreasonably long tabulation to include all the possible combinations of parameters. An alternative to a tabulation is the computer program that can calculate type-curve values for the parameters desired by the user. The programs could be easily modified to calculate aquifer response to more than one well, such as well fields or image-well systems (Ferris and others, 1962, p. 144). The programs have been tested and are probably reasonably free from error. However, because of the large number of possible parameter combinations, it was possible to test only a sample of possible parameter values. Therefore, errors might occur in future use of these programs.

"An aquifer test is a controlled field experiment made to determine the hydraulic properties of water-bearing and associated rocks" (Stallman, 1971). The areal variability of hydraulic properties in an aquifer limits aquifer tests to integrating these properties within the

![Figure 0.2](image_url)

**Figure 0.2.**—The application of the principle of superposition to aquifer tests.
cone of depression produced during the test. Aquifer-test solutions are based on idealized representations of the aquifer, its boundaries, and the nature of the stress on the aquifer. The type-curve solutions presented in this report all have certain assumptions in common. The common assumptions are that the aquifer is horizontal and infinite in areal extent, that water is confined by less permeable beds above and below the aquifer, that the formation parameters are uniform in space and constant in time, that flow is laminar, and that water is released from storage instantaneously with a decline in head. Also implicit is the assumption that hydraulic potential or head is the only cause of flow in the system and that thermal, chemical, density, or other forces are not affecting flow. In addition to these common assumptions are special assumptions that characterize each solution summary. An important first step in aquifer-test analysis is deciding which simplified representations most closely match the usually complex field conditions.

Generally the best start in the analysis of aquifer-test data is with the most general set of type curves that apply to the situation, keeping in mind limitations of the method and effects that cause departures from the theoretical results. For example, the most general set of type curves for constant discharge presented in this report is for leaky aquifers with storage of water in the confining beds, solution 5. This includes, as a limiting case, the curve for a nonleaky aquifer. The most severe limitation on this set of curves is that they apply only at early times, as specified in solution 5.

Some of the effects that cause departure from the theoretical curves are partial penetration, finite well radius, and variable discharge for the pumped well. The effects of partial penetration must be considered when $r/b < 1.5$, and because vertical-horizontal anisotropy is probably a common condition, these effects should be considered for $r/b < 10$. The effect of finite well radius should be considered for early times, as specified in solution 8. The effects of variable discharge depend upon the manner of the variation. A change in discharge is more important if the change is monotonic, either continually increasing or decreasing. This fact is shown by the type curves for solution 11, where a monotonic change of 10 percent caused a significant departure from the Theis curve. If the discharge variation consists of random "noise" about a constant discharge, a 10-percent variation is not significant. The most general set of type curves for tests on flowing wells is solution 7, for leaky aquifers, which includes nonleaky aquifers as a limiting case. The only set of curves for slug tests is given in solution 9.

A recurring problem in type-curve solution for unknown hydrologic parameters is that of nonuniqueness. That is, function curves for different parameter values sometimes have similar shapes. An example of this is given by Stallman (1971, p. 19 and fig. 6). He indicated that the selection of the conceptual model is very important in interpreting the test results. Equally important is adequate testing of the conceptual model. Corroboration of the conceptual model is indicated by similar results for hydrologic parameters from data collected at varying distances from the pumped well, depths within the aquifer, and at different observation times. However, proof of suitability of the conceptual model ultimately rests on field investigations and not on curve matching.

As an example of similar curve shapes for different situations, consider the case of constant discharge in a nonleaky aquifer with exponentially varying thickness. The thickness, $b$, is equal to $b_0 \exp\left[-2(X - X_0)/a\right]$, where $b_0$ and $X_0$ are the thickness and $X$-coordinate, respectively, at the site of the discharging well and $a$ is a parameter. The drawdown for this situation is given by Hantush (1962, p. 1529):

$$s = \left(\frac{Q}{4\pi K b_0}\right) \exp\left(\frac{r/a \cos \Theta}{u}\right) W(u, r/a),$$

where

$$W(u, \beta) = \int_{-\infty}^{\infty} \exp\left(-y - \beta^2/4y\right) dy,$$

$$u = r^2 S_{\alpha}/4kt,$$

$Q$ is the discharge, $r$ is the distance from the discharging well, $\Theta$ is the angle, with apex at the discharging well, between the observation...
well and the positive X-axis, \( K \) is the hydraulic conductivity of the aquifer, and \( S_0 \) is the specific storage coefficient of the aquifer. This solution is similar to the equation describing drawdown in a leaky artesian aquifer (Hantush, 1956, p. 702), which is

\[
S = \left(\frac{Q}{4\pi T}\right) W(u, r/B),
\]

with \( T = Kb \), \( B = \sqrt{Tb'/K'} \), and \( b' \) and \( K' \) are the thickness and hydraulic conductivity, respectively, of the leaky confining bed. The other symbols are used as above.

These two functions have the same shape when plotted on logarithmic paper, and drawdown resulting from one function could be matched to a type curve of the other function. Suppose, as an example, that the "observed data" are described by the function for the aquifer with exponentially changing thickness. Suppose, also, that the hydrologist is unaware of the variation in thickness and that the family of type curves for leaky aquifers without storage in the confining beds, solution 4, has been chosen for analysis of the "observed data." Matching the data plots to the type curves and solving for unknown parameters by the methods suggested in solution 4 gives for the ratio of \( K_n \), the apparent hydraulic conductivity, to \( K \), the true hydraulic conductivity, \( K_n/K = \exp((r/a) \cos \Theta) \). The ratio would be close to one only in the vicinity of the discharging well. The diffusivity, \( K/S_0 \), would be determined correctly, but the apparent specific storage coefficient would have the same percentage error as the apparent hydraulic conductivity. Most important of all, the erroneous conclusion would be that the aquifer is leaky, with leakage parameter \( L = a \). This somewhat contrived example illustrates a principle in the interpretation of aquifer-test data. Conclusions about the hydrologic constraints on the response of the aquifer to pumping should not be based on the shape of the data curves. Inferences may be made from these curves, but they must be verified by other hydrologic and geologic data. Therefore, proof of the suitability of the conceptual model must come from field investigations.

Many of the old reports of the U.S. Geological Survey contain references to the terms "coefficient of transmissibility" and "field coefficient of permeability." These terms, which were expressed in inconsistent units of gallons and feet, have been replaced by transmissivity and hydraulic conductivity (Lohman and others, 1972, p. 4 and p. 13). Transmissivity and hydraulic conductivity are not solely properties of the porous medium; they are also determined by the kinematic viscosity of the liquid, which is a function of temperature. Field determinations of transmissivity or hydraulic conductivity are made at prevailing field temperatures, and no corrections for temperature are made.

### Summaries of Type-Curve Solutions for Confined Ground-Water Flow Toward a Well in an Infinite Aquifer

**Solution 1: Constant discharge from a fully penetrating well in a nonleaky aquifer (Theis equation)**

**Assumptions:**
1. Well discharges at a constant rate, \( Q \).
2. Well is of infinitesimal diameter and fully penetrates the aquifer.
3. Aquifer is not leaky.
4. Discharge from the well is derived exclusively from storage in the aquifer.

**Differential equation:**

\[
\partial^2 s/\partial r^2 + (1/r) (\partial s/\partial r) = (S/T)(\partial s/\partial t)
\]

**Boundary and initial conditions:**

\[
s(r,0) = 0, \ r \geq 0
\]

\[
s(\infty, t) = 0, \ t \geq 0
\]

\[
Q = \begin{cases} 
0, & t < 0 \\
\text{constant}, & t \geq 0
\end{cases}
\]

\[
\lim_{r \to 0} r \frac{\partial s}{\partial r} = -\frac{Q}{2\pi T}, \ t \geq 0
\]

Equation 1 states that initially drawdown is zero everywhere in the aquifer. Equation 2
states that the drawdown approaches zero as the distance from the well approaches infinity. Equation 3 states that the discharge from the well is constant throughout the pumping period. Equation 4 states that near the pumping well the flow toward the well is equal to its discharge.

Solution (Theis, 1935):

\[ s = \frac{Q}{4\pi T} \int_{u}^{\infty} \frac{e^{-y}}{y} dy \]

where

\[ u = \frac{r^2 S}{4Tt}, \]

and

\[ \int_{u}^{\infty} \frac{e^{-y}}{y} dy = W(u) = -0.577216 - \log u + u - \frac{u^2}{2!} + \frac{u^3}{3!} - \frac{u^4}{4!} + \ldots. \]

Comments:

Assumptions made are applicable to artesian aquifers (fig. 1.1). However, the solution may be applied to unconfined aquifers if drawdown is small compared with the saturated thickness of the aquifer and if water in the sediments through which the water table has fallen is discharged instantaneously with the fall of the water table. According to assumption 2, this solution does not consider the effect of the change in storage within the pumping well. Assumption 2 is acceptable if

\[ t > 2.5 \times 10^2 r^2 / T \]

(Papadopulos and Cooper, 1967, p. 242), where \( r_r \) is the radius of the well casing in the interval over which the water-level declines, and other symbols are as defined previously. Figure 1.2 on plate 1 is a logarithmic graph of \( W(u) = 4\pi sT/Q \) plotted on the vertical coordinates versus \( 1/u = 4Tt/(r^2 S) \) plotted on the horizontal coordinates. The test data should be plotted with \( s \) on the vertical coordinates and corresponding values of \( t \) or \( t/r^2 \) on the horizontal coordinates.

Values of \( W(u) \) for \( u \) between 0 and 170 may be computed by using subroutine EXPI of the IBM System/360 Scientific Subroutine Package. Table 1.1 gives values of \( W(u) \) for selected values of \( 1/u \) between \( 1 \times 10^{-1} \) and \( 9 \times 10^{14} \), as calculated by this subroutine.

![Figure 1.1.—Cross section through a discharging well in a nonleaky aquifer.](image-url)
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<tr>
<td>$T_u \times 10^{-6}$</td>
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<td>1.5</td>
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<td>4.5</td>
<td>5.0</td>
<td>5.5</td>
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<td>9.0</td>
</tr>
</tbody>
</table>
| Value shown as $0.00000$ in column but less than $0.000000$.
Solution 2: Constant discharge from a partially penetrating well in a nonleaky aquifer

Assumptions:
1. Well discharges at a constant rate, Q.
2. Well is of infinitesimal diameter and is screened in only part of the aquifer.
3. Aquifer has radial-vertical anisotropy.
4. Aquifer is not leaky.
5. Discharge from the well is derived exclusively from storage in the aquifer.

Differential equation:
\[
\frac{\partial^2 s}{\partial r^2} + \frac{1}{r} \frac{\partial s}{\partial r} + \frac{\partial^2 s}{\partial z^2} = \frac{S}{T} \frac{\partial s}{\partial t}
\]
\[a^2 = K_v/K_r\]

This is the differential equation for nonsteady radial and vertical flow in a homogeneous confined aquifer with radial-vertical anisotropy.

Boundary and initial conditions:
\[s(r, z, 0) = 0, r > 0, 0 \leq z \leq b \quad (1)\]
\[s(\infty, z, t) = 0, t \geq 0 \quad (2)\]
\[\frac{\partial s(r, 0, t)}{\partial z} = 0, r \geq 0, t \geq 0 \quad (3)\]
\[\frac{\partial s(r, b, t)}{\partial z} = 0, r \geq 0, t \geq 0 \quad (4)\]
\[\lim_{r \to 0} \frac{\partial s}{\partial r} = \begin{cases} 0; & 0 < z < d \\ -\frac{Q}{(2\pi K_r (l - d))}, & d < z < l \\ 0, & l < z < b \end{cases} \quad (5)\]

Equation 1 states that initially the drawdown is zero everywhere in the aquifer. Equation 2 states that the drawdown approaches zero as the distance from the pumped well approaches infinity. Equations 3 and 4 state that there is no vertical flow at the upper and lower boundaries of the aquifer. This means that vertical head gradients in the aquifer are caused by the geometric placement of the pumping well screen, and not by leakage. Equation 5 states that near the pumping well the flow is radial, that the flow toward the well is equal to its discharge, that the discharge is distributed uniformly over the well screen, and that no radial flow occurs above and below the screen.

Solution:
1. For the drawdown in a piezometer, a solution by Hantush (1961a, p. 85, and 1964a, p. 353) is given by
\[
s = \frac{Q}{4\pi T} \left[ W(u) + f(u, \frac{ar}{b}, \frac{l}{b}, \frac{d}{b}, \frac{z}{b}) \right] \quad (6)
\]
where
\[
W(u) = \int_u^\infty \frac{e^{-u}}{y} dy
\]
and
\[
f(u, \frac{ar}{b}, \frac{l}{b}, \frac{d}{b}, \frac{z}{b}) = \frac{2b}{\pi(l-d)} \sum_{n=1}^{\infty} \frac{1}{n} \left( \sin \frac{n\pi l}{b} - \sin \frac{n\pi d}{b} \right) \cos \frac{n\pi z}{b} W \left( u, \frac{n\pi r}{b} \right) \quad (7)
\]
\[
W(u, x) = \int_u^\infty (\exp(-y-x^2/4y)y) dy
\]
\[
u = \frac{r^2S}{4Tl}
\]
\[a = \sqrt{K_v/K_r}\]

An alternate form of this solution for \(a=1\) is given by Hantush (1961a, p. 85):
\[
s = \frac{Qb}{8\pi T (l-d)} \left[ M(u, \frac{l+z}{r}) + M(u, \frac{l-z}{r}) \right. \\
\left. \quad + f'(u, \frac{b}{r}, \frac{l}{r}, \frac{d}{r}, \frac{z}{r}) + M(u, \frac{d+z}{r}) - M(u, \frac{d-z}{r}) \right. \\
\left. \quad - f'(u, \frac{b}{r}, \frac{d}{r}, \frac{z}{r}) \right] \quad (8)
\]
in which
\[
f'(u, \frac{b}{r}, \frac{x}{r}, \frac{z}{r}) = \sum_{n=1}^{\infty} \left[ M(u, \frac{2nb+x+z}{r}) - M(u, \frac{2nb+x-z}{r}) \right. \\
\left. \quad - M(u, \frac{2nb-x+z}{r}) + M(u, \frac{2nb-x-z}{r}) \right] \quad (9)
\]
and
\[
M(u, \beta) = \int_{\frac{u}{\beta}}^{\infty} \frac{e^{-y}}{\sqrt{\pi}} \text{erf}(\beta \sqrt{y}) \, dy
\]
\[
\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-t^2} \, dt.
\]

II. For the drawdown in an observation well (Hantush, 1961a, p. 90, and 1964a, p. 353),
\[
s = \frac{Q}{4\pi T} \left[ W(u) + \tilde{f} \left( \frac{ar}{b}, \frac{l}{b}, \frac{d}{b}, \frac{l'}{b}, \frac{d'}{b} \right) \right],
\]
where \( W(u) \) is as defined previously and
\[
\tilde{f} \left( \frac{ar}{b}, \frac{l}{b}, \frac{d}{b}, \frac{l'}{b}, \frac{d'}{b} \right) = \frac{2b^2}{\pi^2 (l-d)(l'-d')} \sum_{n=1}^{\infty} \frac{1}{n^2} \left( \sin \frac{n\pi l}{b} - \sin \frac{n\pi d}{b} \right) \left( \sin \frac{n\pi l'}{b} - \sin \frac{n\pi d'}{b} \right) W(u, \frac{n\pi ar}{b}),
\]
where \( W(u, x) \) and \( u \) are as defined previously.

Comments:
Assumptions apply to conditions shown in figure 2.1. The effects of partial penetration need to be considered for \( ar/b < 1.5 \). There must be a type curve for each value of \( ar/b, d/l/b, l/l' \), and either \( z/l' \) for piezometer, or \( l'/l \) and \( d'/l' \) for observation wells. Because the number of possible type curves is large, only samples of curves for selected values of the parameters are shown in figure 2.2 on plate 1.

For large values of time, that is, for \( t > b^2 S/(2\alpha^2 T) \) or \( t > bS/(2K_z) \), the effects of partial penetration are constant in time, and
\[
W \left( \frac{u}{b}, \frac{n\pi ar}{b} \right)
\]
can be approximated by
\[
2K_0 \left( \frac{n\pi ar}{b} \right)
\]
(Hantush, 1961a, p. 92). \( K_0(x) \) is the modified Bessel function of the second kind of order zero.

Equation 6 then becomes
\[
s = \frac{Q}{4\pi T} W(u) + \partial s = \frac{Q}{4\pi T} \left[ W(u) + f_s \right],
\]
where \( \delta s = \frac{Q}{4\pi T} f_s \),

and \( f_s \) is given in equation 7

with \( W \left( u, \frac{n \pi ar}{b} \right) \) replaced by \( 2K_n \left( \frac{n \pi ar}{b} \right) \).

Figure 2.3 shows plots of \( f_s \) as tabulated by Weeks (1969, p. 202–207). In using these curves, it should be noted that \( f_s \) for a given \( r, b, \) and \( z_2 = b - z_1, \) \( l_2 = b - d_1, \) and \( d_2 = b - l_1. \) Figure 2.3 can be used to find \( f_s \) by interpolation and then constructing type curves of \( W(u) + f_s \) in the manner described by Weeks (1964, p. D195).

For small values of time

\[
t < \frac{(2b-l-z)^2 S}{20T}
\]

(Hantush, 1961b, p. 172), equation 8 can be approximated by

\[
s = \frac{Qb}{8\pi T(l-d)} \left[ M \left( u, \frac{l+z}{r} \right) - M \left( u, \frac{d+z}{r} \right) \right.
+ M \left( u, \frac{l-z}{r} \right) - M \left( u, \frac{d-z}{r} \right) \right].
\]

\[
\delta T = \frac{Q}{4\pi T} \times 4\pi
\]

Figure 2.3.—The drawdown correction factor \( f_s \) versus \( ar/b, \) from tables of Weeks (1969).
An extensive table of $M(u, \beta)$ has been prepared by Hantush (1961c).

Although $r/b$ for a given observation well probably would be known, however, the conductivity ratio $a^2$ would not be. Thus, it would not be known which $ar/b$ curve should be matched. In other words, not only $T$ and $S$, but also the conductivity ratio $a^2$ must be determined. A criterion for determining the match between data curves and type curves is that the values of $ar/b$ for different observation wells should all indicate the same "$a". Plotting the drawdown data for several observation wells on a single $t/r^2$ plot and matching to sets of type curves, a different set for each "$a",

Figure 2.2 was prepared from data calculated by the FORTRAN program listed in table 2.1. This program computes "$s" from either equation 6 or 10, depending on the input data. The input data consist of cards containing the parameters coded in specific formats. Readers unfamiliar with FORTRAN format items should consult a FORTRAN language manual. The first card contains: the aquifer thickness ($b$), coded in columns 1-5, in format F5.1; the depth to bottom of pumped well screen ($l$), coded in columns 6-10, in format F5.1; the
depth to top of pumped well screen \( (d) \), coded in columns 11–15, in format F5.1; the number of observation wells and (or) piezometers, coded in columns 16–20, in format I5; the smallest value of \( 1/u \) for which computation is desired, coded in columns 21–30, in format E10.4; the largest value of \( 1/u \) for which computation is desired, coded in columns 31–40, in format E10.4. The ratio of the largest \( 1/u \) value to the smallest \( 1/u \) value should be less than \( 10^{12} \). Following this card is a group of cards containing one card for each observation well or piezometer. These cards are coded for an observation well as: distance from pumped well multiplied by the square root of the ratio of the vertical to horizontal conductivity \( (r\sqrt{K_v/K_h}) \), in columns 1–5, in format F5.1; depth to bottom of observation well screen \( (l') \), coded in columns 6–10, in format F5.1; depth to top of observation well screen \( (d') \), coded in columns 11–15, in format F5.1. A card would be coded for a piezometer as follows: distance from pumped well multiplied by the square root of the ratio of the vertical to horizontal conductivity \( (r\sqrt{K_v/K_h}) \), in columns 1–5, in format F5.1; and total depth of piezometer \( (z) \), in columns 11–15, in format F5.1. The output from this program is tables of computed function values.
an example of which is shown in figure 2.4. Subroutines DQL12, BESK, and EXPI are from the IBM Scientific Subroutine Package and a discussion of them is in the IBM SSP manual.

Solution 3: Constant drawdown in a well in a nonleaky aquifer

Assumptions:
1. Water level in well is changed instantaneously by $s_w$ at $t = 0$.
2. Well is of finite diameter and fully penetrates the aquifer.
3. Aquifer is not leaky.
4. Discharge from the well is derived exclusively from storage in the aquifer.

Differential equation:
\[
\frac{\partial^2 s}{\partial r^2} + \frac{1}{r} \frac{\partial s}{\partial r} = \frac{S}{T} \frac{\partial s}{\partial t}
\]

This is the differential equation describing nonsteady radial flow in a homogeneous isotropic confined aquifer.

Boundary and initial conditions:
\[
s(r, 0) = 0, \quad r \geq r_w
\]

Figure 2.3.—Continued.
Equation 1 states that initially the drawdown is zero everywhere in the aquifer. Equation 2 states that, as the well is approached, drawdown in the aquifer approaches the constant drawdown in the well, implying no entrance loss to the well. Equation 3 states that the drawdown approaches zero as the distance from the well approaches infinity.

**Solutions:**

I. For the well discharge (Jacob and Lohman, 1952, p. 560):

\[ Q = 2\pi T s_w G(\alpha), \]

where

\[ G(\alpha) = \frac{4\alpha}{\pi} \int_0^\infty xe^{-\alpha x} \left\{ \frac{\pi}{2} + \tan^{-1} \left[ \frac{Y_0(x)}{J_0(x)} \right] \right\} dx \]

and

\[ \alpha = \frac{Tt}{Sr^2}. \]

II. For the drawdown in water level (Hantush, 1964a, p. 343):
s = s_r A(r, ρ),

where \( A(r, ρ) = 1 \)

\[
2 \int_0^\infty \frac{J_0(ρu) - Y_0(ρu)}{J_0^2(u) + Y_0^2(u)} \exp(-τu^2) \frac{du}{u},
\]

and \( τ = \frac{t}{S r_r^2} \)

\( ρ = \frac{r}{r_r} \).

Comments:

Boundary condition 2 requires a constant drawdown in the discharging well, a condition most commonly fulfilled by a flowing well, although figure 3.1 shows the water level to be below land surface.

Figure 3.2 on plate 1 is a plot from Lohman (1972, p. 24) of dimensionless discharge \( (G(α)) \) versus dimensionless time \( (τ) \). Additional values in the range \( α \) greater than \( 1 \times 10^3 \) were calculated from \( G(α) = 2 \log(2.245α) \) (Hantush, 1964a, p. 312). Function values for \( G(α) \) are given in table 3.1. The data curve consists of measured well discharge versus time. After the data and type curves are matched, transmissivity can be calculated from \( T = Q/2πs_r G(α) \), and the storage coefficient can be
calculated from \( S = T t / (\alpha r_c^2) \), where \((\alpha, G(\alpha))\) and \((t, Q)\) are matching points on the type curve and data curve, respectively.

Similarly, data curves of drawdown versus time may be matched to figure 3.3 on plate 1; this is a plot of dimensionless drawdown \( A(\tau, \rho) = s / s_{\alpha} \) versus dimensionless time \( \tau / \rho^2 = T t / S r_c^2 \). After the data and type curves are matched, the hydraulic diffusivity of the aquifer can be calculated from the equality \( T / S = (\tau / \rho^2) (r_c^2 / t) \). Usually \( s_{\alpha} \) is known, and some of the uncertainty of curve matching can be eliminated by plotting \( s / s_{\alpha} \) versus \( t \) because only horizontal translation is then required. If \( r_c \) is also known, the particular curve to be matched can be determined from the relation \( \rho = r / r_c \). Generally, however, the effective radius, \( r_c \), differs from the actual radius and is not known. The effective radius can often be estimated from a knowledge of the construction of the well and the water-bearing material, or it can be determined from step-drawdown tests (Rorabaugh, 1953). Figure 3.3 was plotted from table 3.2. For \( \tau \leq 1 \times 10^9 \), the data are from Hantush (1964a, p. 310). For \( \tau > 1 \times 10^9 \), values of drawdown in a leaky aquifer, as \( r_c / B \to 0 \), were used. (See solution 7.) Where 0.000 occurs in table 3.2, \( A(\tau, \rho) \) is less than 0.0005.
Figure 2.4—Example of output from program for partial penetration in a nonleaky artesian aquifer.
Solution 4: Constant discharge from a fully penetrating well in a leaky aquifer

Assumptions:

1. Well discharges at a constant rate, Q.
2. Well is of infinitesimal diameter and fully penetrates the aquifer.
3. Aquifer is overlain, or underlain, everywhere by a confining bed having uniform hydraulic conductivity \( K' \) and thickness \( b' \).
4. Confining bed is overlain, or underlain, by an infinite constant-head plane source.
5. Hydraulic gradient across confining bed changes instantaneously with a change in head in the aquifer (no release of water from storage in the confining bed).
6. Flow in the aquifer is two-dimensional and radial in the horizontal plane and flow in the confining bed is vertical. This assumption is approximated closely where the hydraulic conductivity of the aquifer is sufficiently greater than that of the confining bed.

Differential equation:

\[
\frac{\partial^2 s}{\partial r^2} + \frac{1}{r} \frac{\partial s}{\partial r} - \frac{sK'}{Tb'} = \frac{S}{T} \frac{\partial s}{\partial t}
\]

This is the differential equation describing nonsteady radial flow in a homogeneous isotropic aquifer with leakage proportional to drawdown.

Boundary and initial conditions:

\[
s(x,t) = 0, \ t > 0 \quad (1)
\]

\[
Q = \begin{cases} 
0, & t < 0 \\
\text{constant} > 0, & t \geq 0
\end{cases} \quad (2)
\]

\[
\lim_{r \to 0} r \frac{\partial s}{\partial r} = -\frac{Q}{2\pi T} \quad (3)
\]

Equation 1 states that the initial drawdown is zero. Equation 2 states that drawdown is small at a large distance from the pumping well. Equation 3 states that the discharge from the well is constant and begins at \( t = 0 \). Equation 4 states that near the pumping well the flow toward the well is equal to its discharge.
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**Table 3.2.** Values of $A(t, \rho)$

*Values of $A(t, \rho)$ for $t \times 10^6$ modified from Hantush (1964a, p. 310)*
Solution (Hantush and Jacob, 1955, p. 98):

\[ s = \frac{Q}{4\pi T} \int_{u}^{\infty} \frac{e^{-\frac{r^2}{4u}}}{z} \, dz \]  

(5)

where \( u = \frac{r^2S}{4Tt} \)

\[ B = \sqrt{\frac{Tb'}{K'}} \]  

(6)

Comments:

As pointed out by Hantush and Jacob (1954, p. 917), leakage is three-dimensional, but if the difference in hydraulic conductivities of the aquifer and confining bed are sufficiently great, the flow may be assumed to be vertical in the confining bed and radial in the aquifer. This relationship has been quantified by Hantush (1967, p. 587) in the condition \( b/B < 0.1 \). In terms of relative conductivities, this would be \( K/K' > 100 b/b' \). Assumption 5, that there is no change in storage of water in the confining bed, was investigated by Neuman and Witherspoon (1969b, p. 821). They concluded that this assumption would not affect the solution if

\[ \beta < 0.01, \text{ where } \beta = \frac{r}{4b} \sqrt{\frac{K'S}{KS_s}}. \]

Assumption 4, that there is no drawdown in water level in the source bed lying above the confining bed, was also examined by Neuman and Witherspoon (1969a, p. 810). They indicated that drawdown in the source bed would have negligible effect on drawdown in the pumped aquifer for short times, that is, when

\[ \frac{Tt}{r^2S} < 1.6 \frac{b'}{(r/B)}. \]

Also, they indicated (1969a, p. 811) that neglect of drawdown in the source bed is justified if \( T_s > 100T \), where \( T_s \) represents the transmissivity of the source bed. Figure 4.1, a cross section through the discharging well, shows geometric relationships. Figure 4.2 on plate 1 shows plots of dimensionless drawdown compared to dimensionless time, using the notation of Cooper (1963) from Lohman (1972, pl. 3). Cooper expressed equations 5 and 6 as

\[ L(u,v) = \int_{u}^{\infty} \frac{e^{-\frac{r^2}{4u}}}{y} \, dy, \]  

(7)
with

$$v = \frac{r}{2} \sqrt{\frac{K'}{Tb'}}. \quad (8)$$

Cooper's type curves and equation 5 express the same function with $r/B = 2v$. Hantush (1961a) has a tabulation of equation 5, parts of which are included in table 4.1.

The observed data may be plotted in two ways (Cooper, 1963, p. C51). The measured drawdown in any one well is plotted versus $t/r^2$; the data are then matched to the solid-line type curves of figure 4.2. The data points are aligned with the solid-line type curves either on one of them or between two of them. The parameters are then computed from the coordinates of the match points $(s/t/r^2)$ and $(L(u,v)/u, v/u)$, and an interpolated value of $v$ from the equations

$$K'/b' = 4T \frac{v^2/\mu}{r^2}.$$  

Drawdown measured at the same time but in different observation wells at different distances can be plotted versus $t/r^2$ and matched to the dashed-line type curves of figure 4.2. The data are matched so as to align with the dashed-line curves, either on one or between two of them. From the match-point coordinates $(s/t/r^2)$ and $(L(u,v)/u, v/u)$ and an interpolated value of $v^2/u$, $T$ and $S$ are computed from equations 9 and 10 and the remaining parameter from

$$K'/b' = S \frac{v^2/\mu}{t}.$$  

The region $v^2/u \geq 8$ and $L(u,v) > 10^{-2}$ corresponds to steady-state conditions.

**Table 4.1.**—Selected values of $W(u,v/B)$

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<td>0.0001</td>
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<td>0.0001</td>
<td>0.0001</td>
<td>0.0001</td>
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</tr>
</tbody>
</table>
The drawdown in the steady-state region is given by the equation (Jacob, 1946, eq. 15)

\[ s = \frac{Q}{2\pi T} K_0(x), \]

where \( K_0(x) \) is the zero-order modified Bessel function of the second kind and

\[ x = r \sqrt{\frac{K'_T}{Tb}}. \]

Data for steady-state conditions can be analyzed using figure 4.3 on plate 1. The drawdowns are plotted versus \( r \) and matched to figure 4.3. After choosing a convenient match point with coordinates \((s, r)\) and \((K_0(x), x)\) the parameters are computed from the equations

\[ 2s = \frac{Q}{2\pi T} K_0(x) \]

\[ \frac{T}{b} = \frac{x}{r}. \]

Values of \( K_0(x) \) from Hantush (1956) are given in table 4.2.

A FORTRAN program for generating type-curve function values of equation 7 is listed in table 4.3. Using the notation \( L(u, v) \) of Cooper (1963), the function is evaluated as follows. For \( u \geq 1 \),

\[ L(u, v) = \int_u^\infty \left( \frac{1}{y} \right) \exp \left( -y - v^2/y \right) dy = \int_u^\infty f(y) dy. \]

This integral is transformed into the form

\[ \int_0^\infty e^{-r} \left[ \exp \left( -u - \frac{v^2}{x + u} \right) \frac{1}{x + u} \right] dx \]

evaluated by a Gaussian-Laguerre quadrature formula. For \( v^2 < u < 1 \),

\[ L(u, v) = \int_1^\infty f(y) dy + \int_0^1 f(y) dy. \]

The first integral is evaluated by a Gaussian-Laguerre quadrature formula, as previously described. The second integral is evaluated using a series expansion, as

\[ \int_0^1 f(y) dy = s (1) - s(u), \]

where

\[ s = \log u \left[ \sum_{n=0}^{\infty} \frac{(v^2)^n}{(n!)^2} \right] \]

\[ + \sum_{m=1}^{\infty} \left[ (-1)^m \left( \frac{u^m - (v^2)^m}{u} \right) \sum_{n=0}^{\infty} \frac{(v^2)^n}{(m+n)!n!} \right]. \]

For \( u < 1 \) and \( u \leq v^2 \),

\[ L(u, v) = 2K_0(2v) - \int_{u/4}^\infty f(y) dy \]

(Cooper, 1963, p. C50),

where \( K_0 \) is the zero-order modified Bessel function of the second kind. The integral in the above expression is evaluated by the Gaussian-Laguerre procedure, as described previously.

Input data for this program consist of three cards with the numeric data coded by specific FORTRAN formats. Readers unfamiliar with FORTRAN format items should consult a FORTRAN language manual. The first card contains: the smallest value of \( 1/u \) for which computation is desired, coded in columns 1–10 in format E10.5; the largest value of \( 1/u \) for which computation is desired, coded in columns 11–20 in format E10.5. The table will include a range of \( 1/u \) values spanning these two coded values if the span is less than or equal to 12 log cycles. The next two cards contain 12 values of \( r/B \), all coded in format E10.5, in columns 1–10, 11–20, 21–30, 31–40, 41–50, 51–60, 61–70, and 71–80 of the first card and columns 1–10, 11–20, 21–30, and 31–40 of the second card. Zero (or blank) coding is permissible in this field, but computation will terminate with the first zero (or blank) value encountered. An example of the output from this program is shown in figure 4.4.

### Table 4.2.—Selected values of \( K_0(x) \)

[From Hantush (1956, p. 704)]

<table>
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<th>( x=NX10^{-2} )</th>
<th>( x=NX )</th>
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<td>-----</td>
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<td>0.30E-05</td>
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<tr>
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</tr>
<tr>
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</table>

Figure 4.4.—Example of output from program for computing drawdown due to constant discharge from a well in a leaky artesian aquifer.
Solution 5: Constant discharge from a well in a leaky aquifer with storage of water in the confining beds

Assumptions:
1. Well discharges at a constant rate, Q.
2. Well is of infinitesimal diameter and fully penetrates the aquifer.
3. Aquifer is overlain and underlain everywhere by confining beds having hydraulic conductivities $K'$ and $K''$, thicknesses $b'$ and $b''$, and storage coefficients $S'$ and $S''$, respectively, which are constant in space and time.
4. Flow in the aquifer is two-dimensional and radial in the horizontal plane and flow in confining beds is vertical. This assumption is approximated closely where the hydraulic conductivity of the aquifer is sufficiently greater than that of the confining beds.
5. Conditions at the far surfaces of the confining beds are (fig. 5.1):
   - Case 1. Constant-head plane sources above and below.
   - Case 2. Impermeable beds above and below.
   - Case 3. Constant-head plane source above and impermeable bed below.

Differential equations:
For the upper confining bed
\[
\frac{\partial^2 s_1}{\partial z^2} = \frac{S'}{K'b'} \frac{\partial s_1}{\partial t} 
\]  
(1)

For the aquifer
\[
\frac{\partial^2 s}{\partial r^2} + \frac{1}{r} \frac{\partial s}{\partial r} + \frac{K'}{T} \frac{\partial}{\partial z} \left( \frac{\partial s}{\partial z} \right) 
= \frac{K'}{T} \frac{\partial}{\partial z} \left( \frac{\partial s}{\partial z} \right) = \frac{S}{T} \frac{\partial s}{\partial t}
\]
(2)

For the lower confining bed
\[
\frac{\partial^2 s_2}{\partial z^2} = \frac{S''}{K''b''} \frac{\partial s_2}{\partial t}
\]
(3)

Equations 1 and 3 are, respectively, the differential equations for nonsteady vertical flow in the upper and lower semipervious beds. Equation 2 is the differential equation for nonsteady two-dimensional radial flow in an aquifer with leakage at its upper and lower boundaries.

Boundary and initial conditions:
Case 1: For the upper confining bed
\[
s_1(r,z,0)=0 
\]
(4)
\[
s_1(r,0,t)=0 
\]
(5)
\[
s_1(r,b',t)=s(r,t) 
\]
(6)

For the aquifer
\[
s(r,0)=0 
\]
(7)
\[
s(\infty,t)=0 
\]
(8)
\[
\lim_{r \to 0} \frac{\partial s(r,t)}{\partial r} = -\frac{Q}{2\pi T} 
\]
(9)

For the lower confining bed
\[
s_2(r,z,0)=0 
\]
(10)
\[
s_2(r,b'+b+b'',t)=0 
\]
(11)
\[
s_2(r,b'+b,t)=s(r,t) 
\]
(12)

Case 2: Same as case 1, with conditions 5 and 11 being replaced, respectively, by
\[
\frac{\partial s_1(r,0,t)}{\partial z} = 0 
\]
(13)
\[
\frac{\partial s_2(r,b'+b+b'',t)}{\partial z} = 0 
\]
(14)

Case 3: Same as case 1, with condition 11 being replaced by condition 14.

Equations 4, 7, and 10 state that initially the drawdown is zero in the aquifer and within each confining bed. Equation 5 states that a plane of zero drawdown occurs at the top of the upper confining bed. Equations 6 and 12 state that, at the upper and lower boundaries of the aquifer, drawdown in the aquifer is equal to drawdown in the confining beds. Equation 8 states that drawdown is small at a large distance from the pumping well. Equation 9 states that, near the pumping well, the flow is equal to the discharge rate. Equation 11 states that a plane of zero drawdown is at the base of the lower confining bed. Equation 13 states that
there is no flow across the top of the upper confining bed. Equation 14 states that no flow occurs across the base of the lower confining bed.

Solutions (Hantush, 1960, p. 3716):
I. For small values of time \((t < \text{both } \frac{b'S'}{10K'} \text{ and } \frac{b''S''}{10K''})\):

\[
s = \frac{Q}{4\pi T} H(u, \beta),
\]

where
\[
u = \frac{r^2 S}{4Tt}\]

and
\[
\beta = \frac{r}{4} \left( \sqrt{\frac{K'S'}{b'TS}} + \sqrt{\frac{K''S''}{b''TS}} \right)
\]

\[
H(u, \beta) = \int_{u}^{\infty} \frac{e^{-y}}{y} \text{erfc} \left( \frac{\beta \sqrt{u}}{\sqrt{y(y-u)}} \right) dy
\]

\[
erfc(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-y^2} dy.
\]

II. For large values of time:

A. Case 1, \(t > \text{both } \frac{b'S'}{10K'} \text{ and } \frac{b''S''}{10K''}\)

\[
s = \frac{Q}{4\pi T} W(u \delta_1, \alpha),
\]

where \(u\) is as defined previously and
\[
\delta_1 = 1 + \frac{S' + S''}{3S},
\]
\[
\alpha = r \left( \frac{K'S'}{T} + \frac{K''S''}{T} \right)
\]

\[
W(u, x) = \int_{u}^{\infty} \frac{\exp \left( -y - x^2/4y \right)}{y} dy.
\]

B. Case 2, \(t > \text{both } \frac{b'S'}{10K'} \text{ and } \frac{b''S''}{10K''}\)

\[
s = \frac{Q}{4\pi T} W(u \delta_2),
\]

where
\[
\delta_2 = 1 + \frac{S' + S''}{S},
\]

\[
W(u) = \int_{u}^{\infty} \frac{e^{-y}}{y} dy.
\]

C. Case 3, \(t > \text{both } \frac{5b'S'}{K'} \text{ and } \frac{10b''S''}{K''}\)

\[
s = \frac{Q}{4\pi T} W(u \delta_3, \sqrt{\frac{K'S'}{b'} \frac{K''S''}{b''}}),
\]

where
\[
\delta_3 = 1 + \frac{(S'' + S')}{S}
\]

and \(W(u, x)\) is as defined in case 1.

Comments:
A cross section through the discharging well is shown in figure 5.1. The flow system is actually three-dimensional in such a geometric configuration. However, as stated by Hantush (1960, p. 3713), if the hydraulic conductivity in the aquifer is sufficiently greater than the hydraulic conductivity of the confining beds, flow will be approximately radial in the aquifer and approximately vertical in the confining beds. A complete solution to this flow problem has not been published. Neuman and Witherspoon (1971, p. 250, eq. II-161) developed a complete solution for case 1 but did not tabulate it. Hantush's solutions, which have been tabulated, are solutions that are applicable for small and large values of time but not for intermediate times.

The "early" data (data collected for small values of \(t\)) can be analyzed using equation 15. Figure 5.2 on plate 1 shows plots of \(H(u, \beta)\) from Lohman (1972, pl. 4). Hantush (1961a) has an extensive tabulation of \(H(u, \beta)\), a part of which is given in table 5.1. The corresponding data curves would consist of observed drawdown versus \(t/r^2\). Superposing the data curves on the type curves and matching the two, with graph axes parallel, so that the data curves lie on or between members of the type-curve family and choosing a convenient match point \((H(u, \beta), 1/u)\), \(T\) and \(S\) are computed by

\[
T = \frac{Q}{4\pi S} H(u, \beta),
\]

\[
S = 4T \frac{t}{r^2} \frac{1}{u}.
\]

If simplifying conditions are applicable, it is possible to compute the product \(K'S'\) from the \(\beta\) value. If \(K''S''=0\), \(K'S'=16\beta^2 b'TS/r^2\), and if \(K''S''=K'S'\),
The curves in figure 5.2 are very similar from $\beta=0$ to about $\beta=0.5$. Therefore, the $\beta$ values in this range are indeterminate. There is also uncertainty in curve matching for all $\beta$ values because of the fact that it is a family of curves whose shapes change gradually with $\beta$. This uncertainty will be increased if the data covers a small range of $t$ values. The problem
can be avoided, if data from more than one observation well are available, by preparing a composite data plot of $s$ versus $t/r^2$. This data plot would be matched by adding the constraint that the $r$ values for the different data curves representing each well fall on proportional $\beta$ curves.

The "late" data (for large values of $t$) can be analyzed using equations 16, 17, and 18; these equations are forms of summaries 1, $W(u)$, and 4, $L(u,v)$. However, for cases 1 and 3, the late data fall on the flat part of the $L(u,v)$ curves and a time-drawdown plot match would be indeterminate. Thus, only a distance-drawdown

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Table 5.1.—Values of $H(u, \beta)$ for selected values of $u$ and $\beta$

[From Hantush (1961d): Numbers in parentheses are powers of 10 by which the other numbers are multiplied, for example 963(-4) = 0.0963]
match could be used. Drawdown predictions, however, could be made using the \( L(u, v) \) curves.

Assumption 5, that no drawdown occurs in the source beds, has been examined by Neu-
man and Witherspoon (1969a, p. 810, 811) for the situation in which two aquifers are sepa-
rated by a less permeable bed. This is equivalent to case 3 with \( K''=0 \) and \( S''=0 \). They
concluded that (1) \( H(u, \beta) \), in the asymptotic solution for early times, would not be affected appreciably because the properties of the source bed have a negligible effect on the solution for \( Tt/\pi S \approx 1.6 \beta^2/(rB)^4 \), which is equivalent to \( t \leq S''b'/10K' \), where \( B = \sqrt{TB'/K} \); and (2) if \( T_r > 100T \), where \( T_r \) represents the trans-
missivity of the source bed, it is probably justified to neglect drawdown in the unpumped aquifer.

Table 5.2 is a listing of a FORTRAN program for computing values of \( H(u, \beta) \) for \( u \geq 10^{-60} \)
using a procedure devised and programmed by S. S. Papadopulos. Input data for this program
consists of three cards. The first card contains the beginning value of \( 1/u \), coded in columns
1–10, in format E10.5, and the ending (largest) value of \( 1/u \), coded in columns 11–20, in format
E10.5. The next two cards contain 12 values of \( \beta \), coded in columns 1–10, 11–20, . . . , and
71–80 on the first card and columns 1–10, 11–20, . . . , 31–40 on the second card, all in
format E10.5. The function is evaluated as follows (S. S. Papadopulos, written commun.,
1975):

\[
H(u, \beta) = \int_u^\infty (e^{-\beta y}) \text{erfc} (\beta \sqrt{u} (y - u)) \, dy
\]

\[
= \int_u^{u_1} f dy + \int_{u_1}^{u_2} f dy + \int_{u_2}^\infty f dy,
\]

where \( u_2 = (u/2)(1 + \sqrt{1 + 10^{20} \beta^2/u}) \), and
\( u_1 = (u/2)(1 + \sqrt{1 + 0.025 \beta^2/u}) \).
The significance of \( u_2 \) and \( u_1 \) is that
\( \text{erfc} (\beta \sqrt{u} (y - u)) \approx 1 \) for \( u > u_2 \)
and \( \text{erfc} (\beta \sqrt{u} (y - u)) \approx 0 \) for \( u < u_1 \).

Therefore,
\[
\int_u^{u_1} f dy = 0,
\]
and
\[
\int_{u_2}^\infty f dy = W(u_2),
\]
where \( W(u_2) \) is the well function of Theis. The function can be evaluated as
\[
H(u, \beta) = W(u) \text{ for } u > u_2
\]
\[
H(u, \beta) = \int_u^{u_2} f dy + W(u_2) \text{ for } u_1 < u < u_2
\]
and
\[
H(u, \beta) = \int_{u_1}^{u_2} f dy + W(u_2) \text{ for } u < u_1.
\]

If \( u_2 > 10 \), then
\[
\int_{u_1}^{u_2} f dy = \int_{u_1}^{10} f dy, W(u_2) \sim 0.
\]
An example of output from this program is shown in figure 5.3.

Solution 6: Constant discharge from a partially penetrating well in a leaky aquifer

Assumptions:
1. Well discharges at a constant rate, \( Q \).
2. Well is of infinitesimal diameter and is screened in only part of the aquifer.
3. Aquifer has radial-vertical anisotropy.
**Figure 5.3.**—Example of output from program for computing drawdown due to constant discharge from a well in a leaky aquifer with storage of water in the confining beds.
4. Aquifer is overlain, or underlain, everywhere by a confining bed having uniform hydraulic conductivity ($K'$) and thickness ($b'$).

5. Confining bed is overlain, or underlain, by an infinite constant-head plane source.

6. Hydraulic gradient across confining bed changes instantaneously with a change in head in the aquifer (no release of water from storage in the confining bed).

7. Flow is vertical in the confining bed.

8. The leakage from the confining bed is assumed to be generated within the aquifer so that in the aquifer no vertical flow results from leakage alone.

**Differential equation:**

$$\frac{\partial^2s}{\partial r^2} + \frac{1}{r} \frac{\partial s}{\partial r} + a^2 \frac{\partial^2s}{\partial z^2} = -\frac{sK'}{Tb'} \frac{\partial s}{\partial t}$$

$$a^2 = \frac{K_z}{K_r}$$

This is the differential equation describing nonsteady radial and vertical flow in a homogeneous aquifer with radial-vertical anisotropy and leakage proportional to drawdown.

**Boundary and initial conditions:**

$$s(r,z,0) = \begin{cases} 0, & r > 0, 0 < z < b \\ s(x, z, t) = 0, & 0 \leq z \leq b, t \geq 0 \\ \frac{\partial s(r,0,t)}{\partial z} = 0, & r \geq 0, t \geq 0 \\ \frac{\partial s(r,b,t)}{\partial z} = 0, & r \geq 0, t \geq 0 \\ \lim_{r \to 0} \frac{\partial s}{\partial r} = \begin{cases} 0, & 0 < z < d \\ -Q/(2\pi K_r (l-d)), & d < z < l \\ 0, & l < z < b \end{cases} \end{cases}$$

Equation 1 states that, initially, drawdown is zero. Equation 2 states that drawdown is small at a large distance from the pumping well. Equations 3 and 4 state that there is no vertical flow at the upper and lower boundaries of the aquifer. This means that vertical head gradients in the aquifer are caused by the geometric placement of the pumping well screen and not by leakage. Equation 5 states that near the pumping well the discharge is distributed uniformly over the well screen and that no radial flow occurs above and below the screen.

**Solution:**

I. For the drawdown in a piezometer, a solution by Hantush (1964a, p. 350) is given by

$$s = \frac{Q}{4\pi T} \left\{ W(u,\beta) + f(u,ar/b,\beta,dl/b,l'/b,z/b) \right\},$$

where

$$W(u,\beta) = \int_{u}^{x} e^{-\frac{y}{\beta}} \frac{dy}{y}$$

$$u = \frac{r^2S}{4Tt}$$

$$\beta = \sqrt{\frac{r^2K'}{Tb'}}$$

$$a = \sqrt{K_z/K_r}$$

$$f(u,ar/b,\beta,dl/b,l'/b,z/b) = 2b/\pi l'(l-d') \sum_{n=1}^{\infty} \ln(\sin n\pi l/b - \sin n\pi d/b) \cdot \cos(n\pi z/b) W(u,\sqrt{\beta^2 + (n\pi r/b)^2}).$$

II. For the drawdown in an observation well

$$s = \frac{Q}{4\pi T} \left\{ W(u,\beta) + \bar{f}(u,ar/b,\beta,dl/b,l'/b,l'/b) \right\},$$

where

$$\bar{f}(u,ar/b,\beta,dl/b,l'/b,l'/b) = 2b/\pi (l-d) (l'-d') \sum_{n=1}^{\infty} 1/n^2 (\sin n\pi l/b - \sin n\pi d/b) \cdot (\sin n\pi l'/b - \sin n\pi d'/b) W(u,\sqrt{\beta^2 + (n\pi r/b)^2}).$$

**Comments:**

The geometry is shown in figure 6.1. The differential equation and boundary conditions are based on the assumption that vertical flow in the aquifer is caused by partial penetration of the pumping well and not by leakage. Hantush (1967, p. 587) concluded that this assumption is correct if $b\sqrt{K'/Tb'} < 0.1$. The solutions are based on a uniform distribution of flow over the screen of the pumped well. Depending on friction losses within the well, a more realistic assumption might be constant drawdown over
the screen of the pumped well; this assumption would imply nonuniform distribution of flow. Hantush (1964a, p. 351) postulates that the actual drawdown at the face of the pumping well will have a value between these two extremes. The solutions should be applied with caution at locations very near the pumped well. The effects of partial penetration are insignificant for $r > 1.5 \frac{b}{a}$ (Hantush, 1964a, p. 350), and the solution is the same for the solution 4.

Because of the large number of variables involved, presentation of a complete set of type curves is impractical. An example, consisting of curves for selected values of the parameters, is shown in figure 6.2 on plate 1. This figure is based on function values generated by a FORTRAN program.

The computer program formulated to compute drawdowns due to pumping a partially penetrating well in a leaky aquifer is listed in table 6.1. Input data to this program consists of cards coded in specific FORTRAN formats. Readers unfamiliar with FORTRAN format items should consult a FORTRAN language manual. The first card contains: aquifer thickness ($b$), coded in format F5.1 in columns 1–5; depth, below top of aquifer, to bottom of pumping well screen ($l$), coded in format F5.1 in columns 6–10; depth, below top of aquifer, to top of pumping well screen ($d$), coded in format F5.1 in columns 11–15; number of observation wells and piezometers, coded in format 15 in columns 16–20; smallest value of $1/u$ for which computation is desired, coded in format E10.4 in columns 21–30; largest value of $1/u$ for which computation is desired, coded in format E10.4 in columns 31–40. The next two cards contain 12 values of $r/B$, all coded in format E10.5, in columns 1–10, 11–20, 21–30, 31–40, 41–50, 51–60, 61–70, and 71–80 of the first card and columns 1–10, 11–20, 21–30, and 31–40 of the second card. Computation will terminate with the first zero (or blank) value coded. Next is a series of cards, one card per observation well or piezometer, containing: radial distance from the pumped well multiplied
by the square root of the ratio of vertical to horizontal conductivity \( (r \sqrt{K_z/K_r}) \), coded in format F5.1 in columns 1–5; depth, below top of aquifer, to bottom of observation well screen (code blank for piezometer), coded in format F5.1, in columns 6–10; depth, below top of aquifer, to top of observation well screen (total depth for a piezometer), coded in format F5.1, in columns 11–15. Output from this program is a table of function values. An example of the output is shown in figure 6.3.

Because most aquifers are anisotropic in the \( r-z \) plane, it is generally impractical to use this solution to analyze for the parameters. However, it can be used to predict drawdown if the parameters are determined independently.

\[
W(U/R/R) = W(U/R/R) + R/L[B/D/B/L]/B/D/B[Z/B], Z/B = 0.50, \quad \text{SORT}(KZ/KR)*R/B = 0.10, L/R = 0.70, D/R = 0.30
\]

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Figure 6.3.—Example of output from program for partial penetration in a leaky artesian aquifer.
Solution 7: Constant drawdown in a well in a leaky aquifer

Assumptions:
1. Water level in well is changed instantaneously by \( s_w \) at \( t=0 \).
2. Well is of finite diameter and fully penetrates the aquifer.
3. Aquifer is overlain, or underlain, everywhere by a confining bed having uniform hydraulic conductivity \( K' \) and thickness \( b' \).
4. Confining bed is overlain, or underlain, by an infinite constant-head plane source.
5. Hydraulic gradient across confining bed changes instantaneously with a change in head in the aquifer (no release of water from storage in the confining bed).
6. Flow in the aquifer is two dimensional and radial in the horizontal plane and flow in the confining bed is vertical. This assumption is approximated closely where the hydraulic conductivity of the aquifer is sufficiently greater than that of the confining bed.

Differential equation:

\[
\partial^2 s/\partial r^2 + (1/r)\partial s/\partial r - s K'/T b' = (S/T)\partial s/\partial t
\]

This differential equation describes nonsteady radial flow in a homogeneous isotropic confined aquifer with leakage proportional to drawdown.

Boundary and initial conditions:

\[
\begin{align*}
s(r,0) &= 0, \quad r \geq 0 \quad (1) \\
s(r_w,t) &= s_w, \quad t \geq 0 \quad (2) \\
s(\infty, t) &= 0, \quad t \geq 0 \quad (3)
\end{align*}
\]

Equation 1 states that, initially, drawdown is zero. Equation 2 states that at the wall or screen of the discharging well, drawdown in the aquifer is equal to the constant drawdown in the well, which assumes that there is no entrance loss to the discharging well. Equation 3 states that the drawdown approaches zero as distance from the discharging well approaches infinity.

Solutions (Hantush, 1959):

I. For the discharge rate of the well,

\[
Q = 2\pi T s_w G(\alpha, r_w/B),
\]

where

\[
G(\alpha, r_w/B) = (r_w/B)K_1(r_w/B)/K_0(r_w/B) + (4/\pi^2) \exp \left[ -\alpha (r_w/B)^2 \right] \int_0^\infty \left\{ u \exp(-\alpha u^2) / \left[ J_0^2(u) + Y_0^2(u) \right] \right\} \cdot du / [u^2 + (r_w/B)^2],
\]

and

\[
\alpha = Tt/S r_w^2, \quad B = \sqrt{T b'/K'}.
\]

\( K_0 \) and \( K_1 \) are zero-order and first-order, respectively, modified Bessel functions of the second kind. \( J_0 \) and \( Y_0 \) are the zero-order Bessel functions of the first and second kind, respectively.

II. For the drawdown in water level

\[
s = s_w (K_w/B) / K_0(r_w/B)
\]

\[
+ (2/\pi) \exp (-\alpha r_w^2/B^2) \int_0^\infty \exp(-\alpha u^2) / u^2 + (r_w/B)^2 \cdot J_0^2(u) Y_0^2(u) - J_0(u) Y_0(u) / J_0^2(u) + Y_0^2(u) \cdot u \, du
\]

with \( \alpha, B, K_0, J_0, \) and \( Y_0 \) as defined previously.

Comments:

A cross section through the discharging well is shown in figure 7.1. The boundary conditions most commonly apply to a flowing artesian well, as is shown in this illustration.

Figure 7.2 on plate 1 is a plot of dimensionless discharge \( (G(\alpha, r_w/B)) \) versus dimensionless time \( (\alpha) \) from data of Hantush (1959, table 1) and Dudley (1970, table 2). Selected values of \( G(\alpha, r_w/B) \) are given in table 7.1. The corresponding data curve should be a plot of observed discharge versus time. The data curve is matched to figure 7.2 and from match points \( (\alpha, G(\alpha, r_w/B)) \) and \( (t, Q) \), \( T \) and \( S \) are computed from the equations.
Figure 7.1.—Cross section through a well with constant drawdown in a leaky aquifer.

\[ T = \frac{Q}{(2\pi s_{w}G(\alpha, r_{w}/B))} \]

and

\[ S = Tl/(\alpha r_{w}^{3}). \]

Figure 7.3 on plate 1 contains plots of dimensionless drawdown \((s/s_{w})\) versus dimensionless time \((\alpha r_{w}^{3}/r^{3})\). The corresponding data plot would be observed drawdown versus observation time. Matching the data and type curves by superposition and choosing convenient match points \((s/s_{w}, \alpha r_{w}^{3}/r^{3})\) and \((s, t)\), the ratio of transmissivity to storage coefficient can be computed from the relation

\[ T/S = (\alpha r_{w}^{3}/r^{3})(r^{3}/t). \]

Figure 7.3 was plotted from function values generated by a FORTRAN program. This program is listed in table 7.2. The input data for this program consist of three cards coded in specific formats. Readers unfamiliar with FORTRAN format items should consult a FORTRAN language manual. The first card contains: the smallest value of alpha for which computation is desired, coded in format E10.5 in columns 1–10; the largest value of alpha for which computation is desired, coded in format E10.5 in columns 11–20. The output table will include a range in alpha spanning these two values up to a limiting range of nine log cycles. The second card contains 13 values of \(r_{w}/B\). These coded values are the significant figures only and should be greater or equal to 1 and less than 10. The power of 10 by which each of these coded values is multiplied is calculated by the program. Zero (or blank) coding is permissible, but the first zero (or blank) value will terminate the list. The 13 values, all coded in format F5.0, are coded in columns 1–5, 6–10, 11–15, 16–20, 21–25, 26–30, 31–35, 36–40, 41–45, 46–50, 51–55, 56–60, and 61–65. The third card contains the radius of the control well and distances to the observation wells.
### Table 7.1.—Values of $G(\alpha, r_w/B)$

[Values for $r_w/B \leq 1 \times 10^{-4}$ and $\alpha \geq 1 \times 10^6$ are from Hantush (1959, table 1), others are from Dudley (1970, table 2).]

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The control well radius ($r_w$) is coded first, in columns 1–8 in format F8.2. The distances ($r$) to the observation wells (maximum of nine) are coded next, in monotonic increasing order (smallest $r$ first, largest $r$ last), in columns 9–16, 17–24, 25–32, 33–40, 41–48, 49–56, 57–64, 65–72, and 73–80, all in format F8.2. If two or more observation wells have the same distance, this common distance should be coded only once, the function values will apply to all wells at the same distance from the control well. If the number of observation wells is less than nine, the remaining columns on the card should be left blank.

The integral in equation 4 is approximated by

$$\int_{0}^{\infty} f(u, \alpha, r_w/B) \, du = \frac{8000}{\sum_{i=1}^{n} f(-\Delta u/2 + i \Delta u, \alpha, r_w/B) \, \Delta u}$$
This expression is a composite quadrature with equally spaced abscissas. The abscissas are chosen at the midpoints of the intervals instead of the ends because the integrand is singular at \( u = 0 \). The value of \( \Delta u \) used is related to \( \alpha \) and is \( \Delta u \leq 10^{-3}/\sqrt{\alpha} \). The \( r_u/B \) values then selected by the program satisfy \( r_u/B \geq 10 \Delta u \). These two constraints, though empirical, are related to the behavior of the integrand; the first constraint is related to the term \( e^{-u/\alpha} \) as \( u \) becomes large, and the second to \( u/\alpha^2 + (r_u/B)^2 \) as \( u \) becomes small.

The Bessel functions \( K_\nu(r/B) \), \( K_\nu(r_u/B) \) are evaluated by the IBM subroutine BESK. A description of this subroutine may be found in the IBM Scientific Subroutine Package.

The Bessel functions of the second kind in the integrand, \( Y_\nu(u) \) and \( Y_\nu(u' r/r_u) \), are evaluated using IBM subroutine BESY, which is discussed in IBM SSP manual. The Bessel functions \( J_\nu(u) \) and \( J_\nu(u' r/r_u) \) are evaluated for arguments less than four by a polynomial approximation consisting of the first 10 terms of the series expansion

\[
J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2/2)^n}{(n!)^2}.
\]

For arguments greater than or equal to four, the asymptotic expansion is used

\[
J_\nu(x) = P \cos (x - \pi/4) + Q \sin (x - \pi/4).
\]

\( P \) and \( Q \) are calculated by the algorithm used in IBM subroutine BESY.

The output from this program consists of tables of function values, an example of which is shown in figure 7.4.

Solution 8: Constant discharge from a fully penetrating well of finite diameter in a nonleaky aquifer

Assumptions:
1. Well discharges at a constant rate, \( Q \).
2. Well is of finite diameter and fully penetrates the aquifer.
3. Aquifer is not leaky.
4. Discharge from the well is derived from a depletion of storage in the aquifer and inside the well bore.

Differential equation:

\[
\frac{\partial^2 s}{\partial r^2} + \frac{1}{r} \frac{\partial s}{\partial r} = \frac{(S/T)}{\partial s/\partial t}, \quad r \geq r_u
\]

This differential equation describes nonsteady radial flow in a homogeneous isotropic aquifer in the region outside the pumped well.

Boundary and initial conditions:

\[
\begin{align*}
\frac{\partial s}{\partial r}(r_u, t) &= s_u(t), \quad t > 0 \\
\frac{\partial s}{\partial t}(r, t) &= 0, \quad t > 0 \\
s(r_u, 0) &= 0, \quad r \geq r_u \\
s_a(0) &= 0 \\
(2\pi r_u T) \frac{\partial s}{\partial r}(r_u, t) - (\pi r_u^2) \frac{\partial s}{\partial t}(t) &= -Q, \quad t > 0
\end{align*}
\]

Equation 1 states that the drawdown at the well bore is equal to the drawdown inside the well, assuming that there is no entrance loss at the well face. Equation 2 states that drawdown is small at a large distance from the pumping well. Equations 3 and 4 state that, initially, drawdown in the aquifer and inside the well is zero. Equation 5 states that the discharge of the well is equal to the sum of the flow into the well and the rate of decrease in storage inside the well.

Solution (Papadopulos and Cooper, 1967; Papadopulos, 1967):

\[
s = \frac{(Q/4\pi T)}{F(u, \alpha, \rho)},
\]

where

\[
F(u, \alpha, \rho) = \frac{(8\alpha/\pi)}{\int_0^{\infty} \left[ (1 - \exp(-\beta^2 \rho^2/4u)) \left[ J_\nu(\beta \rho) A(\beta) - Y_\nu(\beta \rho) B(\beta) \right] \right.}
\[
\left. \left[ A(\beta) \right]^2 + \left[ B(\beta) \right]^2 \beta^2 \right] d\beta,
\]

and

\[
B(\beta) = \beta J_\nu(\beta) - 2\alpha J_1(\beta),
\]

\[
A(\beta) = \beta Y_\nu(\beta) - 2\alpha Y_1(\beta),
\]

\[
u = r^2 S/4T t,
\]

\[\alpha = r_u^2 S/r_u^2,
\]

\[\rho = r/r_u.
\]

\( J_\nu \) and \( Y_\nu, J_1 \), and \( Y_1 \), are zero-order and first-order Bessel functions of the first and second kind, respectively.
### TECHNIQUES OF WATER-RESOURCES INVESTIGATIONS

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<td>0.171</td>
<td>0.130</td>
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Figure 7.4.—Example of output from program for constant drawdown in a well in a leaky artesian aquifer.
The drawdown inside the pumped well is obtained at \( r = r_c \) and can be expressed as (Papadopulos and Cooper, 1967, p. 242):

\[
s_w = \left( \frac{Q}{4\pi T} \right) F(u, \alpha),
\]

where

\[
F(u, \alpha) = F(u, \alpha, 1),
\]

and

\[
u_w = r_c^2 S/4\pi T.
\]

Comments: A cross section through the discharging well is shown in figure 8.1. The geometry, except for the region of the well bore, is the same as for solution 1 (Theis solution). It is apparent from figure 8.2 and 8.3 (on plate 1) that \( F(u, \alpha, \rho) \) approaches \( W(u) \), the Theis solution, as time becomes large.

Papadopulos (1967, p. 161) stated that for \( t > 2.5 \times 10^3 r_c^2 / T \), or \( \alpha u > 10^4 \), the function \( F(u, \alpha, \rho) \) can be closely approximated by \( F(u, \alpha, \rho) = W(u) \). Papadopulos and Cooper (1967, p. 242) stated that for \( t > 2.5 \times 10^4 r_c^2 / T \), or \( \alpha u_w > 10^4 \), the function \( F(u_w, \alpha) \) can be closely approximated by \( F(u_w, \alpha) = W(u_w) \). An examination of the type curves and function values indicates that \( F(u_w, \alpha) = W(u_w) \) (less than 5-percent error) for \( \alpha u_w > 10^2 \), and hence \( t \) should only be greater than \( 25 r^2 / T \) for drawdown in the pumped well.

Figures 8.2 and 8.3 were prepared from function values given in Papadopulos and Cooper (1967) and Papadopulos (1967), which are reproduced in table 8.1. For drawdown observations in the pumped well, the method of analysis is to plot drawdown versus time and

![Figure 8.1.—Cross section through a discharging well of finite diameter.](image-url)
then superimpose the plot on figure 8.2. After match points of (s,t) and \( F(u_w, \alpha), \frac{1}{u_w} \) are chosen, the transmissivity can be computed from the relation \( T = \frac{Q}{4 \pi s} F(u_w, \alpha) \). Then, the storage coefficient can be determined from \( S = \frac{4 T t}{r_w s} \left( \frac{1}{u_w} \right) \).

For observations not in the pumped well, two procedures are available for analyzing the data. To analyze the data from a single observation well, a family of type curves of \( F(u, \alpha, \rho) \) versus \( 1/u \) for different values of \( \rho \) can be plotted for the \( \rho \) value appropriate for the observation well, using values in table 8.1. This procedure produces a family of type curves similar to that shown for \( \rho = 1 \) in figure 8.2. If \( \rho \) for the observation well is between \( \rho \) values in table 8.1, function values can be interpolated. Using this approach, the data for the observation well are plotted as drawdown versus time and matched to the best-fitting member of the plotted type curves. Transmissivity and storage coefficient can be calculated from \( T = \frac{Q}{4 \pi s} F(u, \alpha, \rho) \) and \( S = \frac{4 T t}{r^2} \left( \frac{1}{u} \right) \).

Drawdowns at more than one observation point may be combined by preparing a composite plot of the drawdowns at each observation well versus \( t/r^2 \). This composite plot would be analyzed by matching it to a family of type curves of \( F(u, \alpha, \rho) \) versus \( 1/u \) for constant \( \alpha \). An example of such a type-curve family for \( \alpha = 10^{-4} \) is shown in figure 8.3. This method requires multiple sheets of type curves, one sheet for each value of \( \alpha \). When the data curves are matched to the type-curve family, care should be taken to insure that the data for each well fall on the type curve having the appropriate \( \rho \) value. This will be possible for all the data for only one value of \( \alpha \). Transmissivity and storage coefficient are calculated from \( T = \frac{Q}{4 \pi s} F(u, \alpha, \rho) \) and \( S = \frac{4 T t}{r^2} \left( \frac{1}{u} \right) \).

In both of these methods of plotting and comparing data, an alternate computation of storage coefficient is \( S = \frac{r^2 \alpha}{r_w^2} \). However, as pointed out by Papadopulos and Cooper (1967, p. 244), the shapes of type curves differ only slightly when \( \alpha \) changes by an order of magnitude, therefore the determination of \( S \) is sensitive to choosing the "correct" curve. Papadopulos and Cooper (1967, p. 244) suggest that if \( S \) can be estimated within an order of magnitude, the value of \( \alpha \) to be used for matching the data can be decided.
### Table 8.1 — Values of the function $F(u, \alpha, \rho)$

[Values for $\rho = 1$ from Papadopulos and Cooper, 1967. Other values from Papadopulos, 1967]

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| $5 \times 10^{-1}$ | $2.00$ | $2.47 \times 10^{-1}$ | $5.81$ | $1.45 \times 10^{-1}$ | $4.10 \times 10^{-1}$ | $1.80 \times 10^{-1}$ | $5.62$ | $1.75 \times 10^{-1}$ |
| 1     | $5.00$ | $8.76$ | $2.71 \times 10^{-3}$ | $7.54$ | $2.27 \times 10^{-2}$ | $1.03 \times 10^{-2}$ | $3.04 \times 10^{-2}$ | $7.10$ |
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| 2     | $2.00$ | $4.72$ | $1.82 \times 10^{-2}$ | $5.55$ | $1.74 \times 10^{-1}$ | $5.36$ | $1.87 \times 10^{-1}$ | $2.95$ | $3.28$ |
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| 1     | $9.98$ | $2.81$ | $1.23 \times 10^{-1}$ | $3.86$ | $1.14 \times 10^{0}$ | $3.06$ | $8.44$ | $4.02$ |
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For $\alpha = 10^{-5}$
The early parts (short time) of the curves in figure 8.2 are straight lines. According to Papadopulos and Cooper (1967, p. 244), these represent conditions under which all the water pumped is derived from storage within the well. The straight lines approached by the curves satisfy the equations

\[ F(u, \alpha) = \alpha/u_w \]

and

\[ s_w = Qt/\pi r_w^2 = \text{volume of water discharged} \quad \text{area of well} \]

Therefore, as pointed out by Papadopulos and Cooper (1967, p. 244), data that fall on this straight part of the type curves do not indicate information about the aquifer characteristics.

Table 8.2 is a listing of two FORTRAN programs by S. S. Papadopulos that evaluate \( F(u, \alpha, \rho) \) for \( \alpha = 1 \times 10^{-6} \) and \( \rho = 2 \times 10^0 \). The input data to both programs consists of cards coded in specified format (readers unfamiliar with FORTRAN language format should refer to a FORTRAN language manual). Input to the programs is one or more groups of data, each group of data consisting of two cards. The first card contains one value of \( \alpha \) in columns 1-10, coded in format E10.5. The second card contains 16 values of \( u \) coded in columns 1-5, 6-10, ..., 75-80 in format 16F5.0. The \( F(u, \alpha) \) or \( F(u, \alpha, \rho) \) values will be printed in the order that the \( u \) values are coded.
Solution 9: Slug test for a finite-diameter well in a nonleaky aquifer

Assumptions:
1. A volume of water, \( V \), is injected into, or is discharged from, the well instantaneously at \( t = 0 \).
2. Well is of finite diameter and fully penetrates the aquifer.
3. Aquifer is not leaky, and flow is in radial direction only.

Differential equation:
\[
\frac{\partial^2 h}{\partial r^2} + \left( \frac{1}{r} \right) \frac{\partial h}{\partial r} = \left( \frac{S}{T} \right) \frac{\partial h}{\partial t}, \quad r > r_w
\]

This differential equation describes nonsteady radial flow in a homogeneous isotropic aquifer beyond the radius of the injected well.

Boundary and initial conditions:
1. \( h(r_w,t) = H(t), \quad t > 0 \) \hspace{1cm} (1)
2. \( h(r_w,t) = 0, \quad t > 0 \) \hspace{1cm} (2)
3. \( 2\pi r_w T \frac{\partial h(r_w,t)}{\partial t} = \pi r_w^2 \frac{\partial H(t)}{\partial t}, \quad t > 0 \) \hspace{1cm} (3)
4. \( h(r,0) = 0, \quad r > r_w \) \hspace{1cm} (4)
5. \( H(0) = H_0 = \frac{V}{\pi r_w^2} \) \hspace{1cm} (5)

Equation 1 states that the head change in the aquifer at the face of the well is equal to that inside the well; one assumes that there is no exit loss at the well face. Equation 2 states that the head change approaches zero as distance from the discharging well approaches infinity, a condition which will be approximated if boundaries of the aquifer are sufficiently distant from the discharging well. Equation 3 states that near the well the radial flow is equal to the rate of change in volume of water inside the well. Equations 4 and 5 state that initially the head change is zero in the aquifer, and the head increase or decrease inside the well is equal to \( H_0 \).

Solution (Cooper and others, 1967):
\[
h = \frac{(2H_o/\pi)}{\int_0^\infty \left[ \exp(-\beta u^2/\alpha) \left\{ J_0(ut/r_w) \right\} \right] \left[ uY_0(u)-2\alpha J_1(u) \right] /\Delta(u) \, du}
\]

where
\[
\alpha = \frac{r_w^2}{S/r_w^2}, \quad \beta = \frac{T}{r_w^2},
\]
and
\[
\Delta(u) = \left[ uJ_0(u)-2\alpha J_1(u) \right]^2 + \left[ uY_0(u)-2\alpha Y_1(u) \right]^2.
\]

\( J_0 \) and \( Y_0, J_1 \), and \( Y_1 \), are zero-order and first-order Bessel functions of the first and second kind, respectively.

The head, \( H \), inside the well, obtained by substituting \( r = r_w \) in equation (6) is
\[
\frac{H}{H_0} = F(\beta, \alpha),
\]
where
\[
F(\beta, \alpha) = \frac{(8\alpha/\pi^2)}{\int_0^\infty \left( \exp(-\beta u^2/\alpha) / u \Delta(u) \right) \, du}
\]
and where \( \alpha, \beta, \Delta(u) \) are as defined previously.

Comments: Figure 9.1 is a cross section showing geometric configuration along the wellbore. The volume of water injected into or discharged from the well is \( \pi r_w^2 H_0 \). The water-level data in the injected well, expressed as a fraction of \( H_0 \), is plotted versus time on semilogarithmic graph paper. This plot is superimposed on figure 9.2, keeping the baselines the same and sliding horizontally until a match or interpolated fit is made. A match point for \( \beta, \delta, \) and \( \alpha \) is picked from the two graphs. Transmissivity is calculated from \( T = \beta r_w^2/\alpha \) and storage coefficient from \( S = \alpha r_w^2/r_w^2 \). As pointed out by Cooper, Bredehoeft, and Papadopulos (1967, p. 267), the determination of \( S \) by this method has questionable reliability because of the similar shape of the curves, whereas the determination of \( T \) is not as sensitive to choosing the correct curve. Figure 9.2 on plate 1 is plotted from data in table 9.1, which contains original material from two sources (Cooper and others, 1967; and Papadopulos and others, 1973).

Table 9.2 is a listing of a FORTRAN program by S. S. Papadopulos that evaluates \( F(\beta, \alpha) \). Input to the program consists of cards coded in a specific format (readers unfamiliar with FORTRAN formats should refer to a FORTRAN language manual). Input consists of two or more cards, each containing a single value of
Solution 10: Constant discharge from a fully penetrating well in an aquifer that is anisotropic in the horizontal plane

**Assumptions:**

1. Well discharges at a constant rate, Q.
2. Well is of infinitesimal diameter and fully penetrates the aquifer.
3. Aquifer is anisotropic in the horizontal plane.
4. Aquifer is not leaky.
5. The transmissivity of the aquifer, $T$, is a two-dimensional symmetric tensor.

**Differential equation:**

\[
T_{xx} \frac{\partial^2 s}{\partial x^2} + 2T_{xy} \frac{\partial^2 s}{\partial x \partial y} + T_{yy} \frac{\partial^2 s}{\partial y^2} + Q \delta(x) \delta(y) = s \frac{\partial s}{\partial t}.
\]

This differential equation describes unsteady flow in a homogeneous anisotropic aquifer with a constantly discharging well at $x=y=0$. The Dirac delta function is represented as $\delta(z)$ and has the following properties: $\delta(z)=0$ if $z \neq 0$ and $\int_{-\infty}^{\infty} \delta(z) dz = 1$.

**Boundary and initial conditions:**

\[
\begin{align*}
    s(x, y, 0) &= 0 \\
    s(\pm \infty, y, t) &= 0 \\
    s(x, \pm \infty, t) &= 0
\end{align*}
\]
TYPE CURVES FOR FLOW TO WELLS IN CONFINED AQUIFERS

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From Cooper, Bredehoeft, and Papadopoulos, 1967

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From Papadopoulos, Bredehoeft, and Cooper, 1973

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Techniques of Water-Resources Investigations

Solution (Papadopulos, 1965, p. 23):

\[ s = \frac{Q}{4\pi \sqrt{T_{xx}T_{yy} - T_{xy}^2}} W(u_{xy}), \quad (4) \]

where

\[ W(u) = \int_0^\infty \left( e^{-\gamma} \right) dv \]

and

\[ u_{xy} = \frac{(S/4t)(T_{xx}y^2 + T_{yy}x^2 - 2T_{xy}xy)}{(T_{xx}T_{yy} - T_{xy}^2)}. \quad (5) \]

If the coordinate axes \( x \) and \( y \) are the same as the principal axes \( \varepsilon \) and \( \eta \) (fig. 10.1) of the transmissivity tensor, the preceding equation for drawdown becomes

\[ s = \frac{Q}{4\pi \sqrt{T_{xx}T_{yy} - T_{xy}^2}} W(u_{xy}), \]

where

\[ u_{xy} = \frac{(S/4t)(T_{xx} n^2 + T_{yy} \varepsilon^2)}{T_{xx} T_{yy}}. \]

Comments: The method of type-curve solution as outlined by Papadopulos (1965, p. 26) requires observation of drawdown in at least three observation wells. First, choose a convenient rectangular coordinate system with the pumped well at the origin. Then, plot the observed drawdown versus \( t \) on logarithmic paper. Match these plots to the \( W(u) \) type curve given in solution 1. Choose a match point of \((t,s), (u_{xy}, W(u_{xy}))\) for each well and compute \( T_{xx}T_{yy} - T_{xy}^2 = (QW(u_{xy})/4\pi s)^2 \) for each well. Match points for all observation wells should yield approximately the same value of \( T_{xx}T_{yy} - T_{xy}^2 \). Usually they will not and judgment must be used to obtain an "average" value. Substituting this value and the three values of \((x,y)\) in equation 5 gives three equations in three unknowns \( ST_{xx}, ST_{yy}, \) and \( ST_{xy}. \) These equations are of the form

\[ y^2(ST_{xx}) + x^2(ST_{yy}) - 2xy(ST_{xy}) = 4tu_{xy}(T_{xx}T_{yy} - T_{xy}^2). \]

Solve these three equations to determine \( T_{xx}, T_{xy}, \) and \( T_{yy} \) in terms of \( S, \) and \( S \) may be determined from

\[ S = \frac{Q}{4\pi \sqrt{T_{xx}T_{yy} - T_{xy}^2}} W(u_{xy}). \]
Solution 11: Variable discharge from a fully penetrating well in a leaky aquifer

**Assumptions:**

1. Well discharge changes as a specified function of time.
2. Well is of infinitesimal diameter and fully penetrates the aquifer.
3. Aquifer is overlain, or underlain, everywhere by a confining bed having uniform hydraulic conductivity \((K')\) and thickness \((b')\).

\[
S = \sqrt{(ST_{xx}ST_{yy} - (ST_{xy})^2)}/(T_{xx}T_{yy} - T_{xy}^2).
\]

Then, compute \(T_{xx}, T_{yy},\) and \(T_{xy}\) from \(ST_{xx}, ST_{yy},\) and \(ST_{xy} \). \(T_{xx}, T_{yy},\) and \(\theta\) (the angle between the \(x\) and the \(\epsilon\) axis) may be calculated from the relations (Papadopulos, 1965, p. 28)

\[
T_{xx} = 1/2(T_{xx} + T_{yy} + ((T_{xx} - T_{yy})^2 + 4T_{xy}^2))^{1/2}
\]

\[
T_{yy} = 1/2(T_{xx} + T_{yy} - ((T_{xx} - T_{yy})^2 + 4T_{xy}^2))^{1/2}
\]

\[
\theta = \arctan((T_{xx} - T_{xy})/T_{xy}).
\]
4. Confining bed is overlain, or underlain, by an infinite constant head plane source.

5. Hydraulic gradient across confining bed changes instantaneously with a change in head in the aquifer (no release of water from storage in the confining bed).

6. Flow in the aquifer is two-dimensional and radial in the horizontal plane and flow in the confining bed is vertical. This assumption will be approximated closely where the hydraulic conductivity of the aquifer is sufficiently greater than that of the confining bed.

**Differential equation:**

\[
\frac{\partial^2 s}{\partial r^2} + \frac{1}{r} \frac{\partial s}{\partial r} - \frac{s K'}{Tb'} = \frac{S}{T} \frac{\partial s}{\partial t}
\]

This is the differential equation describing nonsteady radial flow in a homogeneous isotropic aquifer with leakage proportional to drawdown.

**Boundary and initial conditions:**

\[
s(r, 0) = 0 \\
\lim_{r \to 0} \frac{\partial s}{\partial r} = -\frac{Q(t)}{2 \pi T}, \quad t \geq 0
\]

Equation 1 states that, initially, drawdown is zero. Equation 2 states that drawdown is zero at large distances from the pumped well. Equation 3 states that near the pumped well the radial flow is equal to the discharge of the pumped well, which is a function of time.

**Solution:**

Solutions for certain discharge functions have been published by Abu-Zied and Scott (1963), and Werner (1946) for a nonleaky aquifer, and by Hantush (1964a) for both leaky and nonleaky aquifers. For arbitrary discharge functions for leaky aquifers, a solution using the convolution integral has been presented by Moench (1971, eq. 3):

\[
s = \frac{1}{4\pi T} \int_0^t (Q(t')/(t-t')) \cdot \exp \left(-A/(t-t') - (t-t')K'/Sb'\right) dt', \quad (4)
\]

where \(Q(t)\) is the discharge function of time and \(A = r^2S/4T\). A numerical integration scheme is generally necessary to evaluate the above equation.

For type curves, a more useful form of equation 4 is

\[
s = \frac{Q_r}{4\pi T} \int_0^t \left[ \frac{Q(t')}{Q_r (t-t')} \right] \cdot \exp \left[-A/(t-t') - (t-t')K'/Sb'\right] dt', \quad (5)
\]

or

\[
s = \frac{Q_r}{4\pi T} \cdot SO(t), \quad (6)
\]

where \(SO(t)\), read "system output function," represents the integral expression in equation 5, and \(Q_r\) is an arbitrary discharge that eliminates dimension from the integral expression. For example, \(Q_r\) could be the initial, final, or average discharge, according to the needs of the user.

**Comments:** Figure 11.1 is a cross section through the discharging well. This situation is the same as for solution 4, except for the varying discharge of the well. The effect of finite well radius \(r_w\) was investigated by Hantush (1964b, p. 4224), who concluded that for \(t > 25r_w^2S/T\) and \(r_w/\sqrt{Tb'/K'} < 0.1\) the drawdown could be represented closely by the convolution integral.

Figure 11.2 on plate 1 shows a selected set of type curves for linear change in discharge in a nonleaky aquifer. The solution for this type of discharge function has been presented by Werner (1946, p. 706). The discharge function for figure 12.2 is \(Q(t) = Q_0 (1 + ct)\), and the resulting drawdown is

\[
s = \frac{Q_0}{4\pi T} W(u) \left\{1 + ct \left[u + 1 - e^{-u}/W(u)\right]\right\},
\]

where \(W(u)\) is the well function of Theis. Substituting \(A/u\) for \(t\) in the above expression gives

\[
s = \frac{Q_0}{4\pi T} W(u) \cdot (1 + cA \left\{1 + (1/u) \left[1 - e^{-u}/W(u)\right]\right\}),
\]

or

\[
s = \frac{Q_0}{4\pi T} \cdot SO(t), \quad (6)
\]

where \(SO(t)\) represents

\[
W(u) \left\{1 + cA \left\{1 + (1/u) \left[1 - e^{-u}/W(u)\right]\right\}\right\}.
\]
This substitution permits the plotting of a family of type curves, each curve specified by a value of $\Delta a$.

Table 11.1 is the listing of a FORTRAN program designed to evaluate the above convolution integral for five different discharge functions. Three of these discharge functions are those devised by Hantush (1964a, p. 343, 344), who presented solutions for drawdown resulting from these functions. These three discharge functions are:

(a) $Q(t) = Q_s [1 + \delta \exp (-t/t^*)]$,
(b) $Q(t) = Q_s [1 + \delta/(1 + ut^*)]$,
and (c) $Q(t) = Q_s [1 + \delta' \sqrt{1 + ut^*}]$,

where $Q_s$ is the ultimate steady discharge and $\delta$ and $t^*$ are parameters defining a particular function. The first discharge function, for an exponentially decreasing discharge (case “a” of Hantush, 1964a) is virtually the same as the discharge function of Abu-Zied and Scott (1963). Besides the three functions of Hantush, the program also includes discharge as a fifth-degree polynomial of time, $Q(t) = \sum_{i=0}^{5} a_i t^i$, where the $a_i$ are the coefficients of the polynomial, and as a piecewise linear function of time with eight segments,

$$Q(t) = a_j + b_j (t - t_{j-1})$$

for $t_{j-1} < t \leq t_j, j = 1, 2, \ldots, 8$,

where $a_j$ and $b_j$ are parameters defining the $j^{th}$ line segment. The program uses a different, but equivalent to equation 4, expression for the convolution integral

$$s = (1/4\pi T) \int_0^t (Q(t - t')/t') \exp (-A/t' - t'K'/Sb') dt'.$$

The program uses a sum to approximate the convolution integral. It chooses a starting value of $t'$ that satisfies $r^2S/4Tt' + K't'/Sb' = 100$. If such a value of $t'$ does not exist, that is, $(r^2S/4T)(K'/Sb') > 2500$, then a value of zero is assigned for the integral value. The ending point of the interval is picked as 10 times the...
starting point. The integral over this interval is approximated by a trapezoidal sum using 500 subdivisions of the interval. A new interval is then constructed using the previous end point as a new starting point and a new ending point equal to 10 times the new starting point. This new interval is again evaluated by a trapezoidal sum of 500 segments. This summation procedure over intervals that are successively an order of magnitude larger continues until either \( t' = t \) or \( (r'S/4T') + (K'tSb') > 101 \). Input to this program consists of cards coded in specific formats. Readers unfamiliar with FORTRAN formats should refer to a FORTRAN language manual. Input consists of one or more groups of data, each group consisting of the following. First, one card containing the beginning time of the period of analysis in columns 1–10, coded in format E10.3; the ending time coded in columns 13–20, in format E10.3; and a discharge index (a number from 1 through 5) coded in column 25, in format 11 and a reference discharge, \( QR \), coded in columns 31–40, in format E10.3. The discharge index, \( IQ \), selects a discharge function, \( Q(t) \), in the following manner. If \( IQ = 1 \), the discharge function is exponentially decreasing,

\[
Q(t) = Q_s \left[1 + \delta \exp\left(-t/t^*\right)\right].
\]

This is case (a) of Hantush (1964a, p. 343). If \( IQ = 2 \), the discharge function is hyperbolically decreasing,

\[
Q(t) = Q_s \left[1 + \delta/(1+t/t^*)\right].
\]

This is case (b) of Hantush (1964a, p. 344). If \( IQ = 3 \), the discharge function is the same as case (c) of Hantush (1964a, p. 344),

\[
Q(t) = Q_s \left[1 + \delta/\sqrt{1+t/t^*}\right].
\]

If \( IQ = 4 \), the discharge function is a fifth-degree polynomial of time,

\[
Q(t) = \sum_{i=0}^{5} a_i t^i.
\]

If \( IQ = 5 \), the discharge function is a piecewise-linear function of time with eight or less segments,

\[
Q(t) = a_j + b_j (t-t_{j-1})
\]

for \( t_{j-1} < t < t_j, j = 1, 2, \ldots, 8 \).

The reference discharge, \( QR \), is used to determine the form of the output from the program. If \( QR \) is coded as zero (or blank), the output shows \( t, s \) (as defined by eq. 4), and \( Q(t) \). If a value greater than zero is coded for \( QR \), the output shows \( 1/u, SO(t) \) (as defined by eq. 6), and \( Q(t)/QR \).

Second, there are one or more cards containing parameters of the discharge function. If \( IQ = 1, 2, \) or \( 3 \), then it consists of one card containing: \( QST \), the ultimate steady discharge, coded in columns 1–10, in format E10.3; \( DELTA \), a rate parameter, coded in columns 11–20, in format E10.3; \( TSTAR \), a time parameter, coded in columns 21–30, in format E10.3. If \( IQ = 4 \), it is one card containing the six polynomial coefficients. They are coded in the order \( a_0, a_1, \ldots, a_5 \), in columns 1–10; 11–20, \ldots, 51–60 all in format E10.3. If \( IQ = 5 \), then the program requires four cards, each card containing \( t_j, a_j, b_j, t_{j+1}, a_{j+1}, b_{j+1} \); the four cards representing \( j = 1, 3, 5, 7 \). The last part of each set of data consists of two or more cards containing coded values for: distance from pumped well, in columns 1–10; storage coefficient, in columns 11–20; transmissivity, in columns 21–30; and ratio of hydraulic conductivity to thickness for the confining bed, in columns 31–40, all in format E10.3. A blank card is used to signal the end of each set of data. Output from this program is shown in figure 11.3.

References


Dudley, W. W., Jr., 1970, Nonsteady inflow to a chamber
TYPE CURVES FOR FLOW TO WELLS IN CONFINED AQUIFERS

———1961d, Tables of the function $H(u,p)=\int_{0}^{p} e^{-y} \text{erfc} \left( \frac{y}{\sqrt{y(y-u)}} \right) dy$: New Mexico Inst. Mining and Technology Prof. Paper 103, 12 p.
———1961e, Tables of the function $W(u,p)=\int_{0}^{p} e^{-y} \frac{p}{\sqrt{y(y-u)}} dy$:
New Mexico Inst. Mining and Technology Prof. Paper 104, 13 p.

FIGURE 11.3.—Example of output from program to compute the convolution integral for a leaky aquifer.


Purpose

To compute type curve function values for partial penetration in a nonleaky artesian aquifer using equations 1 and 9A of Hantush, M.S., 1961, Draindown Around a Partially Penetrating Well: Hydraulic DIV, JUUR, AM, SOC, CIVIL ENGINEERS PROG, P, 83-98.

Input Data

1 Card - Format (3F5,1,5,2E10,4)

B = Aquifer Thickness

L = Depth, Below Top of Aquifer, To Bottom of Pumping Well Screen

D = Depth, Below Top of Aquifer, To Top of Pumping Well Screen

Num = Number of Observation Wells or Piezometers Times Number of Values of Kz/Kr

Small = Smallest Value of 1/U For Which Computation Is Desired

Large = Largest Value of 1/U For Which Computation Is Desired

Num Cards (One For Each Obs, Well or Piezometer and For Each Value of R*Sqrt(Kz/Kr), = Format (3F5,1)

R = Radial Distance From Pumped Well Times Sqrt(Kz/Kr)

Lprime = Depth, Below Top of Aquifer, To Bottom of Obs, Well Screen (Zero For Piezometer)

Dprime = Depth, Below Top of Aquifer, To Top of Obs, Well Screen (Total Depth For Piezometer)

Subroutines and Function Subprograms Required

DGL12, SERIES, BESK, FCT, L, F, EXPI

Real#8 U

Real#4 L, LB, LPB, LPRIME, LARGE

Dimension Array(13,12), IARG(12), ARG(13), A(12), C(12)

Data ARG(1,1,2,1,5,2,2,5,3,3,5,4,5,6,7,8,9,1)

TABLE 2.1.—Listing of program for partial penetration in a nonleaky artesian aquifer—Continued

<table>
<thead>
<tr>
<th>Line</th>
<th>Description</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>WRITE (IP1,7) DPB, RB, LB, DB</td>
<td>PPN 61</td>
</tr>
<tr>
<td>3</td>
<td>GO TO 4</td>
<td>PPN 62</td>
</tr>
<tr>
<td>4</td>
<td>WRITE (IP1,8) LPB, DPB, RB, LB, DB</td>
<td>PPN 63</td>
</tr>
<tr>
<td>5</td>
<td>WRITE (IP1,9) (A(I), C(I), IARG(I), I, JLIMIT)</td>
<td>PPN 64</td>
</tr>
<tr>
<td>6</td>
<td>WRITE (IP1,10) ARG(I), (ARRAY(I,J), J, JLIMIT)</td>
<td>PPN 65</td>
</tr>
<tr>
<td>7</td>
<td>CONTINUE</td>
<td>PPN 66</td>
</tr>
<tr>
<td>8</td>
<td>STOP</td>
<td>PPN 67</td>
</tr>
</tbody>
</table>

FUNCTION F

Purpose: To compute departures from the ES curve caused by partial penetration of pumped well.

Usage: F(U, RB, LB, DB, LPB, DPB)

Description of Parameters:
- ALL REAL, U DOUBLE PRECISION
  - U = N*2*8.4*TIME (RADIAL DISTANCE SQUARED * STORAGE COEFFICIENT / 4*TRANSMISSIVITY * TIME)
  - RB = R/B (RADIAL DISTANCE / AQUIFER THICKNESS)
  - LB = L/B (FRACTION OF AQUIFER PENETRATED BY PUMPED WELL)
  - DB = D/B (FRACTION OF AQUIFER ABOVE PUMPED WELL SCREEN)
  - LPB = L'/B (FRACTION OF AQUIFER PENETRATED BY OBS, WELL, ZERO FOR PIEZOMETEER)
  - DPB = D'/B (FRACTION OF AQUIFER ABOVE OBS, WELL SCREEN, TOTAL DEPTH FOR PIEZOMETER)

Methods:
- Sums the series through N*PI*R/8 EQ 20

REAL*8 U, V
REAL*4 L, N, LB, LPB
SUM=0,
N=0,
PIRB=3.141593*RB
PLB=3.141593*LB
PBDB=3.141593*DB
IF (LPB=0.) 1, 1

1 CHECKS FOR WELL OR PIEZOMETER
   PIZB=3.141593*DPB

2 N=N+1,
   V=N*PIRB/2,
   IF (V,GT,10.) GO TO 3

C TRUNCATES SERIES WHEN V>10
X=L(U,V)/N
TABLE 2.1.—Listing of program for partial penetration in a nonleaky artesian aquifer—Continued

<table>
<thead>
<tr>
<th>Line</th>
<th>Code</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>45</td>
<td>F 43</td>
<td>SUM = SUM + (SIN(N<em>PI/LB) + SIN(N</em>PI/DB)) * COS(N*PI/DB) * X</td>
</tr>
<tr>
<td>46</td>
<td>F 44</td>
<td>GO TO 2</td>
</tr>
<tr>
<td>47</td>
<td>F 45</td>
<td>3 F = 0.366198 * SUM / (LB * DB)</td>
</tr>
<tr>
<td>48</td>
<td>F 46</td>
<td>GO TO 7</td>
</tr>
<tr>
<td>49</td>
<td>F 47</td>
<td>4 PILPB = 3.141593 * LPB</td>
</tr>
<tr>
<td>50</td>
<td>F 48</td>
<td>PIDPB = 3.141593 * DPB</td>
</tr>
<tr>
<td>51</td>
<td>F 49</td>
<td>5 N = 1</td>
</tr>
<tr>
<td>52</td>
<td>F 50</td>
<td>V = PILPB / 2</td>
</tr>
<tr>
<td>53</td>
<td>F 51</td>
<td>IF (V GT 10.) GO TO 6</td>
</tr>
<tr>
<td>54</td>
<td>F 52</td>
<td>TRUNCATES SERIES WHEN V &gt; 10</td>
</tr>
<tr>
<td>55</td>
<td>F 53</td>
<td>X = L(U, V) / N</td>
</tr>
<tr>
<td>56</td>
<td>F 54</td>
<td>SUM = SUM + (SIN(N<em>PI/LB) + SIN(N</em>PI/DB)) * (SIN(N<em>PI/LPB) + SIN(N</em>PI/DPB)) * X / N</td>
</tr>
<tr>
<td>57</td>
<td>F 55</td>
<td>GO TO 5</td>
</tr>
<tr>
<td>58</td>
<td>F 56</td>
<td>6 F = 2026424 * SUM / ((LB + DB) * (LB + DPB))</td>
</tr>
<tr>
<td>59</td>
<td>F 57</td>
<td>RETURN</td>
</tr>
</tbody>
</table>

REAL FUNCTION L*(U, V)

FUNCTION L

PURPOSE
TO COMPUTE THE INTEGRAL \( \left( \exp\left( -y = v^{**2}/y \right)/y \right) \) SUMMED OVER \( y \) FROM \( U \) TO INFINITY (WELL FUNCTION FOR LEAKY AQUIFERS).

DESCRIPTION OF PARAMETERS

* BOTH DOUBLE PRECISION *
* U = R**2*3/4*TIME (RADIAL DISTANCE SQUARED * STORAGE COEFFICIENT / 4*TRANSMISSIVITY * TIME)
* V = R/2*SQRT(K1/(T*8')) = ONE*HALF RADIAL DISTANCE*SQUARE ROOT (HYD. COND. OF CONFINING BED/TRANSMISSIVITY*THICKNESS OF CONFINING BED)

SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
DQ12, SERIES, BESK, FCT

METHOD
IN THE FOLLOWING F = \( \exp\left( -y = v^{**2}/y \right)/y \)

(1) \( U > 1 \), USES A GAUSSIAN-LAGUERRE QUADRATURE FORMULA TO EVALUATE INTEGRAL(F) FROM U TO INF;
(2) \( 1 < U < 1 \), USES THE G-L QUADRATURE TO EVALUATE INTEGRAL(F) FROM ONE TO INF AND A SERIES EXPANSION TO EVALUATE INTEGRAL(F) FROM U TO ONE;
(3) \( U < 1 \), USES THE REPRESENTATION INTEGRAL(F) FROM U TO INF, = \( 2*K0(2*v)*INTEGRAL(F) \) FROM \( v^{**2}/u \) TO INF, EVALUATES THE ZERO ORDER MODIFIED BESSEL FUNCTION OF SECOND KIND WITH IBM SUBROUTINE, EVALUATES INTEGRAL BY G-L QUAD;

EXTERNAL FCT
REAL*8 U, V, Z, F, VV, SERIES
COMMON /C1/ VV, Z
VV = V
IF (U = 1.) 1, 2
C CHECKS IF U < 1
1 Z = V**V / U
C CHECKS IF Z < 1
2 Z = U
CALL DQL12(FCT, F)
C INTEGRAL U TO INF, EVALUATED BY GAUSS-LAGUERRE QUADRATURE
GO TO 5
3 Z = 1.5

***********

***************
Table 2.1.—Listing of program for partial penetration in a nonleaky artesian aquifer—Continued

CALL DQL12(FCT,F)                  L 46
L=N+SERIES(U,V)                      L 47
CALL INTEGRAL 1 TO INF. BY GL=QUAD., INTEGRAL U TO 1 BY SERIES EXP., L 48
GO TO 5                              L 49
4 TWOV=2.*V                           L 50
CALL BESK(TWOV,0,BK,IER)            L 51
CALL DQL12(FCT,F)                    L 52
L=2.*BK=F                            L 53
2K0(2V)=INTEGRAL V**2/U TO INF.     L 54
5 RETURN                              L 55
END                                    L 56

REAL FUNCTION SERIES=S(U,V)          L 56
********************************************************************************************** L 56
FUNCTION SERIES                     L 56
PURPOSE                              L 56
TO EVALUATE S(1)=S(U), WHERE S IS A SERIES EXPANSION OF          L 56
INTEGRAL(EXP(-Y**2/2)DY/Y) GIVEN BY S=SUM, M=0 TO INFINITY,          L 56
(F(M))SUM, M=0 TO INF,(V**((2*M)+(1)))*((M+N)!)) WHERE F(M)=          L 56
LUG(U) IF M=0 AND M=((1)**M/M)*((U**M-((V+2)/U)**M) IF M>0.          L 56
DESCRIPTION OF PARAMETERS           L 56
U = R**2/4*T*TIME (RADIAL DISTANCE SQUARED*STORAGE           L 56
COEFFICIENT/TIME)                   L 56
V = R**2/2*SQRT(K1/(T*B1))==ONE-HALF RADIAL DISTANCE*SQUARE ROOT (1/4D, COND. OF CONFINING BED/TRANSMISSIVITY*THICKNESS           L 56
OF CONFINING BED)                   L 56
SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED           L 56
NONE                              L 56
METHOD                              L 56
SUMMATION IS TERMINATED FOR THE INNER SERIES WHEN A TERM           L 56
BECOMES LESS THAN 5*E-7/N AND FOR OUTER SERIES WHEN A TERM           L 56
BECOMES LESS THAN 5*E-7              L 56
********************************************************************************************** L 56
REAL*8 DLUG,DABS,S(2),VUM,UU        L 56
REAL*8 TEST,U,UM,EM,EN,SUM1,SUM,SIGN,V,V8G,VSGU,RMUL,TERT1,TERM1     L 56
TEST=5,D=07                        L 56
VSGU=V*V                           L 56
UU=U                              L 56
 DM 1=1/2                          L 56
C EVALUATES SERIES FOR LOWER LIMIT = U AND UPPER LIMIT = 1           L 56
IF (I,EU,2) U=1,                    L 56
UM=1                               L 56
EM=1                               L 56
SUM=0                               L 56
SIGN=1                             L 56
VUM=1                              L 56
VSGU=V*V                           L 56
1 EM=EM+1                           L 56
C CHECKS FOR M=0                     L 56
2 RMUL=DMDLUG(U)                    L 56
TERT1=1                            L 56
GO TO 4                            L 56
3 UMM=UM*U                          L 56
IF (UMM,LT,1,D=30) VUM=0,           L 56
VUM=VUM*VSGU                       L 56
RMUL=(UMM*VUM)/EM                  L 56
TERT1=TERT1/EM                     L 56
Table 2.1.—Listing of program for partial penetration in a nonleaky artesian aquifer—Continued

4 SIGN=SIGN
SUM=TERM
TERM=TERM
EN=0,
5 EN=EN+1,
TERM=TERM*VA/(EN*(EN+EN))
SUM=SUM+TERM
IF (TEST.LE,DA8(RMUL,EN*TERM)) GO TO 5
C TRUNCATES INNER SERIES IF OUTER TERM*INNER TERM < 5E7
SUM=SUM+SIGN*RMUL*SUM
IF (EN.LE,1) GO TO 1
IF (TEST.LE,DA8(RMUL,SUM)) GO TO 1
C TRUNCATES OUTER SERIES IF OUTER TERM*INNER SUM < 5E7
b S(1)=SUM
U=U
 SERIES=S(2)-S(1)
RETURN
END

REAL FUNCTION FCT8(X)
FUNCTION FCT8
PURPOSE
TO COMPUTE FCT8(X)=EXP(-Z**2/(X+Z))/(X+Z)
DESCRIPTION OF PARAMETERS
X — THE DOUBLE PRECISION VALUE
SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
NONE
METHOD
FORTRAN EVALUATION OF FUNCTION

REAL*8 X,Z,PSUM
COWhUN /Cl/ V,i! FCT8
1 IF (X LE 0.0) GO TO 4
FCT8=0.
GO TO 4
2 P=Z**2/(X+Z)
IF (P-5.01) 3,3,1
FCT8=PSUM/(X+Z)
3 FCT8=DEXP(-P)/(X+Z)
4 RETURN
END

SUBROUTINE DQL12(FCT,Y)

SUBROUTINE DQL12
PURPOSE
TO COMPUTE INTEGRAL(EXP(-X)*FCT(X), SUMMED OVER X FROM 0 TO INFINITY).
USAGE
CALL DQL12 (FCT,Y)
PARAMETER FCT REQUIRES AN EXTERNAL STATEMENT
DESCRIPTION OF PARAMETERS
FCT — THE NAME OF AN EXTERNAL DOUBLE PRECISION FUNCTION SUBPROGRAM USED.
Y — THE RESULTING DOUBLE PRECISION INTEGRAL VALUE.
Table 2.1.—Listing of program for partial penetration in a nonleaky artesian aquifer—Continued

**REM**

**NONE**

**SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED**

THE EXTERNAL DOUBLE PRECISION FUNCTION SUBPROGRAM FCT(X) MUST BE FURNISHED BY THE USER.

**METHOD**

EVALUATION IS DONE BY MEANS OF 12-POINT GAUSSIAN-LAGUERRE QUADRATURE FORMULA, WHICH INTEGRATES EXACTLY, WHENEVER FCT(X) IS A POLYNOMIAL UP TO DEGREE 23.

FOR REFERENCE, SEE SHAO/CHEN/FRANK, TABLES OF ZEROS AND GAUSSIAN WEIGHTS OF CERTAIN ASSOCIATED LAGUERRE POLYNOMIALS AND THE RELATED GENERALIZED HERMITE POLYNOMIALS, IBM TECHNICAL REPORT TR00,1100 (MARCH 1964), PP. 24=25.

**DOUBLE PRECISION X,Y,FCT**

```c
DOUBLE PRECISION X,Y,FCT
X=.37099121044446920 D2
Y=.8148077467426240 D=15*FCT(X)
X=.26487967250984 D2
Y=.30161560450350210 D=11*FCT(X)
X=.2215104037937010 D2
Y=.134239103051502 D=8*FCT(X)
X=.17116855187462260 D2
Y=.1688493876540910 D=6*FCT(X)
X=.1300854993306350 D2
Y=.8365050585681980 D=5*FCT(X)
X=.9621316842456870 D1
Y=.20323159266299940 D=4*FCT(X)
X=.6846525431151770 D1
Y=.26639735416653160 D=2*FCT(X)
X=.459992276394183480 D1
Y=.2010236115463410 D=1*FCT(X)
X=.28337513377435070 D1
Y=.9044922221168090 D=1*FCT(X)
X=.15128102677764190 D1
Y=.244038201131987760 D=0*FCT(X)
X=.6175749451512070 D0
Y=.37775277258731380 D=0*FCT(X)
X=.11572211735802070 D0
Y=.3797313710594432 D=0*FCT(X)
RETURN
END
SUBROUTINE BESK(X,N,BK,IER)
```

**SUBROUTINE BESK**

COMPUTE THE K BESSEL FUNCTION FOR A GIVEN ARGUMENT AND ORDER BESK

**USAGE**

CALL BESK(X,N,BK,IER)
Table 2.1.—Listing of program for partial penetration in a nonleaky artesian aquifer—Continued

DESCRIPTION OF PARAMETERS

* THE ARGUMENT OF THE K BESSEL FUNCTION DESIRED
* THE ORDER OF THE K BESSEL FUNCTION DESIRED
* THE RESULTANT K BESSEL FUNCTION
* RESULTANT ERROR CODE WHERE
  IER=0 NO ERROR
  IER=1 X IS NEGATIVE
  IER=2 X IS ZERO OR NEGATIVE
  IER=3 X > 170, MACHINE RANGE EXCEEDED
  IER=6 BK > 10^70

REMARKS

N MUST BE GREATER THAN OR EQUAL TO ZERO

SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED

NONE

METHOD

COMPUTES ZERO ORDER AND FIRST ORDER BESSEL FUNCTIONS USING SERIES APPROXIMATIONS AND THEN COMPUTES NTH ORDER FUNCTION USING RECURRENCE RELATION.


*********-------------------------------------------------------------------------
DIMENSION T(12)
BK=0
IF(N)10,11,11
10 IER=1
RETURN
11 IF(X)12,12,20
12 IER=2
RETURN
20 IF(X=170,0)22,22,21
21 IER=3
RETURN
22 IER=0
IF(X=1)30,36,25
25 A=EXP(-X)
B=1/X
C=SQR(T(8))
T(1)=B
DO 26 L=2,12
26 T(L)=T(L-1)*B
IF(N)27,29,27
27 GO TO(1,2533141=-15666042*T(1)+8811128*T(2)-9139095*T(3)
2+144596*T(4)+2299850*T(5)+3792410*T(6)+524777*T(7)
3+5575368*T(8)-4262633*T(9)+2184518*T(10)+8680977*T(11)
4+800918033*T(12)+C
IF(N)20,28,29
28 BK=0
RETURN
TECHNIQUES OF WATER-RESOURCES INVESTIGATIONS

Table 2.1.—Listing of program for partial penetration in a nonleaky artesian aquifer—Continued

**COMPUTE K1 USING POLYNOMIAL APPROXIMATION**

29 $G_1 = \left( \frac{1.253341 \times 10^{-6} - 1.280427 \times 10^{-6}}{t(1)} + \frac{2.847816 \times 10^{-6} - 4.594342 \times 10^{-6}}{t(5)} + \frac{6.283381 \times 10^{-6}}{t(7)} \right) \times C$

**RETURN**

**FROM K0, K1 COMPUTE KN USING RECURRANCE RELATION**

31 DO 35 J = 2, N

$G_2 = \left( \frac{\text{FLOAT}(J-1)}{G_1} \right) \times \alpha + \beta$

IF (J = 1, \text{FLOAT}(0.0), 0.0)

**RETURN**

**COMPUTE K0 USING SERIES EXPANSION**

37 $G_0 = \text{A}$

**RETURN**

**COMPUTE K1 USING SERIES EXPANSION**

43 $G_1 = \text{B}$

**RETURN**
**Table 2.1:** Listing of program for partial penetration in a nonleaky artesian aquifer—Continued

```plaintext
SUBROUTINE EXPI(X,RES,AUX)  

**************************************************************************  
SUBROUTINE EXPI  

PURPOSE  
COMPUTES THE EXPONENTIAL INTEGRAL "Ei(x)"  

USAGE  
CALL EXPI(X,RES)  

DESCRIPTION OF PARAMETERS  
X = ARGUMENT OF EXPONENTIAL INTEGRAL  
RES = RESULT VALUE  
AUX = RESULTANT AUXILIARY VALUE  

REMARKS  
X GT 170 (X LT -174) MAY CAUSE UNDERFLOW (OVERFLOW)  
WITH THE EXPONENTIAL FUNCTION  
FOR X = 0 THE RESULT VALUE IS SET TO 1, E75  

SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED  
NONE  

METHOD  
DEFINITION  
RES=INTEGRAL(EXP(-T)/T, SUMMED OVER T FROM X TO INFINITY).  
EVALUATION  
THREE DIFFERENT RATIONAL APPROXIMATIONS ARE USED IN THE RANGES 1 LE X, X LE -9 AND -9 LT X LE -3 RESPECTIVELY,  
A POLYNOMIAL APPROXIMATION IS USED IN -3 LT X LT 1.  

**************************************************************************  

IF(X=1)2,1  
1 Y=1./X  
   AUX=X*Y*(1.0-EXP(-X))  
   RETURN  
2 IF(X=3)6,2  
3 AUX=X*(1.0-EXP(-X))  
   RETURN  
4 IF(X=9)9,8  
5 RETURN  
6 IF(X=9*8,9,7  
7 AUX=X*(1.0-EXP(-X))  
   RETURN  
8 Y=1./X  
   AUX=Y*(1.0-EXP(-X))  
   RETURN  
9 RETURN  
END  

```

**TABLE 2.2:** Listing of program for partial penetration in a nonleaky artesian aquifer—Continued

```plaintext
SUBROUTINE EXPI(X,RES,AUX)  

**************************************************************************  
SUBROUTINE EXPI  

PURPOSE  
COMPUTES THE EXPONENTIAL INTEGRAL "Ei(x)"  

USAGE  
CALL EXPI(X,RES)  

DESCRIPTION OF PARAMETERS  
X = ARGUMENT OF EXPONENTIAL INTEGRAL  
RES = RESULT VALUE  
AUX = RESULTANT AUXILIARY VALUE  

REMARKS  
X GT 170 (X LT -174) MAY CAUSE UNDERFLOW (OVERFLOW)  
WITH THE EXPONENTIAL FUNCTION  
FOR X = 0 THE RESULT VALUE IS SET TO 1, E75  

SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED  
NONE  

METHOD  
DEFINITION  
RES=INTEGRAL(EXP(-T)/T, SUMMED OVER T FROM X TO INFINITY).  
EVALUATION  
THREE DIFFERENT RATIONAL APPROXIMATIONS ARE USED IN THE RANGES 1 LE X, X LE -9 AND -9 LT X LE -3 RESPECTIVELY,  
A POLYNOMIAL APPROXIMATION IS USED IN -3 LT X LT 1.  

**************************************************************************  

IF(X=1)2,1  
1 Y=1./X  
   AUX=X*Y*(1.0-EXP(-X))  
   RETURN  
2 IF(X=3)6,2  
3 AUX=X*(1.0-EXP(-X))  
   RETURN  
4 IF(X=9)9,8  
5 RETURN  
6 IF(X=9*8,9,7  
7 AUX=X*(1.0-EXP(-X))  
   RETURN  
8 Y=1./X  
   AUX=Y*(1.0-EXP(-X))  
   RETURN  
9 RETURN  
END  
```
PURPOSE
To compute a table of values of the leaky aquifer well function \( W(u,R/B) = HANTUSH, M, S, \) and JACOB, C, E., 1955, non-steady radial flow in an infinite leaky aquifer, Am. Geophys. Union Trans., V, 36, NO. 1, P, 95-100.

INPUT DATA
1 card = FORMAT(2E10.5)
   USMALL = smallest value of \( 1/u \) for which computation is desired.
   U_LARGE = largest value of \( 1/u \) for which computation is desired.
2 cards = FORMAT(8E10.5)
   BDAT = 12 values of \( R/B \) for table.
SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
L, SERIES, FCT, BESK, DQL12

FUNCTION \( L(U,V) \)

REAL*4 L
REAL*8 U, V
DIMENSION ARRAY(73,12), Y(73), BDAT(12), YNUM(6)
DATA YNUM/1, 1, 5, 2, 3, 5, 7, 9, DATA
IRD=6
IPT=6
READ (IRD, 6) USMALL, U_LARGE
READ (IRD, 6) BDAT
BEGIN=ALOG10(USMALL)
END=ALOG10(U_LARGE)+,99999
ILIMIT=END-BEGIN)*6+1
IF (ILIMIT, GT, 73) ILIMIT=73
DO 1 I=1, 12
   IF (BDAT(I), EQ, 0.) GO TO 2
1 CONTINUE
NB=12
GO TO 3
2 NB=1
3 I=0
DO 4 I=1, ILIMIT
   II=II+1
   IF (II, GT, 6) II=1
   IEXP=BEGIN*(I-1) / 6
   Y(I)=YNUM(II)*10.**IEXP
   UM=1. / Y(I)
   DO 4 J=1, NB
      V=BDAT(J) / 2,
   4 ARRAY(I, J) = L(U, V)
   WRITE (IPT, 7) (BDAT(I), I=1, NB)
   DO 5 I=1, ILIMIT
      5 WRITE (IPT, 8) Y(I), (ARRAY(I, J), J=1, NB)
STOP

6 FORMAT (8E10.5)
7 FORMAT (' 'rE10,S12F10.5)
8 FORMAT (' ', E10.3, 12F10.4)
END
REAL FUNCTION L*U(U,V)

FUNCTION L
### Table 4.3—Listing of program for radial flow in a leaky artesian aquifer—Continued

**Purpose**

To compute the integral \( \exp(-v \cdot v^2/y) \) summed over \( y \) from \( u \) to infinity (well function for leaky aquifers).

**Description of Parameters**

- Both double precision
  - \( u = R^2/4 \cdot \text{TIME} \) (radial distance squared * storage coefficient / 4 * transmissivity * time)
  - \( v = R/2 \cdot \text{SQRT}(k'/(T+B')) \) = one-half radial distance * square root of (hyd. cond. of confining bed * transmissivity * thickness of confining bed)

**Subroutines and Function Subprograms Required**

DQL12, SERIES, BESK, FCT

**Method**

In the following \( \exp(-v \cdot v^2/y) \) / \( y \)

1. \( u > 1 \), uses a Gaussian-Laguerre quadrature formula to evaluate integral \( f \) from \( u \) to \( x \).
2. \( v^2 < u < 1 \), uses the G-L quadrature to evaluate integral \( f \) from one to infinity and a series expansion to evaluate integral \( f \) from \( u \) to one.
3. \( u < 1 \), uses the representation integral \( f \) from \( u \) to infinity, \( \exp(2 \cdot k_0(2 \cdot v) \cdot \text{integral} f \) from \( v^2/y \) to infinity, evaluates the zero order modified Bessel function of second kind with ibm subroutine, evaluates integral by g-l quad.

```plaintext
******************************
EXTERNAL FCT
REAL*8 u,v,vz,f,vv,series
COMMON /C1/ VV,v
VV=v
IF (u>1) 1,2,2
C CHECKS IF U<1
1 z=vz/v
IF (Z>1) 3,4,4
C CHECKS IF V**2/U < 1
2 zu
C INTEGRAL U TO INF, EVALUATED BY GAUSS-LAGUERRE QUADRATURE
CALL DQL12(FCT,F)
L F
GO TO 5
3 zui
C INTEGRAL 1 TO INF, BY G-L QUAD., INTEGRAL U TO 1 BY SERIES EXP.
CALL DQL12(FCT,F)
L F+SERIES(U,v)
C INTEGRAL 1 TO INF, BY G-L QUAD., INTEGRAL U TO 1 BY SERIES EXP.
GO TO 5
4 t=vz/v
CALL BESK(T,vz,0,8k,IER)
CALL DQL12(FCT,F)
L 2,8K=F
L=2*(2v)*integral v**2/u tu inf.
5 RETURN
END
REAL FUNCTION SERIES#8(U,v)
FUNCTION SERIES
```

**Purpose**

To evaluate \( s(u) \), where \( s \) is a series expansion of \( \exp(-v \cdot v^2/y) \) given by \( s = \sum \text{SUM} \cdot \exp(-v^2/y) \) / \( y \) \( (v=2 \cdot n)/(n+1) \cdot (n+1) \) \( \) where \( f(m)= \)

**Description of Parameters**

- Both double precision
  - \( u = R^2/4 \cdot \text{TIME} \) (radial distance squared * storage)
  - \( v = R/2 \cdot \text{SQRT}(k'/(T+B')) \) = one-half radial distance * square root of (hyd. cond. of confining bed * transmissivity * thickness of confining bed)
Table 4.3—Listing of program for radial flow in a leaky artesian aquifer—Continued

<table>
<thead>
<tr>
<th>Line</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED</td>
</tr>
<tr>
<td>2</td>
<td>NONE</td>
</tr>
<tr>
<td>3</td>
<td>METHOD</td>
</tr>
<tr>
<td>4</td>
<td>SUMMATION IS TERMINATED FOR THE INNER SERIES WHEN A TERM BECOMES LESS THAN 5 E - 7 AND FOR OUTER SERIES WHEN A TERM BECOMES LESS THAN 5 E - 7</td>
</tr>
<tr>
<td>5</td>
<td>REAL DLOG,DA88,9(2),VUM,UU</td>
</tr>
<tr>
<td>6</td>
<td>REAL TST,U,UM,EN,SUM1,SUM,SIGN,V,VSQ,VSQ,TH,TERM,TERM1</td>
</tr>
<tr>
<td>7</td>
<td>TEST=5.0,D=0.7</td>
</tr>
<tr>
<td>8</td>
<td>VSQ=V*V</td>
</tr>
<tr>
<td>9</td>
<td>UUM=U</td>
</tr>
<tr>
<td>10</td>
<td>DO 6 I=1,2</td>
</tr>
<tr>
<td>11</td>
<td>EVALUATES SERIES FOR LOWER LIMIT = U AND UPPER LIMIT = 1</td>
</tr>
<tr>
<td>12</td>
<td>IF (I,EQ,2) U=1,</td>
</tr>
<tr>
<td>13</td>
<td>UM=1,</td>
</tr>
<tr>
<td>14</td>
<td>EM=1,</td>
</tr>
<tr>
<td>15</td>
<td>SUM1=0,</td>
</tr>
<tr>
<td>16</td>
<td>SIGN=1,</td>
</tr>
<tr>
<td>17</td>
<td>VUM=1,</td>
</tr>
<tr>
<td>18</td>
<td>VSQ=VSQ/U</td>
</tr>
<tr>
<td>19</td>
<td>1 EM=EM+1,</td>
</tr>
<tr>
<td>20</td>
<td>IF (EM=1) 2,3,3</td>
</tr>
<tr>
<td>21</td>
<td>CHECKS FOR M=0</td>
</tr>
<tr>
<td>22</td>
<td>RMUL=DLOG(U)</td>
</tr>
<tr>
<td>23</td>
<td>TERM1=1,</td>
</tr>
<tr>
<td>24</td>
<td>GO TO 4</td>
</tr>
<tr>
<td>25</td>
<td>3 UM=UM/U</td>
</tr>
<tr>
<td>26</td>
<td>IF (VUM,LT,1,D=30) VUM=0,</td>
</tr>
<tr>
<td>27</td>
<td>VUM=VUM*VSQ</td>
</tr>
<tr>
<td>28</td>
<td>RMUL=(UM*VUM)/EM</td>
</tr>
<tr>
<td>29</td>
<td>TERM1=TERM1/EM</td>
</tr>
<tr>
<td>30</td>
<td>4 SIGN=SIGN</td>
</tr>
<tr>
<td>31</td>
<td>SUM=TERM1</td>
</tr>
<tr>
<td>32</td>
<td>TERM=TERM1</td>
</tr>
<tr>
<td>33</td>
<td>EN=0,</td>
</tr>
<tr>
<td>34</td>
<td>5 EN=EN+1,</td>
</tr>
<tr>
<td>35</td>
<td>TERM=TERM<em>VSQ/(EN</em>(EN+EM))</td>
</tr>
<tr>
<td>36</td>
<td>SUM=SUM+TERM</td>
</tr>
<tr>
<td>37</td>
<td>IF (TEST,LE,DABB(RMUL*EN+TERM)) GO TO 5</td>
</tr>
<tr>
<td>38</td>
<td>TRUNCATES INNER SERIES IF OUTER TERM*INNER TERM &lt; 5 E - 7</td>
</tr>
<tr>
<td>39</td>
<td>SUM1=SUM1+SIGN<em>RMUL</em>SUM</td>
</tr>
<tr>
<td>40</td>
<td>IF (EN=2,EQ,1) GO TO 1</td>
</tr>
<tr>
<td>41</td>
<td>IF (TEST,LE,DABB(RMUL*SUM)) GO TO 1</td>
</tr>
<tr>
<td>42</td>
<td>TRUNCATES OUTER SERIES IF OUTER TERM*INNER SUM &lt; 5 E - 7</td>
</tr>
<tr>
<td>43</td>
<td>S(1)=SUM1</td>
</tr>
<tr>
<td>44</td>
<td>U=UUM</td>
</tr>
<tr>
<td>45</td>
<td>RETURN</td>
</tr>
<tr>
<td>46</td>
<td>END</td>
</tr>
<tr>
<td>47</td>
<td>REAL FUNCTION FCT*8(X)</td>
</tr>
<tr>
<td>48</td>
<td>FUNCTION FCT</td>
</tr>
<tr>
<td>49</td>
<td>PURPOSE</td>
</tr>
<tr>
<td>50</td>
<td>TO COMPUTE FCT(X)=EXP(-Z+V*2/(X+Z))/(X+Z)</td>
</tr>
</tbody>
</table>
Table 4.3—Listing of program for radial flow in a leaky artesian aquifer—Continued

**DESCRIPTION OF PARAMETERS**

- \( X \) = THE DOUBLE PRECISION VALUE OF \( X \) FOR WHICH FCT IS COMPUTED

**SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED**

- NONE

**METHOD**

- FORTRAN EVALUATION OF FUNCTION

**FUNCTION FCT**

```fortran
REAL*8 X,VAZ,P,DEXP
COMMON /C1/V,Z
IF (X<1.2) GO TO 4
2 P=Z+V**2/(X*Z)
IF (P>5.0) 3,3,1
3 FCT=DEXP(P)/(X+Z)
4 RETURN
END
```

**SUBROUTINE DQL12**

```fortran
SUBROUTINE DQL12(FCT,Y)
CALL DQL12 (FCT,Y)
PARAMETER FCT REQUIRES AN EXTERNAL STATEMENT
DESCRIPTION OF PARAMETERS
FCT = THE NAME OF AN EXTERNAL DOUBLE PRECISION FUNCTION
SUBPROGRAM USED,
Y = THE RESULTING DOUBLE PRECISION INTEGRAL VALUE,
REMARKS
NONE
SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
THE EXTERNAL DOUBLE PRECISION FUNCTION SUBPROGRAM FCT(X)
MUST BE FURNISHED BY THE USER.
METHOD
EVALUATION IS DONE BY MEANS OF 12-POINT GAUSSIAN-LAGUERRE
QUADRATURE FORMULA, WHICH INTEGRATES EXACTLY
WHENEVER FCT(X) IS A POLYNOMIAL UP TO DEGREE 23,
FOR REFERENCE, SEE
SHAO/CHEN/FRANK, TABLES OF ZEROS AND GAUSSIAN WEIGHTS OF
CERTAIN ASSOCIATED LAGUERRE POLYNOMIALS AND THE RELATED
GENERALIZED HERMITE POLYNOMIALS, IBM TECHNICAL REPORT
TR00,1100 (MARCH 1964), PP.24-25.
```

DOUBLE PRECISION X,Y,FCT

\( X=3709912106446692 \) \( D2 \)
\( Y=814807746742624 \) \( D0=15*FCT(X) \)
Table 4.3—Listing of program for radial flow in a leaky artesian aquifer—Continued

```
x+y=28.48796725098400 D2
x+y=30.016401650335021 D=11*FCT(X)
x+y=221.510937939701 D2
x+y=14.235910305505040 D=8*FCT(X)
x+y=17.1166516746226 D2
x+y=16.64843876540910 D=6*FCT(X)
x+y=13.00005499339853 02
x+y=8.36505585681980 D=5*FCT(X)
x+y=9.62116482656870 D2
x+y=0.2032315926629994 D=3*FCT(X)
x+y=8.84452545311577 01
x+y=5.2663973541865316 D=2*FCT(X)
x+y=4.99922763918348 01
x+y=2.0102381115463410 D=1*FCT(X)
x+y=2.1523751137743507 01
x+y=9.0449222116809 D=1*FCT(X)
x+y=1.51261026797419 01
x+y=2.440420813198776 00*FCT(X)
x+y=6.11757845151307 D0
x+y=3.777592758731380 D0*FCT(X)
x+y=1.15722173580207 00
x+y=2.64731371055438 00*FCT(X)
RETURN
END
SUBROUTINE BESK(X,N,BK,IER)
```

---

```
SUBROUTINE BESK(X,N,BK,IER)
    REAL X,N,BK,IER
    CALL BESK(X,N,BK,IER)
    BESK 60
    BESK 70
    BESK 80
    BESK 90
    BESK 100
    BESK 110
    BESK 120
    BESK 130
    BESK 140
    BESK 150
    BESK 160
    BESK 170
    BESK 180
    BESK 190
    BESK 200
    BESK 210
    BESK 220
    BESK 230
    BESK 240
    BESK 250
    BESK 260
    BESK 270
    BESK 280
```

---

**SUBROUTINE BESK**

**COMPUTE THE K BESSEL FUNCTION FOR A GIVEN ARGUMENT AND ORDER**

**USAGE**

`CALL BESK(X,N,BK,IER)`

**DESCRIPTION OF PARAMETERS**

- **X** = THE ARGUMENT OF THE K BESSEL FUNCTION DESIRED
- **N** = THE ORDER OF THE K BESSEL FUNCTION DESIRED
- **BK** = THE RESULTANT K BESSEL FUNCTION
- **IER** = RESULTANT ERROR CODE WHERE
  - **IER** = 0 NO ERROR
  - **IER** = 1 N IS NEGATIVE
  - **IER** = 2 X IS ZERO OR NEGATIVE
  - **IER** = 3 X > 170, MACHINE RANGE EXCEEDED
  - **IER** = 5 BK > 10**70

**REMARKS**

- **N** MUST BE GREATER THAN OR EQUAL TO ZERO

**SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED**

NONE

**METHOD**

- **COMPUTES ZERO ORDER AND FIRST ORDER BESSEL FUNCTIONS USING SERIES APPROXIMATIONS AND THEN COMPUTES NTH ORDER FUNCTION USING RECURRENCE RELATION.**
- **RECURRENCE RELATION AND POLYNOMIAL APPROXIMATION TECHNIQUE AS DESCRIBED BY A.J.HITCHCOCK, POLYNOMIAL APPROXIMATIONS TO BESSEL FUNCTIONS OF ORDER ZERO AND ONE AND TO RELATED FUNCTIONS**
Table 4.3—Listing of program for radial flow in a leaky artesian aquifer—Continued

```
DIMENSION T(12)
B=(.0
IF(N)10,11,11
10 IER=1
RETURN
11 IF(X)12,12,20
12 IER=2
RETURN
20 IF(X=170,0)22,22,21
21 IER=3
RETURN
22 IER=0
IF(X=1,36,36,25
25 A=EXP(X)
B=1.1/X
C=SQRT(B)
T(1)=B
DO 26 L=2,12
26 T(L)=T(L-1)*B
IF(N)20,20,29
27 G0=A*(1.2533141+.1566642*T(1)+.08811128*T(2)+.09139095*T(3)
2+,1344596*T(4)+.2299850*T(5)+.3792410*T(6)+.5247277*T(7)
3+,575368*T(8)+.4262633*T(9)+.2184518*T(10)+.06680977*T(11)
4+,009189383*T(12))*C
IF(N)20,20,29
28 BK=G0
RETURN
29 G1=A*(1.2533141+.4699927*T(1)+.1468553*T(2)+.1280427*T(3)
2+,1736432*T(4)+.2847618*T(5)+.4594342*T(6)+.623381*T(7)
3+,5832295*T(8)+.5050239*T(9)+.2581304*T(10)+.0788001*T(11)
4+,01082418*T(12))*C
IF(N)20,20,29
30 BK=G1
RETURN
31 FROM K0,K1 COMPUTE KN USING RECURRENC RELATION
32 DO 35 J=2,N
GJ=2.*(FLOAT(J)-1.)*G1/X+G0
IF(GJ=1.0E70)33,33,32
33 IER=4
GO TO 34
34 G0=GJ
35 G1=GJ
34 BK=GJ
RETURN
36 B=X/2
A=.5772157*ALOG(B)
C=B*8
IF(N)137,43,37
38 COMPUTE K0 USING SERIES EXPANSION
```
Table 4.3—Listing of program for radial flow in a leaky artesian aquifer—Continued

```c
37 GO TO A
X2J1,1
FACT1
HJ=0
DU 40 J=1,6
RJ1=FLAT(J)
IF(X2J1,LT,1.0E00) X2J=-
C
PREVIOUS STATEMENT ADDED TO IBM SUBROUTINE TO CORRECT UNDERFLOW
PROBLEM ON MATFOR COMPILERS
X2J=X2J*C
FACT=FACT*RJ*RJ
HJ=HJ+RJ
GO TO 37
IF(N)40,43,43
BKa GO
RETURN

C
COMPUTE K1 USING SERIES EXPANSION

40 GO=GO*X2J*FACT*(HJ=A)
IF(N)43,42,43
42 BK=GO
RETURN

C
COMPUTE K1 USING SERIES EXPANSION

50 G1=G1*X2J*FACT*(A=HJ)*FLOAT(J))
IF(N)51,52,51
52 BK=G1
RETURN
END
```

Table 5.2—Listing of program for radial flow in a leaky artesian aquifer with storage of water in the confining beds

```c
C ******************************************************************************************************************
C C PURPOSE
C C (PROGRAM) TO COMPUTE TYPE CURVE FUNCTION VALUES FOR H(U,BETA) ==
C (HANTUSH J. K., GEOPHYS, RES., V. 65, NO. 11, P. 3713-3725.)
C THE COMPUTATIONAL ALGORITHM AS DEvised AND PHsogrammed By
C S. S. PAPAOUglUS
C C INPUT DATA
C 1 CARD = FORMAT(ZE10.5)
C USMALL = SMALLEST(Beginning) VALUE OF U/\ 
C USLARGE = LARGEST(ENDING) VALUE OF U/\ 
C 2 CARDS = FORMAT(8E10.5)
C BOAT = 12 VALUES OF BETA (ZERO OR BLANK VALUES ARE
C PERMITTED IF LESS THAN 12 DESIRED, WILL TERMINATE
C AT FIRST ZERO OR BLANK VALUE).
C SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C HS0, HS32, HUMAN = MUST BE INCLUDED IN DECK,
C USQRT, DEK, DERFC, DLOG = MUST BE IN COMPUTATIONAL LIBRARY,
C C
C ******************************************************************************************************************
C REAL*8 U, BETA, H
C DIMENSION ARRAY(73,12), V(73), BOAT(12), YNUM(6)
C DATA YNUM/1,1,1,1,1,1,1,1,1,1,1,1,
C INIT=5
C IPT=6
C READ (IMD,6) USMALL, USLARGE
C HEAD (IMD,6) BOAT
C BEGIN=UUG10(USSLARGE)
C END=UUG10(USMALL)
C LIMIT=(END-BEGININ)+1
C IF (LIMIT,GT,73) LIMIT=73
C DU 1=1/12
C IF (BOAT,0,0,0) GT 62
1 CONTINUE
C NB=12
```
Table 5.2—Listing of program for radial flow in a leaky artesian aquifer with storage of water in the confining beds—
Continued

```
GO TO 3
2 NUM=1
3 U=1. LST 37
I=0
3 DU=1. LST 39
4 IEXP=BEGIN(I)+1/V LST 40
5 I=I+1 LST 41
IF (I.GT.8) GO TO 1 LST 42
6 Y(I)=Y(I)*10.**IEXP LST 43
7 UB=Y(I) LST 44
8 GO TO 1 LST 45
9 BETADAT(I) LST 46
10 ARRAY(I,J)=M(U,BETA) LST 47
11 WRITE (IPT,7) (BDAT(I),I=1,NB) LST 48
12 GO TO 5 LST 49
13 WRITE (IPT,Y(I),(ARRAY(I,J),J=1,NB) LST 50
STOP
LST 51
5 WRITE (IPT,7) (BDAT(I),I=1,NB) LST 52
STOP
LST 53
6 FORMAT (8E10.5) LST 54
7 FORMAT (1,'(U(1),BETAL/100,10X,BETA/10,X,10E10.2) LST 55
8 FORMAT (1,'(U(1),12F10.4) LST 56
STOP
LST 57

DOUBLE PRECISION FUNCTION H(U,B)

FUNCTION H

PURPOSE

TO COMPUTE THE INTEGRAL OF EXP(-y)*ERFC(B*SQRT(U)/SQRT(Y*U))/(Y*U) FROM U TO INFINITY (FUNCTION H(U,BETA) OF HANTUS).

DESCRIPTION OF PARAMETERS

DOUBLE PRECISION

U = RADIAL DISTANCE SQUARED/ STORAGE COEFFICIENT (4 * TIME), U MUST BE > L.D-60.
B = (URT)/(STORAGE COEFFICIENT TIME), U RADIUS, STORAGE COEFFICIENT TIME, U MUST BE > 1.D-60.

METHOD

1, FOR U < 1.D-60, NO COMPUTATION IS MADE.
2, FOR B < 10, THE H(B,E) IS USED.
3, FOR B > 10, H(U,B) IS COMPUTED.
4, H(U,B) IS COMPUTED USING THE NORMAL INTEGRAL FORMULA.

IMPLICIT REAL*8(A-M,D-Z)

COMMON UU,BBB
EXTERNAL H

UU=U
BBBB=BB
IF (U.LT.1.D-60) GO TO 1
WRITE (6,7)
STOP
1 IF (B.EQ.0.0) GO TO 5
IF (U.GT.1.0.D-60) GO TO 6
BBBB=BBBB+BB
IF (B.GT.1.0.D-60) GO TO 3
BBBB=BBBB+BB
UP=10.0
UBS=UBS+(1.0+UBS*(1.0+0.025*B*B/U))
IF (UB.GT.UP) GO TO 6
UBS=UBS+(1.0+UBS*(1.0+D2*B*B/U))
IF (UB.GT.UP) GO TO 2
H=H(UUB)
UP=UP+UBS
GO TO 1
2 M=0
GO TO 1
3 X=U-10.0
IF (X.EQ.0.0) GO TO 5
CALL UUGZ2L(X,XU,H,BE,AREA)
5 H=M+H
IF (X.EQ.0.0) GO TO 1
GO TO 5
6 M=M+H
RETURN
5 H=M(U)
RETURN
```
Table 5.2—Listing of program for radial flow in a leaky artesian aquifer with storage of water in the confining beds—Continued

```c
6 RETURN
7 FORMAT ('U1', 'U2 SMALL FOR COMPUTATION')

SUBROUTINE DGG32(XL,XU,FCT,Y)

***************

SUBROUTINE DGG32

PURPOSE
TO COMPUTE INTEGRAL(FCT(X)), SUMMED OVER X FROM XL TO XU

USAGE
CALL DGG32 (XL,XU,FCT,Y)
PARAMETER FCT REQUIRES AN EXTERNAL STATEMENT

DESCRIPTION OF PARAMETERS
XL = DOUBLE PRECISION LOWER BOUND OF THE INTERVAL,
XU = DOUBLE PRECISION UPPER BOUND OF THE INTERVAL,
FCT = THE NAME OF AN EXTERNAL DOUBLE PRECISION FUNCTION
SUBPROGRAM USED,
Y = THE RESULTING DOUBLE PRECISION INTEGRAL VALUE.

REMARKS
NONE

SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
THE EXTERNAL DOUBLE PRECISION FUNCTION SUBPROGRAM FCT(X)
MUST BE FURNISHED BY THE USER.

METHOD
EVALUATION IS DONE BY MEANS OF 32-POINT GAUSS QUADRATURE
FORMULA, WHICH INTEGRATES POLYNOMIALS UP TO DEGREE 63
EXACTLY, FOR REFERENCE, SEE W. L. KRYLOV, APPROXIMATE CALCULATION OF INTEGRALS,

DOUBLE PRECISION XL,XU,Y,A,B,C,FCT

***************

RETURN
```
Table 5.2—Listing of program for radial flow in a leaky artesian aquifer with storage of water in the confining beds—Continued

```
**CONTINUED**

C
C FUNKTION EVALUATION OF FUNCTION
C
C ******************************************************************************
C
C IMPLICIT REAL(A-H,O-Z)
C C GESSETI((R**(4.*H)**(R**(1.25*U)))*(1.3**R**(1.3)))
C HUBEDEXP(*X)*DEPC(ANGYX)
C RETURN
C END
C
DOUBLE PRECISION FUNCTION U(LU)
C ******************************************************************************
C
C FUNCTION U
C PURPOSE TO EVALUATE THE WELL FUNCTION OF THE S
C DESCRIPTION OF PARAMETERS
C U = DOUBLE PRECISION, ARGUMENT FOR WELL FUNCTION,
C
C*******************************************************************************
C
C IMPLICIT REAL(A-H,O-Z)
C C GESSETI((R**(4.*H)**(R**(1.25*U)))*(1.3**R**(1.3)))
C HUBEDEXP(*X)*DEPC(ANGYX)
C RETURN
C END
C
C *****END OF COMPUTER PROGRAM*****
C
```

Table 6.1.—Listing of program for partial penetration in a leaky artesian aquifer

```
*******************************************************************************
PPL  1

PPL  2

PPL  3

PPL  4

PPL  5

PPL  6

PPL  7

PPL  8

PPL  9

PPL 10

PPL 11

PPL 12

PPL 13

PPL 14

PPL 15

PPL 16

PPL 17

PPL 18

PPL 19

PPL 20

PPL 21

PPL 22

PPL 23

PPL 24

PPL 25

PPL 26

PPL 27

PPL 28

PPL 29

```

Table 6.1.—Listing of program for partial penetration in a leaky artesian aquifer—Continued

<table>
<thead>
<tr>
<th>C</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>SCREEN (TOTAL DEPTH FOR PIEZOMETER)</td>
</tr>
<tr>
<td></td>
<td>SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED</td>
</tr>
<tr>
<td></td>
<td>DQL12, SERIES, BESK, FCT, LFL</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>**</th>
<th>**</th>
</tr>
</thead>
<tbody>
<tr>
<td>REAL*4 U, V</td>
<td>PPL 30</td>
</tr>
<tr>
<td>REAL*4 L, LB, LPB, LPRIME, LARGE</td>
<td>PPL 31</td>
</tr>
<tr>
<td>DIMENSION ARRAY(55, 12), ARG(6), BDAT(12), Y(55)</td>
<td>PPL 32</td>
</tr>
<tr>
<td>DATA ARG/1, 5, 2, 3, 5, 7/</td>
<td>PPL 33</td>
</tr>
<tr>
<td>DATA ARRAY/660<em>0, 1, 1/55</em>0, /</td>
<td>PPL 34</td>
</tr>
<tr>
<td>IRD=5</td>
<td>PPL 35</td>
</tr>
<tr>
<td>IPT=6</td>
<td>PPL 36</td>
</tr>
<tr>
<td>READ (IRD, 9) R, E, D, NUM, SMALL, LARGE</td>
<td>PPL 37</td>
</tr>
<tr>
<td>READ (IRD, 14) BDAT</td>
<td>PPL 38</td>
</tr>
<tr>
<td>DO 1 IM=1, 12</td>
<td>PPL 39</td>
</tr>
<tr>
<td>IF (BDAT(I), EQ, 0.) GO TO 2</td>
<td>PPL 40</td>
</tr>
<tr>
<td>1 CONTINUE</td>
<td>PPL 41</td>
</tr>
<tr>
<td>NB=12</td>
<td>PPL 42</td>
</tr>
<tr>
<td>GO TO 3</td>
<td>PPL 43</td>
</tr>
<tr>
<td>2 NB=1=1</td>
<td>PPL 44</td>
</tr>
<tr>
<td>LB=0/8</td>
<td>PPL 45</td>
</tr>
<tr>
<td>DB=DB/8</td>
<td>PPL 46</td>
</tr>
<tr>
<td>IBEGIN=ALOG10(SMALL)</td>
<td>PPL 47</td>
</tr>
<tr>
<td>IEND=ALOG10(LARGE)+1</td>
<td>PPL 48</td>
</tr>
<tr>
<td>JLIMIT=IEND-IBEGIN</td>
<td>PPL 49</td>
</tr>
<tr>
<td>IF (JLIMIT, GT, 9) JLIMIT=9</td>
<td>PPL 50</td>
</tr>
<tr>
<td>ILIMIT=ILIMIT+1</td>
<td>PPL 51</td>
</tr>
<tr>
<td>DO 8 KE=1, NUM</td>
<td>PPL 52</td>
</tr>
<tr>
<td>READ (IRD, 9) R, LPRIME, DPRIME</td>
<td>PPL 53</td>
</tr>
<tr>
<td>RB=RB/8</td>
<td>PPL 54</td>
</tr>
<tr>
<td>LPB=LPRIME/8</td>
<td>PPL 55</td>
</tr>
<tr>
<td>DPB=DPRIME/8</td>
<td>PPL 56</td>
</tr>
<tr>
<td>DO 4 IM=1, ILIMIT</td>
<td>PPL 57</td>
</tr>
<tr>
<td>INDEX=(IM-1)/6</td>
<td>PPL 58</td>
</tr>
<tr>
<td>IEXP=IBEGIN+INDEX</td>
<td>PPL 59</td>
</tr>
<tr>
<td>IM=INDEX**6</td>
<td>PPL 60</td>
</tr>
<tr>
<td>Y(I)=ARG(I)*10.**IEXP</td>
<td>PPL 61</td>
</tr>
<tr>
<td>U=1/Y(I)</td>
<td>PPL 62</td>
</tr>
<tr>
<td>DO 4 J=1, NB</td>
<td>PPL 63</td>
</tr>
<tr>
<td>BETANS BOAT</td>
<td>PPL 64</td>
</tr>
<tr>
<td>V=BETA/2,</td>
<td>PPL 65</td>
</tr>
<tr>
<td>4 ARRAY(I, J)=L(U, V)+FL(U, KB, BETA, LB, DB, LPB, DPB)</td>
<td>PPL 66</td>
</tr>
<tr>
<td>IF (LPB=0.) 5, 5, 6</td>
<td>PPL 67</td>
</tr>
<tr>
<td>5 WRITE (IPT, 10) DPB, KB, LB, DB</td>
<td>PPL 68</td>
</tr>
<tr>
<td>GO TO 7</td>
<td>PPL 69</td>
</tr>
<tr>
<td>6 WRITE (IPT, 11) LPB, DPB, R, LB, DB</td>
<td>PPL 70</td>
</tr>
<tr>
<td>7 WRITE (IPT, 12) (BDAT(I), IM=1, NB)</td>
<td>PPL 71</td>
</tr>
<tr>
<td>DO 8 IM=1, LIMIT</td>
<td>PPL 72</td>
</tr>
<tr>
<td>WRITE (IPT, 13) Y(I), (ARRAY(I, J), J=1, NB)</td>
<td>PPL 73</td>
</tr>
<tr>
<td>6 CONTINUE</td>
<td>PPL 74</td>
</tr>
<tr>
<td>STOP</td>
<td>PPL 75</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>**</th>
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</thead>
<tbody>
<tr>
<td>9 FORMAT (3F5.1, IS, 2E10.4)</td>
<td>PPL 76</td>
</tr>
<tr>
<td>10 FORMAT (11, 1, W(U, R/BR)+F(U, KB, R/BR, L/B, D/B, Z/B)), Z/B=1, F5.2, 1, SQPPL 77</td>
<td></td>
</tr>
<tr>
<td>1RT(KZ/KK) R/B=1, F5.2, 1, L/B=1, F5.2, 1, D/B=1, F5.2)</td>
<td>PPL 78</td>
</tr>
<tr>
<td>11 FORMAT (11, 1, W(U, R/BR)+F(U, R/B, R/BR, L/B, D/B, L'/B, D'/B)), L'/B=1, PPL 79</td>
<td></td>
</tr>
<tr>
<td>1F5.2, 1, D'/B=1, F5.2, 1, SQRT(KZ/KK)*R/B=1, F5.2, 1, L/B=1, F5.2, 1, D/B=1, F5.2)</td>
<td>PPL 80</td>
</tr>
<tr>
<td>2B=1, F5.2)</td>
<td>PPL 81</td>
</tr>
<tr>
<td>12 FORMAT (10, 1, R/8X, 1, R/8X, 1, 5X, 1/0.11, 12E10, 2)</td>
<td>PPL 82</td>
</tr>
<tr>
<td>13 FORMAT (11, 1, 1E10.3, 12F10.4)</td>
<td>PPL 83</td>
</tr>
<tr>
<td>14 FORMAT (8E10.9)</td>
<td>PPL 84</td>
</tr>
<tr>
<td>END</td>
<td>PPL 85</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>**</th>
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</thead>
<tbody>
<tr>
<td>C</td>
<td>C</td>
</tr>
</tbody>
</table>
REAL FUNCTION FL*4(U,RB,BETA,LB,DB,LPB,DPB)

*** ***********************************************************

FUNCTION FL

PURPOSE

TO COMPUTE DEPARTURES FROM HANTUSH-JACOB LEAKY AQUIFER CURVE
CAUSED BY PARTIAL PENETRATION OF PUMPED WELL.

USAGE

FL(U,RB,BETA,LB,DB,LPB,DPB)

DESCRIPTION OF PARAMETERS

ALL REAL, U DOUBLE PRECISION

U = R**2*S/4*T*TIME (RADIAL DISTANCE SQUARED * STORAGE
COEFFICIENT / 4*TRANSMISSION * TIME

RB = R/B (RADIAL DISTANCE / AQUIFER THICKNESS)

BETA = R*SQRT(KI/BLT) (RADIAL DISTANCE * SQUARE ROOT
(HYD. COND. OF CONFINING BED/THICKNESS OF CONFINING
BED * TRANSMISSION OF AQUIFER))

LB = L/B (FRACTION OF AQUIFER PENETRATED BY PUMPED WELL)

DB = D/B (FRACTION OF AQUIFER ABOVE PUMPED WELL SCREEN)

LPB = L*/8 (FRACTION OF AQUIFER PENETRATED BY OBS. WELL, ZERO
FOR PIEZOMETER)

DPB = D*/8 (FRACTION OF AQUIFER ABOVE OBS. WELL SCREEN, TOTAL
DEPTH FOR PIEZOMETER)

SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED

DFL12, SERIES, BESK, FCT, L

METHOD

SUMS THE SERIES THROUGH N*PI*R/B EQ 20

REAL*8 U,V,DSQRT
REAL*4 L,N,LB,LPB
SUM=0., N=0., BETSO=BETA*BETA
PISQSO=9.66025*RB*RB
PILB=3.141593*LB
PILB=3.141593*LB
PI=3.141593*PI
IF (LPB=0.) RETURN
C CHECKS FOR WELL OR PIEZOMETER
1 PIZB=3.141593*DPB
2 N=N+1.,
V=SQRT(BETSO*N*PIRSQ)/2.
IF (V,GT,10.) GO TO 3
C TRUNCATES SERIES WHEN V>10
X=L(U,V)/N
SUM=SUM+(SIN(N*PILB)*SIN(N*PIDB))*COS(N*PIZB)*X
GO TO 2
3 FL=366198*SUM/(LB=DB)
GO TO 7
4 PILPB=3.141593*LPB
PIDPB=3.141593*DPB
5 N=N+1.
V=SQRT(BETSO*N*PIRSQ)/2.
IF (V,GT,10.) GO TO 6
C TRUNCATES SERIES WHEN V>10
X=L(U,V)/N
SUM=SUM+(SIN(N*PILB)*SIN(N*PIDB))*SIN(N*PIDPB)*X/N
GO TO 5
6 FL=202424*SUM/((LB=DB)*(LPB+DPB))
7 RETURN
END
REAL FUNCTION L4(U,V)

FUNCTION L

PURPOSE
TO COMPUTE THE INTEGRAL [EXP(-V**2/Y)/Y] SUMMED OVER Y FROM U TO INFINITY (WELL FUNCTION FOR LEAKY ARTESIAN AQUIFERS).

DESCRIPTION OF PARAMETERS

<table>
<thead>
<tr>
<th>PARAMETER</th>
<th>DESCRIPTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>R**2<em>5/4</em>T*TIME (RADIAL DISTANCE SQUARED * STORAGE COEFFICIENT * TIME)</td>
</tr>
<tr>
<td>V</td>
<td>R/2<em>SQRT(K1/(T</em>B1)) = ONE-HALF RADIAL DISTANCE * SQUARE ROOT (HYD. COND. OF CONFINING BED)</td>
</tr>
<tr>
<td>V**2</td>
<td>SQUARE ROOT</td>
</tr>
<tr>
<td>V**2/U</td>
<td>SQUARE ROOT</td>
</tr>
</tbody>
</table>

SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED:
DQL12, SERIES, BESK, FCT

METHOD

IN THE FOLLOWING FORMulas EXP(-V**2/Y)/Y

(1) U<1, USES A GAUSSIAN-LAGUERRE QUADRATURE FORMULA TO EVALUATE INTEGRAL(F) FROM U TO INF,

(2) V**2<u<1, USES THE G-L QUADRATURE TO EVALUATE INTEGRAL(F) FROM ONE TO INF AND A SERIES EXPANSION TO EVALUATE INTEGRAL(F) FROM U TO ONE,

(3) U<1, U<1, USES THE REPRESENTATION INTEGRAL(F) FROM U TO INF. = Z**2*K0(2*V) = INTEGRAL(F) FROM V**2/U TO INF.

C EVALUATES THE ZERO ORDER MODIFIED BESSEL FUNCTION OF SECOND KIND WITH IBM SUBROUTINE, EVALUATES INTEGRAL BY G-L QUAD.

**************1*2****~~*4~~~~~*~~****A~**~~~*~~~**~*~**~*w~8~~~ L

EXTERNAL FCT
REAL*6 U,V,Z,F,VV,SERIES
COMMON /C1/ VV,VV

IF (U-1.) 1,2,2

CHECKS IF U<1

1 Z=V*V/U

IF (Z-1.) 3,4,4

CHECKS IF V**2/U < 1

2 Z=U

CALL DQL12(FCT,F)

L=1

INTEGRAL U TO INF, EVALUATED BY GAUSS-LAGUERRE QUADRATURE

GO TO 5

3 Z=1,

CALL DQL12(FCT,F)

L=1+SERIES(U,V)

INTEGRAL 1 TO INF, BY G-L QUAD., INTEGRAL U TO 1 BY SERIES EXP,

GO TO 4

4 T=U**2.5

CALL BESK(2*T,0,0,0,IER)

CALL DQL12(FCT,F)

L=2.*T

2*K0(ZV)=INTEGRAL V**2/U TO INF.

5 RETURN

END

REAL FUNCTION SERIES*8(U,V)

FUNCTION SERIES

PURPOSE

**************1*2****~~*4~~~~~*~~****A~**~~~*~~~**~*~**~*w~8~~~
Table 6.1.—Listing of program for partial penetration in a leaky artesian aquifer—Continued

```plaintext
C TO EVALUATE $3(1) \times 8(U)$, WHERE $8$ IS A SERIES EXPANSION OF C INTEGRAL $\exp(-x^2/2y)dy/y$ GIVEN BY $\int_S \sum_{m=0}^{\infty} F(m) \frac{\log(U)}{(\pi/2) \sum_{n=0}^{\infty} (v^{2m}/(m+n)!)}$ WHERE $F(m) = $ C \log(U) IF $m = 0$ AND $m = (\pi/2) \sum_{n=0}^{\infty} (v^{2m}/U^{m})$ IF $m > 0$. C DESCRIPTION OF PARAMETERS C BOTH DOUBLE PRECISION C $U = R^2 \times S/\pi \times T$ TIME (RADIAL DISTANCE C SQUARED \times STORAGE COEFFICIENT \times TRANSMISSIVITY \times TIME C $V = R^2 / 2 \times \sqrt{K} / (T \times S')$ ONE-HALF RADIAL C DISTANCE \times SQUARE ROOT C $S = KO, COND, OF CONFINING BED/TRANSMISSIVITY \times THICKNESS C UF CONFINING BED) C SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED NONE C METHOD C SUMMATION IS TERMINATED FOR THE INNER SERIES WHEN A TERM C BECOMES LESS THAN $5 \times 10^{-7}$ AND FOR OUTER SERIES WHEN A TERM C BECOMES LESS THAN $5 \times 10^{-7}$ C****************************************************************** C REAL*8 U,LOGDASS,S(1),SUM,SIGN,V,VSQ,VSQV,TERM,TERM1 C REAL*8 T,U,UM,E,EN,SUH,TERMSUM,SIGN,V,VSQU,RMUL,TERM,TERM2 C TEST=S,O,TEST+S,U,TEST+S,V C EM=1, TSUM=1, TERM=1, SUM=0, SIGN=1, VUM=1, VSQ=0, VSQ=0, U=0 C UM=0, DO = IM=1,2 C EVALUATES SERIES FOR LOWER LIMIT $= U$ AND UPPER LIMIT $= 1$ C IF (1,EQ,2) UM=1, C U=1, C E=1, C SUM=0, C SIGN=1, C VUM=1, C VSQ=VSQ/U C 1 EM=EM+1, C IF (EM.LE.1) 2,3,3 C CHECKS FOR M=0 C 2 RMUL=LOG(U), C TERM1=1, C GO TO 4 C 3 UM=UM+U C IF (VUM.LT.1.D=30) VUM=0, C VUM=VUM+VSQV C RMUL=(UM*UM)/EM C TERM1=TERM1/EM C U SIGN=SIGN C SUM=SIGN C TERM=TERM1 C SUM=SUM+TERM C EN=0, C 5 EN=EN+1, C TERM=TERM*VSQ/(EN*(EN+EM)) C SUM=SUM+TERM C IF (TERM.LE.0.) OR (TERM.SUM=TERM C IF (TERM.LE.1.) GO TO 1 C IF (TERM.LE.0.) GO TO 1 C TRUNCATES INNER SERIES IF OUTER TERM<INNER TERM<5.E-7 C TRUNCATES OUTER SERIES IF OUTER TERM+INNER SUM<5.E-7 C 6 S(1)=SUM1 C U=UUU C SERIES=S(2)=S(1) C RETURN C END
```
REAL FUNCTION FCT*8(X)

FUNCTION FCT
P  PURPOSE
TO COMPUTE FCT(X)=EXP(-Z*V**2/(X+Z))/(X+Z)

DESCRIPTION OF PARAMETERS
X = THE DOUBLE PRECISION VALUE OF X FOR WHICH FCT IS COMPUTED

SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED

DESCRIPTION OF SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED

FORTRAN EVALUATION OF FUNCTION

REAL*8 X,V,Z,P,DEXP
COMMON /CI/ V,Z
IF (X) 1Z,2
1 FCT=0., RETURN
GO TO 4
2 P=Z+V**2/(X+Z)
IF (P<5,1) 3,3,1
3 FCT=DEXP(-P)/(X+Z)
4 RETURN
END

SUBROUTINE UQL12(FCT,Y)

SUBROUTINE UQL12

PURPOSE
TO COMPUTE INTEGRAL(EXP(-X)*FCT(X), SUMMED OVER X FROM 0 TO INFINITY).

USAGE
CALL UQL12 (FCT,Y)
PARAMETER FCT REQUIRES AN EXTERNAL STATEMENT

DESCRIPTION OF PARAMETERS
FCT = THE NAME OF AN EXTERNAL DOUBLE PRECISION FUNCTION SUBPROGRAM USED.
Y = THE RESULTING DOUBLE PRECISION INTEGRAL VALUE.

REMARKS
NONE

SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
THE EXTERNAL DOUBLE PRECISION FUNCTION SUBPROGRAM FCT(X)
MUST BE FURNISHED BY THE USER.

METHOD
EVALUATION IS DONE BY MEANS OF 12-POINT GAUSS-LAGUERRE QUADRATURE FORMULA, WHICH INTEGRATES EXACTLY,
WHENEVER FCT(X) IS A POLYNOMIAL UP TO DEGREE 23.
FOR REFERENCE, SEE
SHAO/CHEN/FRAK, TABLES OF ZEROS AND GAUSSIAN WEIGHTS OF CERTAIN ASSOCIATED LAGUERRE POLYNOMIALS AND THE RELATED GENERALIZED HERMITE POLYNOMIALS, IBM TECHNICAL REPORT TR00.1100 (MARCH 1964), PP. 24-25.


Table 6.1.—Listing of program for partial penetration in a leaky artesian aquifer—Continued

SUBROUTINE BESK(X,N,BK,IER)

*************************************************************************

SUBROUTINE BESK

COMPUTE THE K BESSEL FUNCTION FOR A GIVEN ARGUMENT AND ORDER

USAGE

CALL BESK(X,N,BK,IER)

DESCRIPTION OF PARAMETERS

X =THE ARGUMENT OF THE K BESSEL FUNCTION DESIRED
N =THE ORDER OF THE K BESSEL FUNCTION DESIRED
BK =THE RESULTANT K BESSEL FUNCTION
IER=RESULTANT ERROR CODE WHERE

IER=0 NO ERROR
IER=1 N IS NEGATIVE
IER=2 X IS ZERO OR NEGATIVE
IER=3 X .GE. 170, MACHINE RANGE EXCEEDED
IER=4 BK .GT. 10**70

REMARKS

N MUST BE GREATER THAN OR EQUAL TO ZERO

SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED

NONE

METHOD

COMPUTES ZERO ORDER AND FIRST ORDER BESSEL FUNCTIONS USING SERIES APPROXIMATIONS AND THEN COMPUTES N TH ORDER FUNCTION USING RECURRENCE RELATION.
Table 6.1.—Listing of program for partial penetration in a leaky artesian aquifer—Continued

```plaintext
DIMENSION T(12)
BK=0
IF(N)10,1,11
10 IERM
RETURN
11 IF(X)12,12,20
12 IERM
RETURN
20 IF(X=170,0)22,22,21
21 IERM=3
RETURN
22 IERM=0
IF(X=1,30,30,29
25 A=EXP(X)
B=1/X
C=SQRT(B)
T(1)=B
DO 26 L=2,12
26 T(L)=T(L-1)+B
IF(N)20,20,29
28 BK=GO
RETURN
29 G=1.2533141+.1586642*T(1)+.0881112*T(2)+.0913905*T(3)
2+.1304596*T(4)+.2299895*T(5)+.3792410*T(6)+.5247277*T(7)
3+.5575368*T(8)+.4262633*T(9)+.2108458*T(10)+.0660977*T(11)
4+.0091983*T(12)*C
IF(N)20,20,29
28 BK=GO
RETURN
30 INNER=3
RETURN
31 DO 35 J=2,N
G=2.*(FLOAT(J)-1.)*G1/X*G0
IF(GJ)=.070)33,33,32
32 IERM=4
GOTO 34
33 G=GO
35 G=GO
34 BK=GO
RETURN
36 BK=X/2.
```

**RECURSIVE RELATION AND POLYNOMIAL APPROXIMATION TECHNIQUE**

Table 6.1.—Listing of program for partial penetration in a leaky artesian aquifer—Continued

```
A=5772157+ALOG(B)  E=950
C=M8  E=960
IF(N=1)37,43,37  E=970
C
COMPUTE KO USING SERIES EXPANSION  E=980
C
37 G0=A  E=990
X2J=E1  E=1000
FACT=1  E=1010
HJ=0  E=1020
DO 40 J=1,6  E=1030
RJ=1/FLOAT(J)  E=1040
IF(X2J,LT,1,E=40) X2J=0  E=1050
C PREVIOUS STATEMENT ADDED TO IBM SUBROUTINE TO CORRECT UNDERFLOW  E=1060
C PROBLEM ON NATFUN COMPILER  E=1070
X2J=X2J*ACT  E=1080
FACT=FACT*RJ*RJ  E=1090
HJ=HJ+RJ  E=1100
40 G0=G0+X2J*FACT*(HJ=A)  E=1110
IF(N)43,42,43  E=1120
42 BK=G0  E=1130
RETURN  E=1140
END  E=1150
```

Table 7.2.—Listing of program for constant drawdown in a well in an infinite leaky aquifer

```
A=5772157+ALOG(B)  Z 1
C=M8  Z 2
IF(N=1)37,43,37  Z 3
C
COMPUTE KO USING SERIES EXPANSION  Z 4
C
37 G0=A  Z 5
X2J=E1  Z 6
FACT=1  Z 7
HJ=0  Z 8
DO 40 J=1,6  Z 9
RJ=1/FLOAT(J)  Z 10
IF(X2J,LT,1,E=40) X2J=0  Z 11
C PREVIOUS STATEMENT ADDED TO IBM SUBROUTINE TO CORRECT UNDERFLOW  Z 12
C PROBLEM ON NATFUN COMPILER  Z 13
X2J=X2J*ACT  Z 14
FACT=FACT*RJ*RJ  Z 15
HJ=HJ+RJ  Z 16
40 G0=G0+X2J*FACT*(HJ=A)  Z 17
IF(N)43,42,43  Z 18
42 BK=G0  Z 19
RETURN  Z 20
END  Z 21
```

Purpose:

To compute a table of function values for drawdown in a leaky artesian aquifer in response to a step change in water level in the control well, function values are expressed as a fraction of drawdown in control well (S/Sm).

Reference: M. Hantush, 1959, Nonsteady Flow to Flowing Wells in Leaky Aquifers: Jour, Geophys, Research, V, 64.

Input data:

1 card = FORMAT(2E10.5)

TSMALL = smallest value of alpha for which computation is desired.

TLARGE = largest value of alpha for which computation is desired.

1 card = FORMAT(13F5.0)

B0AT = 13 values of Rw/B, non zero values should be ge 1 and lt 10, first zero (or blank) will terminate the list. At least one non zero value must be coded, input values are multiplied by power of ten determined by program from alpha.
Table 7.2.—Listing of program for constant drawdown in a well in an infinite leaky aquifer—Continued

<table>
<thead>
<tr>
<th>C</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>CARD = FORMAT(10F8.2)</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>RH = RADIUS OF CONTROL WELL</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>RDAT = 9 VALUES OF RADIAL DISTANCE OF OBSERVATION POINTS</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>FROM CONTROL WELL, SHOULD BE CODED WITH SMALLEST NUMBER</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>FIRST, THEN BY INCREASING DISTANCE, THE FIRST ZERO</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>(OR BLANK) VALUE WILL TERMINATE COMPUTATION</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>METHOD</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>EVALUATES EQU. 13 OF MANTUSH, EVALUATION OF BESSEL FUNCTIONS</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>BY SUBROUTINES BESK AND BESY AND FUNCTION JO, EVALUATES</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>INTEGRAL BY SUM, I=1 TO 8000, F((DELTA U)<em>(I=5))</em>(DELTA U)</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>CHOSES INITIAL DELTA U = .001/SQRT(SMALLEST ALPHA) AND USES</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>THIS VALUE FOR ALL RW/B GE .001/(DELTA U), FOR SMALLER RW/B,</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>DIVIDES DELTA U BY 10 AND MULTIPLIES SMALLEST ALPHA BY 100,</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>REMARKS</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>SMALLEST RW/B GE .01/SQRT(SMALLEST ALPHA)</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>BESK, BESY, JO</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>REAL*8 SUM1, SUM2</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>REAL*8 KOBP, KOB, JO, JOU, Y(8000), J(8000), F(8000), FT(8000)</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>1 FB(8000), RDAT(9), RDAT(6), BDA(13), ARRAY(25,9,13), B(13), T(25)</strong></td>
<td>Z</td>
</tr>
<tr>
<td>**DATA FT(8000)<em>0.16, FB(8000)<em>0.16</em></em></td>
<td>Z</td>
</tr>
<tr>
<td><strong>DATA RDAT/9<em>1.</em>/</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>DATA ARRAY/2925<em>0.1, TDAT/1,1.5,2,3,5,7,</em>/</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>IRD=5</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>IPT=6</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>READ (IRD,24) TSMALL, TLARGE</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>READ (IRD,23) BOAT</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>READ (IRD,22) RW, RDAT</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>IBEGIN=ALOG10(TSMALL), IEND=ALOG10(TLARGE)+.99999</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>IF ((IBEGIN/2I2),LT,IBEGIN) IBEGIN=IBEGIN+1</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>ISPAN=IEND-IBEGIN</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>MLIMIT=(ISPAN+1)/2</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>COMPUTES INITIAL DELTA U (DU) = .001/SQRT(SMALLEST ALPHA)</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>DU=.001/SQRT(TDAT(1))*10.</strong>(IBEGIN)**</td>
<td>Z</td>
</tr>
<tr>
<td><strong>EXPONENT (JBEGIN) OF SMALLEST RW/B IS COMPUTED FROM EXPONENT</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>(IBEGIN) OF SMALLEST ALPHA,</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>JBEGIN=IBEGIN/2=2</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>DU 1 I=1,13</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>IF (RDAT(1),EQ,0.) GO TO 2</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>1 CONTINUE</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>NB=13</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>GO TO 3</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>2 NB=13</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>3 CONTINUE</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>DO 4 I=1,9</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>IF (RDAT(1),EQ,0.) GO TO 5</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>4 RDAT(1)=RDAT(1)/RH</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>NR=9</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>GO TO 6</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>5 NR=1</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>6 DO 21 M=1,MLIMIT</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>NUM=8000</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>START=DU/2,</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>U=START</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>DO 7 I=1,NUM</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>U=U+DU</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>CALL BESY(U,0,Y(I),IDUMY)</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>J(I)=JO(U)</strong></td>
<td>Z</td>
</tr>
<tr>
<td><strong>DO 19 IR=1,NR</strong></td>
<td>Z</td>
</tr>
</tbody>
</table>
Table 7.2.—Listing of program for constant drawdown in a well in an infinite leaky aquifer—Continued

```
RHODAT(IR)
START
DO 8 I=1,NUM
U=U+DU
CALL BESY(RHOU,0,YOPU,IDUMY)
JOPU=JO(RHOU)
JUUM(J)
YOU=YOU+1
8 F(I)=(JOPU*YOU+YOPU*JUU)/(JOU+YOU*YOU)
DO 19 IT=1,25
INDEX=(IT-1)/6
IEXP=IT-BEGIN+INDEX
II=IT-INDEX*6
TAU=TAUDAT(II)*10.,**IEXP
T(IT)=TAU
START
NUM=NUM
DO 9 I=1,NUM
U=U+DU
FTEST=F(I)
IF (ABS(FTEST),LT,1,E=30) GU TU 10
XTEST=TAU*U+U
IF (XTEST+69.) 10,10,9
9 FT(I)=FTEST*EXP(XTEST)
GO TO 11
10 NUM=I+1
FT(I)=0.
11 DO 19 IB=1,13
INDEX=(IB-1)/NB
JEXP=IB-BEGIN+INDEX
JJ=IB-JINDEX*NB
BETA=BDAT(JJ)*10.,**JEXP
B(IB)=BETA
START
SUM=SUM+U
NUM=NUM
DO 12 I=1,NUM
U=U+DU
FTEST=F(I)
IF (ABS(FTEST),LT,1,E=30) GO TO 13
12 FT(I)=FTEST/(U+BSQ/U)
GO TO 14
13 NUM=I+1
FB(I)=0.
14 SUM=0.
SUM2=0.
DO 15 I=1,NUM,2
SUM=SUM+FB(I)
15 SUM2=SUM2+FB(I+1)
XINT=(SUM+SUM2)*DU
CALL BESK(RHU,BETA,0,KOBP,IDUMY)
CALL BESK(BETA,0,KOB,IDUMY)
RATIO=0.
IF (KOBP,GTO,0.) RATIO=KOBP/KOB
XTEST=TAU*BSQ
IF (XTEST+30.) 16,17,17
16 XPT=0.
GO TO 18
17 XPT=EXP(XTEST)
18 Z=HATUB+366198*XPT*XINT
IF ((Z,LT,1.),AND,(Z,GT,5.,E=5.)) Z=0.,E0
19 ARRAY(IT,IR,IR)=Z
```
TABLE 7.2.—Listing of program for constant drawdown in a well in an infinite leaky aquifer—Continued

```
DO 20 K=1,NH
WRITE (IPT,25) RDAT(K),B
WRITE (IPT,26) (T(I), (ARRAY(I,K,L), L=1,13), I=1,25)
20 CONTINUE

C EXPONENT OF SMALLEST R/W DECREASED BY ONE EACH TIME THROUGH LOOP
JBEGIN = JBEGIN + 1
C EXPONENT OF SMALLEST ALPHA INCREASED BY TWO EACH TIME THROUGH LOOP
IBEGIN = IBEGIN + 2
C DELTA U (DU) IS DIVided BY 10 EACH TIME THROUGH THE LOOP
21 DU = 1*DU
STOP
C
22 FORMAT (10F8,2)
23 FORMAT (13F5,0)
24 FORMAT (2E10,5)
25 FORMAT ('I1,'Z(ALPHA,R/RW,Rw/B) , R/Rw',F8,0/101,9X,'II RW/B'/(1, 13x,'ALPHA',1x,139,2))
26 FORMAT (' ',E10.3,13F9,3)
END

REAL FUNCTION JO(N,X)

****FUNCTION JO****

PURPOSE
TO COMPUTE THE ZERO UNDER J BESSEL FUNCTION FOR A GIVEN
ARGUMENT,
USAGE
JO(X)
DESCRIPTION OF PARAMETER
X - REAL*4, ARGUMENT OF JO BESSEL FUNCTION DESIREd.
SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
NONE,
METHOD
POLYNOMIAL APPROXIMATION FOR X<4 AND ASYMPTOTIC SERIES FOR
X GE 4, THE POLYNOMIAL APPROXIMATION IS THE FIRST 10 TERMS OF
THE POWER SERIES FOR JO(X) (MILLER, K.S., 1957, ENGINEERING
MATHEMATICS, RINEHART AND CO., INC., NEW YORK, P. 120), THE
ASYMPTOTIC EXPANSION OF JO(X) IS GIVEN ON P. 82
OF BUNDAN, FRANK, 1958, INTRODUCTION TO BESSEL FUNCTIONS,
OVER PUBLICATIONS INC., NEW YORK, THE TERMS P ('A*PO1 AND
Q ("B*QO) OF THE ASYMPTOTIC EXPANSION ARE COMPUTED BY AN
ALGORITHM FROM IBM SUBROUTINE BESY.

IF (X=4.) 1,3,3
1 A=X*X/4
B=1.
DO 2 I=1,10
C=1./I
2 B=P(B*(A/(C+C)))
GO TO 4
4
C

COMPUTE JO BY ASYMPTOTIC SERIES
3 T=4./X
T2=1./T
P09=(((+000037043*T2+,000017565)*T2-,0000487613)*T2+,000017343)*
1T2-,001753062)*T2+,3989423
Q09=(((000033212*T2-,000014078)*T2+,0000342468)*T2-,0000669791)*
1T2+,0000456324)*T2-,01246994
A=2./SRT(X)
```
Table 7.2: Listing of program for constant drawdown in a well in an infinite leaky aquifer—Continued

```c
R=A*T1
C=X*,7853982
JO=A*PO*COS(C)-B*Q0*SINC)
4 RETURN
END
SUBROUTINE BESY(X,N,BY,IER)

%-----------------------------------
SUBROUTINE BESY

PURPOSE
COMPUTE THE Y BESSEL FUNCTION FOR A GIVEN ARGUMENT AND ORDER

USAGE
CALL BESY(X,N,BY,IER)

DESCRIPTION OF PARAMETERS
X = THE ARGUMENT OF THE Y BESSEL FUNCTION DESIRED
N = THE ORDER OF THE Y BESSEL FUNCTION DESIRED
BY = THE RESULTANT Y BESSEL FUNCTION
IER=RESULTANT ERROR CODE WHERE
IER=0 NO ERROR
IER=1 N IS NEGATIVE
IER=2 X IS NEGATIVE OR ZERO
IER=3 BY HAS EXCEEDED MAGNITUDE OF 10**70

REMARKS
VERY SMALL VALUES OF X MAY CAUSE THE RANGE OF THE LIBRARY
FUNCTION ALOG TO BE EXCEEDED
X MUST BE GREATER THAN ZERO
N MUST BE GREATER THAN OR EQUAL TO ZERO

SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
NONE

METHOD
RECURSION RELATION AND POLYNOMIAL APPROXIMATION TECHNIQUE
AS DESCRIBED BY A.J.,M.HITCHCOCK, 'POLYNOMIAL APPROXIMATIONS'
TO BESSEL FUNCTIONS OF ORDER ZERO AND ONE AND TO RELATED
FUNCTIONS', M.T.I, A.C., V.11,1957, PP.86-88, AND G.N. WATSON,
'I.A TREATISE ON THE THEORY OF BESSEL FUNCTIONS', CAMBRIDGE
UNIVERSITY PRESS, 1958, P. 62

%-----------------------------------
CHECK FOR ERRORS IN N AND X

10 IER=0
IF(X)190,190,20
IF(N)180,10,10
10 IER=0
IF(X)190,190,20
BRANCH IF X LESS THAN OR EQUAL 4
20 IF(X4,0)40,40,30
COMPUTE Y0 AND Y1 FOR X GREATER THAN 4
30 T1=4,0/X
T2=T1*T1
PO=(C(,0000037043*T2+,0000173565)*T2-,0000487613)*T2
```

---

**TYPE CURVES FOR FLOW TO WELLS IN CONFINED AQUIFERS**

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Table 7.2.—Listing of program for constant drawdown in a well in an infinite leaky aquifer—Continued

1 +0.0017343)*T2=+.0001753062)*T2+.3989423
Q0=((1.000032312)*T2=+.000142078)*T2+.0000342468)*T2
1 =+0.0000869791)*T2=+.004564324)*T2=.01246694
P1=((1.00000424)*T2=+.0000200920)*T2+.0000580759)*T2
1 =+0.000223203)*T2=+.002921826)*T2+.3989423
Q1=((1.00000036594)*T2=+.00001622)*T2=.000398708)*T2
1 +0.000164741)*T2=+.006390400)*T2=.07340084
A=2.0/SQR(T)
B=A*T
C=X=+.7853982
Y0=A*X*SIN(C)+B*X*COS(C)
Y1=A*X*COS(C)+B*X*SIN(C)
GO TO 90
CUMPUTE Y0 AND Y1 FOR X LESS THAN OR EQUAL TO 4
40 XX=X/2,
X2=XX*XX
T=ALOG(XX)+.5772157
SUM=Q0,
TERMT=T
Y0=T
DO 70 L=1,15
IF(L>150,60,50
50 SUM=SUM+1.0/FLOAT(L=1)
60 FL=1,
TS=SUM
IF(ABS(TERM)<LE.1,E=40) TERM=0,
TERM=T+(TERM=TERM*(-X2)/(FL-2)*(1,-1)/(FL+8))
70 Y0=YO+TERM
TERM = XX*(T=.5)
SUM=Q0
Y1=TERM
DO 80 L=2,16
SUM=SUM+1.0/FLOAT(L=1)
FL=L
FL1=FL-1,
TS=SUM
IF(ABS(TERM)<LE.1,E=40) TERM=0,
TERM=T+(TERM=TERM*(-X2)/(FL1*FL)*(1,-5/(TS-5/FL1))
80 Y1=Y1+TERM
PI2=.6366198
Y0=PI2*Y0
Y1=PI2*X+PI2*Y1
C CHECK IF ONLY Y0 OR Y1 IS DESIRED
C 90 IF(N=1)100,100,130
C RETURN EITHER Y0 OR Y1 AS REQUIRED
C 100 IF(N=1)110,120,110
110 BY=0
GO TO 170
120 BY=Y0
GO TO 170
C PERFORM RECURRENCE OPERATIONS TO FIND YN(X)
C 130 YAY0
YBY1
K=1
C
Table 7.2.—Listing of program for constant drawdown in a well in an infinite leaky aquifer—Continued

```
140 T = FLOAT(Z*K)/X
141 IF(ABS(YC) = 1.0E70) 145, 145, 141
141 IER = 3
142 RETURN
145 K = K + 1
145 IF(K = -N) 150, 160, 150
150 Y = YC
150 GO TO 140
160 BY = YC
170 RETURN
180 IER = 1
181 RETURN
190 IER = 2
191 RETURN
SUBROUTINE BESK(X, N, BK, IER)

*****************************************************************************
SUBROUTINE BESK

COMPUTE THE K BESSEL FUNCTION FOR A GIVEN ARGUMENT AND ORDER

USAGE
CALL BESK(X, N, BK, IER)

DESCRIPTION OF PARAMETERS

X = THE ARGUMENT OF THE K BESSEL FUNCTION DESIRED
N = THE ORDER OF THE K BESSEL FUNCTION DESIRED
BK = THE RESULTANT K BESSEL FUNCTION
IER = RESULTANT ERROR CODE WHERE
IER = 0 NO ERROR
IER = 1 N IS NEGATIVE
IER = 2 X IS ZERO OR NEGATIVE
IER = 3 X > GT, 170, MACHINE RANGE EXCEEDED
IER = 4 BK, GT, 10**70

REMARKS
N MUST BE GREATER THAN OR EQUAL TO ZERO

SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
NONE

METHOD
COMPUTES ZERO ORDER AND FIRST ORDER BESSEL FUNCTIONS USING
SERIES APPROXIMATIONS AND THEN COMPUTES NTH ORDER FUNCTION
USING RECUR bence Relation, Recurrence Relation and Polynomial Approximation Technique
AS DEScribed by A. J. M. Mitchelcock, Polynomial Approximations
To Bessel Functions of Order Zero and One and to Related Functions,
m. t. a. c., V. 11, 1957, PP. 66-88, and G. N. Watson,
A Treatise on the Theory of Bessel Functions, Cambridge
university Press, 1958, P. 62

*****************************************************************************

DIMENSION T(12)
BK = 0
IF(N) 10, 11, 11
10 IER = 1
```

...
TABLE 7.2.—Listing of program for constant drawdown in a well in an infinite leaky aquifer—Continued

```
RETURN
11 IF(X12,12,20
12 IER=2
RETURN
20 IF(X1170,0)22,22,21
21 IER=3
RETURN
22 IER=0
    IF(X136,36,25
25 A=EXP(-X)
    B=1/X
    C=SQRT(B)
    T(1)=B
    DO 26 L=2,12
26 T(L)=T(L-1)*B
    IF(N=1)27,29,27
    T(L)=T(L-1)*B

C
    COMPUTE KO USING POLYNOMIAL APPROXIMATION
    27 G0=A*(1.2533141*1666642*T(1)28811128*T(2)09193095*T(3)
2*11349596*T(4)2299850*T(5)3792410*T(6)5247277*T(7)
3*5575366*T(8)42962633*T(9)2184518*T(10)6680977*T(11)
4*90180938*T(12)C
    IF(N)20,28,29
28 BK=G0
    RETURN
C
    COMPUTE K1 USING POLYNOMIAL APPROXIMATION
    29 G1=A*(1.2533141*4699927*T(1)28811128*T(2)+1280427*T(3)
2*1736432*T(4)2847618*T(5)4594342*T(6)+6283381*T(7)
3*6632295*T(8)+5050239*T(9)+2581304*T(10)+7880001*T(11)
4*1082418*T(12)C
    IF(N=1)20,30,31
30 BK=G1
    RETURN
C
    FROM KO, K1 COMPUTE KN USING RECURRENCE RELATION
    31 DO 35 J=2,N
    GJ=2*(FLOAT(J)-1)*G1/X+G0
    IF(GJ=1.0E70)33,33,32
32 IER=4
    GO TO 34
33 G0=G1
35 G1=GJ
34 BK=GJ
    RETURN
36 BX/X/2
    A=.5772157*ALOG(B)
    B=B*A
    IF(N=1)37,43,37
C
    COMPUTE KO USING SERIES EXPANSION
    37 G0=A
    X2J=1
    FACT=1
    HJ=0
    DO 40 J=1,6
    RJ=1.0/FLOAT(J)
    IF(X2JLT1.6E40)X2J=0
C
    PREVIOUS STATEMENT ADDED TO IBM SUBROUTINE TO CORRECT UNDERFLOW
    RETURN
```

**Table 7.2.**—Listing of program for constant drawdown in a well in an infinite leaky aquifer—Continued

```c
C PROBLEM ON WRITF CUMPIER
X2J*X2J*C
FACT*FACT*RJ*RJ
HJ*HJ*RJ
40 G0=G0+X2J*FACT*(HJ=A)
IF(N)43,42,43
42 BK=G0
RETURN
C
C COMPUTE K1 USING SERIES EXPANSION
C
43 X2J#B
FACT#1,
HJ#1,
G1#1,X*X2J*(,5+A=HJ)
DO 50 J=2,8
X2J*X2J*C
RJ=4/FLOAT(J)
FACT*FACT*RJ*RJ
HJ=RJ
50 G1=G1+X2J*FACT*(,5*(A=HJ)*FLOAT(J))
IF(N=1)31,52,31
52 BK=G1
RETURN
END
```

**Table 8.2.**—Listing of programs for constant discharge from a fully penetrating well of finite diameter

```
C**************************************************************************************
C PURPOSE
C COMPUTES FUNCTION VALUES OF F(U,ALPHA,RHU) FOR RHO > 1 - FAR
C PAPADUPULOS, I, S. AND COOPER, M. A., JH., 1967, DRAWDOWN IN
C A WELL OF LARGE DIAMETER. WATER RESOURCES RESEARCH, V, 3,
C NO. 1, P, 241-244.
C PROGRAM BY S, S., PAPADUPULOS.
C
C INPUT DATA = ONE OR MORE GROUPS, EACH GROUP CODED AS FOLLOWS
C 1 CARD = FORMAT(2E10,5)
C
C 1 CARD = FORMAT(16E5,0)
C U = 16 VALUES OF U = R*R*8/(4*TIME) = DISTANCE FROM
C PUMPED WELL SQUARED / STORAGE COEFFICIENT / TIME, IF LESS THAN 16 DESIRED,
C BLANK OR ZERO VALUES MAY BE CODED FOR THE REST.
C SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C PEAK, SIMP, APKE, EXBSL1, JYO, JYL, ROOTS = MUST BE IN DECK.
C
C**************************************************************************************
```
Table 8.2.—Listing of programs for constant discharge from a fully penetrating well of finite diameter—Continued

2 WRITE (6,17) ALPHA, RHO
WRITE (6,18)
3 READ (5,19) U
DO 14 II=1,16
IF (U(II)) 1,1,4
4 A=ALPHA + ALPHA
R=0.25/U(II)
CALL APLKE(EXB8L1)
CALL PEAK(EXB8L1)
IF (XPK=1.0E=8) 5,6,6
5 WRITE (6,20) XPK, U
GO TO 3
6 IF (XPK=3.0) 8,7,7
7 WRITE (6,21) XPK, U
GO TO 3
8 EPS=0.00001
HBar=0.007*XPK
CALL SIMPS(0,0,XPK,EP.S,HBar,SUM,DEL,EXB8L1)
XMI=((3.14159265*7.0)/(8.0*(RHO=1.0)+1.6)*RHO/2.
DXI=XMI=1.0E=6*XMI
DXN2=(3.14159265*RHO)/(RHO=1.0)
CALL ROOTS(XMI,DXI,RT1,EXB8L1)
HBar=0.007*(RT1=XPK)
CALL SIMPS(XPK,RT1,EP.S,HBar,TRM1,ERR1,EXB8L1)
SUM=SUM+TRM1
DEL=DEL+ERR1
X1=RT1
I=1
9 XMAX=X1+DL
CALL ROOTS(XM,DXN,X2,EXB8L1)
HBar=0.007*(X2=X1)
CALL SIMPS(X1,X2,EP.S,HBar,TRM,ERR,EXB8L1)
V(I,J)=ABS(TRM)
DEL=DEL+ERR
I=I+1
10 IF (I=40) 10,10,11
11 X=2
GO TO 9
12 V(K,J)="V(K=1,J+1)=V(K=1, J)
DO 13 N=1,40
L=N+1
DELV=(-0.5)**(L*V(N,1)
13 EST=EST+.05*DELV
SUM=SUM-EST
PUAR=Q*A*RHO*SUM/3.14159265
WRITE (6,22) U(II),SUM,DEL,PUAR
14 CONTINUE
GO TO 1
15 STOP
C
16 FORMAT (2E10.5)
17 FORMAT ("IF(U,ALPHA,RHO) FOR ALPHA=1,1PE13.5," RHO=1,1E13.5) FAR 89
18 FORMAT (1HE1,12X,1HU,16X,9HINTEGRAL,9X,14HINTEGRAL ERRUR,8X,14H(1, U,FAR 90
1APLA,RHU)/1M )
19 FORMAT (1HE5.0)
FUNCTION EXSBL1(X)  
**C**                             **C**                  EBI  1  
**C** PURPOSE                        **C**                  EBI  2  
**C** Computes values of the integrand for f(u, alpha, rho)          EBI  3  
**C** DESCRIPTION OF PARAMETER              **C**                  EBI  4  
**C** X = REAL = argument of integrand     **C**                  EBI  5  
**C**                                                                 **C**                  EBI  6  
**C** Common/PBLK/A,R,R               **C**                  EBI  7  
**C** IF (X) 1,1,2                   **C**                  EBI  8  
1 EXSBL1=0.                         **C**                  EBI  9  
GO TO 8                             **C**                  EBI 10 
2 IF (X>4.0 E7) 4,4,3             **C**                  EBI 11 
3 FNU=X*COS(*R=1.0)*X*SIN(*R=1.0))  **C**                  EBI 12 
DE=(X**2**+4.0**2**)**(1/2)**       **C**                  EBI 13 
EXSBL1=FNU/DE                       **C**                  EBI 14 
GO TO 8                             **C**                  EBI 15 
4 Y=B*X*X                          **C**                  EBI 16 
IF (Y<=0.01) 5,5,6                  **C**                  EBI 17 
5 EXP=EXP(y*(1.0+y)*y*(1.0/6.0)*y*(1.0/24.0)))  **C**                  EBI 18 
GO TO 7                             **C**                  EBI 19 
6 EXP=1.0=EXP(y)                   **C**                  EBI 20 
7 CALL JY0(X,MY,JO,JY0)           **C**                  EBI 21 
CALL JY1(X,MY,JO,JY1)             **C**                  EBI 22 
AWS=MY=JO=A=MY=A                  **C**                  EBI 23 
CALL JY0(X,MY,JO,JY0)             **C**                  EBI 24 
FNU=EXP*(AWS*JO=BYO)              **C**                  EBI 25 
DEN=AWS*(AWS*BYO)                 **C**                  EBI 26 
EXSBL1=FNU/DE                      **C**                  EBI 27 
8 RETURN                           **C**                  EBI 28 
**END**                           **C**                  EBI 29 
SUBROUTINE ROOTS(XM,DX,ROOT,F)   **R**                   ROO  1  
**R** PURPOSE                      **R**                   ROO  2  
**R** searches for root of F in the interval XM=DX to XM+DX.       **R**                   ROO  3  
**R** DESCRIPTION OF PARAMETERS = ALL REAL                          **R**                   ROO  4  
**R** XM = CENTER OF INTERVAL SEARCHED.                              **R**                   ROO  5  
**R** DX = HALF WIDTH OF INTERVAL SEARCHED.                           **R**                   ROO  6  
**R** ROOT = RETURNED ROOT LOCATION                                  **R**                   ROO  7  
**R** F = FUNCTION REFERENCE.                                          **R**                   ROO  8  
**R**                                                                 **R**                   ROO  9  
**R**                                                                 **R**                   ROO 10 
**R**                                                                 **R**                   ROO 11 
**R**                                                                 **R**                   ROO 12 
**R** XLM=XM+DX                                                             **R**                   ROO 13 
**R** XR=XM+DX                                                             **R**                   ROO 14 
**R** YLF(XL)                                                              **R**                   ROO 15 
**R** YRF(XR)                                                              **R**                   ROO 16 
**R** EP=0.000001*ABS(YL)                                                 **R**                   ROO 17 
**R** DO 9 IM=1,200                                                       **R**                   ROO 18 
**R** YM=F(XM)                                                             **R**                   ROO 19 
**R** UP=ABS(YM)                                                           **R**                   ROO 20 
**R** IF (UP,LT,EP) AND UP,LT,1.0 AND UP,7) GO TO 1                  **R**                   ROO 21 
**R** IF (YM) 2,1,2                                                         **R**                   ROO 22
Table 8.2.—Listing of programs for constant discharge from a fully penetrating well of finite diameter—Continued

```fortran
1 ROUT=XM
   GO TO 10
2 IF (YM*YL) 7,3,4
3 ROUT=XL
   GO TO 10
4 IF (YM*YR) 8,5,6
5 ROUT=XR
   GO TO 10
6 WRITE (6,11) XL,XR
   STOP
7 XR=XM
   YR=YM
   GO TO 9
8 XL=XM
   YL=YM
9 XM=(XL+XR)/2.0
   ROUT=XM
10 RETURN
C
11 FORMAT (1H,10X,27HINO ROOT IN INTERVAL XM=DX =,1PE20,8,5,11HAND X
1M+DX =,1PE20,8/)
END

SUBROUTINE APEKE(EX8SL)

C********************~********C***~~~~~*~~~~~~~***~~*~~~~~~**~~~~~~~~~PE
C
PURPOSE
GETS FIRST APPROXIMATION TO PEAK POSITION
C

COMMON XPK,YPK
XPK=0.0
YPK=0.0
DO 2 I=1,17
   XM=10.0**(I-9)
   Y=EX8SL(X)
2 IF (Y<YPK) 3,3,1
   1 XPK=X
   YPK=Y
2 CONTINUE
3 RETURN
END

SUBROUTINE PEAK(EX8SL)

C*******************~********C***~~~~~*~~~~~~~***~~*~~~~~~**~~~~~~~~~PE
C
PURPOSE
ATTEMPTS TO FIND POSITION OF MAXIMUM FOR INTEGRAND
C

COMMON XPK,YPK
YPK=EX8SL(XPK)
DO 13 L=1,200
   DX=0.01*L
   XL=XPK+DX
   YL=EX8SL(XL)
   XR=XPK+DX
   YR=EX8SL(XR)
   DEN=XR+YL-YPK
   IF (DEN) 1,9,1
9  Y=XPK=0.5*(YR+YL)*DX/DEN
   1 X=XM 3,4,4
2 IF (X) 3,4,4
```
Table 8.2.—Listing of programs for constant discharge from a fully penetrating well of finite diameter—Continued

3 \( x=0,0 \)
4 \( y=EXBSL(x) \)
   IF (YM=Y) 6,6,5
5 \( y=yh \)
6 IF (YL=Y) 8,8,7
7 \( y=yl \)
8 \( x=xl \)
9 \( y=ypk \)
10 \( x=xpk+dx*dx \)
   GO TO 2
11 \( y=ypk-dx*dx \)
   GO TO 2
12 \( y=ypk \)
13 \( x=xpk \)
14 \( y=xpk \)
15 \( x=xpk \)
16 \( y=xpk \)
17 \( x=xpk \)
18 \( y=xpk \)
19 \( x=xpk \)
20 \( y=xpk \)
21 \( x=xpk \)
22 \( y=xpk \)
23 \( x=xpk \)
24 \( y=xpk \)
25 \( x=xpk \)
26 \( y=xpk \)
27 \( x=xpk \)
28 \( y=xpk \)
29 \( x=xpk \)
30 \( y=xpk \)
31 \( x=xpk \)
32 \( y=xpk \)
33 \( x=xpk \)
34 \( y=xpk \)
35 \( x=xpk \)
36 \( y=xpk \)
37 \( x=xpk \)
38 \( y=xpk \)

SUBROUTINE SIMPS(Q,R,EPSP,MBAR,AREA,DEL,F)
C***********************************************************************
C PURPOSE
C TO DETERMINE THE INTEGRAL OF A FUNCTION, F, FROM Q TO R,
C USING SIMPSON'S RULE,
C DESCRIPTION OF PARAMETERS
C ALL REAL
C Q = LOWER LIMIT OF INTEGRAL
C R = UPPER LIMIT OF INTEGRAL
C EPS = DESIRED ACCURACY
C MBAR = MINIMUM DIVISION OF THE INTERVAL
C AREA = COMPUTED VALUE OF INTEGRAL BETWEEN Q AND R
C DEL = COMPUTED ESTIMATE OF ERROR
C F = THE INTEGRAND (FUNCTION REFERENCE)
C METHOD
C USES SIMPSON'S RULE TO COMPUTE A SUM APPROXIMATING THE INTEGRAL
C USES INITIAL H=(R-Q)/2, COMPUTES A SEQUENCE OF SUMS BY HALVING
C EACH TIME, COMPUTES ESTIMATE OF ERROR (DEL) AS (PREVIOUS
C SUM - CURRENT SUM)/15, COMPUTATION STOPS WHEN 1) MHBAR,
C 2) ABS(DEL)<ABS(EPS*CURRENT SUM), IF MBAR IS LE 0,
C THEN MHBAR=0.007*(R-Q),
C R MUST BE GREATER THAN Q
C************************************************************
C MHR=0
C IF (H) 1,1,2
1 AREA=0,0
2 DEL=0,0
3 GO TO 10
C R MUST BE GREATER THAN Q
2 SP=0.0E35
3 SI=0,0
4 IF (F) 3,3,4
5 HBAR=0.007*H
6 SI=SI+F(H)
7 IF (HBAR) 3,3,4
8 S2=0,0
9 X=X+0.5*H
10 S2=S2+4.0*F(X)
11 X=X+H
12 IF (X=H) 5,5,6
13 SC=(SI+S2+S3)*H*0.1666667
C***********************************************************************
Table 8.2—Listing of programs for constant discharge from a fully penetrating well of finite diameter—Continued

```
DEL=0.00000007*SP*SC
IF (ABS(DEL)=ABS(EPS*SC)) 7,0,8
7 AREA=SC=DEL
GO TO 10
8 SP=SC
GO TO 4
10 RETURN
END
SUBROUTINE JY0(X,JO,Y0)
C************************************************************************
C PURPOSE
C COMPUTES BESSEL FUNCTIONS OF THE FIRST AND SECOND KIND,
C ZERO ORDER, FOR POSITIVE ARGUMENTS,
C SEE NBS AMS 55, P. 360-370.
C DESCRIPTION OF PARAMETERS = ALL REAL
C X= ARGUMENT, MUST BE >0
C JO = RETURNED FUNCTION VALUE, JO(X)
C Y0 = RETURNED FUNCTION VALUE, Y0(X)
C************************************************************************
REAL JO YO
IF (X=3.0) 1,2,3
1 IF (X) 4,4,2
2 ZM=0.33333333*2
JO=1.0-ZM*(2,2499997-ZM*(1.2656206-ZM*(0.3163866-ZM*(0.0444479-ZM*(0.00000121-ZM))))))
Y0=0.63661977*ALOG(0.5*X)*JO+0.3674691*ZM*(0.60559366*ZM*(0.74350386*Y0)
142*ZM*(0.25300117*ZM*(0.04261214*ZM*(0.00427887*ZM*(0.00024844*ZM)))
RETURN
1 ZM=3.0/X
F=0.79788465-ZM*(0.776=6*ZM*(0.059274+ZM*(0.00039512-ZM*(0.00137237-ZM))
1*0.00037605=0.00014476*ZM)))))
P=0.78539816-ZM*(0.00186397*ZM*(0.0003954-ZM*(0.00262573-ZM*(0.000341*Y0)
125*ZM*(0.0029333=0.00013589*ZM)))
G=SRT(1.0/X)
JO=U*F*COS(X*P)
Y0=U*SIN(X*P)
RETURN
END
SUBROUTINE JY1(X,JI,Y1)
C************************************************************************
C PURPOSE
C COMPUTES BESSEL FUNCTIONS OF THE FIRST AND SECOND KIND,
C FIRST ORDER, FOR POSITIVE ARGUMENTS,
C SEE NBS AMS 55, P. 370.
C DESCRIPTION OF PARAMETERS = ALL REAL
C X= ARGUMENT, MUST BE >0
C J1 = RETURNED FUNCTION VALUE, J1(X)
C Y1 = RETURNED FUNCTION VALUE, Y1(X)
C************************************************************************
REAL J1 Y1
IF (X=3.0) 1,2,3
1 IF (X) 4,4,2
2 ZM=0.33333333*2
```

Table 8.2.—Listing of programs for constant discharge from a fully penetrating well of finite diameter—Continued

| J1 = x^0.5 * z * (0.5 + 2 * 0.56249989 * r * 0.21093573 * z * (0.03954289 * z * (0.00443319 = y1) |
| 12 = (0.0031671 * 0.00001109 * z)) |
| J1 = 0.63661977 * ALOG(0.5 * x) + J1 + (0.0063661977 * (0.212091 + z * (0.1682709) + y1) |
| 1 = z * (0.3166827 * z * (0.03123951 * z * (0.00409796 * 0.0027873 * z)) / x) |
| RETURN |
| J1 = 0.79788458 * z * (0.1546 + z * (0.01659667 * z * (0.00017105 * z * (0.00249511 + y1) |
| 1 = z * (0.00113553 * 0.00020033 * z)) |
| P = 0.78539186 * z * (0.00000565 * z * (0.00637879 * z * (0.00007434 * y1) |
| 1 = 0.00007824 * 0.00002916 * z) |
| 4 = RETURN |

Purpose

Computes function values of F(uw, alpha) = FUA

Papadopoulos, S., and Cooper, M., Jr., 1967, Drawdown in
A well of large diameter: Water Resources Research, v. 3,
No. 1, p. 241-244, FUA

Program by S. Papadopoulos,

Input data = one or more groups, each group coded as follows

1 card = format (16, 5)

S = (alpha) * R^2 + S/RC^2 = radius of well (screen)

OR open bore in aquifer squared * storage

COEFFICIENT / RADIUS OF CASING (OVER INTERVAL UF

WATER LEVEL CHANGE) SQUARED

1 card = format (16, 5)

U = 16 values of uw = R^2 + S/(4 * T * TIME) = radius uf

Pumped well squared * storage coefficient /

4 = transmissivity * time, if less than 16 desired,

Blank or zero values may be coded for the rest.

Subroutines and function subprograms required

PEAK, SIMP, APEKE, EX88L2, JY0, JY1 = must be included in deck.

Common XPK, YPK

Common/PBLK/A, B

External EX88L2

Dimension U(16)

EPS = 0.0001

1 READ (5, 13, END=12) S

IF (S) 1, 1, 2

2 READ (5, 14) U

WRITE (6, 15) S

DO 11 I = 1, 16

U = U(I)

IF (U) 1, 1, 3

3 B = 0.25 / uw

A = S + B

call apeke(EX88L2)

call peak(EX88L2)

IF (XPK = 1, 0, 8) 4, 5, 5

4 WRITE (6, 16) U, S, XPK, YPK

go to 11

5 IF (XPK = 1, 0, 8) 7, 7, 6

6 WRITE (6, 17) U, S, XPK, YPK

go to 11

7 MCHAR = 0, 0, 0.07 * XPK

FUA 1

FUA 2

FUA 3

FUA 4

FUA 5

FUA 6

FUA 7

FUA 8

FUA 9

FUA 10

FUA 11

FUA 12

FUA 13

FUA 14

FUA 15

FUA 16

FUA 17

FUA 18

FUA 19

FUA 20

FUA 21

FUA 22

FUA 23

FUA 24

FUA 25

FUA 26

FUA 27

FUA 28

FUA 29

FUA 30

FUA 31

FUA 32

FUA 33

FUA 34

FUA 35

FUA 36

FUA 37

FUA 38

FUA 39

FUA 40

FUA 41

FUA 42

FUA 43

FUA 44

FUA 45

FUA 46
Table 8.2.—Listing of programs for constant discharge from a fully penetrating well of finite diameter—Continued

CALL SIMPS(0.,0.,XPK,EPS,HBAR,SUM,DEL,EXBSL2) FUA 47
X2=XPK FUA 48
DX=XPK FUA 49
8 DX=10.0*DX FUA 50
X1=X2 FUA 51
X2=X1+DX FUA 52
Y=EXBSL2*(X2) FUA 53
HBAR=0.007*DX FUA 54
CALL SIMPS(X1,X2,EPS,HBAR,TRM,ERR,EXBSL2) FUA 55
SUM=SUM+TRM FUA 56
DEL=DEL+ERR FUA 57
IF (X2=1.0E9) 9,10,10 FUA 58
9 Y1=1.5707963/X2**4 FUA 59
IF (ABS(Y1)/YT=0.5E-6) 10,8,8 FUA 60
10 EST=0.52359876/X2**3 FUA 61
SUM=SUM+EST FUA 62
FUX=S*24277.9*SUM FUA 63
WRITE (6,18) UW,SlJM,UEl,FU*S,XpK,VPK FUA 64
11 CONTINUE FUA 65
GO TO 1 FUA 66
12 STOP FUA 67
C
13 FORMAT (E10.5) FUA 68
14 FORMAT (16E5.0) FUA 69
15 FORMAT ('+',F(UW,ALPHA) FOR ALPHA=1.,1PE14,5/10.1,7X,1uw1,12X,'INTEGRATE') FUA 70
16 FORMAT (1H,1PE14,7,9X,34HVALUES OF DUMMY VARIABLE TOO SMALL,1PE25FUA 71
17 FORMAT ('H',1PE14,7,9X,34HVALUES OF DUMMY VARIABLE TOO LARGE,1PE25FUA 72
18 FORMAT ('H',1PE14,5,1PE17,5) FUA 73
END FUA 74
FUNCTION EXBSL2(X) FUA 75
C**********************************************************************FUA 76
C
C                  PURPOSE
C
C                  COMPUTES VALUES OF THE INTEGRAND FOR F(UW,ALPHA)
C
C                  DESCRIPTION OF PARAMETER
C
C                  X=HEAL=ARGUMENT OF INTEGRAND
C
C**********************************************************************FUA 77
COMMON/PBLK/A,B FUA 78
IF (X) 1,1,2 FUA 79
1 EXBSL2=0. FUA 80
GO TO 8 FUA 81
2 IF (X=1.E+7) 4,4,3 FUA 82
3 EXBSL2=1.5707963/X**4 FUA 83
GO TO 8 FUA 84
4 Y=E**X**X FUA 85
IF (Y=0.1) 5,5,6 FUA 86
5 FNUM=Y*(1.+Y*(5+Y*((1.0/6.)=Y*(1.0/24.)))) FUA 87
GO TO 7 FUA 88
6 FNUM=E**Y FUA 89
7 CALL JYO(X,BJ0,BYO) FUA 90
CALL JYO1(X,BJ1,BY1) FUA 91
DENO=(X*BJO=A*B1)**2+(X*BYO*A*BY1)**2)**3 FUA 92
EXBSL2=FNUM/DENO FUA 93
8 RETURN FUA 94
END FUA 95
TYPE CURVES FOR FLOW TO WELLS IN CONFINED AQUIFERS

TABLE 9.2.—Listing of program to compute change in water level due to sudden injection of a slug of water into a well

**Purpose**

COMPUTES FUNCTION VALUES OF F(BETA,ALPHA) = THE SLUG TEST FUNCTION. FOR VALUES OF BETA RANGING FROM 0.001 TO 1000.0 BY INCREMENTS OF 0.001, THE AVERAGE COMPUTATION TIME IS ABOUT 30 SECONDS PER VALUE OF ALPHA ON IBM 360/155.

**Input Data**

1 OR MORE CARDS = FORMAT(F16.5) A = (ALPHA) = (R**2*S/KC**2 - RADIUS OF WELL (SCREEN OR OPEN BORE IN AQUIFER) SQUARED / STORAGE COEFFICIENT)

**Subroutines and Function Subprograms Required**

PRX, DJY0, DJY1, DSMPS = MUST BE INCLUDED IN DECK

**Method**

THIS PROGRAM CALCULATES THE SLUG TEST FUNCTION, F(BETA,ALPHA), FOR VALUES OF BETA RANGING FROM 0.001 TO 1000.0 BY INCREMENTS OF 0.001. AVERAGE COMPUTATION TIME IS ABOUT 30 SECONDS PER VALUE OF ALPHA ON IBM 360/155.

**Double Precision Variables**

A, B, PI, ZZ, EPS, Y, X1, X2, TERM, FAB, DATAN, DEL, HBAR

**Common Blocks**

A, B, PI

**Subroutines and Function Subprograms**

PRX, DJY0, DJY1, DSMPS

**Data**

PI=4.43167/25 (FBI)

**Constants**

EPS=0.00001

1 IF (A LE 0.0) GO TO 5

6 FORMAT (F16.5)

7 FORMAT (F16.5)
TABLE 9.2.— Listing of program to compute change in water level due to sudden injection of a slug of water into a well—
Continued

DOUBLE PRECISION FUNCTION PRX(X)

C******************************************************

POURPOSE

COMPUTE VALUES OF THE INTEGRAND FOR F(BETA,ALPHA)

DESCRIPTION OF PARAMETER

X = DOUBLE PRECISION = ARGUMENT OF INTEGRAND

DOUBLE PRECISION A,B,PI,XX,X,C,F1,F2,J0,J1,JY

COMMON A,B,PI

XX=D$WRT(A*X/B)

IF (X) 6,1,2

PRX=(PI*PI)/(lb,*A*S)

GO TO 6

IF (XX,LT,150,1 GO TO 3

PRXa0.0

GO TU 6

3 IF (XX,GT,0,0001) GO TO 4

C*DO EX(5,772156649D=01)/2,

F1=PI*X*(1.+A)

F2=PI*DLOG(C*C*A*X/B)+4.,*B

PRX=(B*PI*DEXP(=-X))/((A*(F1*F1+F2*F2))

GO TO 6

4 IF (XX,.LT,50,0) GO TO 5

PRX=(PI*DEXP(=-X))/((2.,*XX*(X+4.,*A*B))

GO TU 6

5 CALL DJY0(XX,J0,YO)

CALL DJy1(XX,Jl,Yl)

F1=(XX*JO-Z,*A*Jl)

F2=(XX*YO-Z,*A*yl)

PRX=DEXP(=-X)/((X*(F1*F1+F2*F2))

RETURN

END

SUBROUTINE DJYO(X,J0,YO)

C******************************************************

PURPOSE

COMPUTES BESSEL FUNCTIONS OF THE FIRST AND SECOND KIND, J0 AND J1,

ZERO ORDER, FOR POSITIVE ARGUMENTS.

DESCRIPTION OF PARAMETERS = ALL DOUBLE PRECISION

X = ARGUMENT, MUST BE > 0

J0 = RETURNED FUNCTION VALUE, J0(X)

YO = RETURNED FUNCTION VALUE, YO(X)

DOUBLE PRECISION Z,J0,YO,F,P,Q,u,W,X,DLOG,DCOS,DSIN,DSQRT

IF (X=3.,*0) 1,2,3

1 IF (X) 4,412

2 Z=(X/3,0)*2

J0=1,0*Z*(2,2499997*Z*(1,2656208*Z*(0,3163866*Z*(0,0444479*Z*(0,00000000000D0 17

139444=0.00021+Z))))

W=0.50,0*X

YO=0.63661977*DLOG(W)*J0+0.36746691+Z*(0.06559366+Z*(0.74350384-Z+D0J0 20

1(0.2530117=Z*(0.04261214*Z*(0.00427916=0.00024846*Z))))

RETURN

3 Z=3.,*X

F=0.79788456*Z*(0.777*6+Z*(0.055274*Z*(0.0009512*Z*(0.0137237=Z+D0J0 24

1+(0.00072605=0.00014476*Z)))))

P=0.78539816*Z*(0.04161397*Z*(0.0003054*Z*(0.00262573*Z*(0.00541D0J0 26

125+Z*(0.00029333=0.00013558*Z))))

C
Table 9.2.—Listing of program to compute change in water level due to sudden injection of a slug of water into a well—Continued

\[
\begin{align*}
U &= (1.000)/X \\
Q &= DSQR(T(U)) \\
J_0 &= Q*FDCUS(X*P) \\
Y_0 &= Q*FDSIN(X*P) \\
4 & RETURN
\end{align*}
\]

SUBROUTINE DJY(X,JL,JL)

**Purpose**
Computes Bessel functions of the first and second kind, \(J_1\) and \(Y_1\), for positive arguments, \(x\).

**Description of Parameters**
- \(x\): Argument, must be >0
- \(J_1\): Returned function value, \(J_1(x)\)
- \(Y_1\): Returned function value, \(Y_1(x)\)

**Method**
- Uses \(J_1\) and \(Y_1\) for \(x > 0\) to compute \(J_1(x)\) and \(Y_1(x)\).
- \(J_1(x) = J_1(x)\), \(Y_1(x) = Y_1(x)\).

**Examples**
- For \(x = 1.0\), \(J_1(1.0) \approx 0.244237\), \(Y_1(1.0) \approx 0.069756\).
- For \(x = 2.0\), \(J_1(2.0) \approx 0.018841\), \(Y_1(2.0) \approx 0.048357\).
Table 9.2.—Listing of program to compute change in water level due to sudden injection of a slug of water into a well—Continued

```c
C******************************************************
DOUBLE PRECISION H,HBAR,AREA,DEL,S1,S2,S3,SC,SP,X,A,B,EPS,F,DABS
C AREA OF F FROM A TO B,EPS IS DESIRED ACCURACY, HBAR THE MINIMUM
C ALLOWABLE INTERVAL, DEL THE ESTIMATE OF THE ERROR
H=B=A
IF (H) 1,1,2
1 AREA=0,0
DEL=0,0
GO TO 10
2 SP=1.0D35
S3=0.0
S1=F(A)+F(B)
IF (HBAR) 3,3,4
3 HBAR=0.007*H
4 S2=0.0
X=0.5*H
5 S2=S2+0.0*F(X)
IF (X=B) 6,3,6
6 SC=(S1+S2+S3)*H=0.16666666667
DEL=0.06666666667*(SP*SC)
IF (DABS(DEL)=DABS(EPS*SC)) 7,8,8
7 AREA=SC=DEL
GO TO 10
8 S3=S3+0.5*S2
H=0.5*H
IF (H=HBAR) 9,9,9
9 SP=SC
GO TO 4
10 RETURN
END
```

Table 11.1.—Listing of program to compute the convolution integral for a leaky aquifer

```c
C******************************************************
C PURPOSE
C COMPUTES CHANGES IN WATER LEVEL, H(R,T), IN RESPONSE TO
C VARYING DISCHARGE USING THE CONVOLUTION INTEGRAL FOR
C LEAKY AQUIFERS = EU, 3 OF MOENCH, ALLEN, 1971, GROUND-WATER
C FLUCTUATIONS IN RESPONSE TO ARBITRARY PUMPAGE; GROUND WATER,
C Y=9, NO. 2, P. 243
C INPUT DATA - ONE OR MORE GROUPS, EACH GROUP CODED AS FOLLOWS
C 1 CARD FORMAT(2E10,5,4X,11,5X,1E10)
C NBEGIN = SMALLEST VALUE OF TIME FOR OUTPUT,
C NEND = LARGEST VALUE OF TIME FOR OUTPUT,
C IQ = INDICATES FORM OF DISCHARGE FUNCTION, Q(T),
C IQ=1,2,3 REFER TO DISCHARGE FUNCTIONS IN
C HANTUSH, M. S., 1964, HYDRAULICS OF WELLS IN CHOW,
C VEN TRE, ED., ADVANCES IN HYDROSCIENCE, VOL. II
C ACADEMIC PRESS INC., NEW YORK, P. 281-442,
C IQ=1, Q(T) IS AN EXPONENTIAL FUNCTION, CASE A,
C P. 343 OF HANTUSH,
C IQ=2, Q(T) IS A HYPERBOLIC FUNCTION, CASE B,
C P. 344 OF HANTUSH,
C IQ=3, Q(T) IS AN INVERSE SQUARE ROOT FUNCTION,
C CASE C, P. 344 OF HANTUSH,
```

```c
```
Table 11.1.—Listing of program to compute the convolution integral for a leaky aquifer—Continued

C

C IG=4, Q(T) IS A FIFTH-DEGREE POLYNOMIAL.
C IG=5, Q(T) IS A PIECEWISE LINEAR FUNCTION OF
C TIME (EIGHT SEGMENTS).
C
C QR = REFERENCE DISCHARGE, ZERO OR BLANK FOR PROJECTION.
C 1 OR 4 CARDS, DEPENDING ON IG.
C
C IF IG=1,2,3 = 1 CARD = FORMAT(3E10,3)
C QST = EVENTUAL CONSTANT DISCHARGE,
C DELTA = RATE PARAMETER,
C TSTAR = TIME PARAMETER,
C IF IG=4 = 1 CARD = FORMAT(6E10,3)
C AQ(6) = 6 VALUES THE POLYNOMIAL COEFFICIENTS
C WITH A0 FIRST AND AS LAST.
C IF IG=5 = 4 CARDS = FORMAT(6E10,3)
C T(I),AI(I),BI(I),TI(I+1),AI(I+1),BI(I+1),I=1,3,5,7
C PARAMETERS OF THE PIECEWISE LINEAR FUNCTION
C (8 SEGMENTS), CODED 2 SEGMENTS PER CARD, FIRST
C AND SECOND SEGMENTS ON FIRST CARD, THEN SEQUENTIALLY
C ON SUCCEEDING CARDS, EACH SEGMENT HAS THREE
C PARAMETERS WHICH ARE IN CODING ORDER
C TI = ENDING TIME OF THE SEGMENT,
C BI = RATE OF CHANGE IN DISCHARGE UNHING SEG.
C THE DISCHARGE FUNCTION IN EACH SEGMENT HAS THE
C FORM Q(T) = AI(I)+BI(I)*(T-TI(I-1)), IF LESS THAN 8
C SEGMENTS ARE NEEDED, BLANKS CAN BE CODED FOR
C SUCCEEDING SEGMENTS.
C 2 OR MORE CARDS = FORMAT(6E10,3)
C R = RADIAL DISTANCE FROM PUMPED WELL, BLANK OR ZERO
C SIGNALS PROGRAM AS END TO GROUP OF DATA.
C S = STORAGE COEFFICIENT
C T = TRANSMISSIVITY
C PM = (PI/MI) = HYD. COND. OF CONFINING BED DIVIDED
C BY THICKNESS OF CONFINING BED.
C SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C CONVOL, Q MUST BE INCLUDED IN DECK.

C*************************************************************************

DIMENSION D(12),EX(12),X(6),H(12,6),QS(12,6),CP(12),CT(12)
DIMENSION H1(12),H2(12),Q1(12),Q2(12)
DIMENSION H3(12),H4(12),Q3(12),Q4(12)
COMMON AQ(6),TI(9),AI(9),BI(9),QST,DELTA,TSTAR
DATA CP/12*1/,T1/11,CT/12*1/UT/11,UD/12*1/10**1/1
DATA H1/12*1/8/11,H2/12*1/RT/11,Q1/12*1/1/1,Q2/12*1/Q(T)/1
DATA H3/12*1/9/11,H4/12*1/OT/11,Q3/12*1/0(T)/1/Q4/12**1/QR/1
DATA X/1,1,5,2,3,5,7/1
T(I)@01
N=500
1 READ (5,18)ENDEM17)TBEGIN,TEND,IG,QR
IF (IQ LT 4) READ (5,19)QST,DELTA,TSTAR
IF (IQ EQ 4) READ (5,19)AQ
IF (IQ EQ 5) READ (5,19) (TI(I),AI(I),BI(I),I=2,9)
WRITE (6,20)
2 READ (5,19)R,S,T,PM
IF (R EQ 0) GO TO 1
A=M*M/4*1T
B=P3/8
Y=ALOG10(TBEGIN)
Table 11.1.—Listing of program to compute the convolution integral for a leaky aquifer—Continued

```
IF (Y) 3, 5, 4
3 Y = Y + 0.001
GO TO 5
4 Y = Y + 0.001
5 IBEGIN = Y
   Y = ALOG10 (TEND)
IF (Y) 6, 8, 7
6 Y = Y, 0.001
GO TO 8
7 Y = Y + 0.001
8 IEND = Y
   M = IEND - IBEGIN + 1
   IF (M, GT, 12) M = 12
   DO 10 I = 1, M
      IEX(I) = IBEGIN + I - 1
      V = 10. ** (IBEGIN + I - 1)
   DO 10 J = 1, B
      TIME = X(J) * Y
      IF (QR, GT, 0.5) TIME = TIME
      CALL CONVOL (TIME, A, B, N, IQ, SUM)
      IF (QR, GT, 0.5) GO TO 9
      H(I, J) = SUM / (12, 5664 * T)
      QS(I, J) = Q(TIME, IQ)
   GO TO 10
9 M(I, J) = SUM / QR
   QS(I, J) = Q(TIME, IQ) / QR
10 CONTINUE
   K = M
   IF (M, GE, 6) K = 6
   IF (QR, GT, 0.5) GO TO 11
   WRITE (6, 20) A, B, (CP(I), D(I), IEX(I), I = 1, K)
   WRITE (6, 21) (H1(I), H2(I), Q1(I), QS(I), I = 1, K)
   GO TO 12
11 WRITE (6, 25) A, B, QR, (CT(I), D(I), IEX(I), I = 1, K)
   WRITE (6, 26) (M1(I), M4(I), QS(I), iq(I), I = 1, K)
   DO 13 J = 1, B
      X(J) = (M(I, J), QS(I, J), I = 1, K)
13 CONTINUE
   IF (M, LE, 6) GO TO 2
   K = M
   IF (QR, GT, 0.5) GO TO 14
   WRITE (6, 23) (CP(I), D(I), IEX(I), I = 1, M)
   WRITE (6, 24) (H1(I), H2(I), Q1(I), Q2(I), I = M + 1, M)
   GO TO 15
14 WRITE (6, 27) (CT(I), D(I), IEX(I), I = 1, M)
   WRITE (6, 28) (M1(I), M4(I), QS(I), I = 1, M)
   DO 16 J = 1, B
      X(J) = (M(I, J), QS(I, J), I = 1, M)
16 CONTINUE
   GO TO 2
17 STOP
C
18 FORMAT (2E10.5, 4X, I1, 5X, E10.5)
19 FORMAT (6E10.3)
20 FORMAT ('01', 'R**2*8/(4*TRANS)=', '1PE10.3, ', 'X**/(9*B11)=', 'E10.3/10', '12X, 'TI', '5X, B(2A4, 12, 9X))
21 FORMAT ('11', '4X, B(2A4, 2X, 2A4, 1X))
```

Table 11.1.—Listing of program to compute the convolution integral for a leaky aquifer—Continued

22 FORMAT ('1 ',F4.1,B,6(0PFB,3,1PE11,3))
23 FORMAT ('01,2X,'1',5X,0(2A4,12,9X))
24 FORMAT (1M1)
25 FORMAT ('01,1X,R*2*5/(4*TRANS)*1,1PE10,3',1, K*1/(5*8)*1,E10,3,1)
10R*1,E10,3,01,1X,'1',1/0,4X,6(2A4,12,9X))
26 FORMAT ('01,1X,1/0,4X,6(2A4,12,9X))

SUBROUTINE CONVOL (TIME, A, B, N, I0, SUM)

C******************************************************************************
C
C PURPOSE
C COMPUTES VALUES OF THE CONVOLUTION INTEGRAL FOR LEAKY AQUIFERS, THE INTEGRAL IS, FROM 0 TO T, OF
C Q(T) = T*EXP(-A/T) = RADIAL DISTANCE SQUARED * STORAGE COEFFICIENT / 4 * TRANSMISSIVITY,
C B = P/((S*H)) = HYD. COND. OF CONFINING BED DIVIDED BY
C AQUIFER STORAGE COEFFICIENT * THICKNESS OF CONF. BED,
C N = NUMBER OF INCREMENTS FOR EACH INTERVAL OF THE SUM,
C I0 = INDICATES FORM OF DISCHARGE FUNCTION,
C SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED

C METHOD
C APPROXIMATES INTEGRAL BY SUMMING THE TRAPEZOIDAL RULE APPLIED
C TO A SEQUENCE OF SEGMENTS, LOWER LIMIT OF FIRST SEGMENT IS
C PICKED AT POINT WHERE EXPONENT > -100,
C IF SUCH A POINT DOES NOT EXIST (A*B > 2500) A FUNCTION VALUE
C OF 0 IS RETURNED, UPPER LIMIT = 10 * LOWER LIMIT FOR EACH
C SEGMENT, USES INCREMENT OF DELTA T = (U*L)/N WHERE N IS THE
C NUMBER OF INCREMENTS IN THE CALL, CEASES SUMMATION WHEN
C EXPONENT < -101,

C******************************************************************************

REAL*8 DSUM
REAL*4 NEWT,NEWTP,NEWX,NEWF
DSUM=0,U=0
IS=0
C
C INITIAL T1 COMPUTED FROM A,B
AB=A*B
IF (A,B,GE,2500,0) GOTO 7
IF (A,B,GT,0,0) GOTO 2
1 OLDT=U*A
GO TO 3
2 OLDT=(1,-SQR(1,-A/B/2500,0))%50,A/B
IF (ULDT,EQ,0,0) GOTO 1
INITIAL T=1
3 OLDT=TIME=ULDT
OLDX=-A/OLDT=E-OLDT
OLDF=Q(OLDTP,I0)*EXP(OLDX)/ULDT
C
END OF SUMMATION SEGMENT IS 10 TIMES THE BEGINNING
4 ENDT=10,ULDT
IF (ENDT,LT,TIME) GOTO 5
IF (ULDT,GE,TIME) GOTO 7
IS=1
ENDT=TIME

END
Table 11.1. Listing of program to compute the convolution integral for a leaky aquifer—Continued

C DELTA T' IS COMPUTED FROM LENGTH AND NUMBER OF INCREMENTS
C
5 DELT=(ENDT-OLDT)/N
DO 6 I=1,N
C T' IS INCREMENTED BY DELTA T'
NEW=OLDT+DEL
NEwX=A/NEW+B*NEW
C TERMINATES SUMMATION WHEN EXP(-A/T'+B*T') < 1.37E-44
IF (NEWX,LT,=101.) GO TO 7
NEW=TIME=NEW
NEwF=(NEWT,INT)*EXP(NEWX)*NEW
DSUM=DSUM(NEWF+OLDF)*DEL/T
OLDT=NEW
OLDF=NEW
6 CONTINUE
IF (IS,GT,0) GO TO 7
IF T' < T, BEGINS A NEW SEGMENT
GO TO 4
7 SUM=DSUM/2.0+0
RETURN
END

FUNCTION Q(TIME,IW)

C******************************~*~~*~**~~~~**~~~****~*~*~*~**w*~**~~~~~**
Z PURPOSE
C COMPUTES THE DISCHARGE FUNCTION, Q(T)
C DESCRIPTION OF PARAMETERS
C TIME = REAL = ELAPSED TIME SINCE BEGINNING OF DISCHARGE,
C IW = INTEGER = INDICATES FORM OF DISCHARGE FUNCTION,
C IG=1,2,3, CASES A,B,C, RESPECTIVELY, OF HANLUSH*, 8, 1964, HYDRAULICS OF WELLS IN CHN, VEN TE, ED.,
C ADVANCES IN HYDROSCIENCE, VOL. 1, ACADEMIC PRESS, NEW YORK, P. 343,344,
C IG=4, DISCHARGE IS A FIFTH DEGREE POLYNOMIAL OF TIME,
C IW=5, DISCHARGE IS A PIECEWISE LINEAR FUNCTION OF UP TO 8 SEGMENTS,
C METHOD
C FORTRAN EVALUATION OF FUNCTIONS,
C
C******************************************************
COMMON AQ(6),TI(9),AI(9),BI(9),QST,DELTA,TSTAR
GO TO (1,2,3,4,5), IG
1 Q=QST*(1.+DELTA*EXP(-TIME/TSTAR))
RETURN
2 Q=QST*(1.+DELTA/(1.+TIME/TSTAR))
RETURN
3 Q=QST*(1.+DELTA/SQRT(1.+TIME/TSTAR))
RETURN
4 Q=Q(1.)+TIME*(Q(2.)+TIME*(Q(3.)+TIME*(Q(4.)+TIME*(Q(5.)+TIME*AQ(6) 1)))))
RETURN
5 DO 6 I=2,9
IF (TIME.LE,=TI(I)) GO TO 7
6 CONTINUE
I=9
7 Q=AI(I)*BI(I)*(TIME-TI(I-1))
RETURN
END

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