Techniques of Water-Resources Investigations of the United States Geological Survey

Chapter B3

TYPE CURVES FOR SELECTED PROBLEMS OF FLOW TO WELLS IN CONFINED AQUIFERS

By J. E. Reed

Book 3
APPLICATIONS OF HYDRAULICS

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SYMBOLS AND DIMENSIONS

[Numbers in parentheses indicate the solutions to which the definition applies. If no number appears, the symbol has only one definition in this report]

Symbol	Dimension	Description
a	Dimensionless	$\sqrt{K_z/K_r}$.
b	L	Aquifer thickness.
b'	L	Thickness of confining bed (4, 6, 7, 11); specifically the upper confining bed (5).
b''	L	Thickness of lower confining bed.
d	L	Depth from top of aquifer to top of pumped well screen.
d'	L	Depth from top of aquifer to top of observation-well screen.
H	L	Change in water level in well.
H_0	L	Initial head increase in well.
h	L	Change in water level in aquifer.
K	LT^{-1}	Hydraulic conductivity of aquifer.
K_r	LT^{-1}	Hydraulic conductivity of the aquifer in the radial direction.
K_z	LT^{-1}	Hydraulic conductivity of the aquifer in the vertical direction.
K'	LT^{-1}	Hydraulic conductivity of confining bed (4, 6, 7); specifically the upper confining bed (5).
K''	LT^{-1}	Hydraulic conductivity of lower confining bed.
l	L	Depth from top of aquifer to bottom of pumped well screen.
l'	L	Depth from top of aquifer to bottom of observation-well screen.
Q	L^3T^{-1}	Discharge rate.
$Q(t)$	L^3T^{-1}	Discharge rate.
r	L	Radial distance from center of pumping, flowing, or injecting well.
r_c	L	Radius of well casing or open hole in the interval where the water level changes.
r_w	L	Effective radius of well screen or open hole for pumping, flowing, or injecting well.
S	Dimensionless	Storage coefficient.
S_s	L^{-1}	Specific storage of aquifer.
S_s'	L^{-1}	Specific storage of confining beds.
S'	Dimensionless	Storage coefficient of upper confining bed.
S''	Dimensionless	Storage coefficient of lower confining bed.
s	L	Drawdown in head (change in water level).
s_1	L	Drawdown in upper confining bed.
s_2	L	Drawdown in lower confining bed.
s_w	L	Constant drawdown in discharging well.
T	L^2T^{-1}	Transmissivity.
T_{xx}, T_{xy}, T_{yy}	L^2T^{-1}	Components of the transmissivity tensor in any orthogonal x-, y-axis system.
$T_{\epsilon\epsilon}$, $T_{\eta\eta}$	L^2T^{-1}	Transmissivities along two principal axes, ϵ and η, such that $T_{\epsilon\eta}=0$.
t	T	Time.
t'	Dimensionless	Variable of integration.
u	Dimensionless	$r^2S/4Tt$(2, 6); variable of integration (3, 7, 9).
v	Dimensionless	Variable of integration.
x	Dimensionless	Dummy variable (2, 5); variable of integration (3).
x, y	L	Distances from the pumped well for an arbitrary rectangular coordinate system (10).
y	Dimensionless	Variable of integration (1, 2, 4, 5, 6).
z	L	Depth from top of aquifer, also, specifically, the depth to bottom of a piezometer (2, 6); depth below top of upper confining bed (5).
z	Dimensionless	Dummy variable (10).
α	Dimensionless	Tt/Sr_w^2.
β	Dimensionless	Variable of integration.
Θ	Dimensionless	Angle between x axis and ϵ axis.
ϵ, η	L	Distances from pumped well in a coordinate system colinear with principal axes of transmissivity tensor.
ρ	Dimensionless	r/r_w.
τ	Dimensionless	Tt/Sr_w^2.

TYPE CURVES FOR SELECTED PROBLEMS OF FLOW TO WELLS IN CONFINED AQUIFERS

By J. E. Reed

Abstract

This report presents type curves and related material for 11 conditions of flow to wells in confined aquifers. These solutions, compiled from hydrologic literature, span an interval of time from Theis (1935) to Papadopulos, Bredehoeft, and Cooper (1973). Solutions are presented for constant discharge, constant drawdown, and variable discharge for pumping wells that fully penetrate leaky and nonleaky aquifers. Solutions for wells that partially penetrate leaky and nonleaky aquifers are included. Also, solutions are included for the effect of finite well radius and the sudden injection of a volume of water for nonleaky aquifers. Each problem includes the partial differential equation, boundary and initial conditions, and solutions. Programs in FORTRAN for calculating additional function values are included for most of the solutions.

Introduction

The purpose of this report is to assemble, under one cover and in a standard format, the more commonly used type-curve solutions for confined ground-water flow toward a well in an infinite aquifer. Some of these solutions are only published in several different journals; some of these journals are not readily obtainable. Other solutions which are included in several references (for example, Ferris and others, 1962; Walton, 1962; Hantush, 1964a; Lohman, 1972) are included here for completeness.

The need for a compendium of type curves for aquifer-test analysis was recognized by Robert W. Stallman, who initiated the work on it. However, ill health and the press of other duties prevented him from personally carrying out his concept, but he never ceased to advocate the need for the compendium. Although it is reduced in scope from his original concept, this report should be recognized to be a result of Stallman's foresight and endeavors in the field of ground-water hydrology.

The type-curve method was devised by C. V. Theis (Wenzel, 1942, p. 88) to determine the two unknown parameters, S and T, in the equations

$$s = (Q/4\pi T)W(u)$$

and

$$u = r^2S/(4Tt),$$

where s is the drawdown in water level in response to the pumping rate Q in an aquifer with transmissivity T and storage coefficient S. The distance r from the pumping well, and the elapsed time t since pumping began, combine with S and T to define a dimensionless variable u and corresponding dimensionless response $W(u)$. Briefly, the method consists of plotting a function curve or type curve, such as $(1/u, W(u))$ on logarithmic-scale graph paper, and plotting the time-drawdown (t-s) data on a second sheet having the same scales. This is equivalent to expressing the preceding equations as

$$\log s = \log Q/4\pi T + \log W(u)$$

and

$$\log 1/u = \log t + \log 4T/r^2S.$$

If the two sheets are superimposed and matched, keeping coordinate axes parallel, as shown in figure 0.1, the respective coordinate

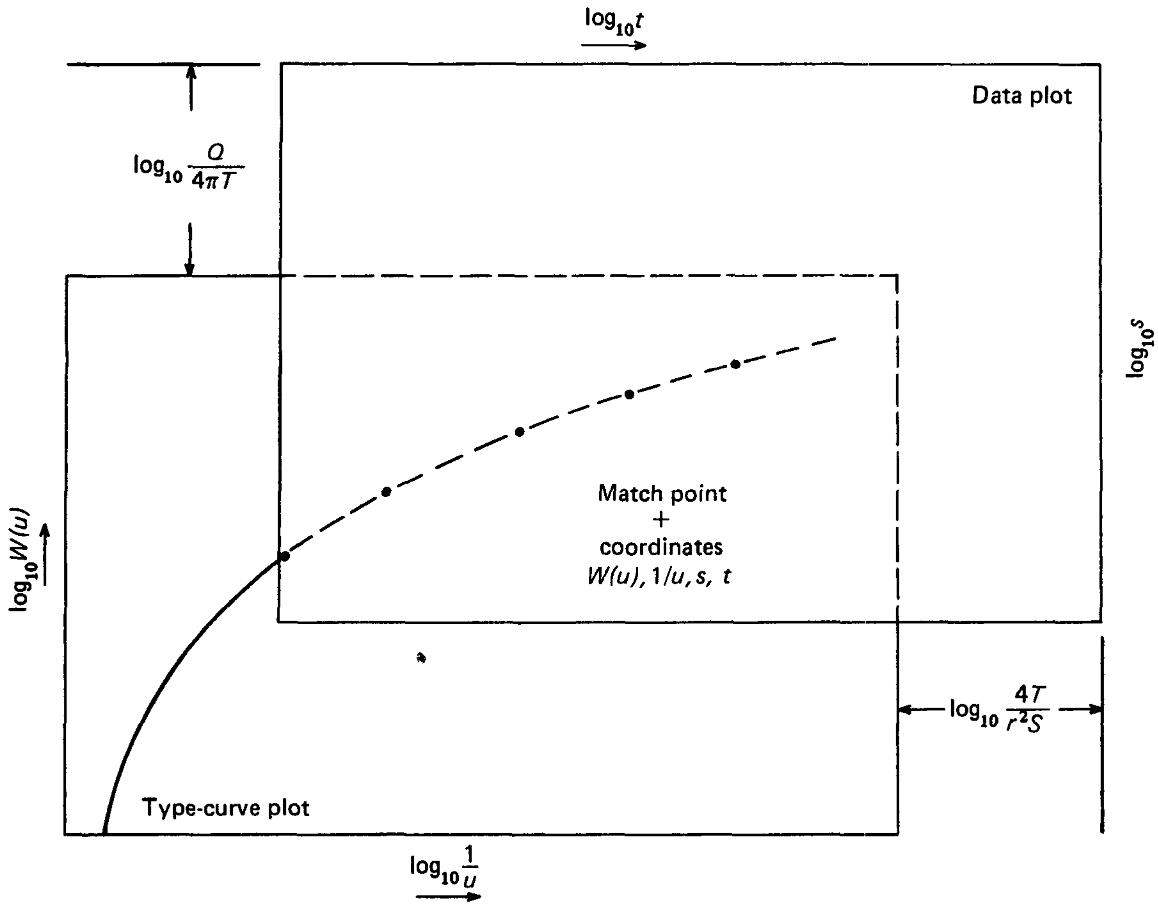

FIGURE 0.1.—Relation of $1/u, W(u)$ type curve and t, s data plot. Modified from Stallman (1971, p. 5, fig. 1).

axes will be related by constant factors: $s/W(u)=C_1$ and $t/(1/u)=C_2$. The values of these two constants are

$$C_1=Q/(4\pi T)$$

and

$$C_2=r^2S/(4T).$$

Thus, a common match point for the two curves may be chosen, and the four coordinate points—$W(u)$, $1/u$, s, and t—recorded for the common match point. T can be obtained from the equation $T=QW(u)/(4\pi s)$, and then S can be solved from the equation $S=4Tut/r^2$, where $W(u)$, $1/u$, s, and t are the match-point values.

It is apparent that the type curves, and data, can be plotted in several ways. That is, the function curve, using $W(u)$ as an example, could be plotted as $(u,W(u))$ with corresponding data plots of $(1/t,s)$ or $(r^2/t,s)$; or could be plotted as $(1/u,W(u))$ with corresponding data plots of (t,s) or $(t/r^2,s)$. The type-curve method is covered more fully by Ferris, Knowles, Brown, and Stallman (1962, p. 94).

The type curves presented in this report are shown on two different plots. One plot has both logarithmic scales with 1.85 inches per log-cycle, such as K and E 467522.[1] The other plot is arithmetic-logarithmic scale with the logarithmic scale 2 inches per log-cycle and the arithmetic scale with divisions at multiples of 0.1, 0.5, and 1.0 inches, such as K and E 466213.

Other methods exist for analysis of aquifer-test data. Among them are methods based on plots of data on semi-log paper, developed by

[1]The use of brand names in this report is for identification purposes only and does not imply endorsement by the U S Geological Survey

Jacob (Ferris and others, 1962, p. 98) and by Hantush (1956, p. 703). These methods are useful, but they are beyond the scope of this report.

Aquifer tests deal with only one component of the natural flow system. The isolation of the effects of one stress upon the system is based upon the technique of superposition. This technique requires that the natural flow system can be approximated as a linear system, one in which total flow is the addition of the individual flow components resulting from distinct stresses.

The use of the principle of superposition is implied in most aquifer-test analyses. The term "superposition," as here applied, is derived from the theory of linear differential equations. If the partial-differential equation is linear (in the dependent variable and its derivatives), two or more solutions, each for a given set of boundary and initial conditions, can be summed algebraically to obtain a solution for the combined conditions. For instance, consider a situation (fig. 0.2) where a well has been pumping for some time at a constant rate Q_0, and the drawdown trend for that pumping rate has been established. Assume that the pumping rate increases by some amount ΔQ at some time t_1. Then the drawdown for that step incrase in rate will be the change in drawdown from that occurring due to the pumpage Q_0.

Programs, written in FORTRAN, for calculating additional function values are included for most of the solutions. Some of the type-curve solutions would require an unreasonably long tabulation to include all the possible combinations of parameters. An alternative to a tabulation is the computer program that can calculate type-curve values for the parameters desired by the user. The programs could be easily modified to calculate aquifer response to more than one well, such as well fields or image-well systems (Ferris and others, 1962, p. 144). The programs have been tested and are probably reasonably free from error. However, because of the large number of possible parameter combinations, it was possible to test only a sample of possible parameter values. Therefore, errors might occur in future use of these programs.

"An aquifer test is a controlled field experiment made to determine the hydraulic properties of water-bearing and associated rocks" (Stallman, 1971). The areal variability of hydraulic properties in an aquifer limits aquifer tests to integrating these properties within the

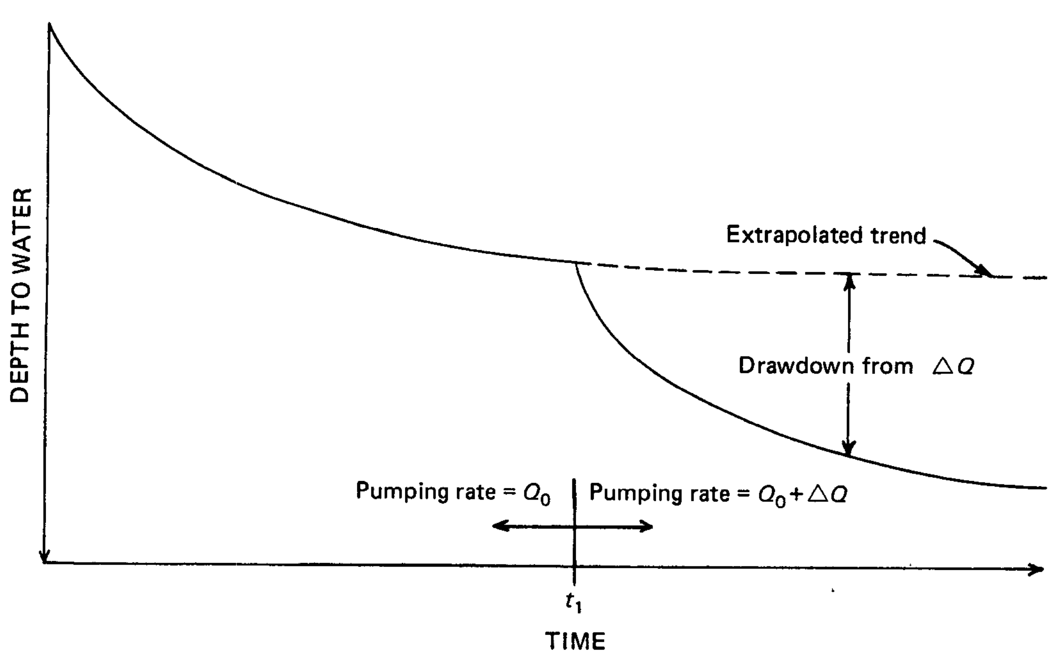

FIGURE 0.2.—The application of the principle of superposition to aquifer tests.

cone of depression produced during the test. Aquifer-test solutions are based on idealized representations of the aquifer, its boundaries, and the nature of the stress on the aquifer. The type-curve solutions presented in this report all have certain assumptions in common. The common assumptions are that the aquifer is horizontal and infinite in areal extent, that water is confined by less permeable beds above and below the aquifer, that the formation parameters are uniform in space and constant in time, that flow is laminar, and that water is released from storage instantaneously with a decline in head. Also implicit is the assumption that hydraulic potential or head is the only cause of flow in the system and that thermal, chemical, density, or other forces are not affecting flow. In addition to these common assumptions are special assumptions that characterize each solution summary. An important first step in aquifer-test analysis is deciding which simplified representations most closely match the usually complex field conditions.

Generally the best start in the analysis of aquifer-test data is with the most general set of type curves that apply to the situation, keeping in mind limitations of the method and effects that cause departures from the theoretical results. For example, the most general set of type curves for constant discharge presented in this report is for leaky aquifers with storage of water in the confining beds, *solution* 5. This includes, as a limiting case, the curve for a nonleaky aquifer. The most severe limitation on this set of curves is that they apply only at early times, as specified in *solution* 5.

Some of the effects that cause departure from the theoretical curves are partial penetration, finite well radius, and variable discharge for the pumped well. The effects of partial penetration must be considered when $r/b<1.5$, and because vertical-horizontal anisotropy is probably a common condition, these effects should be considered for $r/b<10$. The effect of finite well radius should be considered for early times, as specified in *solution* 8. The effects of variable discharge depend upon the manner of the variation. A change in discharge is more important if the change is monotonic, either continually increasing or decreasing. This fact is shown by the type curves for *solution* 11, where a monotonic change of 10 percent caused a significant departure from the Theis curve. If the discharge variation consists of random "noise" about a constant discharge, a 10-percent variation is not significant. The most general set of type curves for tests on flowing wells is *solution* 7, for leaky aquifers, which includes nonleaky aquifers as a limiting case. The only set of curves for slug tests is given in *solution* 9.

A recurring problem in type-curve solution for unknown hydrologic parameters is that of nonuniqueness. That is, function curves for different parameter values sometimes have similar shapes. An example of this is given by Stallman (1971, p. 19 and fig. 6). He indicated that the selection of the conceptual model is very important in interpreting the test results. Equally important is adequate testing of the conceptual model. Corroboration of the conceptual model is indicated by similar results for hydrologic parameters from data collected at varying distances from the pumped well, depths within the aquifer, and at different observation times. However, proof of suitability of the conceptual model ultimately rests on field investigations and not on curve matching.

As an example of similar curve shapes for different situations, consider the case of constant discharge in a nonleaky aquifer with exponentially varying thickness. The thickness, b, is equal to $b_0 \exp[-2(X-X_0)/a]$, where b_0 and X_0 are the thickness and X-coordinate, respectively, at the site of the discharging well and a is a parameter. The drawdown for this situation is given by Hantush (1962, p. 1529):

$$s=(Q/4\pi K b_0) \exp\,(r/a \cos \Theta)\, W(u,r/a),$$

where

$$W(u,\beta)=\int_u^\infty (\exp(-y-\beta^2/4y)/y)\, dy,$$

$$u=r^2 S_s/4Kt,$$

Q is the discharge, r is the distance from the discharging well, Θ is the angle, with apex at the discharging well, between the observation well and the positive X-axis, K is the hydraulic conductivity of the aquifer, and S_s is the specific storage coefficient of the aquifer. This solution is similar to the equation describing drawdown in a leaky artesian aquifer (Hantush, 1956, p. 702), which is

$$s = (Q/4\pi T)\ W(u,r/B),$$

with $T=Kb$, $B=\sqrt{Tb'/K'}$, and b' and K' are the thickness and hydraulic conductivity, respectively, of the leaky confining bed. The other symbols are used as above.

These two functions have the same shape when plotted on logarithmic paper, and drawdown resulting from one function could be matched to a type curve of the other function. Suppose, as an example, that the "observed data" are described by the function for the aquifer with exponentially changing thickness. Suppose, also, that the hydrologist is unaware of the variation in thickness and that the family of type curves for leaky aquifers without storage in the confining beds, *solution* 4, has been chosen for analysis of the "observed data." Matching the data plots to the type curves and solving for unknown parameters by the methods suggested in *solution* 4 gives for the ratio of K_a, the apparent hydraulic conductivity, to K, the true hydraulic conductivity, $K_a/K=\exp((r/a)\cos\Theta)$. The ratio would be close to one only in the vicinity of the discharging well. The diffusivity, K/S_s, would be determined correctly, but the apparent specific storage coefficient would have the same percentage error as the apparent hydraulic conductivity. Most important of all, the erroneous conclusion would be that the aquifer is leaky, with leakage parameter $B = \sqrt{Kbb'/K'} = a$. This somewhat contrived example illustrates a principle in the interpretation of aquifer-test data. Conclusions about the hydrologic constraints on the response of the aquifer to pumping should not be based on the shape of the data curves. Inferences may be made from these curves, but they must be verified by other hydrologic and geologic data. Therefore, proof of the suitability of the conceptual model must come from field investigations.

Many of the old reports of the U.S. Geological Survey contain references to the terms "coefficient of transmissibility" and "field coefficient of permeability." These terms, which were expressed in inconsistent units of gallons and feet, have been replaced by transmissivity and hydraulic conductivity (Lohman and others, 1972, p. 4 and p. 13). Transmissivity and hydraulic conductivity are not solely properties of the porous medium; they are also determined by the kinematic viscosity of the liquid, which is a function of temperature. Field determinations of transmissivity or hydraulic conductivity are made at prevailing field temperatures, and no corrections for temperature are made.

Summaries of Type-Curve Solutions for Confined Ground-Water Flow Toward a Well in an Infinite Aquifer

Solution 1: Constant discharge from a fully penetrating well in a nonleaky aquifer (Theis equation)

Assumptions:

1. Well discharges at a constant rate, Q.
2. Well is of infinitesimal diameter and fully penetrates the aquifer.
3. Aquifer is not leaky.
4. Discharge from the well is derived exclusively from storage in the aquifer.

Differential equation:

$$\partial^2 s/\partial r^2 + (1/r)\,(\partial s/\partial r) = (S/T)(\partial s/\partial t)$$

Boundary and initial conditions:

$$s(r,0) = 0,\ r \geq 0 \qquad (1)$$

$$s(\infty,t) = 0,\ t \geq 0 \qquad (2)$$

$$Q = \begin{cases} 0, t<0 \\ \text{constant} >0,\ t\geq 0 \end{cases} \qquad (3)$$

$$\lim_{r\to 0} r\frac{\partial s}{\partial r} = -\frac{Q}{2\pi T},\quad t\geq 0 \qquad (4)$$

Equation 1 states that initially drawdown is zero everywhere in the aquifer. Equation 2 states that the drawdown approaches zero as the distance from the well approaches infinity. Equation 3 states that the discharge from the well is constant throughout the pumping period. Equation 4 states that near the pumping well the flow toward the well is equal to its discharge.

Solution (Theis, 1935):

$$s = \frac{Q}{4\pi T}\int_u^\infty \frac{e^{-y}}{y}\,dy$$

$$u = \frac{r^2S}{4Tt},$$

where

$$\int_u^\infty \frac{e^{-y}}{y}\,dy = W(u) = -0.577216 - \log_e u + u - \frac{u^2}{2!\,2} + \frac{u^3}{3!\,3} - \frac{u^4}{4!\,4} + \ldots\ldots\ldots$$

Comments:

Assumptions made are applicable to artesian aquifers (fig. 1.1). However, the solution may be applied to unconfined aquifers if drawdown is small compared with the saturated thickness of the aquifer and if water in the sediments through which the water table has fallen is discharged instantaneously with the fall of the water table. According to assumption 2, this solution does not consider the effect of the change in storage within the pumping well. Assumption 2 is acceptable if

$$t > 2.5\times10^2 r_c^2/T$$

(Papadopulos and Cooper, 1967, p. 242), where r_c is the radius of the well casing in the interval over which the water-level declines, and other symbols are as defined previously. Figure 1.2 on plate 1 is a logarithmic graph of $W(u)=4\pi sT/Q$ plotted on the vertical coordinates versus $1/u=4Tt/(r^2S)$ plotted on the horizontal coordinates. The test data should be plotted with s on the vertical coordinates and corresponding values of t or t/r^2 on the horizontal coordinates.

Values of $W(u)$ for u between 0 and 170 may be computed by using subroutine EXPI of the IBM System/360 Scientific Subroutine Package. Table 1.1 gives values of $W(u)$ for selected values of $1/u$ between 1×10^{-1} and 9×10^{14}, as calculated by this subroutine.

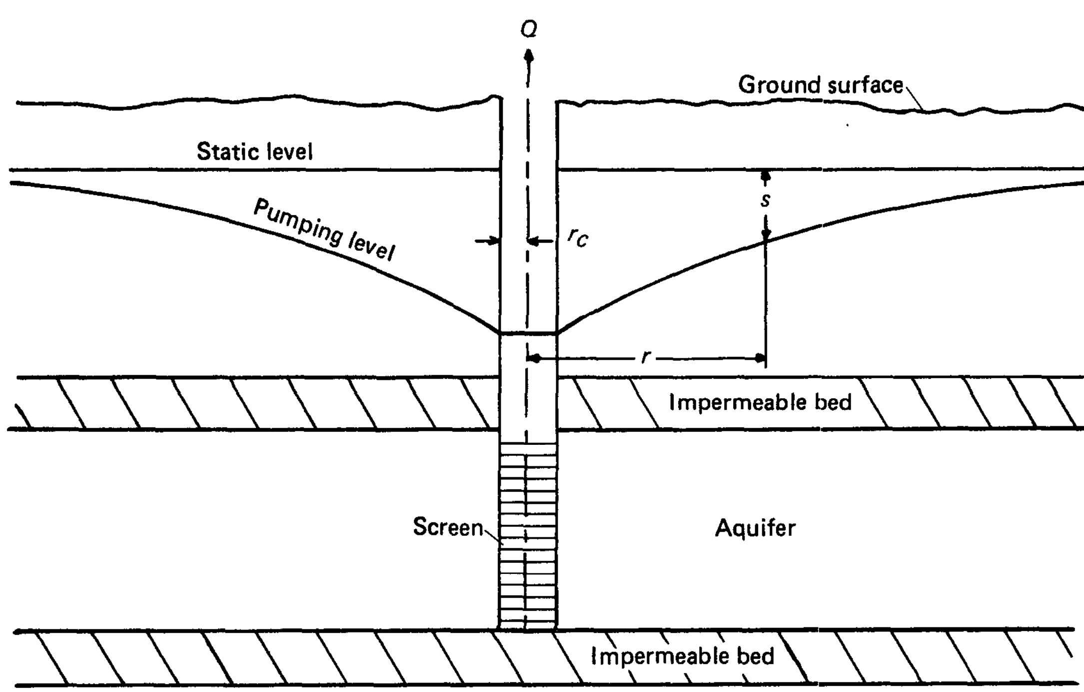

FIGURE 1.1.—Cross section through a discharging well in a nonleaky aquifer.

TABLE 1.1.—*Values of Theis equation* W(u) *for values of* 1/u

$1/u$	$1/u \times 10^{-1}$	1	10	10^2	10^3	10^4	10^5	10^6
1.0	[1]0.00000	0.21938	1.82292	4.03793	6.33154	8.63322	10.93572	13.23830
1.2	.00003	.29255	1.98932	4.21859	6.51369	8.81553	11.11804	13.42062
1.5	.00017	.39841	2.19641	4.44007	6.73667	9.03866	11.34118	13.64376
2.0	.00115	.55977	2.46790	4.72610	7.02419	9.32632	11.62886	13.93144
2.5	.00378	.70238	2.68126	4.94824	7.24723	9.54945	11.85201	14.15459
3.0	.00857	.82889	2.85704	5.12990	7.42949	9.73177	12.03433	14.33691
3.5	.01566	.94208	3.00650	5.28357	7.58359	9.88592	12.18847	14.49106
4.0	.02491	1.04428	3.13651	5.41675	7.71708	10.01944	12.32201	14.62459
5.0	.04890	1.22265	3.35471	5.63939	7.94018	10.24258	12.54515	14.84773
6.0	.07833	1.37451	3.53372	5.82138	8.12247	10.42490	12.72747	15.03006
7.0	.11131	1.50661	3.68551	5.97529	8.27659	10.57905	12.88162	15.18421
8.0	.14641	1.62342	3.81727	6.10865	8.41011	10.71258	13.01515	15.31774
9.0	.18266	1.72811	3.93367	6.22629	8.52787	10.83036	13.13294	15.43551

$1/u$	$1/u \times 10^7$	10^8	10^9	10^{10}	10^{11}	10^{12}	10^{13}	10^{14}
1.0	15.54087	17.84344	20.14604	22.44862	24.75121	27.05379	29.35638	31.65897
1.2	15.72320	18.02577	20.32835	22.63094	24.93353	27.23611	29.53870	31.84128
1.5	15.94634	18.24892	20.55150	22.85408	25.15668	27.45926	29.76184	32.06442
2.0	16.23401	18.53659	20.83919	23.14177	25.44435	27.74693	30.04953	32.35211
2.5	16.45715	18.75974	21.06233	23.36491	25.66750	27.97008	30.27267	32.57526
3.0	16.63948	18.94206	21.24464	23.54723	25.84982	28.15240	30.45499	32.75757
3.5	16.79362	19.09621	21.39880	23.70139	26.00397	28.30655	30.60915	32.91173
4.0	16.92715	19.22975	21.53233	23.83492	26.13750	28.44008	30.74268	33.04526
5.0	17.15030	19.45288	21.75548	24.05806	26.36064	28.66322	30.96582	33.26840
6.0	17.33263	19.63521	21.93779	24.24039	26.54297	28.84555	31.14813	33.45071
7.0	17.48677	19.78937	22.09195	24.39453	26.69711	28.99969	31.30229	33.60487
8.0	17.62030	19.92290	22.22548	24.52806	26.83064	29.13324	31.43582	33.73840
9.0	17.73808	20.04068	22.34326	24.64584	26.94843	29.25102	31.55360	33.85619

[1]Value shown as 0.00000 is nonzero but less than 0.000005.

Solution 2: Constant discharge from a partially penetrating well in a nonleaky aquifer

Assumptions:

1. Well discharges at a constant rate, Q.
2. Well is of infinitesimal diameter and is screened in only part of the aquifer.
3. Aquifer has radial-vertical anisotropy.
4. Aquifer is not leaky.
5. Discharge from the well is derived exclusively from storage in the aquifer.

Differential equation:

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} + a^2\frac{\partial^2 s}{\partial z^2} = \frac{S}{T}\frac{\partial s}{\partial t}$$

$$a^2 = K_z/K_r$$

This is the differential equation for nonsteady radial and vertical flow in a homogeneous confined aquifer with radial-vertical anisotropy.

Boundary and initial conditions:

$$s(r,z,0)=0,\ r\geq 0,\ 0\leq z\leq b \qquad (1)$$

$$s(\infty,z,t)=0,\ t\geq 0 \qquad (2)$$

$$\partial s(r,0,t)/\partial z=0,\ r\geq 0,\ t\geq 0 \qquad (3)$$

$$\partial s(r,b,t)/\partial z=0,\ r\geq 0,\ t\geq 0 \qquad (4)$$

$$\lim_{r\to 0} r\frac{\partial s}{\partial r} = \begin{cases} 0, & 0<z<d \\ -Q/(2\pi K_r(l-d)), & d<z<l \\ 0, & l<z<b \end{cases} \qquad (5)$$

Equation 1 states that initially the drawdown is zero everywhere in the aquifer. Equation 2 states that the drawdown approaches zero as the distance from the pumped well approaches infinity. Equations 3 and 4 state that there is no vertical flow at the upper and lower boundaries of the aquifer. This means that vertical head gradients in the aquifer are caused by the geometric placement of the pumping well screen, and not by leakage. Equation 5 states that near the pumping well the flow is radial, that the flow toward the well is equal to its discharge, that the discharge is distributed uniformly over the well screen, and that no radial flow occurs above and below the screen.

Solution:

I. For the drawdown in a piezometer, a solution by Hantush (1961a, p. 85, and 1964a, p. 353) is given by

$$s = \frac{Q}{4\pi T}\left[W(u) + f\left(u, \frac{ar}{b}, \frac{l}{b}, \frac{d}{b}, \frac{z}{b}\right)\right], \qquad (6)$$

where

$$W(u) = \int_u^\infty \frac{e^{-y}}{y}\,dy$$

and

$$f\left(u, \frac{ar}{b}, \frac{l}{b}, \frac{d}{b}, \frac{z}{b}\right) = \frac{2b}{\pi(l-d)}\sum_{n=1}^{\infty}\frac{1}{n}\cdot$$

$$\left(\sin\frac{n\pi l}{b} - \sin\frac{n\pi d}{b}\right)\cos\frac{n\pi z}{b}\,W\left(u, \frac{n\pi ar}{b}\right) \qquad (7)$$

$$W(u,x) = \int_u^\infty (\exp(-y-x^2/4y)/y)\,dy$$

$$u = \frac{r^2S}{4Tt}$$

$$a = \sqrt{K_z/K_r}$$

An alternate form of this solution for $a=1$ is given by Hantush (1961a, p. 85):

$$s = \frac{Qb}{8\pi T(l-d)}\left[M\left(u, \frac{l+z}{r}\right) + M\left(u, \frac{l-z}{r}\right)\right.$$

$$+ f'\left(u, \frac{b}{r}, \frac{l}{r}, \frac{z}{r}\right) - M\left(u, \frac{d+z}{r}\right) - M\left(u, \frac{d-z}{r}\right)$$

$$\left. - f'\left(u, \frac{b}{r}, \frac{d}{r}, \frac{z}{r}\right)\right], \qquad (8)$$

in which

$$f'\left(u, \frac{b}{r}, \frac{x}{r}, \frac{z}{r}\right) = \sum_1^\infty\left[M\left(u, \frac{2nb+x+z}{r}\right)\right.$$

$$- M\left(u, \frac{2nb-x-z}{r}\right) + M\left(u, \frac{2nb+x-z}{r}\right)$$

$$\left. - M\left(u, \frac{2nb-x+z}{r}\right)\right] \qquad (9)$$

and

$$M(u,\beta)=\int_u^\infty \frac{e^{-y}}{y}\, erf(\beta\sqrt{y})\, dy$$

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}\, dy .$$

II. For the drawdown in an observation well (Hantush, 1961a, p. 90, and 1964a, p. 353),

$$s = \frac{Q}{4\pi T}\left[W(u) + \bar{f}\left(u, \frac{ar}{b}, \frac{l}{b}, \frac{d}{b}, \frac{l'}{b}, \frac{d'}{b}\right)\right], \quad (10)$$

where $W(u)$ is as defined previously and

$$\bar{f}\left(u, \frac{ar}{b}, \frac{l}{b}, \frac{d}{b}, \frac{l'}{b}, \frac{d'}{b}\right) = \frac{2b^2}{\pi^2(l-d)(l'-d')}$$

$$\cdot \sum_{n=1}^{\infty} \frac{1}{n^2}\left(\sin\frac{n\pi l}{b} - \sin\frac{n\pi d}{b}\right)$$

$$\cdot\left(\sin\frac{n\pi l'}{b} - \sin\frac{n\pi d'}{b}\right) W\left(u, \frac{n\pi ar}{b}\right), \quad (11)$$

where $W(u,x)$ and u are as defined previously.

Comments:

Assumptions apply to conditions shown in figure 2.1. The effects of partial penetration need to be considered for $ar/b<1.5$. There must be a type curve for each value of ar/b, d/b, l/b, and either z/b for piezometer, or l'/b and d'/b for observation wells. Because the number of possible type curves is large, only samples of curves for selected values of the parameters are shown in figure 2.2 on plate 1.

For large values of time, that is, for $t>b^2S/(2a^2T)$ or $t>bS/(2K_z)$, the effects of partial penetration are constant in time, and

$$W\left(u, \frac{n\pi ar}{b}\right)$$

can be approximated by

$$2K_0\left(\frac{n\pi ar}{b}\right)$$

(Hantush, 1961a, p. 92). $K_0(x)$ is the modified Bessel function of the second kind of order zero.

Equation 6 then becomes

$$s = \frac{Q}{4\pi T}\, W(u) + \delta s = \frac{Q}{4\pi T}\left[W(u) + f_s\right],$$

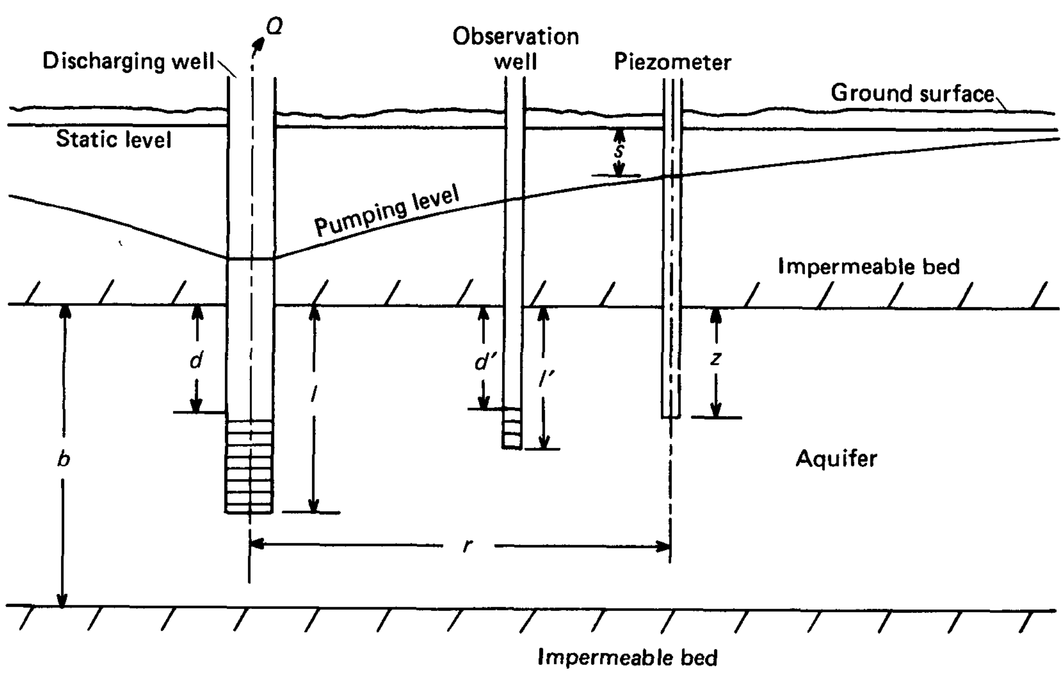

FIGURE 2.1.—Cross section through a discharging well that is screened in a part of a nonleaky aquifer.

where $$\partial s = \frac{Q}{4\pi T} f_s,$$

and f_s is given in equation 7

with $W\left(u, \frac{n\pi ar}{b}\right)$ replaced by $2K_0\left(\frac{n\pi ar}{b}\right)$.

Figure 2.3 shows plots of f_s as tabulated by Weeks (1969, p. 202–207). In using these curves, it should be noted that f_s for a given r, b, and z_1, l_1, d_1 is equal to f_s for the same r, b, and $z_2=b-z_1$, $l_2=b-d_1$, and $d_2=b-l_1$. Figure 2.3 can be used to find f_s by interpolation and then constructing type curves of $W(u)+f_s$ in the manner described by Weeks (1964, p. D195).

For small values of time

$$t< \frac{(2b-l-z)^2S}{20T}$$

(Hantush, 1961b, p. 172), equation 8 can be approximated by

$$s = \frac{Qb}{8\pi T(l-d)}\left[M\left(u, \frac{l+z}{r}\right) - M\left(u, \frac{d+z}{r}\right) + M\left(u, \frac{l-z}{r}\right) - M\left(u, \frac{d-z}{r}\right)\right].$$

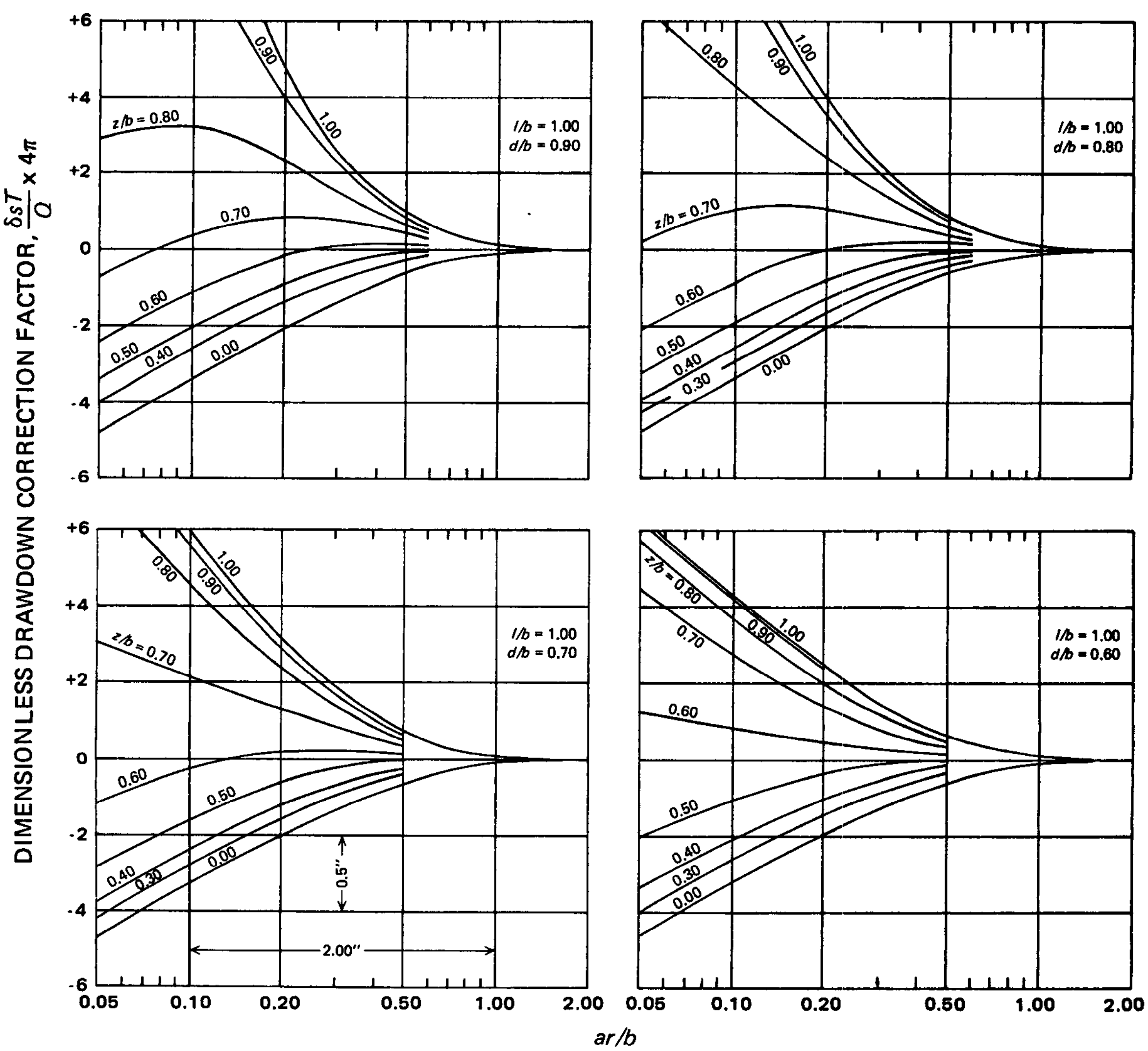

FIGURE 2.3.—The drawdown correction factor f_s versus ar/b, from tables of Weeks (1969).

An extensive table of $M(u,\beta)$ has been prepared by Hantush (1961c).

Although r/b for a given observation well probably would be known, however, the conductivity ratio a^2 would not be. Thus, it would not be known which ar/b curve should be matched. In other words, not only T and S, but also the conductivity ratio a^2 must be determined. A criterion for determining the match between data curves and type curves is that the values of ar/b for different observation wells should all indicate the same "a". Plotting the drawdown data for several observation wells on a single t/r^2 plot and matching to sets of type curves, a different set for each "a", is a useful approach.

Figure 2.2 was prepared from data calculated by the FORTRAN program listed in table 2.1. This program computes "s" from either equation 6 or 10, depending on the input data. The input data consist of cards containing the parameters coded in specific formats. Readers unfamiliar with FORTRAN format items should consult a FORTRAN language manual. The first card contains: the aquifer thickness (b), coded in columns 1–5, in format F5.1; the depth to bottom of pumped well screen (l), coded in columns 6–10, in format F5.1; the

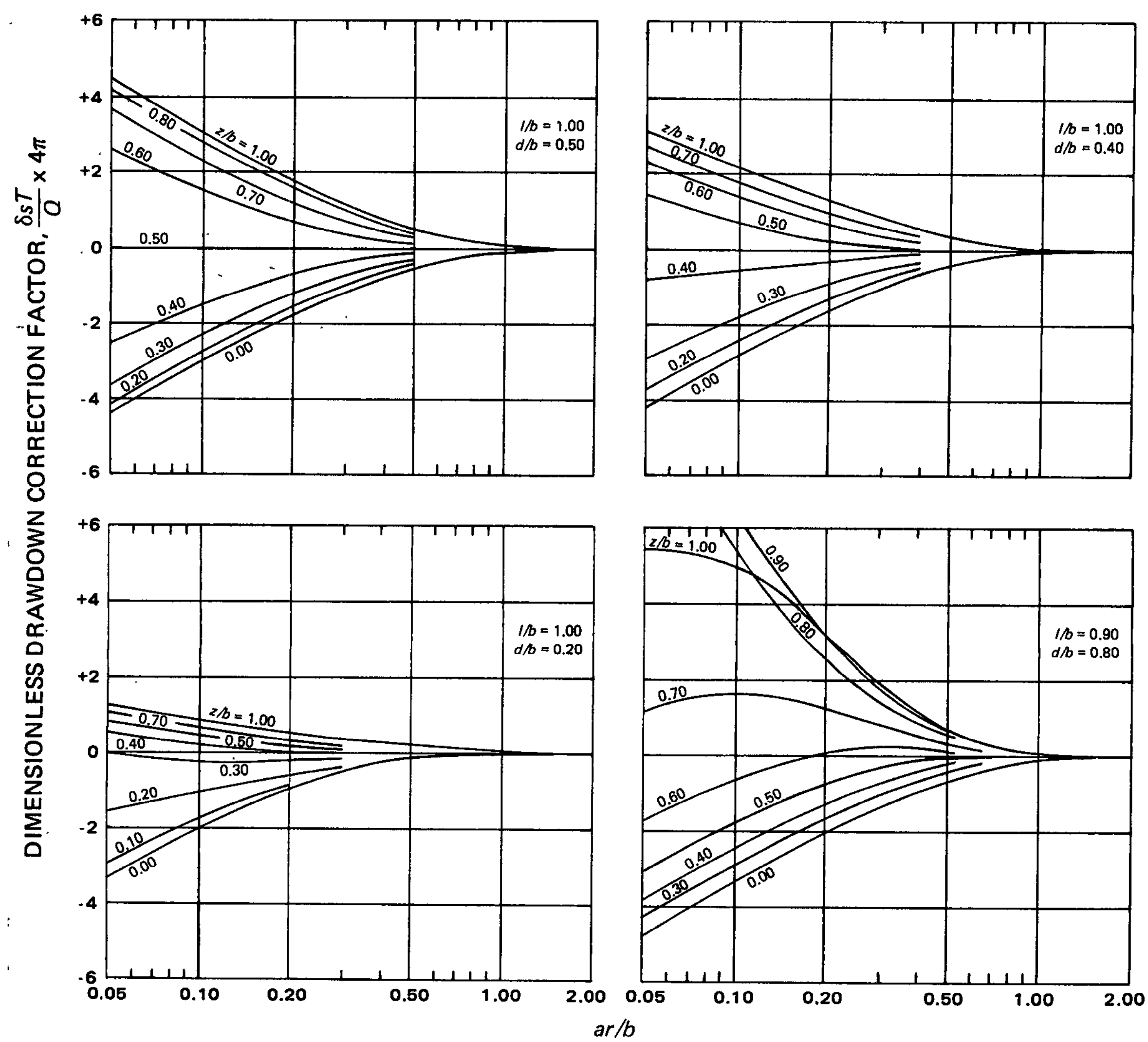

FIGURE 2.3.—Continued.

depth to top of pumped well screen (d), coded in columns 11–15, in format F5.1; the number of observation wells and (or) piezometers, coded in columns 16–20, in format I5; the smallest value of $1/u$ for which computation is desired, coded in columns 21–30, in format E10.4; the largest value of $1/u$ for which computation is desired, coded in columns 31–40, in format E10.4. The ratio of the largest $1/u$ value to the smallest $1/u$ value should be less than 10^{12}. Following this card is a group of cards containing one card for each observation well or piezometer. These cards are coded for an observation well as: distance from pumped well multiplied by the square root of the ratio of the vertical to horizontal conductivity ($r\sqrt{K_z/K_r}$), in columns 1–5, in format F5.1; depth to bottom of observation well screen (l'), coded in columns 6–10, in format F5.1; depth to top of observation well screen (d'), coded in columns 11–15, in format F5.1. A card would be coded for a piezometer as follows: distance from pumped well multiplied by the square root of the ratio of the vertical to horizontal conductivity ($r\sqrt{K_z/K_r}$), in columns 1–5, in format F5.1; and total depth of piezometer (z), in columns 11–15, in format F5.1. The output from this program is tables of computed function values,

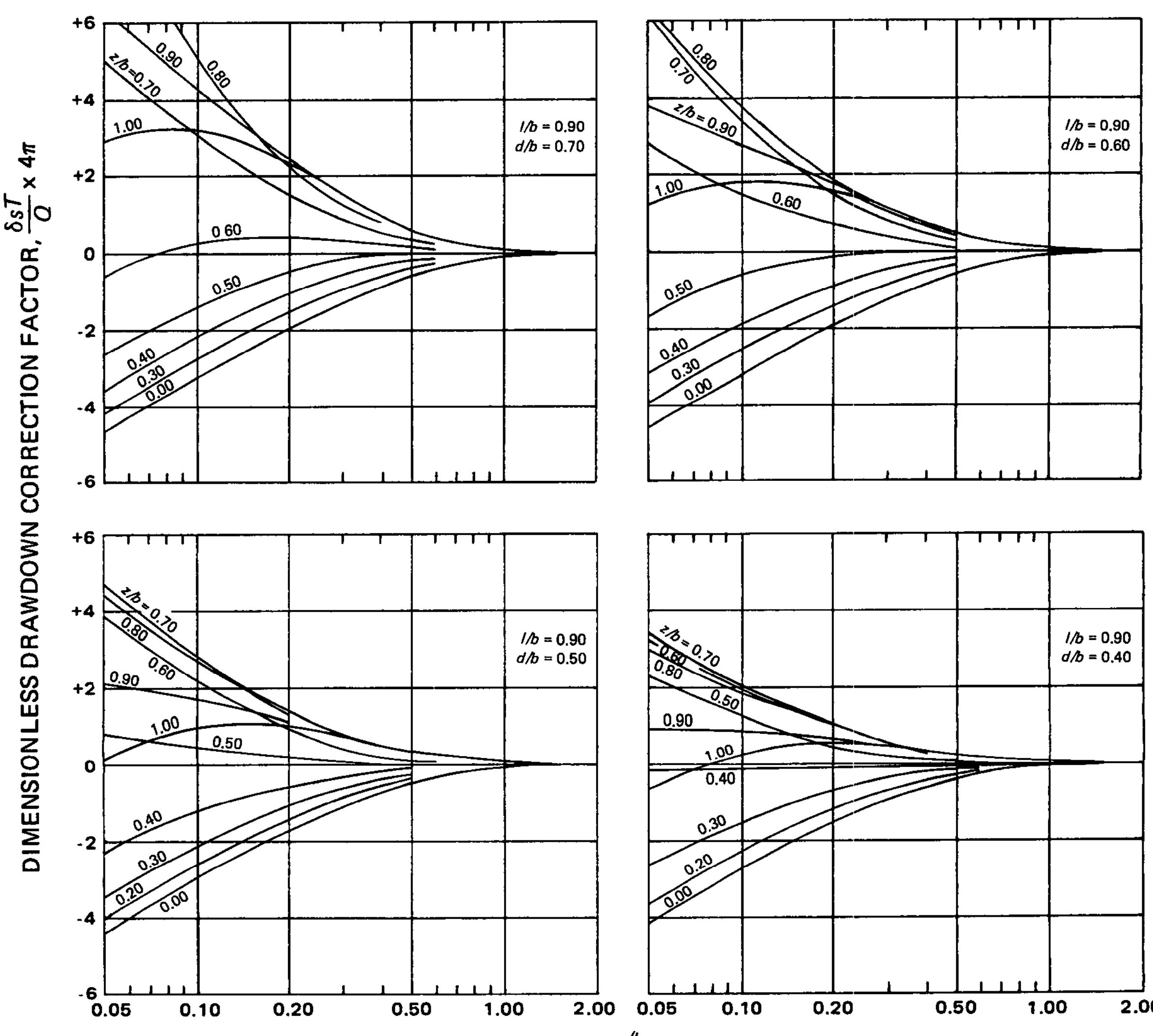

FIGURE 2.3.—Continued.

an example of which is shown in figure 2.4. Subroutines DQL12, BESK, and EXPI are from the IBM Scientific Subroutine Package and a discussion of them is in the IBM SSP manual.

Solution 3: Constant drawdown in a well in a nonleaky aquifer

Assumptions:

1. Water level in well is changed instantaneously by s_w at $t = 0$.
2. Well is of finite diameter and fully penetrates the aquifer.
3. Aquifer is not leaky.
4. Discharge from the well is derived exclusively from storage in the aquifer.

Differential equation:

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} = \frac{S}{T}\frac{\partial s}{\partial t}$$

This is the differential equation describing nonsteady radial flow in a homogeneous isotropic confined aquifer.

Boundary and initial conditions:

$$s(r,0) = 0,\ r \geq r_w \tag{1}$$

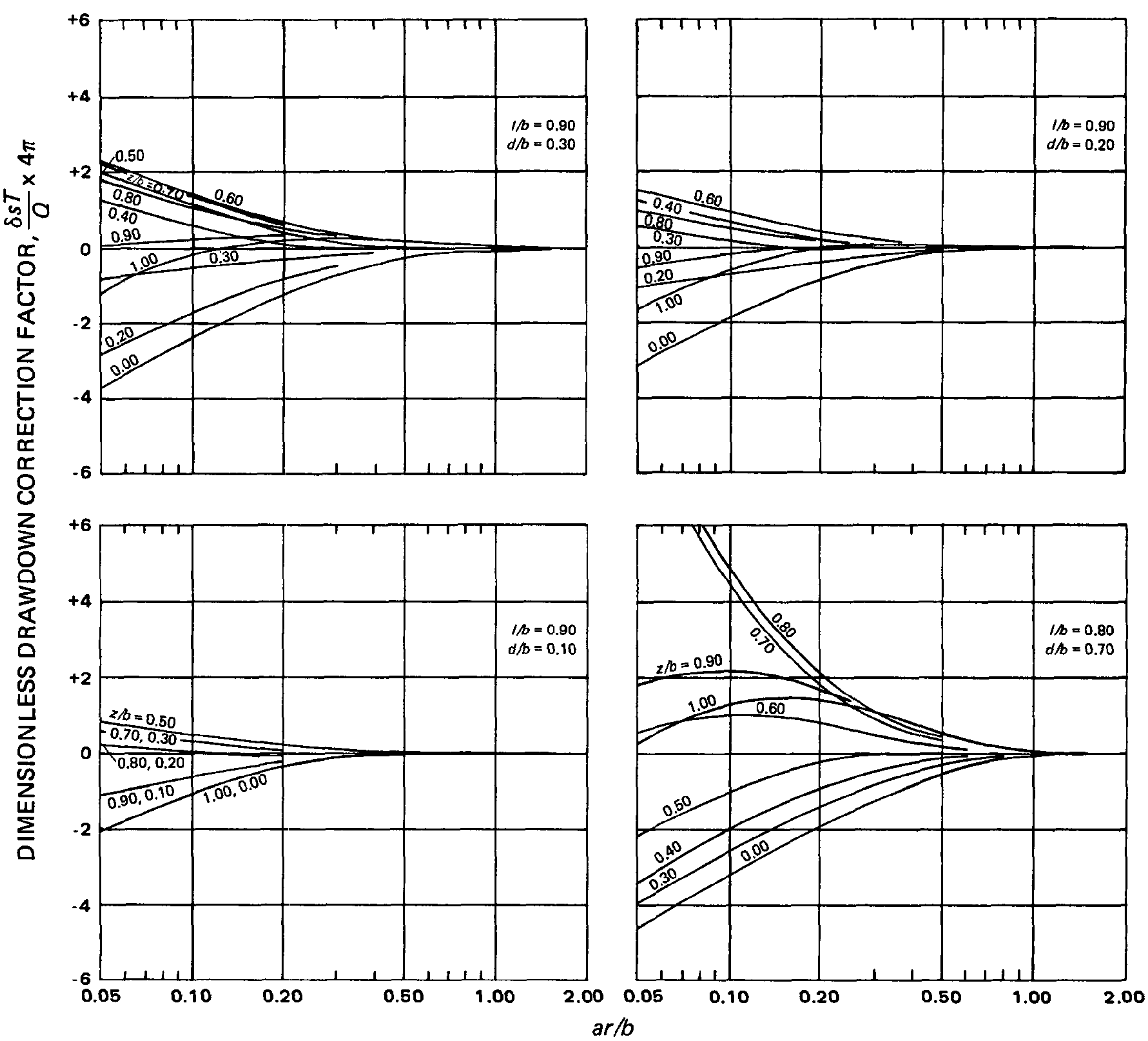

FIGURE 2.3.—Continued.