COMPUTER SIMULATION OF TWO-DIMENSIONAL UNSTEADY FLOWS IN ESTUARIES AND EMBAYMENTS BY THE METHOD OF CHARACTERISTICS--

BASIC THEORY AND THE FORMULATION OF THE NUMERICAL METHOD

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16. Abstracts Two-dimensional unsteady flows of homogeneous density in estuaries and embayments can be described by hyperbolic, quasi-linear partial differential equations involving three dependent and three independent variables. A linear combination of these equations leads to a parametric equation of characteristic form, which consists of two parts: total differentiation along the bicharacteristics and partial differentiation in space. For its numerical solution, the specified-time-interval scheme has been used. The unknown, partial space-derivative terms can be eliminated first by suitable combinations of difference equations, converted from the corresponding differential forms and written along four selected bicharacteristics and a streamline. Other unknowns are thus made solvable from the known variables on the current time plane. The computation is carried to the second-order accuracy by using trapezoidal rule of integration. Means to handle complex boundary conditions are developed for practical application. Computer programs have been written and a mathematical model has been constructed for flow simulation. The favorable computer outputs suggest further exploration and development of model worthwhile.

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COMPUTER SIMULATION OF TWO-DIMENSIONAL UNSTEADY FLOWS IN ESTUARIES AND EMBAYMENTS BY THE METHOD OF CHARACTERISTICS--BASIC THEORY AND THE FORMULATION OF THE NUMERICAL METHOD

By Chintu Lai

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PREFACE

Owing to the increasing water demands of society and the availability of powerful modern computers, the Water Resources Division of the U.S. Geological Survey initiated research into the dynamics of transient open-channel flows more than a decade ago. The writer has been conducting research on the numerical solution and computer simulation of unsteady open-channel flows, beginning with the study and development of basic numerical methods for one-dimensional flows, then proceeding to the formulation of computer programs, the construction of mathematical models, and assessing the utility and potential application of the models to various field problems.

Among the mathematical methods for one-dimensional flow investigated, the writer used most often the method of characteristics for flow simulation (Lai, 1965; Lai, 1967). As a logical and gradual extension to these one-dimensional flow studies, research was begun several years ago into the comptuer solution of two-dimensional flows by the method of characteristics. With the knowledge and the experience gained from one-dimensional flow models and with heavy reliance on the unique numerical scheme developed by Butler (1960) for compressible flows involving three independent variables, the writer formulated a numerical algorithm to write computer programs. An oral presentation, accompanied with visual aids, summarizing preliminary results was made at the 1971 Annual Hydraulics Specialty Conference of the American Society of Civil Engineers held at the University of Iowa.

In preparing the formal report a decision was made to divide the study into a few different phases with a separate report for each phase. Following completion of the first report, however, colleagues informed the writer of a paper by Townson (1974) in the Journal of Hydraulic Research, International Association for Hydraulic Research, in which the method of characteristics was also applied to a two-dimensional water body. Townson's method which also followed the one suggested by Butler (1960) showed a fairly close resemblance to the writer's in the numerical approach. Both methods were prepared independently without mutual communication, and they contain many differences. The most conspicuous ones are those associated with the approach used to numerically integrate the finite difference equations. Townson used a first-order approximation, whereas the writer relied on the second-order approximation using the trapezoidal rule. Using the trapezoidal rule, more terms must be incorporated and some form of iteration technique is required--factors which greatly complicate numerical procedures and computer programming.

This report is the first in a series outlining use of the method of characteristics to solve two-dimensional flow problems. It deals with the basic theory and the numerical method, and covers the material that is in a large part the same as that presented orally in the 1971 Iowa conference.

Chintu Lai
November 1975

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SYMBOLS

Symbols	Dimensions	Description
	M-L-T	
a	L	amplitude
b		subscript indicating bottom
C	$L^{1/2}\ T^{-1}$	Chézy's coefficient
C		constant, constant of integration
c	LT^{-1}	celerity of gravity wave
c_f, c_f'		drag coefficient
F		a function; also $F = F_p+F_r = F_p+F_s+F_f+F_c+F_w$
F_c	LT^{-1}	Coriolis term
F_f	LT^{-1}	friction term
F_p	LT^{-1}	partial space derivative term, $F_p = -Hp$
F_r	LT^{-1}	$= F_s+F_f+F_c+F_w$
		[All F_i terms are in the dimension of momentum (or impulse) per unit mass]
F_s	LT^{-1}	bottom slope term
F_w	LT^{-1}	wind term
F_x, F_y, F_z	LT^{-2}	components of body force per unit mass
$G,\ G_x, G_y$	LT^{-2}	tide-generating force per unit mass
g	LT^{-2}	acceleration of gravity
H	L	depth of water
h		subscript indicating horizontal
$\vec{i}$		unit vector in the x-direction
$\vec{j}$		unit vector in the y-direction

Symbols	Dimensions M-L-T	Description
k	$L^{-2/3}T^2$	coefficient, n^2 (metric) or $\left(\frac{n}{1.486}\right)^2$ (English)
M	L	mean water level
n		Manning's resistance coefficient
$\vec{n}$		unit vector in the normal direction
P_i		aggregation of the terms (known) at point i on $t = t_0 - \Delta t$ plane
p	$ML^{-1}T^{-2}$	pressure intensity, water
p	T^{-1}	aggregation of the terms containing partial derivatives of u and v in x-y plane
p_{wd}	$ML^{-1}T^{-2}$	pressure intensity, wind
p_o	$ML^{-1}T^{-2}$	atmospheric pressure at the water surface
q_x, q_y	L^2T^{-1}	discharge per unit horizontal distance
R	L	hydraulic radius
S_x, S_y		slope of energy gradient
S_o, S_{ox}, S_{oy}		bottom slope
s	L	distance
s		subscript indicating surface or slope
$\vec{s}$	L	displacement vector
T	T	tidal period
t	T	time
t_0	T	initial time or current time
U_i	T	$= \frac{1}{2g}(c_0 \cos \phi_i + c_i \cos \theta_i)$
u	LT^{-1}	x-component of flow velocity
$\bar{u}$	LT^{-1}	vertically averaged flow velocity in x-direction
u'		relative fluctuation of u about $\bar{u}$

Symbols	Dimensions	Description
	M-L-T	
$\vec{V}$	LT^{-1}	flow velocity vector
V_f	LT^{-1}	flow speed; $V_f = \sqrt{\bar{u}^2 + \bar{v}^2}$
V_i	T	$= \frac{1}{2g} (c_0 \sin \phi_i + c_i \sin \theta_i)$
V_w	LT^{-1}	wind speed
v	LT^{-1}	y-component of flow velocity
v		subscript indicating vertical
$\bar{v}$	LT^{-1}	vertically average flow velocity in y-direction
v'		relative fluctuation of v about $\bar{v}$
W_x, W_y	LT^{-2}	wind force per unit mass
w	LT^{-1}	z-component of flow velocity
x	L	Cartesian coordinate; distance measured along x-axis
y	L	Cartesian coordinate; distance measured along y-axis
Z	L	surface elevation of water body; stage
Z_b	L	bottom elevation of water body
z	L	Cartesian coordinate; distance measured along z-axis
α		angle of flow direction with the $+x$-axis; angle between the x-y coordinates and x'-y' coordinates
β		momentum correction factor
γ	$ML^{-2}T^{-2}$	specific weight of fluid
Δt	T	time increment
Δx	L	distance increment in x-direction
Δy	L	distance increment in y-direction

Symbols	Dimensions	Description
	M-L-T	
ζ	T	phase lag
η, η_h, η_v	$ML^{-1}T^{-1}$	eddy viscosity
Θ		angle of wind direction with the $+x$-axis
θ		angle between the line normal to the wave front and the $+x$-direction, parametric direction of wave front
$\lambda, \lambda_2, \lambda_3$	T	combination factor
λ		coefficient, $\lambda = \frac{\tau_b}{\tau_s} + 1.$
μ	$ML^{-1}T^{-1}$	dynamic viscosity
ξ		dimensionless wind stress coefficient, $\xi = \frac{\xi'}{\rho}$
ξ'	ML^{-3}	wind stress coefficient, $\xi' = \frac{c_f' \lambda \rho_a}{2}$
ρ	ML^{-3}	density of water
ρ_a	ML^{-3}	density of air
τ	T	time parameter, $\tau = t - t_0$
$\tau_s, \tau_{sx}, \tau_{sy}$	$ML^{-1}T^{-2}$	surface shear stress
$\tau_b, \tau_{bx}, \tau_{by}$	$ML^{-1}T^{-2}$	bottom shear stress
ϕ		latitude
ϕ		θ value (parametric direction of wave) at a reference (or grid) point on $t = t_0$ plane
Ω	T^{-1}	Coriolis factor, $\Omega = 2\omega_z$
ω	T^{-1}	angular velocity of the earth's rotation
ω_z	T^{-1}	vertical component of the earth's rotation

Symbols	Dimensions M-L-T	Description
∇^2	L^{-2}	Laplacian operator; $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

Symbols with a prime, unless used for specific purposes and defined in this list, indicate the corresponding quantities on the x'-y' plane. Symbols not defined above either are defined in the text or have meanings apparent from the context.

ABSTRACT

The method of characteristics used in one-dimensional unsteady open-channel flows has been extended to two-dimensional shallow-water unsteady flows of homogeneous density that are found in wide rivers, estuaries, embayments, and tidal inlets. The governing differential equations derived for such shallow-water flows are hyperbolic, quasi-linear partial differential equations involving three dependent and three independent variables. A linear combination of these equations leads to an equation of a characteristic form, having the normal direction to the wave front as a parameter and comprising two parts--one representing total differentiation along the bicharacteristics and the other representing partial differentiation in space.

Similar to the writer's previous approach in one-dimensional flow computations, the specified-time-interval scheme, consisting of a sequence of two-dimensional grid planes spaced at time interval Δt along the t-direction, is also employed for the numerical solution of the characteristic equation. The numerical computation is carried to the second-order of accuracy by the direct use of the trapezoidal rule of integration. Suitable combinations of difference equations that are converted from the corresponding differential forms and are written along four selected bicharacteristics and a streamline, eliminate partial derivative terms lying on the advanced time plane. Thus the unknowns are made solvable from the known variables on the current time plane.

Boundary conditions in two-dimensional flow are considerably more complex than the one-dimensional case and require special treatment. Methods are developed to apply a variety of boundary conditions that are frequently encountered in natural or man-made embayments. Using this method, scheme, and technique, computer programs were written and a mathematical model constructed for flow simulation in tidal inlets and coastal seas. The encouraging results of the initial numerical experiments indicate that continued exploration of the method and further development of the model is worthwhile.

INTRODUCTION

The rapid pace of progress in computer science and computer capacity, the advancement of knowledge in numerical methods, the accumulation of experience in one-dimensional unsteady flow solutions, and, more importantly, the ever-growing demand for reliable information on flows in estuaries, embayments, and coastal areas, have encouraged scientists and engineers to explore the feasibility of using numerical simulation techniques to solve two- or three-dimensional unsteady flow problems.

Several papers and reports have appeared showing methods for the simulation of two-dimensional unsteady flows in estuaries and coastal seas

(Hansen, 1962; Leendertse, 1967; Reid and Bodine, 1968; Leendertse, 1970; Masch and Brandes, 1971). These studies have dealt with the direct transformation of the original partial differential equation set to a corresponding finite difference equation set, and its subsequent numerical solution employing implicit or explicit methods or the alternate use of the two.

As the extension work of its one-dimensional unsteady flow analysis, the U.S. Geological Survey began investigating two-dimensional flows several years ago. As with one-dimensional flow studies, the work in two-dimensional flows is carried out in two aspects; that is, the solution of flow equations with different numerical techniques, and the implementation as well as the field application of the flow simulation models.

This report presents the solution of two-dimensional unsteady flow by the method of characteristics and is the extension of earlier work in one-dimensional flow solution by the method of characteristics (Lai, 1965). Rigorous mathematical treatments of hyperbolic partial differential equations by the method of characteristics are available (Butler, 1960; Fox, 1962; Richardson, 1964), and because the method of characteristics has a distinct feature of closely modeling the physical properties of the hydraulic systems, a somewhat different way of deriving and describing the characteristics equations is used in this report. The presentation may not be mathematically very rigorous, but it is believed to be a practical approach more easily understandable for engineers.

The first part of this report deals with a review of the basic partial differential equations describing two-dimensional unsteady flow of homogeneous density in tidal embayments. The depth averaged values of x- and y-components of velocity are used in the equation set, and hence, the flow is two-dimensional in the horizontal sense. A method is developed to transform these basic partial differential equations into the characteristic equation based on physical concepts. Whenever appropriate, the differences in the characteristic equations between one- and two-dimensional flows are pointed out.

A numerical approach using the specified-time-interval scheme is then presented. The numerical integration is carried out using the second-order approximation by the trapezoidal rule. The treatment of boundary conditions is far more complicated in the two-dimensional flow and for that reason the subject is discussed in a separate chapter.

Finally, development of the mathematical model to implement the numerical method and scheme described in this report is outlined, including depiction of the conceptual model structure, necessary initial value data, geometrical and boundary data, and sample computer runs. The scope and diversity of the topics involved in the numerical simulation of unsteady flow of this nature, and the length of time and various stages in which the numerical model is to be constructed, implemented and operated, precludes inclusion of a detailed and thorough treatment of all topics in a single volume. Such details should be and will be reported separately in the future.

DIFFERENTIAL EQUATIONS FOR TWO-DIMENSIONAL

UNSTEADY FLOWS IN SHALLOW WATER

General Considerations

A brief review of partial differential equations describing unsteady flows of homogeneous density in shallow water, such as flows in wide estuaries, tidal inlets, or bays, is given in this chapter. In the derivation of the differential equations, the two-space dimensional form in the horizontal plane is used. The following assumptions are first made and the Eulerian approach is used in the derivations.

1. The water in the sea or bay is not deep compared with the length of the wave and the shallow water theory applies.
2. The water is substantially of homogeneous density.
3. The vertical velocity of flow is small.
4. The vertical acceleration of the fluid particle is very small compared with the acceleration of gravity, g, and, hence, can be neglected.
5. The pressure is hydrostatic (from the above assumption).
6. The frictional resistance coefficient for unsteady flow is the same as that for steady flow, thus can be approximated from the Chézy or Manning equation.
7. Only shear stresses due to horizontal velocity components are significant.
8. The bottom of the embayment is rigid or relatively stable and fixed with respect to time.

For the derivation of the governing partial differential equations, an arbitrary horizontal plane is taken as a datum plane and Cartesian orthogonal co-ordinates with the x-y plane lying on the datum plane are chosen as space coordinates. The east direction is set as the $+x$ direction and the north direction as $+y$ and (using the right-hand screw system) $+z$ is the upward direction.

The derivation of the differential equations begins from the general equation of continuity and the general equations of motion for an incompressible fluid in the classical hydrodynamics (Lamb, 1945; Rouse, 1959). They are, namely, equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{2.1}$$

and equations of motion

$$\rho \frac{Du}{Dt} = \rho F_x - \frac{\partial p}{\partial x} + \mu\Delta^2 u, \tag{2.2}$$

$$\rho \frac{Dv}{Dt} = \rho F_y - \frac{\partial p}{\partial y} + \mu\Delta^2 v, \tag{2.3}$$

$$\rho \frac{Dw}{Dt} = \rho F_z - \frac{\partial p}{\partial z} + \mu\Delta^2 w. \tag{2.4}$$

In these equations, u, v, and w are the components of the velocity and F_x, F_y, and F_z are the components of the body force per unit mass, in the directions of x, y, and z, respectively. The symbol ρ denotes the fluid density, p represents the fluid pressure, and μ is the coefficient of viscosity. Two mathematical notations, $\frac{D}{Dt}$ and ∇^2, stand for the substantial derivative operator

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\ , \tag{2.5}$$

and the Laplace operator

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\ , \tag{2.6}$$

respectively.

Equation of Continuity

Consider a column of water having a cross-section of unit square and extending from the bottom Z_b to the surface Z. The equation of continuity for this column can be obtained by integrating eq 2.1 over the vertical, namely, over $z = Z_b(x,y)$ to $z = Z(x,y,t)$

$$\int_{Z_b}^{Z} \frac{\partial u}{\partial x}\, dz + \int_{Z_b}^{Z} \frac{\partial v}{\partial y}\, dz + w(Z) - w(Z_b) = 0. \tag{2.7}$$

From the Leibnitz rule, at any time t,

$$\frac{\partial}{\partial x}\int_{Z_b}^{Z} udz = u(Z)\,\frac{\partial Z}{\partial x} - u(Z_b)\,\frac{\partial Z_b}{\partial x} + \int_{Z_b}^{Z}\frac{\partial u}{\partial x}\,dz, \qquad (2.8)$$

and

$$\frac{\partial}{\partial y}\int_{Z_b}^{Z} vdz = v(Z)\,\frac{\partial Z}{\partial y} - v(Z_b)\,\frac{\partial Z_b}{\partial y} + \int_{Z_b}^{Z}\frac{\partial v}{\partial y}\,dz. \qquad (2.9)$$

If the equation of the boundary surface is $F(x,y,z,t) = 0$, assuming that any particle on the surface remains on it, then the following condition holds,

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + u\,\frac{\partial F}{\partial x} + v\,\frac{\partial F}{\partial y} + w\,\frac{\partial F}{\partial z} = 0. \qquad (2.10)$$

Because $z = Z(x,y,t)$ at the free surface, the boundary equation for the surface can be written as

$$F = z - Z\,(x,y,t) = 0.$$

Hence, $$\frac{DF}{Dt} = -\frac{\partial Z}{\partial t} - u\,\frac{\partial Z}{\partial x} - v\,\frac{\partial Z}{\partial y} + w = 0,$$

or $$\frac{\partial Z}{\partial t} + u\,\frac{\partial Z}{\partial x} + v\,\frac{\partial Z}{\partial y} - w = 0. \qquad (2.11)$$

Similarly, at the bottom, $z = Z_b(x,y)$, the boundary equation for the bottom is

$$F = z - Z_b(x,y) = 0,$$

and

$$\frac{DF}{Dt} = -u\,\frac{\partial Z_b}{\partial x} - v\,\frac{\partial Z_b}{\partial y} + w = 0,$$

or

$$u\,\frac{\partial Z_b}{\partial x} + v\,\frac{\partial Z_b}{\partial y} = w = 0. \qquad (2.12)$$

Substituting eq 2.8, eq 2.9, eq 2.11 and eq 2.12 into eq 2.7,

$$\frac{\partial Z}{\partial t} + \frac{\partial}{\partial x}\int_{Z_b}^{Z} udz + \frac{\partial}{\partial y}\int_{Z_b}^{Z} vdz = 0, \qquad (2.13)$$

or

$$\frac{\partial Z}{\partial t} + \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = 0, \qquad (2.14)$$

in which

$$q_x = \int_{Z_b}^{Z} u dz, \qquad q_y = \int_{Z_b}^{Z} v dz .$$

Denoting $\overline{u}$ and $\overline{v}$ as the mean values of u and v over the vertical,

$$\overline{u} = \frac{q_x}{Z - Z_b} = \frac{1}{Z - Z_b} \int_{Z_b}^{Z} u dz$$

$$\overline{v} = \frac{q_y}{Z - Z_b} = \frac{1}{Z - Z_b} \int_{Z_b}^{Z} v dz ,$$

eq 2.14, then, becomes

$$\frac{\partial Z}{\partial t} + \frac{\partial}{\partial x} [\overline{u}(Z - Z_b)] + \frac{\partial}{\partial y} [\overline{v}(Z - Z_b)] = 0. \qquad (2.15)$$

For the sake of convenience, the bar notation for the mean will be dropped hereafter.

Equation 2.15 can be expanded as

$$\frac{\partial Z}{\partial t} + (Z - Z_b) \frac{\partial u}{\partial x} + u \frac{\partial Z}{\partial x} - u \frac{\partial Z_b}{\partial x}$$

$$+ (Z - Z_b) \frac{\partial v}{\partial y} + v \frac{\partial Z}{\partial y} - v \frac{\partial Z_b}{\partial y} = 0. \qquad (2.16)$$

The use of notations $H = Z - Z_b$, $S_{ox} = - \frac{\partial Z_b}{\partial x}$, and $S_{oy} = - \frac{\partial Z_b}{\partial y}$

permits the simplification of eq 2.16 to

$$\frac{\partial Z}{\partial t} + u \frac{\partial Z}{\partial x} + v \frac{\partial Z}{\partial y} + H \frac{\partial u}{\partial x} + H \frac{\partial v}{\partial y} + uS_{ox} + vS_{oy} = 0. \qquad (2.17)$$

Equations of Motion

According to the assumptions 4 and 7 made previously, namely,

$$\frac{Dw}{Dt} \approx 0, \qquad \mu \Delta^2 w \approx 0,$$

eq 2.4 may be reduced to

$$\rho F_z - \frac{\partial p}{\partial z} = 0. \tag{2.18}$$

In this equation, although F_z includes the gravitational attraction, the Coriolis force generated by the earth rotation, and the tide-generating force, the vertical component of the Coriolis force can be included in the observed acceleration of gravity, g, and that of the tide-generating force is negligibly small compared with the gravity force, thus, the symbol F_z may simply be replaced by $-g$ (Dronkers, 1964; Leendertse, 1967).

Integrating eq 2.18 in the z-direction, with the assumption of constant density in mind, (Assumption 2),

$$p = -\rho g \int dz + C = -\rho g z + C, \tag{2.19}$$

in which C is the constant of integration. If the atmospheric pressure at the water surface is signified by p_o, then at $z = Z$, $p = p_o$ and $C = p_o + \rho g Z$. With this expression for C, eq 2.19 becomes

$$p = \rho g(Z - z) + p_o. \tag{2.20}$$

The terms $-\frac{\partial p}{\partial x}$ and $-\frac{\partial p}{\partial y}$ in eq 2.2 and eq 2.3, as a result, can be expressed as

$$-\frac{\partial p}{\partial x} = -\rho g \frac{\partial Z}{\partial x} - \frac{\partial p_o}{\partial x}, \tag{2.21}$$

and

$$-\frac{\partial p}{\partial y} = -\rho g \frac{\partial Z}{\partial y} - \frac{\partial p_o}{\partial y}. \tag{2.22}$$

Using the notation 2.5 and assuming p_o is constant, eq 2.2 and eq 2.3 may be rewritten as

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right] = \rho F_x - \rho g\frac{\partial Z}{\partial x} + \mu\Delta^2 u, \tag{2.23}$$

$$\rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right] = \rho F_y - \rho g\frac{\partial Z}{\partial y} + \mu\Delta^2 v. \tag{2.24}$$

As done in the previous section, eq 2.23 and eq 2.24 will be integrated over the vertical. First, add (ρu) X eq 2.1 to eq 2.23; the left-hand side becomes

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho uu)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} + \frac{\partial(\rho uw)}{\partial z}. \tag{2.25}$$

The vertical integration of these terms is

$$\int_{Z_b}^{Z} \frac{\partial(\rho u)}{\partial t} dz + \int_{Z_b}^{Z} \frac{\partial(\rho uu)}{\partial x} dz + \int_{Z_b}^{Z} \frac{\partial(\rho uv)}{\partial y} dz + \int_{Z_b}^{Z} \frac{\partial(\rho uw)}{\partial z} dz \; . \quad (2.26)$$

Applying the Leibnitz's rule again, the above expression becomes

$$\frac{\partial}{\partial t}\int_{Z_b}^{Z} \rho u dz - \rho u(Z)\frac{\partial Z}{\partial t} + \frac{\partial}{\partial x}\int_{Z_b}^{Z} \rho uu dz - \rho u(Z)u(Z)\frac{\partial Z}{\partial x}$$

$$+ \rho u(Z_b)u(Z_b)\frac{\partial Z_b}{\partial x} + \frac{\partial}{\partial y}\int_{Z_b}^{Z} \rho uv dz - \rho u(Z)v(Z)\frac{\partial Z}{\partial y}$$

$$+ \rho u(Z_b)v(Z_b)\frac{\partial Z_b}{\partial y} + \rho u(Z)w(Z). \quad (2.27)$$

If the vertical distribution of the velocity is related to the mean velocity as

$$u(z) = \bar{u}\,[1 + u'(z)], \quad (2.28)$$

$$v(z) = \bar{v}\,[1 + v'(z)], \quad (2.29)$$

then the following relationship should hold:

$$\int_{Z_b}^{Z} u'(z)dz = 0, \quad (2.30)$$

$$\int_{Z_b}^{Z} v'(z)dz = 0. \quad (2.31)$$

The expression 2.27 now can be written as

$$\rho\frac{\partial}{\partial t}(\bar{u}H) - \rho u(Z)\frac{\partial Z}{\partial t} + \rho\frac{\partial}{\partial x}\int_{Z_b}^{Z} \bar{u}\bar{u}[1+2u'(z) + u'^2(z)]\; dz$$

$$- \rho u(Z)u(Z)\frac{\partial Z}{\partial x} + \rho u(Z_b)u(Z_b)\frac{\partial Z_b}{\partial x}$$

$$+ \rho\frac{\partial}{\partial y}\int_{Z_b}^{Z} \bar{u}\bar{v}[1+u'(z) + v'(z) + u'(z)v'(z)]\; dz - \rho u(Z)v(Z)\frac{\partial Z}{\partial y}$$

$$+ \rho u(Z_b)v(Z_b)\frac{\partial Z_b}{\partial y} + \rho u(Z)w(Z). \qquad (2.32)$$

Substitution of eq 2.11, eq 2.12, eq 2.30, and eq 2.31 into the above expression yields a simple form

$$\rho\frac{\partial}{\partial t}(\bar{u}H) + \rho\frac{\partial}{\partial x}(\beta\bar{u}\bar{u}H) + \rho\frac{\partial}{\partial y}(\beta\bar{u}\bar{v}H), \qquad (2.33)$$

in which β is the moment correction factor defined as

$$\beta = \frac{1}{H}\int_{Z_b}^{Z}\left\{1+u'^2(z)\right\}dz = \frac{1}{H}\int_{Z_b}^{Z}\left\{1+u'(z)v'(z)\right\}dz$$

Similarly, the left-hand side of eq 2.24 becomes

$$\rho\frac{\partial}{\partial t}(\bar{v}H) + \rho\frac{\partial}{\partial x}(\beta\bar{u}\bar{v}H) + \rho\frac{\partial}{\partial y}(\beta\bar{v}\bar{v}H). \qquad (2.34)$$

The various forces on the right-hand side of eq 2.23 and eq 2.24 are now sought.

The body forces acting in the horizontal directions are, devoid of the earth's gravitational force, the effect of earth's rotation (the Coriolis effect)* and the tide-generating force. They are expressed as

$$F_x = \Omega v + G_x, \qquad (2.35)$$

$$F_y = -\Omega u + G_y, \qquad (2.36)$$

in which G is the tide-generating force, and Ω is the Coriolis factor, which is a function of the latitude; that is,

$$\Omega = 2\omega_z = 2\omega \sin\phi, \qquad (2.37)$$

where ϕ is the latitude, ω is the angular velocity of the earth rotation, and ω_z is its vertical component (Dronkers, 1964). If the velocity distribution over the vertical is assumed to be fairly constant, the effect of tide-generating forces can be neglected (Leendertse, 1967) and eq 2.25 and eq 2.26 reduce to

* To give some concept of the Coriolis effect, if a wide estuary at the latitude of 40°N flows toward west with velocity of 0.5 m/s, then the slope of water surface in the transverse direction is about 4.77 mm/km northwards. This amounts to about 24 mm rise of water level at the north bank above the stage at the south bank for a 5 km wide estuary.

$$F_x = \Omega v, \tag{2.38}$$

$$F_y = -\Omega u. \tag{2.39}$$

Integrating these over the vertical results in

$$\int_{Z_b}^{Z} F_x dz = \Omega \int_{Z_b}^{Z} v dz = \Omega \bar{v} H, \tag{2.40}$$

$$\int_{Z_b}^{Z} F_y dz = -\Omega \int_{Z_b}^{Z} u dz = -\Omega \bar{u} H. \tag{2.41}$$

Next, the shear stress terms are considered. For a turbulent flow, the eddy viscosity coefficient η is used instead of the dynamic viscosity μ. In the x-direction the shear stress may be expressed as

$$\eta_h \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \eta_v \frac{\partial^2 u}{\partial z^2} , \tag{2.42}$$

and in the y-direction, as

$$\eta_h \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \eta_v \frac{\partial^2 v}{\partial z^2} , \tag{2.43}$$

in which the subscripts h and v indicate horizontal and vertical. Because the terms that contain η_h are small in comparison with the terms that contain η_v, the above expressions are simplified to

$$\eta_v \frac{\partial^2 u}{\partial z^2} \text{ and } \eta_v \frac{\partial^2 v}{\partial z^2} .$$

Integrating these terms with respect to z from the bottom to the surface,

$$\int_{Z_b}^{Z} \eta_v \frac{\partial^2 u}{\partial z^2} dz = \eta_v \left[\left(\frac{\partial u}{\partial z}\right)_{z=Z} - \left(\frac{\partial u}{\partial z}\right)_{z=Z_b} \right], \tag{2.44}$$

$$\int_{Z_b}^{Z} \eta_v \frac{\partial^2 v}{\partial z^2} dz = \eta_v \left[\left(\frac{\partial v}{\partial z}\right)_{z=Z} - \left(\frac{\partial v}{\partial z}\right)_{z=Z_b} \right]. \tag{2.45}$$

Denoting the surface and bottom stresses by τ_{sx}, τ_{sy}, and τ_{bx}, τ_{by}, that is,

$$\tau_{sx} = \eta_v \left(\frac{\partial u}{\partial z}\right)_{z=Z}, \qquad \tau_{sy} = \eta_v \left(\frac{\partial v}{\partial z}\right)_{z=Z},$$

and
$$\tau_{bx} = \eta_v \left(\frac{\partial u}{\partial z}\right)_{z=Z_b}, \qquad \tau_{by} = \eta_v \left(\frac{\partial v}{\partial z}\right)_{z=Z_b}.$$

Equations 2.44 and 2.45 may be written as

$$\int_{Z_b}^{Z} \eta_v \frac{\partial^2 u}{\partial z^2}\, dz = \tau_{sx} - \tau_{bx}, \tag{2.46}$$

$$\int_{Z_b}^{Z} \eta_v \frac{\partial^2 v}{\partial z^2}\, dz = \tau_{sy} - \tau_{by}. \tag{2.47}$$

Bottom friction gives rise to bottom stress and wind resistance contributes to surface stress.

If τ_b is expressed by the relationship

$$\tau_b = c_f \frac{\rho V_f^2}{2}, \tag{2.48}$$

in which c_f is a drag coefficient (dimensionless) and V_f is the flow speed, that is, $V_f = \sqrt{\bar{u}^2 + \bar{v}^2}$, and if c_f is further related to Chézy's coefficient C, a dimensional quantity, by

$$C = \sqrt{\frac{2g}{c_f}}, \tag{2.49}$$

the following relationship is obtained:

$$\tau_b = \frac{g}{C^2} \rho V_f^2 = \frac{\gamma}{C^2} V_f^2 . \tag{2.50}$$

Here, γ signifies the specific weight of the water.

Let the angle between the flow direction and the $+x$-direction be α, then

$$\tau_{bx} = \tau_b \cos \alpha = \frac{\gamma}{C^2} V_f \bar{u}, \tag{2.51}$$

$$\tau_{by} = \tau_b \sin\alpha = \frac{\gamma}{C^2} V_f \bar{v} . \tag{2.52}$$

When Manning n is used instead of Chezy's C, since it is known from the open-channel hydraulics that

$$C = \frac{R^{1/6}}{n} \quad \text{(metric system)}, \tag{2.53}$$

or $$C = 1.486\,\frac{R^{1/6}}{n} \quad \text{(English system)}, \tag{2.54}$$

with R indicating the hydraulic radius, eq 2.51 and eq 2.52 will have the form

$$\tau_{bx} = \frac{k\gamma}{R^{1/3}} V_f \bar{u} = \frac{k\gamma}{H^{1/3}} V_f \bar{u} , \tag{2.55}$$

$$\tau_{by} = \frac{k\gamma}{H^{1/3}} V_f \bar{v} , \tag{2.56}$$

(because in shallow water $R=H$), with $k=n^2$ for the metric system and $k=(\frac{n}{1.486})^2$ for the English system.

The surface stress τ_s may have a form similar to the bottom stress τ_b, namely,

$$\tau_s = c'_f \frac{\rho_a V_w^2}{2} , \tag{2.57}$$

in which ρ_a is the density of the air, c'_f is the drag coefficient at the water surface, and V_w is the wind velocity. Assuming that the shear stress varies linearly in the z-direction,

$$\frac{\partial\tau}{\partial z} = \frac{\tau_s + \tau_b}{Z - Z_b} .$$

Let p_{wd} be the pressure intensity produced by the wind,

$$\frac{\partial p_{wd}}{\partial s} = \frac{\partial\tau}{\partial z} = \left(\frac{\tau_b}{\tau_s} + 1\right) \frac{\tau_s}{H} ,$$

in which s is the distance in the downwind direction. Using the notation

$$\lambda = \left(\frac{\tau_b}{\tau_s} + 1\right),$$

$$\frac{\partial p_{wd}}{\partial s} = \frac{\lambda \tau_s}{H} . \qquad (2.58)$$

From eq 2.57 and eq 2.58

$$\frac{\partial p_{wd}}{\partial s} = \left(\frac{c'_f \lambda \rho_a}{2}\right) \frac{V_w^2}{H} . \qquad (2.59)$$

Integrating over the vertical, the shear stress due to wind becomes

$$\int_{Z_b}^{Z} \frac{\partial \tau}{\partial z}\, dz = \int_{Z_b}^{Z} \frac{\partial p_{wd}}{\partial s}\, dz = \xi' \frac{V_w^2}{H} \int_{Z_b}^{Z} dz = \xi' V_w^2 , \qquad (2.60)$$

in which $\frac{c'_f \lambda \rho_a}{2}$ is represented by a single parameter ξ', which has the dimension of density.

The vertical integration of the pressure terms simply yields $-\rho g \frac{\partial Z}{\partial x} H$ and $-\rho g \frac{\partial Z}{\partial y} H$, or $-\gamma \frac{\partial Z}{\partial x} H$ and $-\gamma \frac{\partial Z}{\partial y} H$.

Assembling all the acceleration and force terms, the vertically integrated forms of eq 2.23 and eq 2.24 can now be written as

$$\rho \left[\frac{\partial}{\partial t}(\bar{u}H) + \frac{\partial}{\partial x}(\beta \bar{u}\bar{u}H) + \frac{\partial}{\partial y}(\beta \bar{u}\bar{v}H)\right] = \rho\Omega \bar{v}H - \gamma \frac{\partial Z}{\partial x} H - \frac{\gamma}{C^2} V_f \bar{u} + \xi' V_w^2 \cos\Theta, \qquad (2.61)$$

$$\rho \left[\frac{\partial}{\partial t}(\bar{v}H) + \frac{\partial}{\partial x}(\beta \bar{u}\bar{v}H) + \frac{\partial}{\partial y}(\beta \bar{v}\bar{v}H)\right] = -\rho\Omega \bar{u}H - \gamma \frac{\partial Z}{\partial y} H - \frac{\gamma}{C^2} V_f \bar{v} + \xi' V_w^2 \sin\Theta, \qquad (2.62)$$

in which Θ is the angle between the wind direction and the $+x$-direction. With the assumption $\beta \simeq 1$ and through the use of $H = Z - Z_b$ and $\xi = \frac{\xi'}{\rho} = \frac{c'_f \lambda}{2} \frac{\rho_a}{\rho}$, eq 2.61 and eq 2.52 expand to

$$H \frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial Z}{\partial t} + 2\bar{u}H \frac{\partial \bar{u}}{\partial x} + \bar{u}^2 \left(\frac{\partial Z}{\partial x} - \frac{\partial Z_b}{\partial x}\right) + H\bar{v} \frac{\partial \bar{u}}{\partial y} + H\bar{u} \frac{\partial \bar{v}}{\partial y} + \bar{u}\bar{v} \left(\frac{\partial Z}{\partial y} - \frac{\partial Z_b}{\partial y}\right)$$

$$= \Omega \bar{v}H - g \frac{\partial Z}{\partial x} H - \frac{g}{C^2} V_f \bar{u} + \xi V_w^2 \cos\Theta, \qquad (2.63)$$

$$H\frac{\partial\bar{v}}{\partial t} + \bar{v}\frac{\partial Z}{\partial t} + H\bar{v}\frac{\partial\bar{u}}{\partial x} + H\bar{u}\frac{\partial\bar{v}}{\partial x} + \bar{u}\bar{v}\left(\frac{\partial Z}{\partial x} - \frac{\partial Z_b}{\partial x}\right) + 2\bar{v}H\frac{\partial\bar{v}}{\partial y} + \bar{v}^2\left(\frac{\partial Z}{\partial y} - \frac{\partial Z_b}{\partial y}\right)$$

$$= -\Omega\bar{u}H - g\frac{\partial Z}{\partial y}H - \frac{g}{C^2}V_f\bar{v} + \xi V_w^2 \sin\Theta. \qquad (2.64)$$

Substituting eq 2.17 into eq 2.63 and eq 2.64, dividing through by H and transposing,

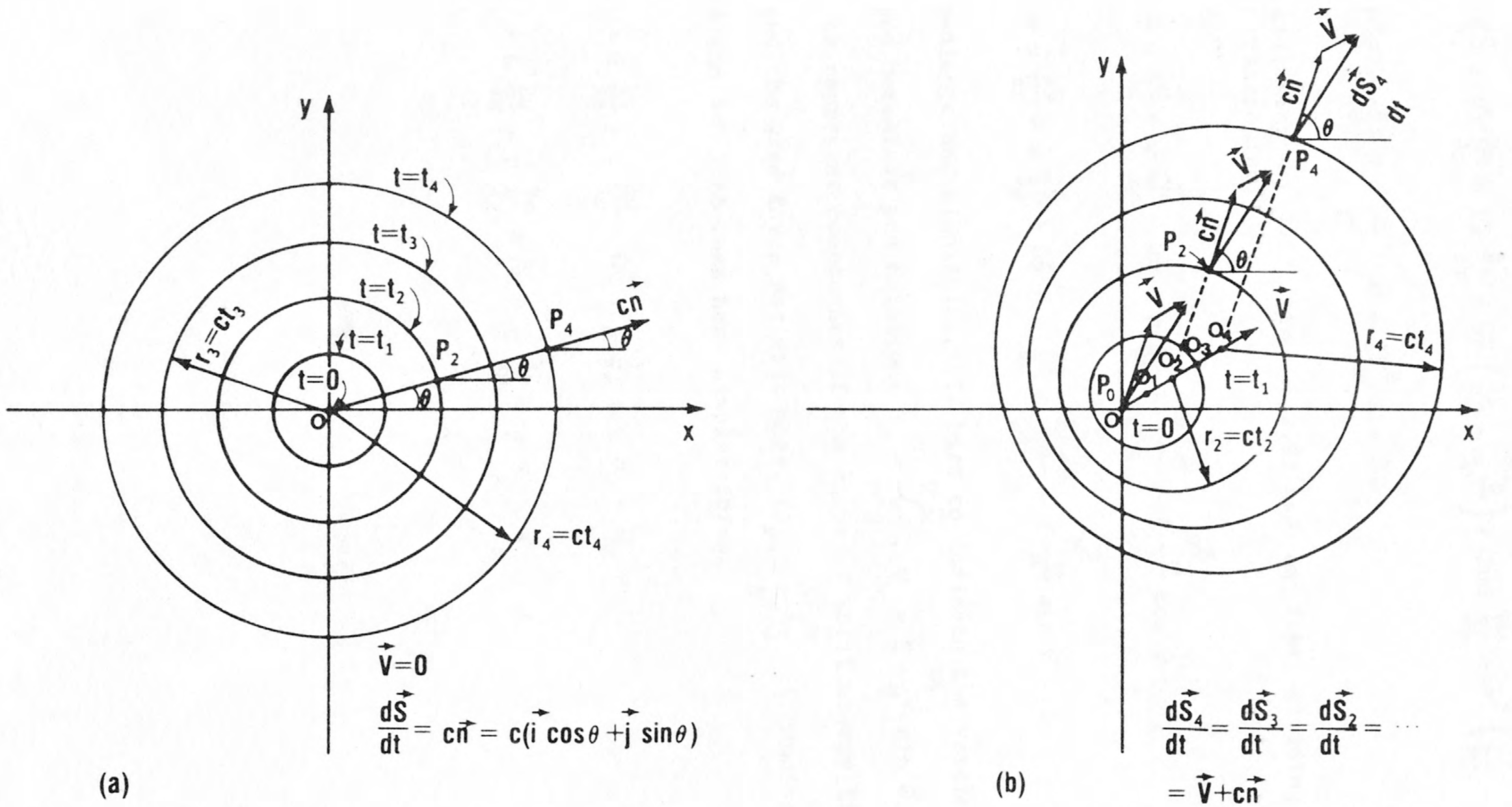
$$\frac{\partial\bar{u}}{\partial t} + u\frac{\partial\bar{u}}{\partial x} + \bar{v}\frac{\partial\bar{u}}{\partial y} - \Omega\bar{v} + g\frac{\partial Z}{\partial x} + \frac{gV_f\bar{u}}{C^2H} - \xi\frac{V_w^2}{H}\cos\Theta = 0, \qquad (2.65)$$

$$\frac{\partial\bar{v}}{\partial t} + \bar{u}\frac{\partial\bar{v}}{\partial x} + \bar{v}\frac{\partial\bar{v}}{\partial y} + \Omega\bar{u} + g\frac{\partial Z}{\partial y} + \frac{gV_f\bar{v}}{C^2H} - \xi\frac{V_w^2}{H}\sin\Theta = 0. \qquad (2.66)$$

For convenience and simplicity, all bars to indicate the vertical means will be dropped hereafter and notations $S_x = \frac{V_f\bar{u}}{C^2H}$, $W_y = \xi\frac{V_w^2}{H}\sin\Theta$, etc., will be used to represent components of the slope of total energy line (= friction slope) and the wind force per unit mass. ($S_x = \frac{kV_f\bar{u}}{H^{4/3}}$ if Manning n is used.) Eq 2.65 and eq 2.66 then have simpler forms,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - \Omega v + g\frac{\partial Z}{\partial x} + g\,S_x - W_x = 0, \qquad (2.67)$$

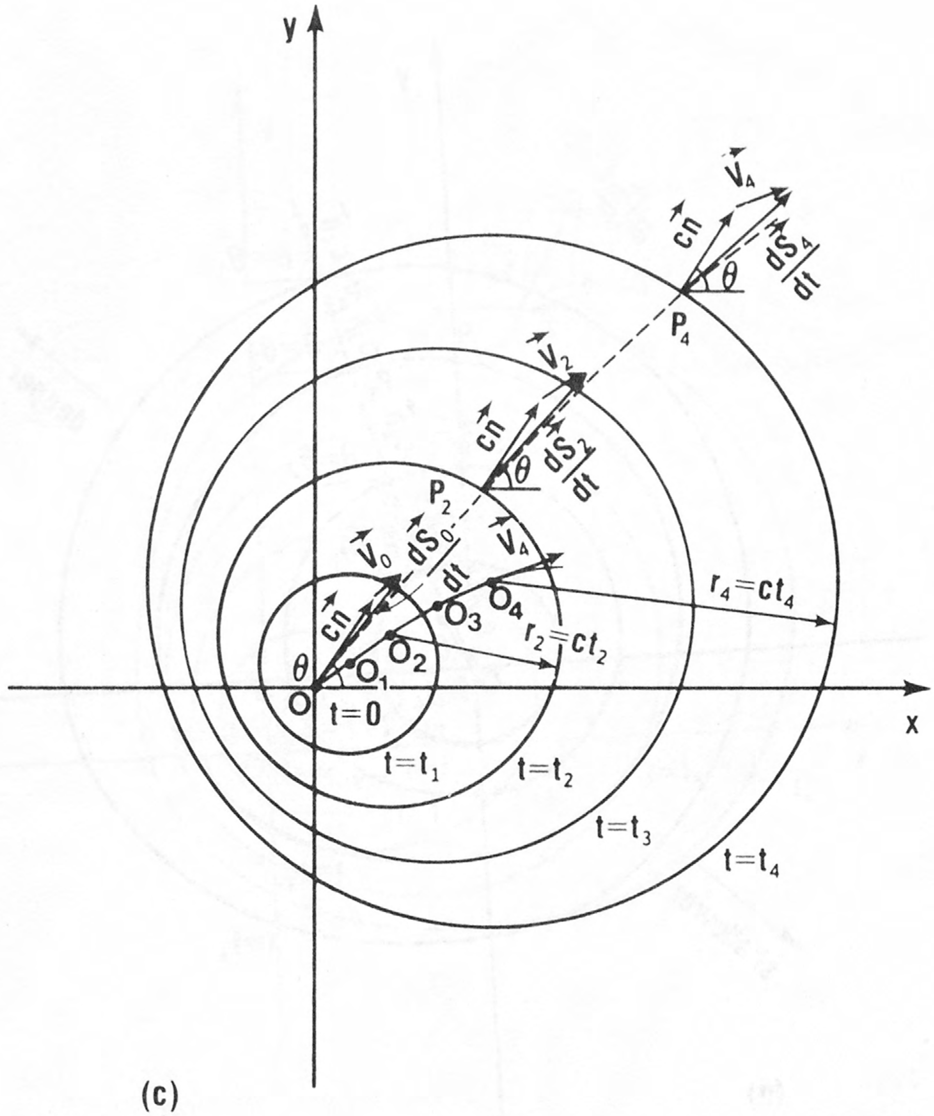
$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \Omega u + g\frac{\partial Z}{\partial y} + g\,S_y - W_y = 0. \qquad (2.68)$$



Still water; constant depth

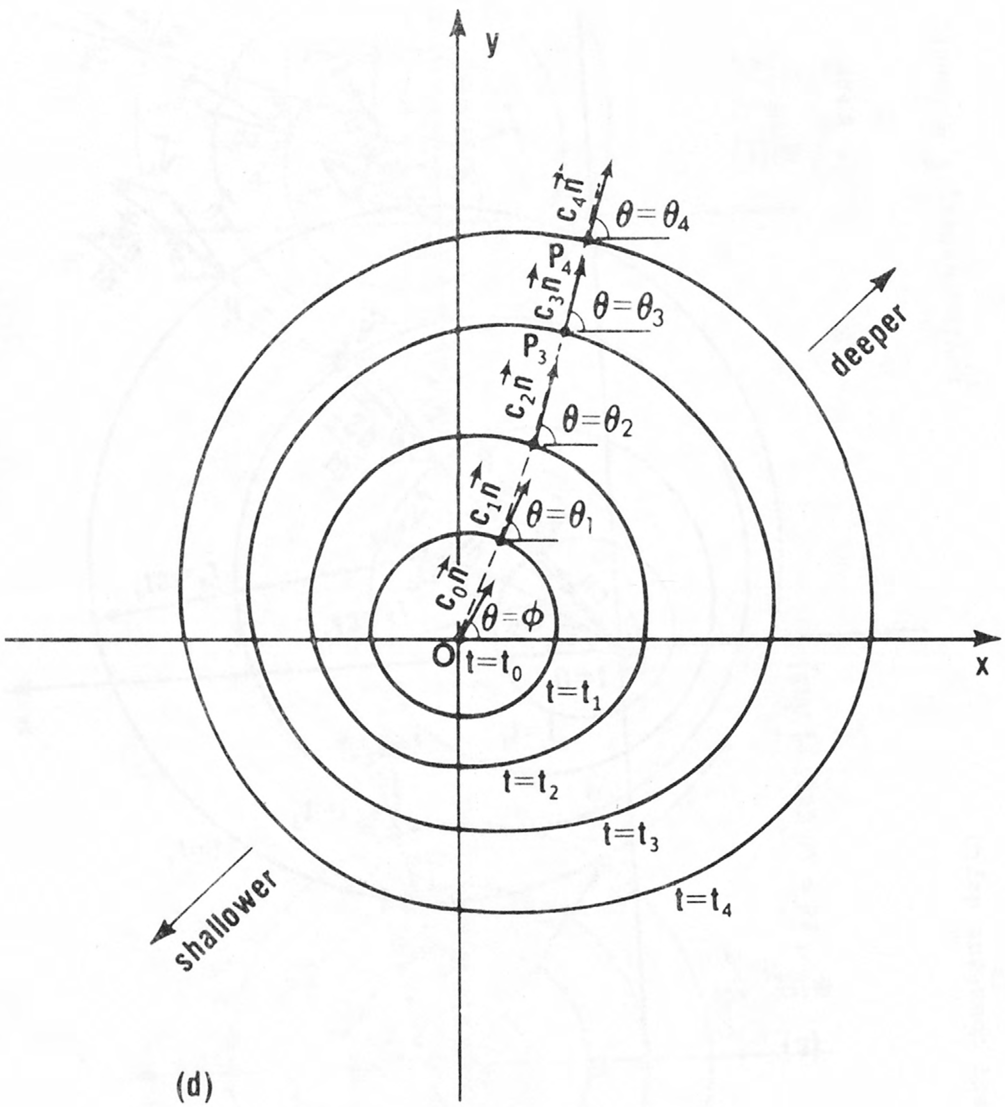
Moving water, $\vec{V}$ = const; constant depth

Figure 1. - The movement of wave front in various two-dimensional water bodies.



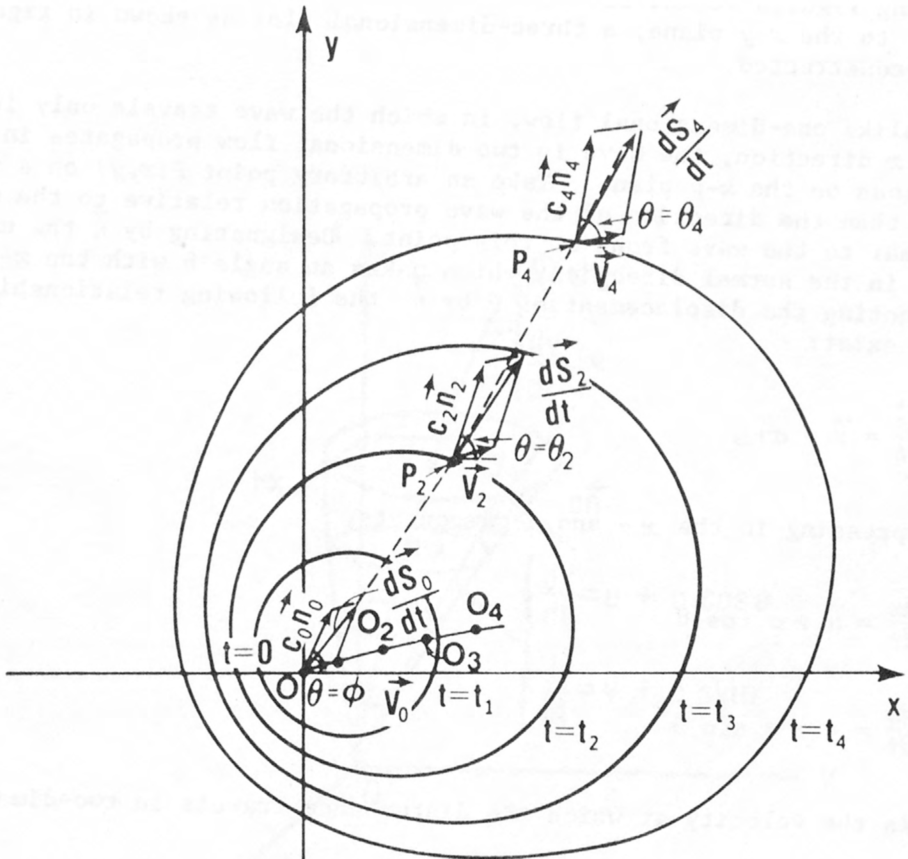
Unsteady flow; constant depth

Figure 1. - The movement of wave front in various two-dimensional water bodies.



Still water; variable depth

Figure 1. - The movement of wave front in various two-dimensional water bodies.



(e)

Unsteady flow; variable depth

Figure 1. - The movement of wave front in various two-dimensional water bodies.

When the disturbance is created in a still water body of variable depth, the wave front will no longer be a circle, because c is no longer a constant but is a function of location, that is, $c = c(H) = c[Z_b(x,y)] = c(x,y)$. Wave fronts at successive time intervals are spaced wider at deeper places and narrower at shallower, as may be seen from figure 1d. For wave propagation in the water of variable depth and variable flow velocity, the resulting wave fronts may be plotted as figure 1e, by combining figures 1c and 1d. If the third axis, t, is erected perpendicular to the x-y plane, a three-dimensional plot as shown in figure 2 can be constructed.

Unlike one-dimensional flow, in which the wave travels only in either $+x$ or $-x$ direction, the wave in two-dimensional flow propagates in all directions on the x-y plane. Take an arbitrary point $P(x,y)$ on a wave front, then the direction of the wave propagation relative to the water is normal to the wave front at this point. Designating by $\vec{n}$ the unit vector in the normal direction, which makes an angle θ with the x-axis, and denoting the displacement of P by $\vec{s}$, the following relationship should exist:

$$\frac{d\vec{s}}{dt} = \vec{V} + c\vec{n} \ , \tag{3.4}$$

or, expressing in the x- and y-components,

$$\begin{cases} \dfrac{dx}{dt} = u + c \cos \theta & (3.5) \\ \\ \dfrac{dy}{dt} = v + c \sin \theta & (3.6) \end{cases}$$

This is the velocity at which the disturbance travels in two-dimensional flow.

Inasmuch as the physical meaning of the characteristic method is to observe the changes of the water surface and flow velocity, (Z,u,v), by following the characteristic direction, namely, by moving at the velocity a disturbance would travel in $\vec{s}$-direction, a differential operator analogous to that of the one-dimensional flow analysis (Lai, 1965) may be devised. It is hoped, at least at the first hand, that such operator would transform the partial differential form into a total differential form as in the one-dimensional case. The operator should have a form as

$$\frac{D_\theta}{Dt} = \frac{\partial}{\partial t} + \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} = \frac{\partial}{\partial t} + (u + c \cos \theta)\frac{\partial}{\partial x} + (v + c \sin \theta)\frac{\partial}{\partial y} \ , \tag{3.7}$$

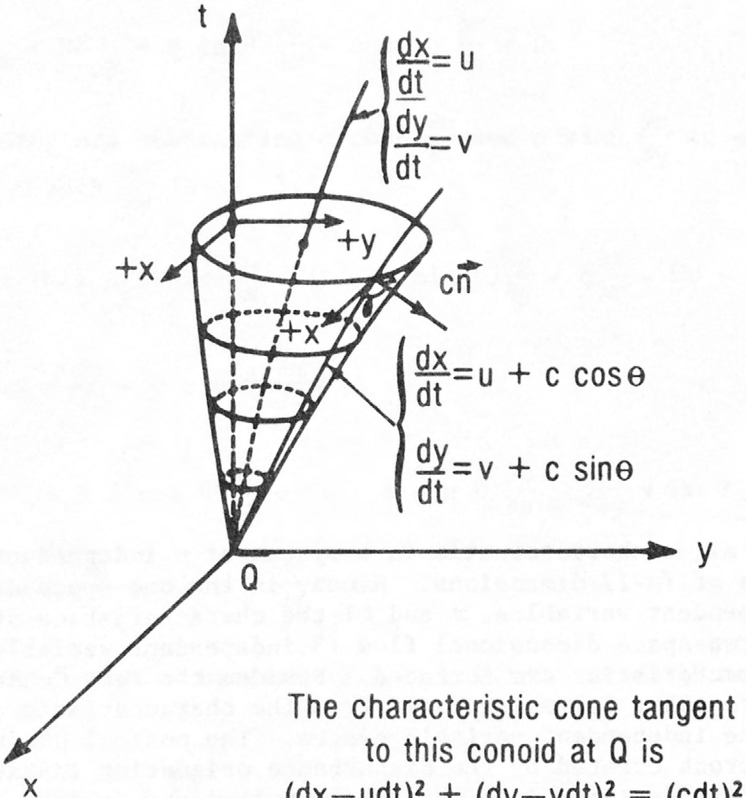


Figure 2. - The characteristic conoid showing the range of influence of Q.

where $\frac{D_\theta}{Dt}$ means differentiation in the direction defined by the parametric equations 3.5 and 3.6, using θ as a parameter. The direction defined by eq 3.5 and eq 3.6 is called the "bicharacteristic direction" (Fox, 1962)*. It is self-evident that when $v = 0$ and $\theta = 0$ or π the above expression reduces to the differential operator along the characteristics in the one-dimensional flow, namely (Lai, 1965),

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (u \pm c) \frac{\partial}{\partial x} \; . \qquad (3.8)$$

---

* In general, a characteristic in a system of n independent variables is a subspace of $(n-1)$ dimensions. Hence, in the one-space dimensional flow (2 independent variables, x and t) the characteristics are curves whereas in the two-space dimensional flow (3 independent variables, x, y, and t) the characteristics are surfaces. Besides the term "characteristic surface", there are other terms generated from the characteristic configuration in the three independent variable system. The conical surface generated by a wave front created by the disturbance orignating instataneously from point Q, as depicted in figure 2, is called the "characteristic conoid"; the rays along which such disturbance travels are defined as the "bicharacteristics", and the cone whose generators are tangential to those of the conoid (or to the bicharacteristics) at Q is named as "characteristic cone". For more mathematical treatments of the multi-dimensional characteristics, rigorous definitions of terminology, and suitable geometrical description of the characteristic space, the reader is referred to some of the literature listed in the references (Butler, 1960; Courant, 1962; Fox, 1962; Richardson, 1964).

Transformation of the Partial Differential Equations along Bicharacteristics

Using the differential operator for the two-dimensional unsteady flow (3.7), the equation set, 3.1 to 3.3, can now be transformed along the bicharacteristics.

Adding and subtracting $c \cos\theta \frac{\partial Z}{\partial x}$ and $c \sin\theta \frac{\partial Z}{\partial y}$ in eq 3.1, gives

$$J_1 = \frac{\partial Z}{\partial t} + (u + c\cos\theta)\frac{\partial Z}{\partial x} + (v + c\sin\theta)\frac{\partial Z}{\partial y} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

$$+ uS_{ox} + vS_{oy} - c\cos\theta\frac{\partial Z}{\partial x} - c\sin\theta\frac{\partial Z}{\partial y} = 0. \qquad (3.9)$$

Similarly, adding and subtracting $c \cos\theta \frac{\partial u}{\partial x}$ and $c \sin\theta \frac{\partial u}{\partial y}$ in eq 3.2 and $c \cos\theta \frac{\partial v}{\partial x}$ and $c \sin\theta \frac{\partial v}{\partial y}$ in eq 3.3,

$$J_2 = \frac{\partial u}{\partial t} + (u + c\cos\theta)\frac{\partial u}{\partial x} + (v + c\sin\theta)\frac{\partial u}{\partial y} + g\frac{\partial Z}{\partial x} - \Omega v + gS_x - W_x$$

$$- c\cos\theta\frac{\partial u}{\partial x} - c\sin\theta\frac{\partial u}{\partial y} = 0, \qquad (3.10)$$

$$J_3 = \frac{\partial v}{\partial t} + (u + c\cos\theta)\frac{\partial v}{\partial x} + (v + c\sin\theta)\frac{\partial v}{\partial y} + g\frac{\partial Z}{\partial y} + \Omega u + gS_y - W_y$$

$$- c\cos\theta\frac{\partial v}{\partial x} - c\sin\theta\frac{\partial v}{\partial y} = 0. \qquad (3.11)$$

Applying the differential operator 3.7, eq 3.9 to eq 3.11 can be condensed to

$$J_1 = \frac{D_\theta Z}{Dt} - c\cos\theta\frac{\partial Z}{\partial x} - c\sin\theta\frac{\partial Z}{\partial y} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + uS_{ox} + vS_{oy} = 0, \qquad (3.12)$$

$$J_2 = \frac{D_\theta u}{Dt} + g\frac{\partial Z}{\partial x} - c\cos\theta\frac{\partial u}{\partial x} - c\sin\theta\frac{\partial u}{\partial y} - \Omega v + gS_x - W_x = 0, \qquad (3.13)$$

$$J_3 = \frac{D_\theta v}{Dt} + g\frac{\partial Z}{\partial y} - c\cos\theta\frac{\partial v}{\partial x} - c\sin\theta\frac{\partial v}{\partial y} + \Omega u + gS_y - W_y = 0. \qquad (3.14)$$

A linear combination of eq 3.12 through eq 3.14 by

$$J = J_1 + \lambda_2 J_2 + \lambda_3 J_3$$

$$= J_1 + (\frac{c}{g}\cos\theta)\, J_2 + (\frac{c}{g}\sin\theta)\, J_3, \qquad (3.15)$$

and the use of $c = \sqrt{gH}$, will yield

$$J = \frac{D_\theta Z}{Dt} + \frac{c}{g}\cos\theta\,\frac{D_\theta u}{Dt} + \frac{c}{g}\sin\theta\,\frac{D_\theta v}{Dt}$$

$$+ H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\cos^2\theta - \frac{\partial u}{\partial y}\sin\theta\cos\theta - \frac{\partial v}{\partial x}\sin\theta\cos\theta - \frac{\partial v}{\partial y}\sin^2\theta\right)$$

$$+ uS_{ox} + vS_{oy} - \frac{c}{g}\Omega(v\cos\theta - u\sin\theta)$$

$$+ c(S_x\cos\theta + S_y\sin\theta) - \frac{c}{g}(W_x\cos\theta + W_y\sin\theta) = 0.$$

Transposing and simplifying,

$$\frac{D_\theta Z}{Dt} + \frac{c}{g}\cos\theta\,\frac{D_\theta u}{Dt} + \frac{c}{g}\sin\theta\,\frac{D_\theta v}{Dt}$$

$$= -H\left[\frac{\partial u}{\partial x}\sin^2\theta - \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\sin\theta\cos\theta + \frac{\partial v}{\partial y}\cos^2\theta\right]$$

$$- uS_{ox} - vS_{oy} - c(S_x\cos\theta + S_y\sin\theta)$$

$$+ \frac{c}{g}\Omega(v\cos\theta - u\sin\theta) + \frac{c}{g}(W_x\cos\theta + W_y\sin\theta). \qquad (3.16)$$

In the one-dimensional flow analysis (Lai, 1965) the corresponding linear combination was

$$J = J_1 + \lambda J_2 = J_1 \pm \frac{c}{g} J_2 ,$$

and this is the case when $\theta = 0$ or π in the relationship 3.15.

Let $F = F_p + F_s + F_f + F_c + F_w$, (3.17) in which

$$F_p = -H\left[\frac{\partial u}{\partial x} \sin^2 \theta - \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \sin \theta \cos \theta + \frac{\partial v}{\partial y} \cos^2 \theta\right] = -Hp, \qquad (3.18)$$

$$F_s = -uS_{ox} - vS_{oy}, \qquad (3.19)$$

$$F_f = -c(S_x \cos \theta + S_y \sin \theta), \qquad (3.20)$$

$$F_c = \frac{c}{g}\Omega\, (v \cos \theta - u \sin \theta), \qquad (3.21)$$

$$F_w = \frac{c}{g}\, (W_x \cos \theta + W_y \sin \theta), \qquad (3.22)$$

then eq 3.16 has a short form

$$dZ + \frac{c}{g} \cos \theta\, du + \frac{c}{g} \sin \theta\, dv = F\, dt, \qquad (3.23)$$

or

$$dZ + \frac{c}{g} \cos \theta\, du + \frac{c}{g} \sin \theta\, dv = F_p dt + F_r dt, \qquad (3.24)$$

along a bicharacteristic. Here $F_r = F_s + F_f + F_c + F_w$, which is the sum of four terms representing the bottom slope, bottom friction, Coriolis effect, and wind effect, respectively. As defined in eq 3.18, p stands for those terms within the brackets, all having the partial derivative forms.

Noteworthy here is that unlike the characteristic relation of the one-dimensional flow, which is entirely in the form of a total differential equation (Lai, 1965), the characteristic relation in the two-dimensional flow, eq 3.24, still retains partial differential terms represented by F_p. Without these terms eq 3.24 would be a simple total differential equation and a numerical integration could be performed directly along the bicharacteristics. The treatment of these additional terms in the numerical solution will be examined in the following paragraphs.

Finite Difference Approximation

Because the parameter θ can assume any value in the range $0<\theta<2\pi$ first consider a point $P(x_o,y_o,t_o)$ on the plane $t = t_o$ and an infinite number of bicharacteristic rays that were all emitted from the plane $t = t_o - \Delta t$ and meet exactly at point P at Δt later. Then, all these bicharacteristics form a characteristic cone (or conoid) and P is its vertex, as shown in figure 3. The conoid depicted in figure 2 may be called as "the range of influence of Q" and that in figure 3 as "the domain of determinancy of P", just as the counterparts found in the one-dimensional flow case.

From point P a different form of eq 3.24 or eq 3.16 can be written along any bicharacteristic between $t = t_o$ plane and $t = t_o - \Delta t$ plane. Denoting the θ value at $P(x_o,y_o,t_o)$ by ϕ, that is, $\theta_o = \phi$, four difference equations along four bicharacteristics $\phi = 0$, $\frac{\pi}{2}$, π, and $\frac{3}{2}\pi$ will be written below. The four bicharacteristics meet the plane $t = t_o - \Delta t$ at points $P_1(x_1,y_1,t_1)$, $P_2(x_2,y_2,t_2)$, ..., for $\phi = 0$, $\frac{\pi}{2}$, ..., respectively, and $t_1 = t_2 = \ldots$. (See fig. 3.) The difference equations, with variables and parameters referred to any point by their corresponding subscripted number, have the form:

$$(Z_0 - Z_1) + U_1 (u_0 - u_1) + V_1(v_0 - v_1) = -\frac{1}{2}\Delta t H_0 \left(\frac{\partial v}{\partial y}\right)_0 - \frac{1}{2}\Delta t H_1 p_1 + \frac{1}{2}\Delta t (F_r)_{01} + \frac{1}{2}\Delta t (F_r)_1 , \quad (3.25)$$

$$(Z_0 - Z_2) + U_2 (u_0 - u_1) + V_2(v_0 - v_2) = -\frac{1}{2}\Delta t H_0 \left(\frac{\partial u}{\partial x}\right)_0 - \frac{1}{2}\Delta t H_2 p_2 + \frac{1}{2}\Delta t (F_r)_{02} + \frac{1}{2}\Delta t (F_r)_2 , \quad (3.26)$$

$$(Z_0 - Z_3) + U_3 (u_0 - u_3) + V_3(v_0 - v_3) = -\frac{1}{2}\Delta t H_0 \left(\frac{\partial v}{\partial y}\right)_0 - \frac{1}{2}\Delta t H_3 p_3 + \frac{1}{2}\Delta t (F_r)_{03} + \frac{1}{2}\Delta t (F_r)_3 , \quad (3.27)$$

$$(Z_0 - Z_4) + U_4 (u_0 - u_4) + V_4(v_0 - v_4) = -\frac{1}{2}\Delta t H_0 \left(\frac{\partial u}{\partial x}\right)_0 - \frac{1}{2}\Delta t H_4 p_4 + \frac{1}{2}\Delta t (F_r)_{04} + \frac{1}{2}\Delta t (F_r)_4 , \quad (3.28)$$

in which

$$U_i = \frac{1}{2g}(c_0 \cos \phi_i + c_i \cos \theta_i),$$

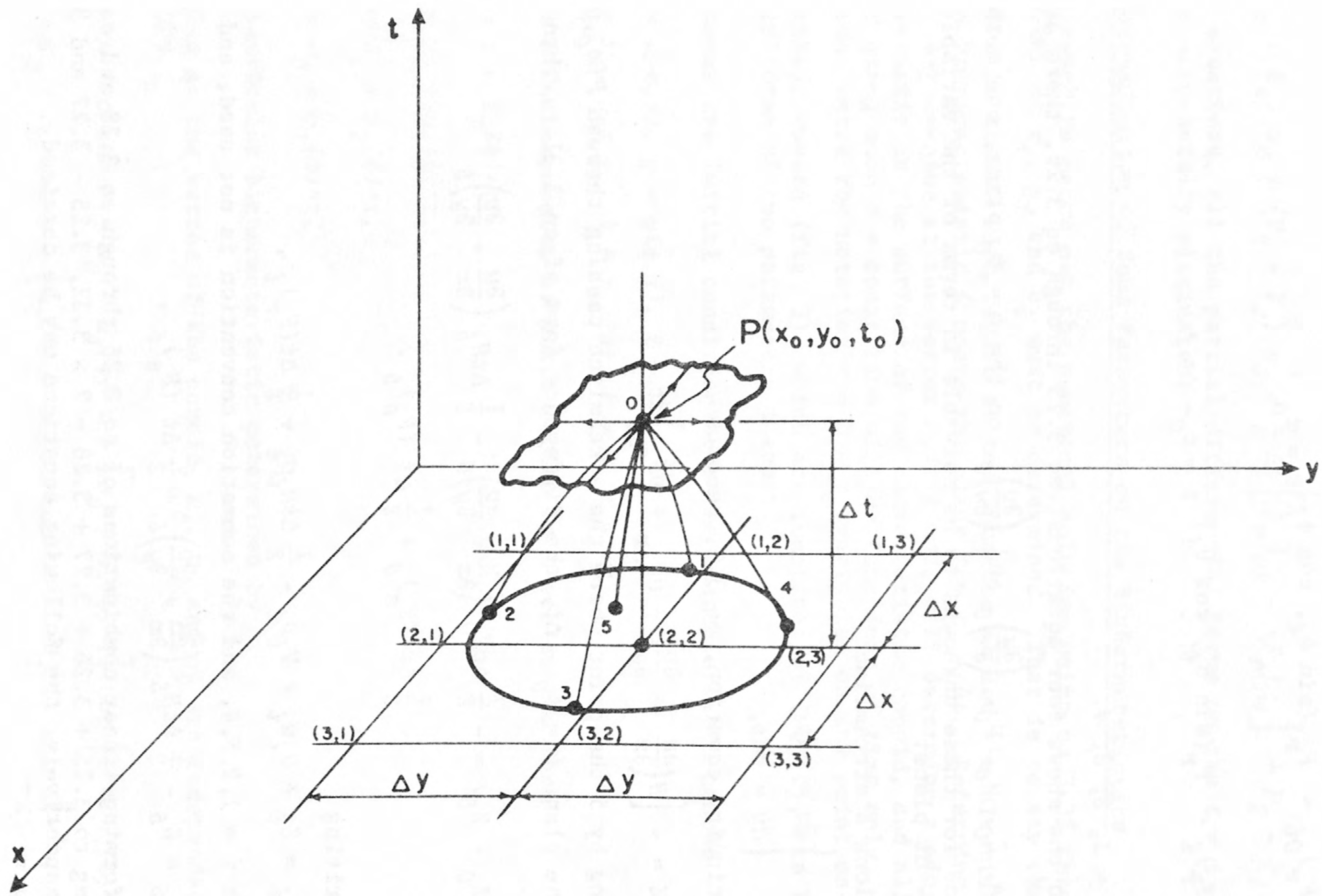


Figure 3. - The specified-time-interval scheme consisting of four bicharacteristics (which depicts the domain of dependence of P) and one streamline.

$$V_i = \frac{1}{2g}(c_0 \sin \phi_i + c_i \sin \theta_i),$$

$$(F_r)_{0i} = F_r(\sin \phi_i, \cos \phi_i)_{t = t_0},$$

and

$$(F_r)_i = F_r(\sin \theta_i, \cos \theta_i)_{t = t_0 - \Delta t},$$

for

$$i = 1, 2, 3, 4.$$

Since in the above four equations, eq 3.25 through eq 3.28, there are five unknowns, Z_0, u_0, v_0, $\left(\frac{\partial u}{\partial x}\right)_0$ and $\left(\frac{\partial v}{\partial y}\right)_0$, on the $t = t_0$ plane, one more equation is needed for these unknowns to be solvable in terms of the variables on the $t = t_0 - \Delta t$ plane.

Along a streamline

$$\begin{cases} dx = udt & (3.29) \\ dy = vdt, & (3.30) \end{cases}$$

the continuity equation, eq 3.1, becomes

$$dZ = -\left[H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + uS_{ox} + vS_{oy}\right]dt. \qquad (3.31)$$

Numbering by 5 the point where the streamline passing through $P(x_0, y_0, t_0)$ meets the plane $t = t_0 - \Delta t$, the difference form of eq 3.31 is then given by

$$(Z_0 - Z_5) = -\frac{1}{2}\Delta t H_0\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_0 - \frac{1}{2}\Delta t H_5\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_5$$
$$+ \frac{1}{2}\Delta t (F_s)_0 + \frac{1}{2}\Delta t (F_s)_5 . \qquad (3.32)$$

Writing

$$P_i = Z_i + U_i u_i + V_i v_i - \frac{1}{2}\Delta t H_i p_i + \frac{1}{2}\Delta t (F_r)_i , \qquad (3.33)$$

in which $i = 1,2,3,4$, and the summation convention is not used, and

$$P_5 = Z_5 - \frac{1}{2}\Delta t H_5\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_5 + \frac{1}{2}\Delta t (F_s)_5 , \qquad (3.34)$$

and performing linear combinations of eq 3.25 through eq 3.28 and eq 3.32 according to 3.25 + 3.26 + 3.27 + 3.28 - 2 x 3.32, 3.25 - 3.27 and 3.26 - 3.28, respectively, the following equations may be obtained.

$$2Z_0 + (U_1 + U_2 + U_3 + U_4) u_0 + (V_1 + V_2 + V_3 + V_4) v_0$$
$$= \frac{1}{2}\Delta t\left[(F_r)_{01} + (F_r)_{02} + (F_r)_{03} + (F_r)_{04} - 2(F_s)_0\right]$$
$$+ P_1 + P_2 + P_3 + P_4 - 2P_5 , \qquad (3.35)$$

$$(U_1 - U_3)\, u_0 + (V_1 - V_3)\, v_0 = \frac{1}{2}\,\Delta t \left[(F_r)_{01} - (F_r)_{03}\right] + P_1 - P_3, \quad (3.36)$$

and

$$(U_2 - U_4)\, u_0 + (V_2 - V_4)\, v_0 = \frac{1}{2}\,\Delta t \left[(F_r)_{02} - (F_r)_{04}\right] + P_2 - P_4. \quad (3.37)$$

In these equations, all the partial differential terms on the $t = t_o$ plane have been successfully eliminated.

Determination of Some Parameters on the Bicharacteristics

In order to carry out the numerical solution using eq 3.35 through eq 3.37, values of x_i, y_i and θ_i must be determined. That is to say that information is needed with regard to the location at and the direction in which the bicharacteristic i meets the plane $t = t_o - \Delta t$. Referring to figure 3, any ϕ-value at the vertex P, $(0 \le \phi < 2\pi)$, will determine a single bicharacteristic on the surface of the characteristic conoid, and all other variables along such ϕ = const line will be functions of time only. In other words, using the notation $\tau = t - t_o$ variables on any point on the characteristic conoid (fig. 3), which are functions of (x,y,t), can be defined in terms of two parameters ϕ and τ.

Because the initial conditions for the characteristic conoid

$$x = x(\phi,\tau),\; y = y(\phi,\tau),\; t = t(\tau)$$

are $x = x_o,\; y = y_o,\; t = t_o,$ at $\theta = \phi,\; \tau = 0;$

the first order expansion of power series in τ yields the form

$$\theta = \phi + \theta_1(\phi)\cdot\tau, \quad (3.38)$$

$$u = u_0 + u_1(\phi)\cdot\tau, \quad (3.39)$$

$$v = v_0 + v_1(\phi)\cdot\tau, \quad (3.40)$$

$$c = c_0 + c_1(\phi)\cdot\tau, \quad (3.41)$$

along a particular bicharacteristic determined by ϕ.

Being at the vertex of the conoid, u_0, v_0 and c_0 are independent of ϕ. From eq 3.5

$$\frac{\partial x}{\partial \tau}(\phi,\tau) = u + c\cos\theta = u_0 + u_1(\phi)\cdot\tau + \left[c_0 + c_1(\phi)\cdot\tau\right]\cos\left[\phi + \theta_1(\phi)\cdot\tau\right] + O(\tau^2)$$

$$= u_0 + c_0\cos\phi + \left[u_1(\phi) + c_1(\phi)\cos\phi - c\theta_1(\phi)\sin\phi\right]\tau + O\left(\tau^2\right). \quad (3.42)$$

Integrating with respect to τ,

$$\int_{x_0}^{x} dx = (u_0 + c_0 \cos\phi)\tau + \frac{1}{2}\left[u_1(\phi) + c_1(\phi)\cos\phi - c\theta_1(\phi)\sin\phi\right]\tau^2 + O(\tau^3)$$

$$= \frac{1}{2}\tau\left[u_0 + c_0\cos\phi + u_0 + u_1(\phi)\tau + c_0\cos\phi + c_1(\phi)\tau\cos\phi\right.$$

$$\left.- c\theta_1(\phi)\tau\sin\phi\right] + O(\tau^3)$$

$$= \frac{1}{2}\tau\left\{u_0 + u + c_0\cos\phi + c\cos\left[\phi + \theta_1(\phi)\tau\right]\right\} + O(\tau^3). \quad (3.43)$$

Replacing τ by $-\Delta t$, x by x_i, ...,

$$x_i = x_0 - \frac{1}{2}\Delta t\,\left[u_0 + u_i + c_0\cos\phi_i + c_i\cos\theta_i\right] + O(\tau^3). \quad (3.44)$$

Similarily,

$$y_i = y_0 - \frac{1}{2}\Delta t\,\left[v_0 + v_i + c_0\sin\phi_i + c_i\sin\theta_i\right] + O(\tau^3). \quad (3.45)$$

Next, the direction of the normal to the wave front, θ_i, can be obtained as follows. From the first line of eq 3.43

$$x(\phi,\tau) = x_0 + \left(u_0 + c_0\cos\phi\right)\tau + \frac{1}{2}\left[u_1(\phi) + c_1(\phi)\cos\phi - c\theta_1(\phi)\sin\phi\right]\tau^2 + O\left(\tau^3\right). \quad (3.46)$$

and, similarly,

$$y(\phi,\tau) = y_0 + \left(v_0 + c_0\sin\phi\right)\tau + \frac{1}{2}\left[v_1(\phi) + c_1(\phi)\sin\phi + c\theta_1(\phi)\cos\phi\right]\tau^2 + O\left(\tau^3\right). \quad (3.47)$$

Differentiating eq 3.46 and eq 3.47 with respect to ϕ, and denoting the differentiation with prime,

$$\frac{\partial x}{\partial\phi}(\phi,\tau) = -c_0\sin\phi\cdot\tau + \frac{1}{2}\left[u_1'(\phi) + c_1'(\phi)\cos\phi - c_1(\phi)\sin\phi - c\theta_1'(\phi)\sin\phi\right.$$
$$\left.- c\theta_1(\phi)\cos\phi - c'\theta_1(\phi)\sin\phi\right]\tau^2 + O\left(\tau^3\right), \quad (3.48)$$

$$\frac{\partial y}{\partial\phi}(\phi,\tau) = +c_0\cos\phi\cdot\tau + \frac{1}{2}\left[v_1'(\phi) + c_1'(\phi)\sin\phi + c_1(\phi)\cos\phi + c\theta_1'(\phi)\cos\phi\right.$$
$$\left.- c\theta_1(\phi)\sin\phi + c'\theta_1(\phi)\cos\phi\right]\tau^2 + O\left(\tau^3\right). \quad (3.49)$$

As has been discussed previously, variables on any point on the characteristic conoid can be defined by two parameters ϕ and τ, then the partial differential operator $\frac{\partial}{\partial\tau}$ means the differentiation along the bicharacteristic (ϕ = const) and $\frac{\partial}{\partial\phi}$ means the differentiation along the wave front (τ = const). Consider a point on any wave front in figure 1, a segment of which is enlarged in figure 4, and the point considered is denoted by P. Consider another point Q near P also on the same wave front. Moving from P to Q, which is in the + ϕ-direction, the value of ϕ changes by amount $\Delta\phi$ and the corresponding changes of θ, x, and y will be $\Delta\theta$,

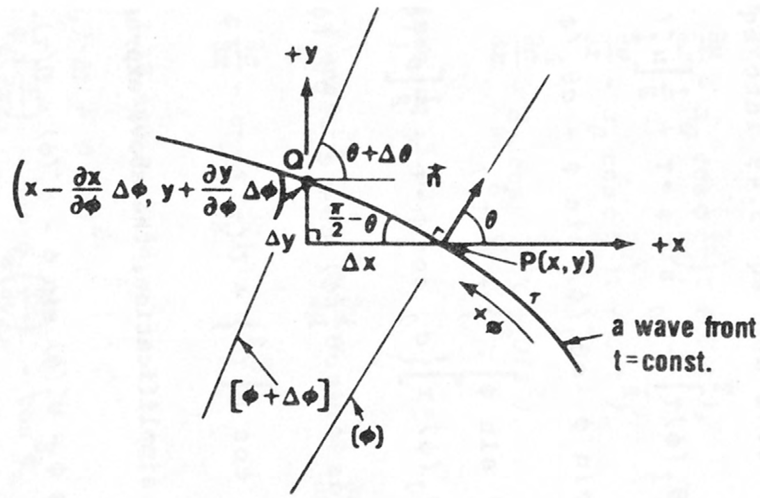


Figure 4. - The change of x, y values along a segment of a wave front.

$\Delta x = -\frac{\partial x}{\partial \phi}\Delta\phi$, and $\Delta y = \frac{\partial y}{\partial \phi}\Delta\phi$, respectively. Now, let Q approach P indefinitely, that is, $\Delta\phi \to 0$; then, the following relationship results:

$$\tan\theta = \lim_{\Delta\phi\to 0}\frac{\Delta x}{\Delta y} = \lim_{\Delta\phi\to 0}\frac{-\frac{\partial x}{\partial\phi}\Delta\phi}{\frac{\partial y}{\partial\phi}\Delta\phi} = \frac{-\frac{\partial x}{\partial\phi}}{\frac{\partial y}{\partial\phi}} . \tag{3.50}$$

From eq 3.38 and eq 3.50,

$$\cos\left[\phi + \theta_1(\phi)\cdot\tau\right]\frac{\partial x}{\partial\phi}(\phi,\tau) = -\sin\left[\phi + \theta_1(\phi)\cdot\tau\right]\frac{\partial y}{\partial\phi}(\phi,\tau). \tag{3.51}$$

Substituting eq 3.48 and eq 3.49 into eq 3.51, and expanding

$$\begin{aligned}
&\left[\cos\phi - \sin\phi\cdot\theta_1(\phi)\tau\right]\Big\{-c_0\sin\phi\cdot\tau + \tfrac{1}{2}\Big[u_1'(\phi) + c_1'(\phi)\cos\phi \\
&\quad - c_1(\phi)\sin\phi - c\theta_1'(\phi)\sin\phi - c\theta_1(\phi)\cos\phi \\
&\quad - c'\theta_1(\phi)\sin\phi\Big]\tau^2\Big\} + O(\tau^3) \\
&= -\left[\sin\phi + \cos\phi\cdot\theta_1(\phi)\cdot\tau\right]\Big\{c_0\cos\phi\cdot\tau + \tfrac{1}{2}\Big[v_1'(\phi) + c_1'(\phi)\sin\phi \\
&\quad + c_1(\phi)\cos\phi + c\theta_1'(\phi)\cos\phi - c\theta_1(\phi)\sin\phi \\
&\quad + c'\theta_1(\phi)\cos\phi\Big]\tau^2\Big\} + O(\tau^3).
\end{aligned}$$

By further expansion and simplification, the above expression reduces to

$$c_0\theta_1(\phi) = -u_1'(\phi)\cos\phi - v_1'(\phi)\sin\phi - c_1'(\phi) + O(\tau). \tag{3.52}$$

Applying the relationships 3.39 through 3.41, eq 3.52 becomes

$$c_0\theta_1(\phi)\cdot\tau = -\cos\phi\frac{\partial u}{\partial\phi} - \sin\phi\frac{\partial v}{\partial\phi} - \frac{\partial c}{\partial\phi} . \tag{3.53}$$

Since

$$\frac{\partial f}{\partial \phi}(x,y) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \phi}$$

$$= -c_0 \sin\phi \cdot \tau \frac{\partial f}{\partial x} + c_0 \cos\phi \cdot \tau \frac{\partial f}{\partial y} + O(\tau^2), \quad (3.54)$$

(See eq 3.48 and eq 3.49)

it follows that

$$\frac{\partial u}{\partial \phi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \phi}$$

$$= -\left(c_0 \sin\phi \frac{\partial u}{\partial x} - c_0 \cos\phi \frac{\partial u}{\partial y}\right)\tau + O(\tau^2), \quad (3.55)$$

$$\frac{\partial v}{\partial \phi} = -\left(c_0 \sin\phi \frac{\partial v}{\partial x} - c_0 \cos\phi \frac{\partial v}{\partial y}\right)\tau + O(\tau^2), \quad (3.56)$$

$$\frac{\partial c}{\partial \phi} = -\left(c_0 \sin\phi \frac{\partial c}{\partial x} - c_0 \cos\phi \frac{\partial c}{\partial y}\right)\tau + O(\tau^2). \quad (3.57)$$

Substituting these into eq 3.53,

$$\theta_1(\phi)\cdot\tau = (\sin\phi\cos\phi \frac{\partial u}{\partial x} - \cos^2\phi \frac{\partial u}{\partial y} + \sin^2\phi \frac{\partial v}{\partial x} - \sin\phi\cos\phi \frac{\partial v}{\partial y}$$

$$+ \sin\phi \frac{\partial c}{\partial x} - \cos\phi \frac{\partial c}{\partial y})\tau + O(\tau^2). \quad (3.58)$$

Therefore, from eq 3.38 and eq 3.58,

$$\theta_i = \phi_i - \theta_1(\phi_i)\cdot\Delta t + O(\Delta t)^2$$

$$= \phi_i - \left[\sin\phi_i\left(\frac{\partial c}{\partial x}\right)_i - \cos\phi_i\left(\frac{\partial c}{\partial y}\right)_i - \cos^2\phi_i\left(\frac{\partial u}{\partial y}\right)_i + \sin\phi_i\cos\phi_i\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)_i\right.$$

$$\left. + \sin^2\phi_i\left(\frac{\partial v}{\partial x}\right)_i\right]\Delta t + O(\Delta t)^2. \quad (3.59)$$

For the fifth point, from eq 3.29 and eq 3.30,

$$x_5 = x_0 - \frac{1}{2}\Delta t\left(u_0 + u_5\right) + O\left(\Delta t\right)^3, \quad (3.60)$$

$$y_5 = y_0 - \frac{1}{2}\Delta t\left(v_0 + v_5\right) + O\left(\Delta t\right)^3. \quad (3.61)$$

Referring to figure 3, all variables and parameters are assumed to be known at each grid point in the flow domain at $t = t_0 - \Delta t$, either given as initial values or evaluated from the previous computation. If the grid size $(\Delta x, \Delta y)$ on the $t = t_0 - \Delta t$ plane is taken as

$$\Delta x \geq \max|u + c \cos \theta| \cdot \Delta t, \tag{3.62}$$

and

$$\Delta y \geq \max|v + c \sin \theta| \cdot \Delta t, \tag{3.63}$$

then, for subcritical flow, all five points, points 1, 2, 3, 4, and 5, will fall in a region formed by four adjacent rectangular meshes as shown in figure 3, the nine grid points of these rectangles being (1,1), (1,2), (1,3), (2,1), ..., (3,3). Point $O(x_0, y_0, t_0)$ coincides with the center point $O'(2,2)$ in the x-y plane at $t = t_0 - \Delta t$.

Thus, once the locations of (x_i, y_i) are determined by eq 3.44 and eq 3.45 or by eq 3.60 and eq 3.61, the variables and parameters at these points on the time plane $t = t_0 - \Delta t$ may be found by interpolation from the known values at the grid points. The three unknowns at $t = t_0$, Z_0, u_0 and v_0, are then computed using eq 3.35 to eq 3.37. Inspection of equations derived in this chapter indicates that iterations are required when the trapezoidal rule of finite-difference approximation is used.

In dealing with numerical solution of the hyperbolic-type partial differential equations, the Courant-Frederick-Lewy (CFL) condition for the difference scheme is often referred. The CFL condition in the two-dimensional flow is much more complicated than that in the one-dimensional flow, and only limited examples, mostly of homogeneous linear cases, have been shown. For more general cases involving nonhomogeneous, nonlinear terms, the CFL condition derived for simplified cases is often substituted, and used only as a guide owing to the absence of explicit criteria.

According to Mitchell (1969), if an explicit finite difference approximation using five grid points [namely, (2,1), (1,2), (2,2), (3,2), (2,3) in figure 3] is applied to the simplest wave equation, (in which, $\frac{dx}{dt} = \frac{dy}{dt} = \underline{+}1$) the CFL condition becomes $(\Delta x = \Delta y)$

$$\frac{\Delta t}{\Delta x} \leq \frac{1}{\sqrt{2}} . \tag{3.64}$$

From the physical concepts of the characteristics, the square of dependence of the difference scheme must contain the circle of dependence of the differential equation within the boundaries of the square. For the square joining four points, (1,2), (2,1), (3,2), (2,3), the radius of the circle included in it is $\leq \frac{\Delta x}{\sqrt{2}}$, and eq 3.64 follows naturally.

For the numerical scheme presented in this chapter, the square of dependence consists of the four corners, (1,1), (3,1), (3,3), (1,3), (See figure 3.) and the circle included in it has a radius $\leq \Delta x$, which is equivalent to conditions

3.62 and 3.63 when $\Delta x = \Delta y$. As mentioned above, because such criteria are to be used only as a guide, whenever in doubt, more conservative values should be chosen.

BOUNDARY CONDITIONS

General Considerations

Boundary conditions for two-dimensional flow in tidal estuaries and embayments are much more complex than those encountered in one-dimensional flow cases. For this reason, it seems that an examination of various boundary geometry types, varying flow conditions, and the nature of boundary data, is in order before entering the specific mathematical treatments of boundary conditions.

Because the physical shapes of many bays, coastal seas, and wide estuaries may be adequately represented by a number of sufficiently small square meshes, the solution of boundary equations may be developed at boundary points on a rectangular grid system oriented in the x- and y-directions. Boundary grid points may be placed along an edge line going in either x- or y-direction, or they may be situated on corners of different angles. The boundary line can be of solid-wall type, which is a physical boundary, or an aritificial boundary line drawn on the water as a limit for the numerical model.

The following conditions are used at the solid (natural) boundary, namely: a) the velocity component normal to the solid wall is zero, and b) the velocity component parallel to the same wall is assumed finite, that is, there is a slip between the water and the wall. The latter condition is a reasonable assumption for the scale of problems generally treated.

When two long stretches of straight boundary lines meet with an angle other than a right angle, it is sometimes more convenient to develop boundary equations for an oblique angle rather than attempting to cover one of the lines with zig-zag shaped grids. Inside the boundary the rectangular x-y grid system is still used, and the values computed along the boundary are matched with those computed at the interior points through suitable coordinate rotation and transformation.

Theoretically, different types of boundary data may be used for the flow computation. Practically, however, the cost involved in the boundary data acquisition almost limits them to stage data only. These stage data are normally collected from gaging stations and organized for the input to the numerical computation. The derivation of boundary equations in the following sections and the handling of the boundary data problems hereafter are mainly adapted to the stage boundary data condition.

Boundary Points Along Edges

Once the study area is covered by a rectangular x-y grid system, many boundary points would usually fall along straight edges, either in x- or

y-direction. Four edge-point alinements may be observed going counterclockwise along the boundary, namely, points along the edges in $(-x)$-direction (I), $(-y)$-direction (II), $(+x)$-direction (III), and $(+y)$-direction (IV) as shown in figure 5.

Along Edge I a) The wall boundary: -- Since $v = 0$ along the edge, by combining bicharacteristics 1, 2, and 3, and the streamline (passing through the edge point) according to eq 3.25 + 2 X eq 3.26 + eq 3.27 - 2 X eq 3.32 and eq 3.25 - eq 3.27, the following two equations result:

$$2Z_0 + \left(U_1 + 2U_2 + U_3\right)u_0 = \frac{1}{2}\Delta t\left[\left(F_r\right)_{01} + 2\left(F_r\right)_{02} + \left(F_r\right)_{03} - 2\left(F_s\right)_0\right]_{v_0=0} + \left(P_1 + 2P_2 + P_3 - 2P_5\right), \quad (4.1)$$

$$\left(U_1 - U_3\right)u_0 = \frac{1}{2}\Delta t\left[\left(F_r\right)_{01} - \left(F_r\right)_{03}\right]_{v_0=0} + \left(P_1 - P_3\right). \quad (4.2)$$

Applying the relationships 3.19 through 3.22, the above two equations can also be expressed as

$$2Z_0 + \left(U_1 + 2U_2 + U_3\right)u_0 = \left(P_1 + 2P_2 + P_3 - 2P_5\right) - \Delta t\left[u_0\left(S_{ox}\right)_0 + \frac{c_0}{g}\Omega u_0 - \frac{c_0}{g}\left(W_y\right)_0\right], \quad (4.3)$$

$$\left(U_1 - U_3\right)u_0 = \left(P_1 - P_3\right) - \Delta t\left[c_0\left(S_x\right)_0 - \frac{c_0}{g}\left(W_x\right)_0\right]. \quad (4.4)$$

b) The open-water boundary: -- Assuming that Z_0 is given, and remembering that $v_0 \neq 0$, from the same combination shown in a),

$$\left(U_1 + 2U_2 + U_3\right)u_0 + \left(V_1 + 2V_2 + V_3\right)v_0 = \left(P_1 + 2P_2 + P_3 - 2P_5\right) - 2Z_0 - \Delta t\left[u_0\left(S_{ox}\right)_0 + v_0\left(S_{oy}\right)_0 + c_0\left(S_y\right)_0 + \frac{c_0}{g}\Omega u_0 - \frac{c_0}{g}\left(W_y\right)_0\right], \quad (4.5)$$

and

$$\left(U_1 - U_3\right)u_0 + \left(V_1 - V_3\right)v_0 = \left(P_1 - P_3\right) - \Delta t\left[c_0\left(S_x\right)_0 - \frac{c_0}{g}\Omega v_0 - \frac{c_0}{g}\left(W_x\right)_0\right]. \quad (4.6)$$

Similarly, the boundary equations along the other edges can be found as follows:

Along Edge II: The combinations, eq 3.26 + 2 X eq 3.27 + eq 3.28 - 2 X eq 3.32 and eq 3.26 - eq 3.28, give

$$2Z_0 + \left(U_2 + 2U_3 + U_4\right)u_0 + \left(V_2 + 2V_3 + V_4\right)v_0 = \frac{1}{2}\Delta t\left[\left(F_r\right)_{02} + 2\left(F_r\right)_{03} + \left(F_r\right)_{04} - 2\left(F_s\right)_0\right] + \left(P_2 + 2P_3 + P_4 - 2P_5\right), \quad (4.7)$$

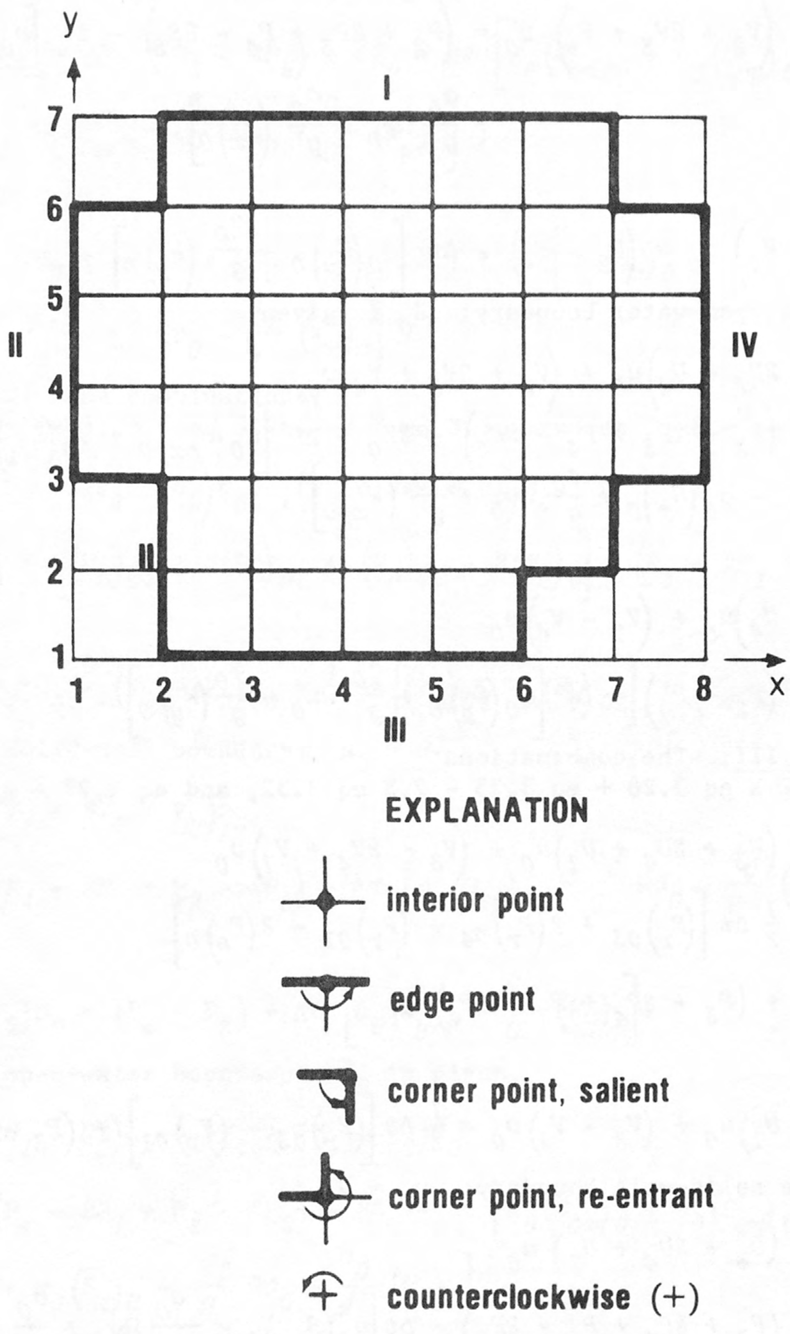


Figure 5. - An x-y grid system showing the interior and the boundary grid points.

and

$$\left(U_2 - U_4\right)u_0 + \left(V_2 - V_4\right)v_0 = \frac{1}{2}\,\Delta t\,\left[\left(F_r\right)_{02} - \left(F_r\right)_{04}\right] + \left(P_2 - P_4\right)\ . \quad (4.8)$$

a) For the solid-wall boundary: $u_0 = 0$,

$$2Z_0 + \left(V_2 + 2V_3 + V_4\right)\ v_0 = \left(P_2 + 2P_3 + P_4 - 2P_5\right) - \Delta t\ \left[v_0\left(S_{oy}\right)_0 + \frac{c_0}{g}\,\Omega v_0 + \frac{c_0}{g}\left(W_x\right)_0\right], \quad (4.9)$$

and

$$\left(V_2 - V_4\right)\ v_0 = \left(P_2 - P_4\right) - \Delta t\ \left[c_0\left(S_y\right)_0 - \frac{c_0}{g}\left(W_y\right)_0\right]\ . \quad (4.10)$$

b) For the open-water boundary: Z_0 is given,

$$\left(U_2 + 2U_3 + U_4\right)u_0 + \left(V_2 + 2V_3 + V_4\right)v_0$$
$$= \left(P_2 + 2P_3 + P_4 - 2P_5\right) - 2Z_0 - \Delta t\ \left[u_0\left(S_{ox}\right)_0 + v_0\left(S_{oy}\right)_0 - c_0\left(S_x\right)_0 + \frac{c_0}{g}\,\Omega v_0 + \frac{c_0}{g}\left(W_x\right)_0\right]\ , \quad (4.11)$$

and

$$\left(U_2 - U_4\right)u_0 + \left(V_2 - V_4\right)v_0$$
$$= \left(P_2 - P_4\right) - \Delta t\ \left[c_0\left(S_y\right)_0 + \frac{c_0}{g}\,\Omega u_0 - \frac{c_0}{g}\left(W_y\right)_0\right]\ . \quad (4.12)$$

<u>Along Edge III</u>: The combinations
eq 3.27 + 2 X eq 3.28 + eq 3.25 - 2 X eq 3.32, and eq 3.27 - eq 3.25, give

$$2Z_0 + \left(U_3 + 2U_4 + U_1\right)u_0 + \left(V_3 + 2V_4 + V_1\right)v_0$$
$$= \frac{1}{2}\,\Delta t\left[\left(F_r\right)_{03} + 2\left(F_r\right)_{04} + \left(F_r\right)_{01} - 2\left(F_s\right)_0\right]$$
$$+ \left(P_3 + 2P_4 + P_1 - 2P_5\right)\ , \quad (4.13)$$

and

$$\left(U_3 - U_1\right)u_0 + \left(V_3 - V_1\right)v_0 = \frac{1}{2}\,\Delta t\left[\left(F_r\right)_{03} - \left(F_r\right)_{01}\right] + \left(P_3 - P_1\right)\ . \quad (4.14)$$

a) For the solid-wall boundary: $v_0 = 0$,

$$2Z_0 + \left(U_3 + 2U_4 + U_1\right)\ u_0$$
$$= \left(P_3 + 2P_4 + P_1 - 2P_5\right) - \Delta t\left[u_0\left(S_{ox}\right)_0 - \frac{c_0}{g}\,\Omega u_0 + \frac{c_0}{g}\left(W_y\right)_0\right], \quad (4.15)$$

and

$$\left(U_3 - U_1\right)u_0 = \left(P_3 - P_1\right) + \Delta t \left[c_0\left(S_x\right)_0 - \frac{c_0}{g}\left(W_x\right)_0\right] . \tag{4.16}$$

b) For the open-water boundary: Z_0 is given,

$$\left(U_3 + 2U_4 + U_1\right)u_0 + \left(V_3 + 2V_4 + V_1\right)v_0$$
$$= \left(P_3 + 2P_4 + P_1 - 2P_5\right) - 2Z_0 - \Delta t \left[u_0\left(S_{ox}\right)_0 + v_0\left(S_{oy}\right)_0 - c_0\left(S_y\right)_0 - \frac{c_0}{g}\Omega u_0 + \frac{c_0}{g}\left(W_y\right)_0\right], \tag{4.17}$$

and

$$\left(U_3 - U_1\right)u_0 + \left(V_3 - V_1\right)v_0 = \left(P_3 - P_1\right) + \Delta t\left[c_0\left(S_x\right)_0 - \frac{c_0}{g}\Omega v_0 - \frac{c_0}{g}\left(W_x\right)_0\right]. \tag{4.18}$$

Along Edge IV: The combinations,
eq 3.28 + 2 X eq 3.25 + eq 3.26 - 2 X eq 3.32, and eq 3.28 - eq 3.26, give

$$2Z_0 + \left(U_4 + 2U_1 + U_2\right)u_0 + \left(V_4 + 2V_1 + V_2\right)v_0$$
$$= \frac{1}{2}\Delta t\left[\left(F_r\right)_{04} + 2\left(F_r\right)_{01} + \left(F_r\right)_{02} - 2\left(F_s\right)_0\right] + \left(P_4 + 2P_1 + P_2 - 2P_5\right), \tag{4.19}$$

and

$$\left(U_4 - U_2\right)u_0 + \left(V_4 - V_2\right)v_0 = \frac{1}{2}\Delta t\left[\left(F_r\right)_{04} - \left(F_r\right)_{02}\right] + \left(P_4 - P_2\right) . \tag{4.20}$$

a) For the solid-wall boundary: $u_0 = 0$,

$$2Z_0 + \left(V_4 + 2V_1 + V_2\right)v_0$$
$$= \left(P_4 + 2P_1 + P_2 - 2P_5\right) - \Delta t\left[v_0\left(S_{oy}\right)_0 - \frac{c_0}{g}\Omega v_0 - \frac{c_0}{g}\left(W_x\right)_0\right], \tag{4.21}$$

and

$$\left(V_4 - V_2\right)v_0 = \left(P_4 - P_2\right) + \Delta t\left[c_0\left(S_y\right)_0 - \frac{c_0}{g}\left(W_y\right)_0\right]. \tag{4.22}$$

b) For the open-water boundary: Z_0 is given,

$$\left(U_4 + 2U_1 + U_2\right)u_0 + \left(V_4 + 2V_1 + V_2\right)v_0$$
$$= \left(P_4 + 2P_1 + P_2 - 2P_5\right) - 2Z_0 - \Delta t\left[u_0\left(S_{ox}\right)_0 + v_0\left(S_{oy}\right)_0 + c_0\left(S_x\right)_0 - \frac{c_0}{g}\Omega v_0 - \frac{c_0}{g}\left(W_x\right)_0\right], \tag{4.23}$$

and

$$\left(U_4 - U_2\right)u_0 + \left(V_4 - V_2\right)v_0$$
$$= \left(P_4 - P_2\right) + \Delta t \left[c_0\left(S_y\right)_0 + \frac{c_0}{g}\Omega u_0 - \frac{c_0}{g}\left(W_y\right)_0\right] . \qquad (4.24)$$

Boundary Points at Corners

There are two types of corners with respect to the region covered by the rectangular grid system, namely, the "salient" corner characterized by a 90° interior angle and the "re-entrant" corner with a 270° interior angle. For example, in figure 5, points (2,1), (6,1), (7,2), (1,3) belong to the former and points (6,2), (2,3), (7,3) belong to the latter. Each corner type can be classified further into four catagories according to its orientation, in a similar way edge boundary points were classified.

Salient Corners

The corner bounded by $\phi = 0$ and $\phi = \frac{\pi}{2}$ will be called the 1st kind and the remaining three kinds will be defined in a similar way, with each corner covering the corresponding quadrant; namely, $\phi = \frac{\pi}{2}$ to $\phi = \pi$ for the 2nd kind, $\phi = \pi$ to $\phi = \frac{3}{2}\pi$ for the 3rd and $\phi = \frac{3}{2}\pi$ to $\phi = 2\pi$ for the 4th. The angle ϕ is measured on the $t = t_0$ plane counterclockwise from the $+x$ axis ($-x$ axis) to the positive (negative) limb of the marching bicharacteristic passing through point 0, the apex of the characteristic conoid which was used as the origin. Each kind may have four possible combinations of the two boundary lines that form the corner. These four cases are: (1) both boundary lines are closed-wall boundary, (2) both, open-water boundary, (3) the first boundary line closed-wall, and the second open-water, and (4) the first open-water and the second closed-wall as shown in figure 6. Boundary equations for these different kinds and cases of corner points are derived as follows:

A. The 1st kind: The corner between $\phi = 0$ and $\phi = \frac{\pi}{2}$

a) Case I. $u_0 = 0$, $v_0 = 0$; Z_0 is unknown;
From combination eq 3.25 + eq 3.26 - eq 3.32

$$Z_0 = \left(P_1 + P_2 - P_5\right) + \frac{1}{2}\Delta t \frac{c_0}{g}\left(W_x + W_y\right)_0 . \qquad (4.25)$$

b) Case II. Z_0 is known; u_0, v_0 are unknown;
From combination eq 3.25 + eq 3.26 - eq 3.32

$$\left(U_1 + U_2\right)u_0 + \left(V_1 + V_2\right)v_0$$
$$= \left(P_1 + P_2 - P_5\right) - Z_0 - \frac{1}{2}\Delta t \left[u_0\left(S_{ox}\right)_0 + v_0\left(S_{oy}\right)_0\right.$$
$$\left. + c_0\left(S_x + S_y\right)_0 + \frac{c_0}{g}\Omega\left(u_0 - v_0\right) - \frac{c_0}{g}\left(W_x + W_y\right)_0\right] . \qquad (4.26)$$

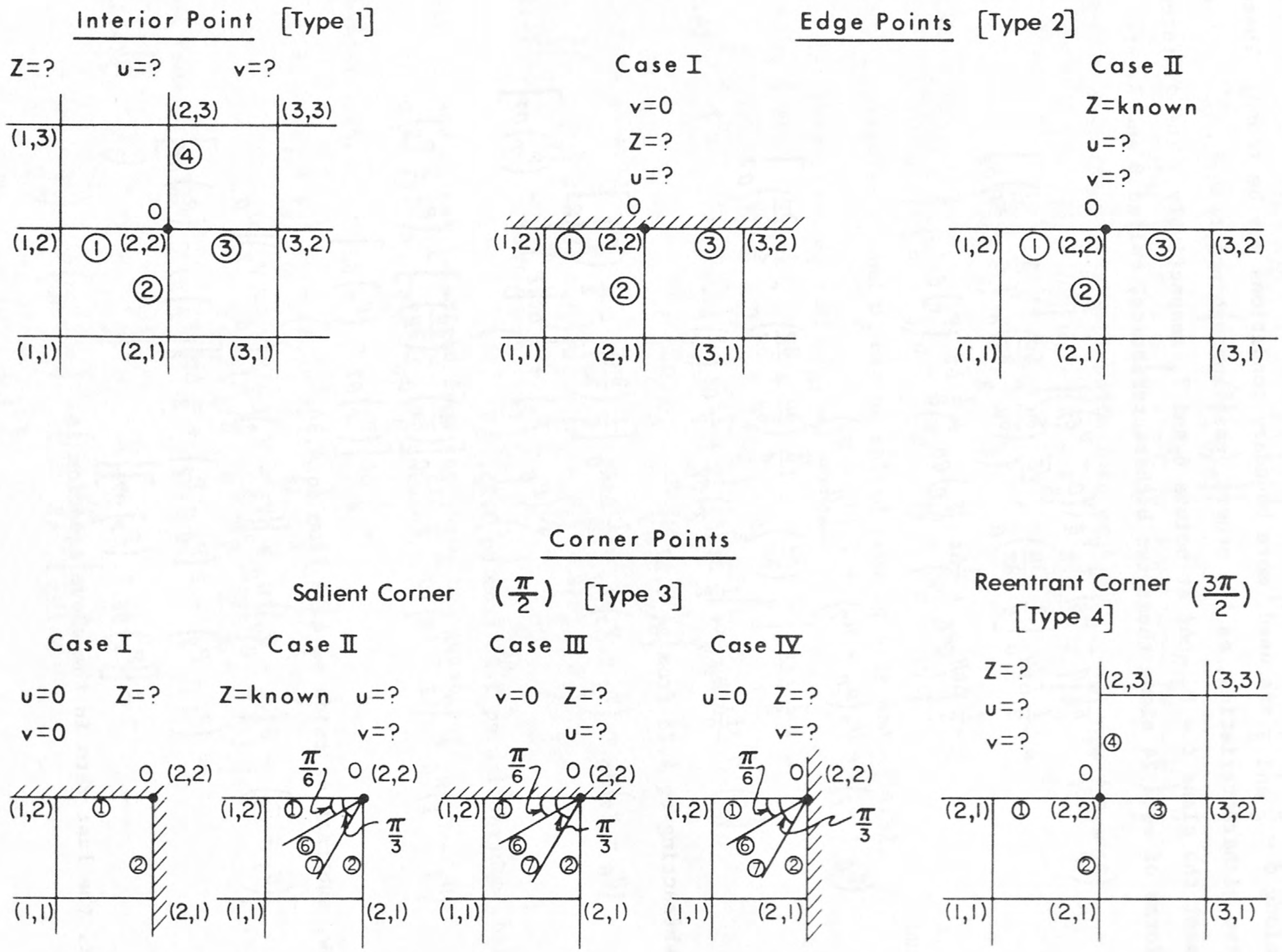


Figure 6. - Four types of grid points used in the schematization of the embayment.

The above equation has two unknowns, u_0 and v_0, and hence, one more equation is necessary for its solution. If two additional bicharacteristics along $\phi = \frac{\pi}{6}$ and $\frac{\pi}{3}$ are used, more boundary conditions may be found. These two bicharacteristics, as the others, pass through point $O(x_0, y_0, t_0)$ and meet the plane $t = t_0 - \Delta t$ at points 6 and 7, respectively. The difference forms of eq 3.24 along these two bicharacteristics, called 6 and 7, are

$$\left(Z_0 - Z_6\right) + U_6\left(u_0 - u_6\right) + V_6\left(v_0 - v_6\right)$$
$$= -\frac{1}{2}\Delta t H_0\left[\frac{1}{4}\left(\frac{\partial u}{\partial x}\right)_0 - \frac{\sqrt{3}}{4}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)_0 + \frac{3}{4}\left(\frac{\partial v}{\partial y}\right)_0\right]$$
$$-\frac{1}{2}\Delta t H_6 p_6 + \frac{1}{2}\Delta t\left(F_r\right)_{06} + \frac{1}{2}\Delta t\left(F_r\right)_6 , \tag{4.27}$$

and

$$\left(Z_0 - Z_7\right) + U_7\left(u_0 - u_7\right) + V_7\left(v_0 - v_7\right)$$
$$= -\frac{1}{2}\Delta t H_0\left[\frac{3}{4}\left(\frac{\partial u}{\partial x}\right)_0 - \frac{\sqrt{3}}{4}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)_0 + \frac{1}{4}\left(\frac{\partial v}{\partial y}\right)_0\right]$$
$$-\frac{1}{2}\Delta t H_7 p_7 + \frac{1}{2}\Delta t\left(F_r\right)_{07} + \frac{1}{2}\Delta t\left(F_r\right)_7 . \tag{4.28}$$

Subtracting eq 4.28 from eq 4.27,

$$\left(U_6 - U_7\right)u_0 + \left(V_6 - V_7\right)v_0 = \frac{1}{2}\Delta t H_0\left[\frac{1}{2}\left(\frac{\partial u}{\partial x}\right)_0 - \frac{1}{2}\left(\frac{\partial v}{\partial y}\right)_0\right]$$
$$+ \left(P_6 - P_7\right) + \frac{1}{2}\Delta t\left[\left(F_r\right)_{06} - \left(F_r\right)_{07}\right]. \tag{4.29}$$

Also, subtracting eq 3.26 from eq 3.25,

$$\left(U_1 - U_2\right)u_0 + \left(V_1 - V_2\right)v_0 = \frac{1}{2}\Delta t H_0\left[\left(\frac{\partial u}{\partial x}\right)_0 - \left(\frac{\partial v}{\partial y}\right)_0\right] + \left(P_1 - P_2\right)$$
$$+ \frac{1}{2}\Delta t\left[\left(F_r\right)_{01} - \left(F_r\right)_{02}\right] . \tag{4.30}$$

Now, subtracting twice eq 4.29 from eq 4.30,

$$\left[\left(U_1 - U_2\right) - 2\left(U_6 - U_7\right)\right]u_0 + \left[\left(V_1 - V_2\right) - 2\left(V_6 - V_7\right)\right]v_0$$
$$= \left(P_1 - P_2\right) - 2\left(P_6 - P_7\right) + \frac{1}{2}\Delta t\left\{\left[\left(F_r\right)_{01} - \left(F_r\right)_{02}\right]\right.$$
$$\left. - 2\left[\left(F_r\right)_{06} - \left(F_r\right)_{07}\right]\right\}. \tag{4.31}$$

But, the last term in the above equation is

$$\frac{1}{2}\Delta t\Big\{-c_0\left(S_x - S_y\right)_0 + \frac{c_0}{g}\Omega\left(v_0 + u_0\right) + \frac{c_0}{g}\left(W_x - W_y\right)_0$$
$$+ c_0\left[\left(\sqrt{3} - 1\right)\left(S_x\right)_0 + \left(1 - \sqrt{3}\right)\left(S_y\right)_0\right] - \frac{c_0}{g}\Omega\left[\left(\sqrt{3} - 1\right)v_0 - \left(1 - \sqrt{3}\right)u_0\right]$$
$$- \frac{c_0}{g}\left[\left(\sqrt{3} - 1\right)\left(W_x\right)_0 + \left(1 - \sqrt{3}\right)\left(W_y\right)_0\right]\Big\}$$
$$= \left(\frac{\sqrt{3}}{2} - 1\right)\Delta t\left[c_0\left(S_x - S_y\right)_0 - \frac{c_0}{g}\Omega\left(u_0 + v_0\right) - \frac{c_0}{g}\left(W_x - W_y\right)_0\right].$$

Hence, substituting this expression into eq 4.31,

$$\left[\left(U_1 - U_2\right) - 2\left(U_6 - U_7\right)\right]u_0 + \left[\left(V_1 - V_2\right) - 2\left(V_6 - V_7\right)\right]v_0$$
$$= \left(P_1 - P_2\right) - 2\left(P_6 - P_7\right)$$
$$- \left(1 - \frac{\sqrt{3}}{2}\right)\Delta t\frac{c_0}{g}\left[g\left(S_x - S_y\right)_0 - \Omega\left(u_0 + v_0\right) - \left(W_x - W_y\right)_0\right]. \qquad (4.32)$$

The two variables u_0 and v_0 can be solved from eq 4.26 and eq 4.32.

c) Case III. $v_0 = 0;$ Z_0, u_0 unknown.

From eq 4.26,

$$Z_0 + \left(U_1 + U_2\right)u_0 = \left(P_1 + P_2 - P_5\right) - \frac{1}{2}\Delta t\left[u_0\left(S_{ox}\right)_0 + c_0\left(S_x\right)_0\right.$$
$$\left. + \frac{c_0}{g}\Omega u_0 - \frac{c_0}{g}\left(W_x + W_y\right)_0\right], \qquad (4.33)$$

and from eq 4.32,

$$\left[\left(U_1 - U_2\right) - 2\left(U_6 - U_7\right)\right]u_0 = \left(P_1 - P_2\right) - 2\left(P_6 - P_7\right)$$
$$- \left(1 - \frac{\sqrt{3}}{2}\right)\Delta t\left[c_0\left(S_x\right)_0 - \frac{c_0}{g}\Omega u_0 - \frac{c_0}{g}\left(W_x - W_y\right)_0\right]. \qquad (4.34)$$

Thus, Z_0 and u_0 can be solved from the above two equations.

d) Case IV. $u_0 = 0;$ Z_0, v_0 unknown.

From eq 4.26,

$$Z_0 + \left(V_1 + V_2\right)v_0 = \left(P_1 + P_2 - P_5\right) - \frac{1}{2}\Delta t\left[v_0\left(S_{oy}\right)_0\right.$$
$$\left. + c_0\left(S_y\right)_0 - \frac{c_0}{g}\Omega v_0 - \frac{c_0}{g}\left(W_x + W_y\right)_0\right], \qquad (4.35)$$

and from eq 4.32,

$$\left[\left(V_1 - V_2\right) - 2\left(V_6 - V_7\right)\right]v_0$$
$$= \left(P_1 - P_2\right) - 2\left(P_6 - P_7\right) + \left(1 - \frac{\sqrt{3}}{2}\right)\Delta t\left[c_0\left(S_y\right)_0\right.$$
$$\left. + \frac{c_0}{g}\Omega v_0 + \frac{c_0}{g}\left(W_x - W_y\right)_0\right]. \qquad (4.36)$$

Again, the above two equations permit the solution of Z_0 and v_0.

The boundary equations for the other three kinds of salient corners can be found in a similar way. They are listed as follows:

B. The 2nd kind: The corner between $\phi = \frac{\pi}{2}$ and $\phi = \pi$;

a) Case I. $u_0 = 0,\ v_0 = 0;$ Z_0 is unknown;

$$Z_0 = \left(P_2 + P_3 - P_5\right) - \frac{1}{2}\Delta t \frac{c_0}{g}\left(W_x - W_y\right)_0 . \tag{4.37}$$

b) Case II. Z_0 known; u_0, v_0 unknown;

$$\left(U_2 + U_3\right)u_0 + \left(V_2 + V_3\right)v_0 = \left(P_2 + P_3 - P_5\right) - Z_0 - \frac{1}{2}\Delta t \left[u_0\left(S_{ox}\right)_0 + v_0\left(S_{oy}\right)_0 - c_0\left(S_x - S_y\right)_0 + \frac{c_0}{g}\Omega\left(u_0 + v_0\right) + \frac{c_0}{g}\left(W_x - W_y\right)_0\right], \tag{4.38}$$

and

$$\left[\left(U_2 - U_3\right) - 2\left(U_8 - U_9\right)\right]u_0 + \left[\left(V_2 - V_3\right) - 2\left(V_8 - V_9\right)\right]v_0 = \left(P_2 - P_3\right) - 2\left(P_8 - P_9\right) - \left(1 - \frac{\sqrt{3}}{2}\right)\Delta t \frac{c_0}{g}\left[g\left(S_x + S_y\right)_0 + \Omega\left(u_0 - v_0\right) - \left(W_x + W_y\right)_0\right] . \tag{4.39}$$

where $\phi_8 = \frac{2\pi}{3}$ and $\phi_9 = \frac{5\pi}{6}$.

c) Case III. $u_0 = 0;$ Z_0, v_0 unknown;

$$Z_0 + \left(V_2 + V_3\right)v_0 = \left(P_2 + P_3 - P_5\right) - \frac{1}{2}\Delta t\left[v_0\left(S_{oy}\right)_0 + c_0\left(S_y\right)_0 + \frac{c_0}{g}\Omega v_0 + \frac{c_0}{g}\left(W_x - W_y\right)_0\right] , \tag{4.40}$$

and

$$\left[\left(V_2 - V_3\right) - 2\left(V_8 - V_9\right)\right]v_0 = \left(P_2 - P_3\right) - 2\left(P_8 - P_9\right) - \left(1 - \frac{\sqrt{3}}{2}\right)\Delta t\left[c_0\left(S_y\right)_0 - \frac{c_0}{g}\Omega v_0 - \frac{c_0}{g}\left(W_x + W_y\right)_0\right] . \tag{4.41}$$

d) Case IV. $v_0 = 0;$ Z_0, u_0 unknown;

$$Z_0 + \left(U_2 + U_3\right)u_0 = \left(P_2 + P_3 - P_5\right) - \frac{1}{2}\Delta t\left[u_0\left(S_{ox}\right)_0 - c_0\left(S_x\right)_0 + \frac{c_0}{g}\Omega u_0 + \frac{c_0}{g}\left(W_x - W_y\right)_0\right], \tag{4.42}$$

and

$$\left[\left(U_2 - U_3\right) - 2\left(U_8 - U_9\right)\right]u_0 = \left(P_2 - P_3\right) - 2\left(P_8 - P_9\right) - \left(1 - \frac{\sqrt{3}}{2}\right)\Delta t\left[c_0\left(S_x\right)_0 + \frac{c_0}{g}\Omega u_0 - \frac{c_0}{g}\left(W_x + W_y\right)_0\right]. \tag{4.43}$$

C. The 3rd kind: The corner between $\phi = \pi$ and $\phi = \frac{3\pi}{2}$;

a) Case I. $u_0 = 0$, $v_0 = 0$; Z_0 is unknown;

$$Z_0 = \left(P_3 + P_4 - P_5\right) - \frac{1}{2}\Delta t\,\frac{c_0}{g}\left(W_x + W_y\right)_0. \tag{4.44}$$

b) Case II. Z_0 known; u_0, v_0 unknown;

$$\left(U_3 + U_4\right)u_0 + \left(V_3 + V_4\right)v_0 = \left(P_3 + P_4 - P_5\right) - Z_0 - \frac{1}{2}\Delta t\left[u_0\left(S_{ox}\right)_0 + v_0\left(S_{oy}\right)_0 - c_0\left(S_x + S_y\right)_0 + \frac{c_0}{g}\Omega\left(v_0 - u_0\right) + \frac{c_0}{g}\left(W_x + W_y\right)_0\right], \tag{4.45}$$

$$\left[\left(U_3 - U_4\right) - 2\left(U_{10} - U_{11}\right)\right]u_0 + \left[\left(V_3 - V_4\right) - 2\left(V_{10} - V_{11}\right)\right]v_0 = \left(P_3 - P_4\right) - 2\left(P_{10} - P_{11}\right) + \left(1 - \frac{\sqrt{3}}{2}\right)\Delta t\,\frac{c_0}{g}\left[g\left(S_x - S_y\right)_0 - \Omega\left(v_0 + u_0\right) - \left(W_x - W_y\right)_0\right], \tag{4.46}$$

where $\phi_{10} = \frac{7}{6}\pi$ and $\phi_{11} = \frac{4}{3}\pi$.

c) Case III. $v_0 = 0$; Z_0, u_0 unknown;

$$Z_0 + \left(U_3 + U_4\right)u_0 = \left(P_3 + P_4 - P_5\right) - \frac{1}{2}\Delta t\left[u_0\left(S_{ox}\right)_0 - c_0\left(S_x\right)_0 - \frac{c_0}{g}\Omega u_0 + \frac{c_0}{g}\left(W_x + W_y\right)_0\right], \tag{4.47}$$

$$\left[\left(U_3 - U_4\right) - 2\left(U_{10} - U_{11}\right)\right] u_0 = \left(P_3 - P_4\right) - 2\left(P_{10} - P_{11}\right) + \left(1 - \frac{\sqrt{3}}{2}\right) \Delta t \left[c_0 \left(S_x\right)_0 - \frac{c_0}{g} \Omega u_0 - \frac{c_0}{g} \left(W_x - W_y\right)_0\right]. \quad (4.48)$$

d) Case IV. $u_0 = 0$; Z_0, v_0 unknown;

$$Z_0 + \left(V_3 + V_4\right) v_0 = \left(P_3 + P_4 - P_5\right) - \frac{1}{2} \Delta t \left[v_0 \left(S_{oy}\right)_0 - c_0 \left(S_y\right)_0 + \frac{c_0}{g} \Omega v_0 + \frac{c_0}{g} \left(W_x + W_y\right)_0\right], \quad (4.49)$$

$$\left[\left(V_3 - V_4\right) - 2\left(V_{10} - V_{11}\right)\right] v_0 = \left(P_3 - P_4\right) - 2\left(P_{10} - P_{11}\right) - \left(1 - \frac{\sqrt{3}}{2}\right) \Delta t \left[c_0 \left(S_y\right)_0 + \frac{c_0}{g} \Omega v_0 + \frac{c_0}{g} \left(W_x - W_y\right)_0\right]. \quad (4.50)$$

D. The 4th kind: The corner between $\phi = \frac{3}{2}\pi$ and 2π;

a) Case I. $u_0 = 0$, $v_0 = 0$; Z_0 unknown;

$$Z_0 = \left(P_4 + P_1 - P_5\right) + \frac{1}{2} \Delta t \frac{c_0}{g} \left(W_x - W_y\right)_0 . \quad (4.51)$$

b) Case II. Z_0 known; u_0, v_0 unknown;

$$\left(U_4 + U_1\right) u_0 + \left(V_4 + V_1\right) v_0 = \left(P_4 + P_1 - P_5\right) - Z_0 - \frac{1}{2} \Delta t \left[u_0 \left(S_{ox}\right)_0 + v_0 \left(S_{oy}\right)_0 + c_0 \left(S_x - S_y\right)_0 - \frac{c_0}{g} \Omega \left(u_0 + v_0\right) - \frac{c_0}{g} \left(W_x - W_y\right)_0\right], \quad (4.52)$$

$$\left[\left(U_4 - U_1\right) - 2\left(U_{12} - U_{13}\right)\right] u_0 + \left[\left(V_4 - V_1\right) - 2\left(V_{12} - V_{13}\right)\right] v_0 = \left(P_4 - P_1\right) - 2\left(P_{12} - P_{13}\right) + \left(1 - \frac{\sqrt{3}}{2}\right) \Delta t \frac{c_0}{g} \left[g \left(S_x + S_y\right)_0 + \Omega \left(v_0 - v_0\right) \left(W_x + W_y\right)_0\right]. \quad (4.53)$$

where $\phi_{12} = \frac{5}{3}\pi$ and $\phi_{13} = \frac{11}{6}\pi$.

c) Case III. $u_0 = 0$; Z_0, v_0 unknown;

$$Z_0 + \left(V_4 + V_1\right)v_0 = \left(P_4 + P_1 - P_5\right) - \frac{1}{2}\Delta t\left[v_0\left(S_{oy}\right)_0 - c_0\left(S_y\right)_0 - \frac{c_0}{g}\,\Omega v_0 - \frac{c_0}{g}\left(W_x - W_y\right)_0\right], \tag{4.54}$$

$$\left[\left(V_4 - V_1\right) - 2\left(V_{12} - V_{13}\right)\right] v_0 = \left(P_4 - P_1\right) - 2\left(P_{12} - P_{13}\right) + \left(1 - \frac{\sqrt{3}}{2}\right)\Delta t\left[c_0\left(S_y\right)_0 - \frac{c_0}{g}\,\Omega v_0 - \frac{c_0}{g}\left(W_x + W_y\right)_0\right]. \tag{4.55}$$

d) Case IV. $v_0 = 0$; Z_0, u_0 unknown;

$$Z_0 + \left(U_4 + U_1\right)u_0 = \left(P_4 + P_1 - P_5\right) - \frac{1}{2}\,\Delta t\left[u_0\left(S_{ox}\right)_0 + c_0\left(S_x\right)_0 - \frac{c_0}{g}\,\Omega u_0 - \frac{c_0}{g}\left(W_x - W_y\right)_0\right], \tag{4.56}$$

$$\left[\left(U_4 - U_1\right) - 2\left(U_{12} - U_{13}\right)\right] u_0 = \left(P_4 - P_1\right) - 2\left(P_{12} - P_{13}\right) + \left(1 - \frac{\sqrt{3}}{2}\right)\Delta t\left[c_0\left(S_x\right)_0 + \frac{c_0}{g}\,\Omega u_0 - \frac{c_0}{g}\left(W_x + W_y\right)_0\right]. \tag{4.57}$$

Reentrant Corners

As shown in figure 6, the four bicharacteristics, $\phi = 0, \frac{\pi}{2}, \pi, \frac{3}{2}\pi$, will be all available at the reentrant corner (270° angle), the equations derived in the preceding chapter, eq 3.25 through eq 3.28, and eq 3.32, are also usable in the flow computation here. Because there are only three rectangles within the region on the $t = t_0 - \Delta t$ plane and one of the nine grid points (1,2), (1,2), ..., (3,3) is missing, variables and parameters at points (x_i, y_i) have to be derived from the values at the remaining eight points.

Oblique Boundary

When a wall boundary is not parallel to x- or y- axis, the problem may be handled by setting a new co-ordinate system, x' and y', oriented parallel and perpendicular to the wall. The unknown boundary values are evaluated for points along the edge, using the boundary equations previously derived. The values thus obtained are then matched with the original x-y system through the co-ordinate transformation.

For example, if the wall makes an angle α with the x-axis, as shown in figure 7, then the following relationships hold:

$$u' = u\cos\alpha + v\sin\alpha, \tag{4.58}$$

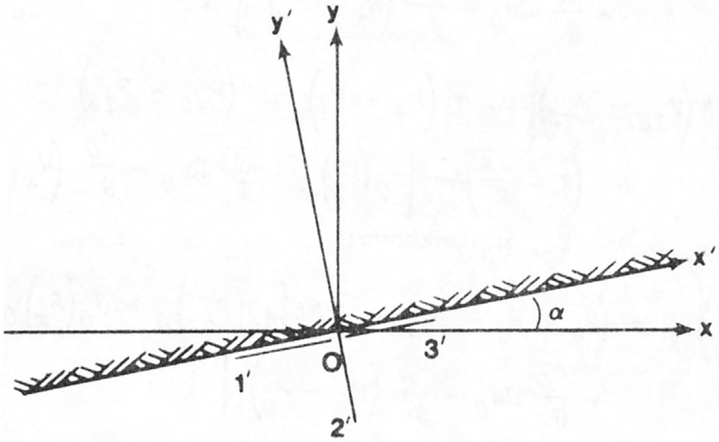


Figure 7. - The oblique boundary.

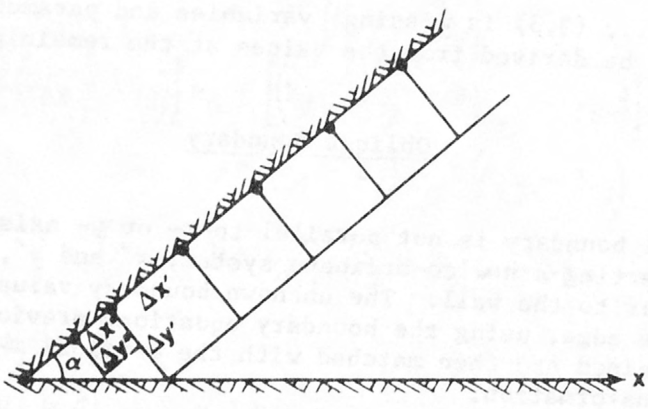


Figure 8. - Grids near an oblique solid-wall corner.

$$v' = -u \sin\alpha + v \cos\alpha, \tag{4.59}$$

$$Z' = Z, \tag{4.60}$$

in which the prime refers to the $x' - y'$ co-ordinate system. The boundary equations at point 0 then become

$$2Z_0 + (U_1' + 2U_2' + U_3')u_0' = (P_1' + 2P_2' + P_3' - 2P_5') - \Delta t\left[u_0'(S_{ox'})_0 + \frac{c_0}{g}\Omega u_0' - \frac{c_0}{g}(W_{y'})_0\right], \tag{4.61}$$

$$(U_1' - U_3')u_0' = (P_1' - P_3') - \Delta t\left[c_0(S_{x'})_0 - \frac{c_0}{g}(W_{x'})_0\right], \tag{4.62}$$

in which

$$U_i' = \frac{1}{2g}\left(c_0 \cos\phi_i' + c_i' \cos\theta_i'\right)$$

and

$$V_i' = \frac{1}{2g}\left(c_0 \sin\phi_i' + c_i' \sin\theta_i'\right),$$

for $i = 1', 2', 3', 4'$,

$$\phi_1', \phi_2', \ldots = 0, \frac{\pi}{2}, \ldots$$

and $\phi_1, \phi_2, \ldots = \alpha, \alpha + \frac{\pi}{2}, \ldots$.

It is not necessary to consider an oblique condition for the open-water boundary, because the boundary line in the open water is an arbitrary one and one can always choose a boundary parallel to x- or y- axis.

Near a solid-wall corner, the above described computation may be carried out up to the last rectangle of $(\Delta x')(\Delta y')$ that is still within the flow domain. Points closer to the corner than the last point computed can be handled by the use of smaller rectangles with Δt reduced accordingly, (see figure 8). The size of supplemental grids should be chosen in such a way that an integral multiple of the smaller Δt meets with the original Δt.

At the corner point, which is a stagnition point, u_0 and v_0 are both zero, and Z_0 may be approximated by extrapolating the neighboring points when they are sufficiently close.

Conceptual Structure of the Mathematical Model

Based on the mathematical method and the boundary conditions described in the foregoing chapters computer programs can be written and assembled to form a mathematical model which should be capable of simulating the unsteady shallow-water flows. The entire embayment, estuary, or other water body in which the two-dimensional flow is to be simulated, is first covered by an x-y grid system, and the stage, velocity components and other parameters are then calculated at each grid point from the given initial and boundary data. The same procedures are repeated on each time level using the proceding results as knowns, thus advancing stepwise in t-direction with the interval Δt.

The mathematical model consists of four active subroutines, two supervisory programs, a few supporting subroutines including boundary value handling, and a number of lower-level subprograms which take care of many detail terms and factors in the computation. The conceptual structure of the model is illustrated in figure 9.

The four "active" or "operational" subroutines are the programs that actually perform the flow computation for each grid point for each time step. The first subroutine computes the flow for the interior points, the second one for the edge points, the third for the salient corner points, and the fourth for the re-entrant corner points. The oblique boundary points were not treated in this first stage of programming. Figure 6 displays the diagram of those grid points that are handled by the above four subroutines. The diagram shows the position of each grid point type in the x-y rectangular meshes, nature of boundaries, knowns and unknowns at the grid points, and sets of bicharacteristics to be used in the flow computations. Only one orientation (1st kind) is shown for each boundary point, because other orientations (2nd, 3rd, and 4th kinds) are simple rotations of this diagram.

The four subroutines are controlled and called by a managerial subroutine which, in turn, is supervised by the main program. The managerial subroutine manages the computational process for the entire flow region for one time step, Δt. The managerial subroutine receives instructions and information on the current time level from the main program, it receives appropriate boundary values on the advanced time level from one of the supporting subroutines, and processes the flow computation to find unknowns of the advanced time level. Computations are carried out point by point on the x-y grid according to the grid-point type. Thus, at each grid point of the computational region, one of the four active subroutiens must be called on duty to perform the actual compuations. At the end of one time-step computation, the program transposes the computed results from the advanced time level to the current time level and returns the control to the main program.

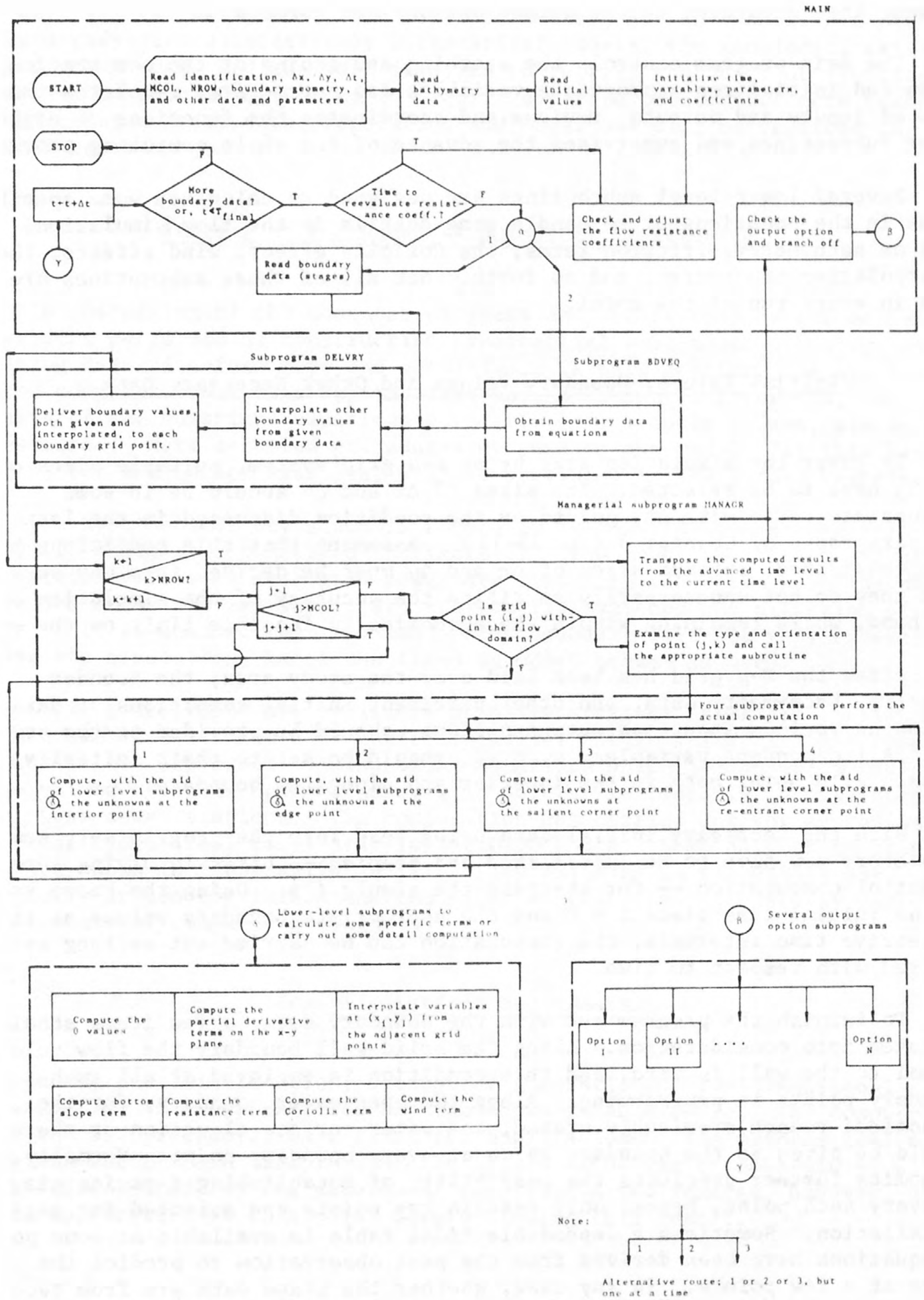


Figure 9. - Schematic flow diagram and conceptual organization chart of the computer program system.

Supporting subroutines organize the input data and allocate them to proper boundary points, and organize and interpret the computed results and present them according to the output options and formats.

The main program controls the starting and ending of the computation, reads and initializes appropriate variables and parameters, regulates the flow of inputs and outputs, reviews and coordinates the functions of higher-order subroutines, and supervises the advance of the whole computing process.

Several lower-level subroutines are designed to calculate some specific terms in the equations or to handle some details in the flow simulation, such as bathymetry, friction terms, the Coriolis effect, wind effects, the interpolation procedures, and so forth. Not all of these subroutines are used in every run of the model.

Initial Values, Boundary Values and Other Necessary Data

To cover the simulation area by an x-y grid system, suitable sizes of Δx and Δy have to be selected. The sizes of Δx and Δy should be in some appropriate ratio with Δt, guided by the condition discussed in the last few paragraphs of Chapter 3 (p. 33-35). Assuming that this conditions has been considered, optimum sizes of Δx and Δy must be decided in a way such that they do not unnecessarily sacrifice the accuracy of the simulation on one hand, while remaining within the economically feasible limit on the other.

After the x-y grid has been laid over the study area, the boundary geometry, bathymetry data, and other pertinent initial conditions or data, including various input and output options, should be provided to the program set. All dependent variables, u, v, Z, should be set to their initial values at each grid point both in the interior and along the boundary.

With the necessary initial data being read into the program set, some parameters now have to be initialized and readied -- often involving some amount of computation -- for starting the simulation. Using the known values on the initial time plane $t = 0$ and the appropriate boundary values at the successive time intervals, the computation can be carried out as long as desired with respect to time.

To furnish the program set with the boundary data, a few items should be taken into consideration. Along the solid-wall boundary the flow velocity normal to the wall is zero, and this condition is employed at all such boundary points in programming. Along the open-water boundary, for the economical reason previously stated, the water-surface elevation or the stage should be given as the boundary value at every boundary point. Normally, economics further precludes the possibility of establishing a gaging station at every such point, hence, only certain key points are selected for gage installation. Sometimes a dependable tidal table is available at some points, or equations have been derived from the past observation to predict the stage at a few points. In any case, whether the stage data are from recording gages or from other sources, such data are available only at a limited number of grid points along the water boundary. The rest have to be interpolated from these key points.

Although the collection, organization and preparation of data for boundary values, bathymetry, and other parameters are an inseparable part of the unsteady-flow simulation by mathematical models, the knowledge, skill, and experience needed, and the work involved to fulfill that part of simulation are often of such an extent as to require separate treatment. More description and discussion about data preparation and model use will be reported in further reports.

Testing of the Model -- Numerical Experiments

The debugging of the computer programs and the testing of the mathematical model were performed by constructing hypothetical embayments or using some existing data of actual bays and estuaries. In this section, the application of the computer model to a hypothetical bay, as shown in figure 10, and the results of the numerical experiments, mostly in a graphical form, are briefly illustrated. More detailed procedures of running the model, for this bay as well as for some other examples, will be presented in the future reports.

The hypothetical bay shown in figure 10 is covered by a square grid of 12 by 11, which is schematized in figure 11. The type, the bottom elevation and initial values of each grid point are listed in table 1. All land points, which are above the sea level, are arbitrarily assigned the elevation of 100 meters. It is assumed that the land-water boundary will not vary excessively during the simulation, hence the fixed boundary scheme is used.

Stage hydrographs are available at seven stations as marked in figure 10 and are numbered 1 through 7 counterclockwise. Stage values in this computer simulation are expressed by sinusoidal equations which are shown in table 2. Stage values of other open-boundary grid points are linearly interpolated from these seven stations. The computation was carried out for approximately one tidal cycle (real time) starting at 0900 hours. Some outputs in graphical form are shown in figures 12a to 12i. Flow patterns in the entire bay, especially around the island and the river mouths were successfully depicted with the aid of computer graphics.

Some Limitations and Remarks

In the preceding section, a computer simulation of two-dimensional unsteady flow in a hypothetical bay was shown as an example. The computational algorithm and procedure, as well as the application of the mathematical model using the above-shown example and other field estuaries or embayments, will be described in detail in the subsequent reports. A few remarks, however, are deemed appropriate at this point before closing this report.

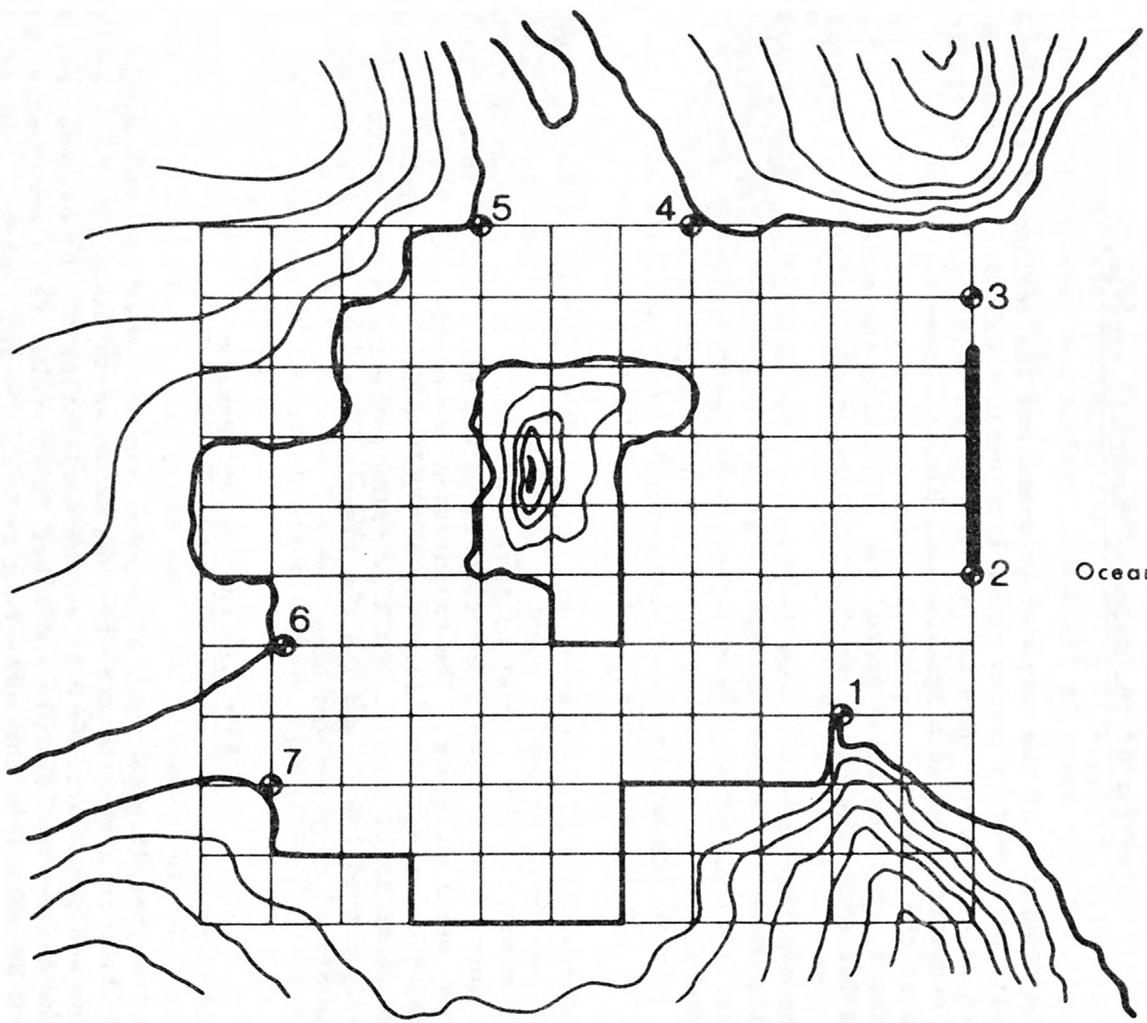


Figure 10. - A hypothetical bay.

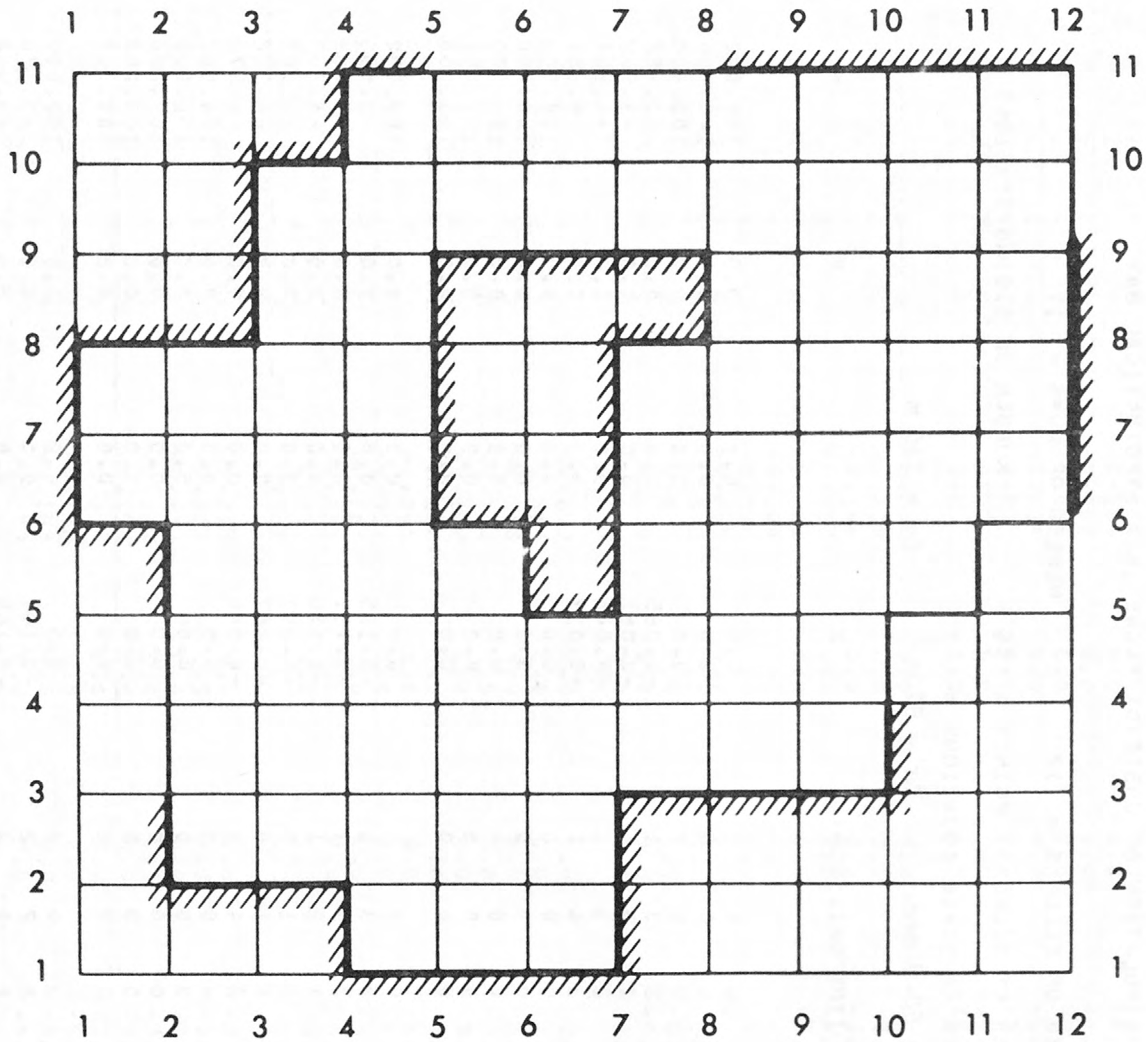


Figure 11. - The schematization of a hypothetical bay shown in figure 10.

Table 1. Initial data for flow computation in a hypothetical bay.

COMPUTATION OF UNSTEADY FLOWS IN A BAY, HARBOR, INLET, OR TIDAL ESTUARY

COMPUTER SIMULATION OF UNSTEADY FLOWS IN A HYPOTHETICAL BAY

NUMBER OF COLUMNS = 12 NUMBER OF ROWS = 11

NUMBER OF BOUNDARY POINTS = 56 NUMBER OF GAGING STATIONS = 0

NUMBER OF STAGE EQUATIONS = 7

Δt = 60.0 sec. Δx = 610m Δy = 610 m

ROW	COL	TYPE	ORI.	BDRY	Z	U	V	ZB	FRC
1									
	1	0	0	0	0.0	0.0	0.0	100.000	0.0
	2	0	0	0	0.0	0.0	0.0	100.000	0.0
	3	0	0	0	0.0	0.0	0.0	100.000	0.0
	4	3	3	1	-0.085	0.0	0.0	-3.962	0.0350
	5	2	3	1	-0.061	0.0	0.0	-3.962	0.0350
	6	2	3	1	-0.046	0.0	0.0	-4.267	0.0340
	7	3	4	1	-0.037	0.0	0.0	-4.267	0.0340
	8	0	0	0	0.0	0.0	0.0	100.000	0.0
	9	0	0	0	0.0	0.0	0.0	100.000	0.0
	10	0	0	0	0.0	0.0	0.0	100.000	0.0
	11	0	0	0	0.0	0.0	0.0	100.000	0.0
	12	0	0	0	0.0	0.0	0.0	100.000	0.0
2									
	1	0	0	0	0.0	0.0	0.0	100.000	0.0
	2	3	3	1	-0.165	0.0	0.0	-3.658	0.0400
	3	2	3	1	-0.137	0.0	0.0	-3.962	0.0390
	4	4	2	1	-0.110	0.0	0.0	-4.267	0.0380
	5	1	1	0	-0.091	0.0	0.0	-4.267	0.0370
	6	1	1	0	-0.067	0.0	0.0	-4.572	0.0360
	7	2	4	1	-0.046	0.0	0.0	-4.572	0.0350
	8	0	0	0	0.0	0.0	0.0	100.000	0.0
	9	0	0	0	0.0	0.0	0.0	100.000	0.0
	10	0	0	0	0.0	0.0	0.0	100.000	0.0
	11	0	0	0	0.0	0.0	0.0	100.000	0.0
	12	0	0	0	0.0	0.0	0.0	100.000	0.0
3									
	1	0	0	0	0.0	0.0	0.0	100.000	0.0
	2	2	2	2	-0.197	0.0	0.0	-3.658	0.0400
	3	1	1	0	-0.168	0.0	0.0	-3.962	0.0400
	4	1	1	0	-0.137	0.0	0.0	-4.267	0.0390
	5	1	1	0	-0.110	0.0	0.0	-4.572	0.0390
	6	1	1	0	-0.085	0.0	0.0	-4.877	0.0380
	7	4	3	1	-0.061	0.0	0.0	-4.877	0.0370
	8	2	3	1	-0.046	0.0	0.0	-4.877	0.0360

Table 1. - Initial data for flow computation in a hypothetical bay. - Continued

ROW	COL	TYPE	ORI.	BDRY	Z (m)	U (m/s)	V (m/s)	ZB (m)	FRC "n"
	9	2	3	1	-0.030	0.0	0.0	-5.182	0.0350
	10	3	4	1	-0.015	0.0	0.0	-5.182	0.0350
	11	0	0	0	0.0	0.0	0.0	100.000	0.0
	12	0	0	0	0.0	0.0	0.0	100.000	0.0
4									
	1	0	0	0	0.0	0.0	0.0	100.000	0.0
	2	2	2	2	-0.213	0.0	0.0	-3.962	0.0380
	3	1	1	0	-0.183	0.0	0.0	-4.267	0.0380
	4	1	1	0	-0.152	0.0	0.0	-4.572	0.0370
	5	1	1	0	-0.122	0.0	0.0	-4.877	0.0370
	6	1	1	0	-0.104	0.0	0.0	-4.877	0.0360
	7	1	1	0	-0.082	0.0	0.0	-4.877	0.0360
	8	1	1	0	-0.061	0.0	0.0	-5.182	0.0350
	9	1	1	0	-0.034	0.0	0.0	-5.486	0.0350
	10	2	4	2	0.0	0.0	0.0	-5.486	0.0340
	11	0	0	0	0.0	0.0	0.0	100.000	0.0
	12	0	0	0	0.0	0.0	0.0	100.000	0.0
5									
	1	0	0	0	0.0	0.0	0.0	100.000	0.0
	2	2	2	1	-0.230	0.0	0.0	-3.658	0.0400
	3	1	1	0	-0.198	0.0	0.0	-3.962	0.0390
	4	1	1	0	-0.152	0.0	0.0	-4.267	0.0380
	5	1	1	0	-0.122	0.0	0.0	-4.572	0.0370
	6	4	4	1	-0.104	0.0	0.0	-4.572	0.0370
	7	4	1	1	-0.076	0.0	0.0	-4.572	0.0370
	8	1	1	0	-0.055	0.0	0.0	-4.877	0.0360
	9	1	1	0	-0.030	0.0	0.0	-5.486	0.0360
	10	4	3	2	0.0	0.0	0.0	-5.791	0.0350
	11	3	4	2	0.030	0.0	0.0	-5.791	0.0340
	12	0	0	0	0.0	0.0	0.0	100.000	0.0
6									
	1	3	3	1	-0.198	0.0	0.0	-3.048	0.0400
	2	4	2	1	-0.213	0.0	0.0	-3.353	0.0400
	3	1	1	0	-0.213	0.0	0.0	-3.353	0.0390
	4	1	1	0	-0.183	0.0	0.0	-3.962	0.0380
	5	4	4	1	-0.152	0.0	0.0	-3.658	0.0380
	6	3	1	1	-0.122	0.0	0.0	-3.658	0.0380
	7	2	2	1	-0.046	0.0	0.0	-4.572	0.0370
	8	1	1	0	-0.091	0.0	0.0	-4.877	0.0360
	9	1	1	0	-0.107	0.0	0.0	-5.182	0.0360
	10	1	1	0	-0.061	0.0	0.0	-5.486	0.0350
	11	4	3	2	0.030	0.0	0.0	-5.486	0.0340
	12	3	4	2	0.062	0.0	0.0	-5.486	0.0340
7									
	1	2	2	1	-0.213	0.0	0.0	-2.743	0.0400
	2	1	1	0	-0.213	0.0	0.0	-3.048	0.0390
	3	1	1	0	-0.213	0.0	0.0	-3.048	0.0380
	4	1	1	0	-0.229	0.0	0.0	-3.658	0.0380
	5	2	4	1	-0.198	0.0	0.0	-3.353	0.0370
	6	0	0	0	0.0	0.0	0.0	100.000	0.0
	7	2	2	1	-0.122	0.0	0.0	-4.267	0.0370
	8	1	1	0	-0.183	0.0	0.0	-4.572	0.0360
	9	1	1	0	-0.183	0.0	0.0	-4.877	0.0360
	10	1	1	0	-0.152	0.0	0.0	-4.877	0.0350
	11	1	1	0	-0.091	0.0	0.0	-5.182	0.0350
	12	2	4	1	-0.030	0.0	0.0	-4.877	0.0340
8									
	1	3	2	1	-0.229	0.0	0.0	-2.743	0.0400
	2	2	1	1	-0.244	0.0	0.0	-2.743	0.0400
	3	4	1	1	-0.259	0.0	0.0	-2.743	0.0390
	4	1	1	0	-0.274	0.0	0.0	-3.048	0.0390

Table 1. - Initial data for flow computation in a hypothetical bay. - Continued

ROW	COL	TYPE	ORI.	BDRY	Z (m)	U (m/s)	V (m/s)	ZB (m)	FRC "n"
	5	2	4	1	-0.259	0.0	0.0	-3.353	0.0380
	6	0	0	0	0.0	0.0	0.0	100.000	0.0
	7	3	2	1	-0.213	0.0	0.0	-3.962	0.0370
	8	4	1	1	-0.305	0.0	0.0	-4.267	0.0360
	9	1	1	0	-0.274	0.0	0.0	-4.572	0.0360
	10	1	1	0	-0.213	0.0	0.0	-4.572	0.0350
	11	1	1	0	-0.152	0.0	0.0	-4.877	0.0350
	12	2	4	1	-0.122	0.0	0.0	-4.572	0.0360
9									
	1	0	0	0	0.0	0.0	0.0	100.000	0.0
	2	0	0	0	0.0	0.0	0.0	100.000	0.0
	3	2	2	1	-0.305	0.0	0.0	-3.048	0.0390
	4	1	1	0	-0.335	0.0	0.0	-3.353	0.0380
	5	4	3	1	-0.335	0.0	0.0	-3.353	0.0380
	6	2	3	1	-0.366	0.0	0.0	-3.658	0.0370
	7	2	3	1	-0.381	0.0	0.0	-3.962	0.0370
	8	4	2	1	-0.366	0.0	0.0	-3.962	0.0360
	9	1	1	0	-0.320	0.0	0.0	-4.267	0.0360
	10	1	1	0	-0.290	0.0	0.0	-4.572	0.0360
	11	1	1	0	-0.259	0.0	0.0	-4.877	0.0360
	12	2	4	1	-0.229	0.0	0.0	-4.877	0.0360
10									
	1	0	0	0	0.0	0.0	0.0	100.000	0.0
	2	0	0	0	0.0	0.0	0.0	100.000	0.0
	3	3	2	1	-0.366	0.0	0.0	-3.048	0.0390
	4	4	1	1	-0.396	0.0	0.0	-3.353	0.0380
	5	1	1	0	-0.396	0.0	0.0	-3.658	0.0380
	6	1	1	0	-0.427	0.0	0.0	-3.962	0.0370
	7	1	1	0	-0.442	0.0	0.0	-4.267	0.0360
	8	1	1	0	-0.427	0.0	0.0	-4.267	0.0360
	9	1	1	0	-0.396	0.0	0.0	-4.572	0.0350
	10	1	1	0	-0.366	0.0	0.0	-4.877	0.0350
	11	1	1	0	-0.351	0.0	0.0	-4.877	0.0350
	12	2	4	2	-0.344	0.0	0.0	-5.182	0.0350
11									
	1	0	0	0	0.0	0.0	0.0	100.000	0.0
	2	0	0	0	0.0	0.0	0.0	100.000	0.0
	3	0	0	0	0.0	0.0	0.0	100.000	0.0
	4	3	2	1	-0.469	0.0	0.0	-3.353	0.0380
	5	2	1	2	-0.465	0.0	0.0	-3.658	0.0370
	6	2	1	2	-0.466	0.0	0.0	-3.658	0.0360
	7	2	1	2	-0.466	0.0	0.0	-3.962	0.0350
	8	2	1	2	-0.468	0.0	0.0	-3.962	0.0360
	9	2	1	1	-0.396	0.0	0.0	-3.962	0.0360
	10	2	1	1	-0.381	0.0	0.0	-4.267	0.0360
	11	2	1	1	-0.335	0.0	0.0	-4.572	0.0360
	12	3	1	3	-0.305	0.0	0.0	-4.877	0.0360

Note: ZB = Z_b, FRC = Manning *n*, ORI. = Orientation, COL = Column, BDRY = Boundary case;

The metric units in this table were converted from the customary units in the original table.

TABLE 2 Boundary values for flow computation in a hypothetical bay.

Boundary value equation:

$$Z = a \sin \left[\frac{2\pi(t-\zeta)}{T}\right] + M,$$

in which

a = amplitude,

ζ = phase lag,

M = mean water level,

t = time, in seconds, and

T = tidal period (in this example, T = 12.4 hr = 44,640 s).

Sta. No.	a (m)	ζ (s)	M (m)
1	1.524	0	0
2	1.509	60	0
3	1.478	180	0
4	1.448	1200	0.091
5	1.433	1320	0.122
6	1.372	1800	0.213
7	1.356	1860	0.229

Note: The metric units in this table were converted from the customary units in the original table.

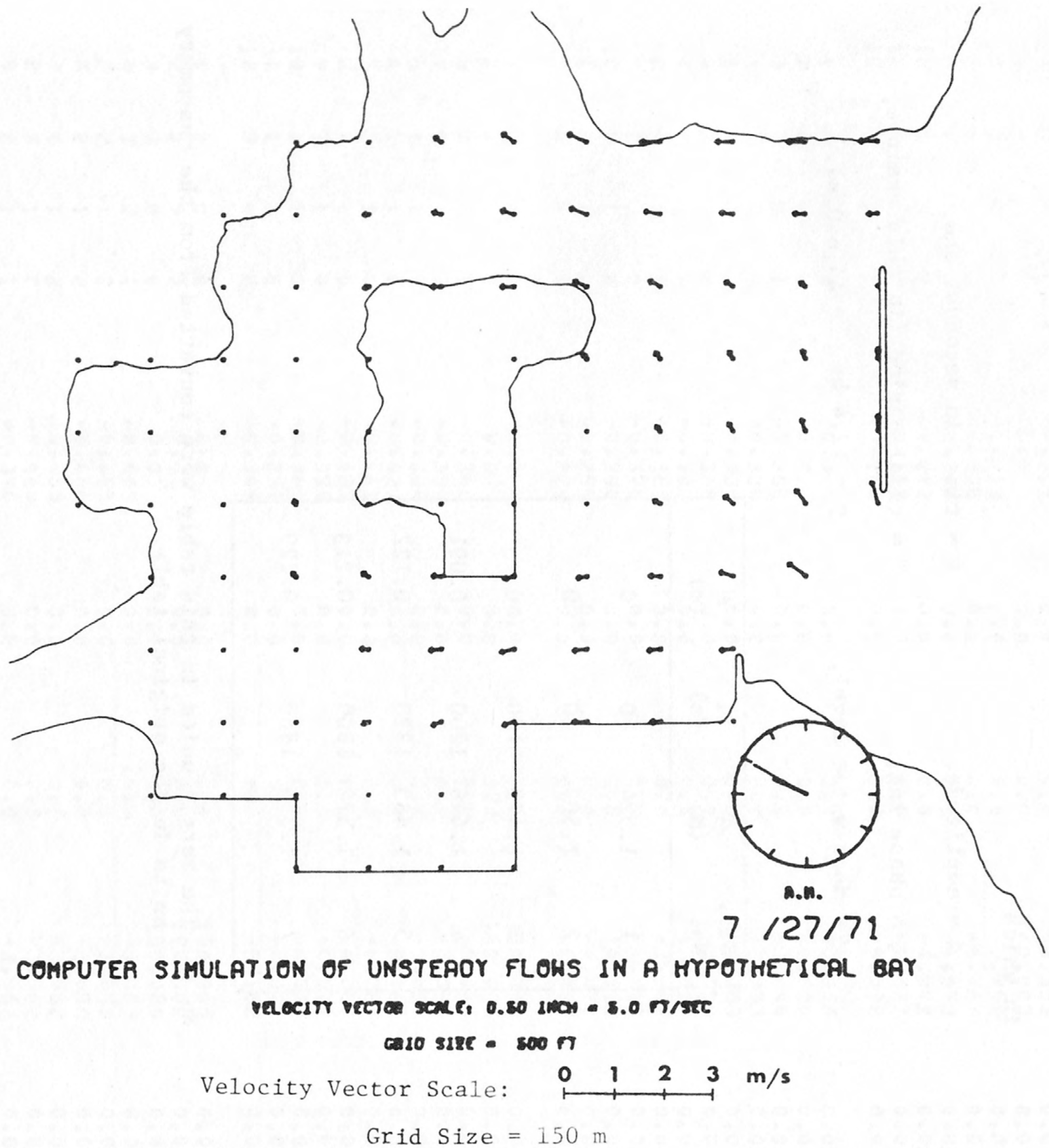


Figure 12a. - Sample computer outputs of flow simulation in a hypothetical bay.

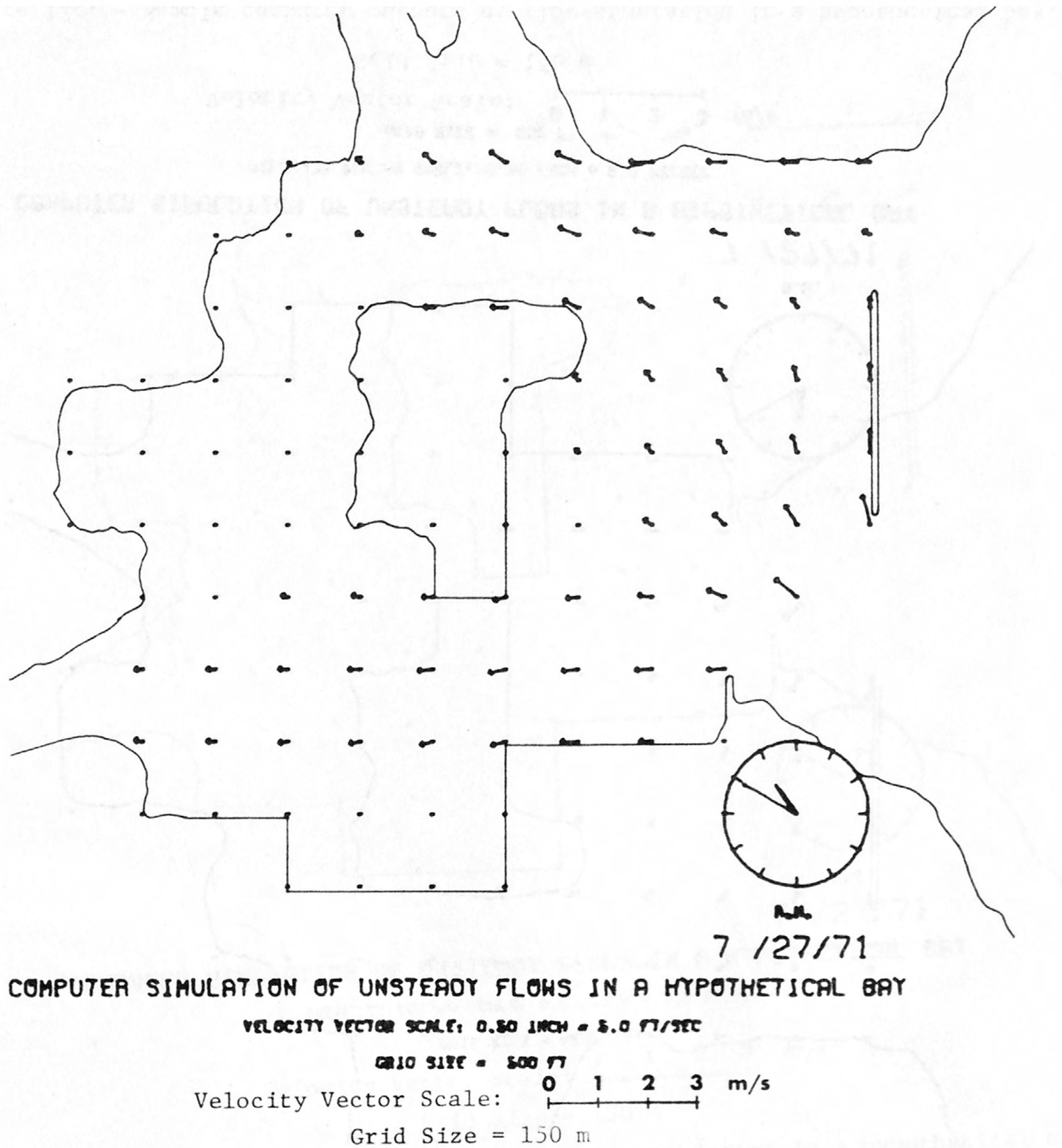
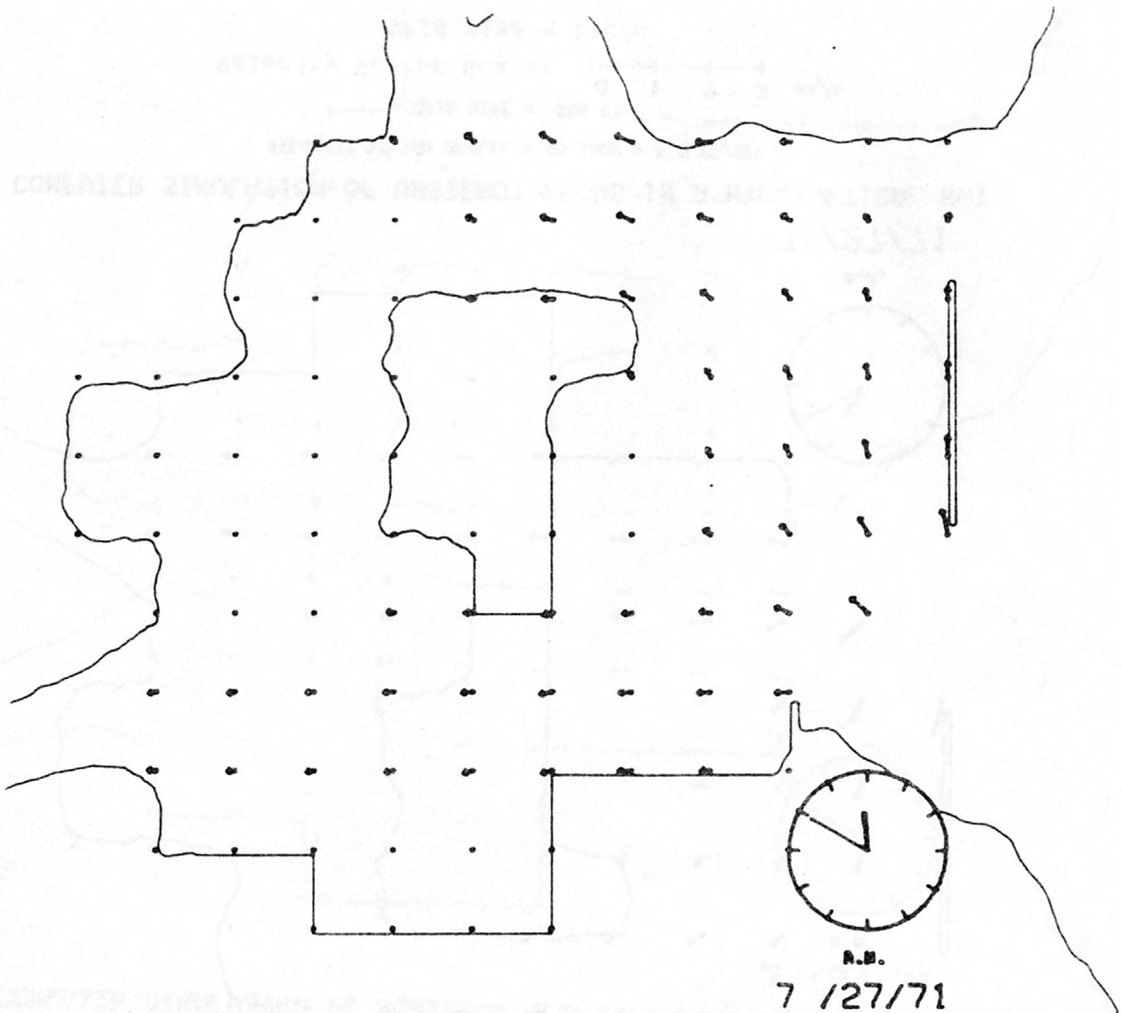


Figure 12b. - Sample computer outputs of flow simulation in a hypothetical bay.



COMPUTER SIMULATION OF UNSTEADY FLOWS IN A HYPOTHETICAL BAY

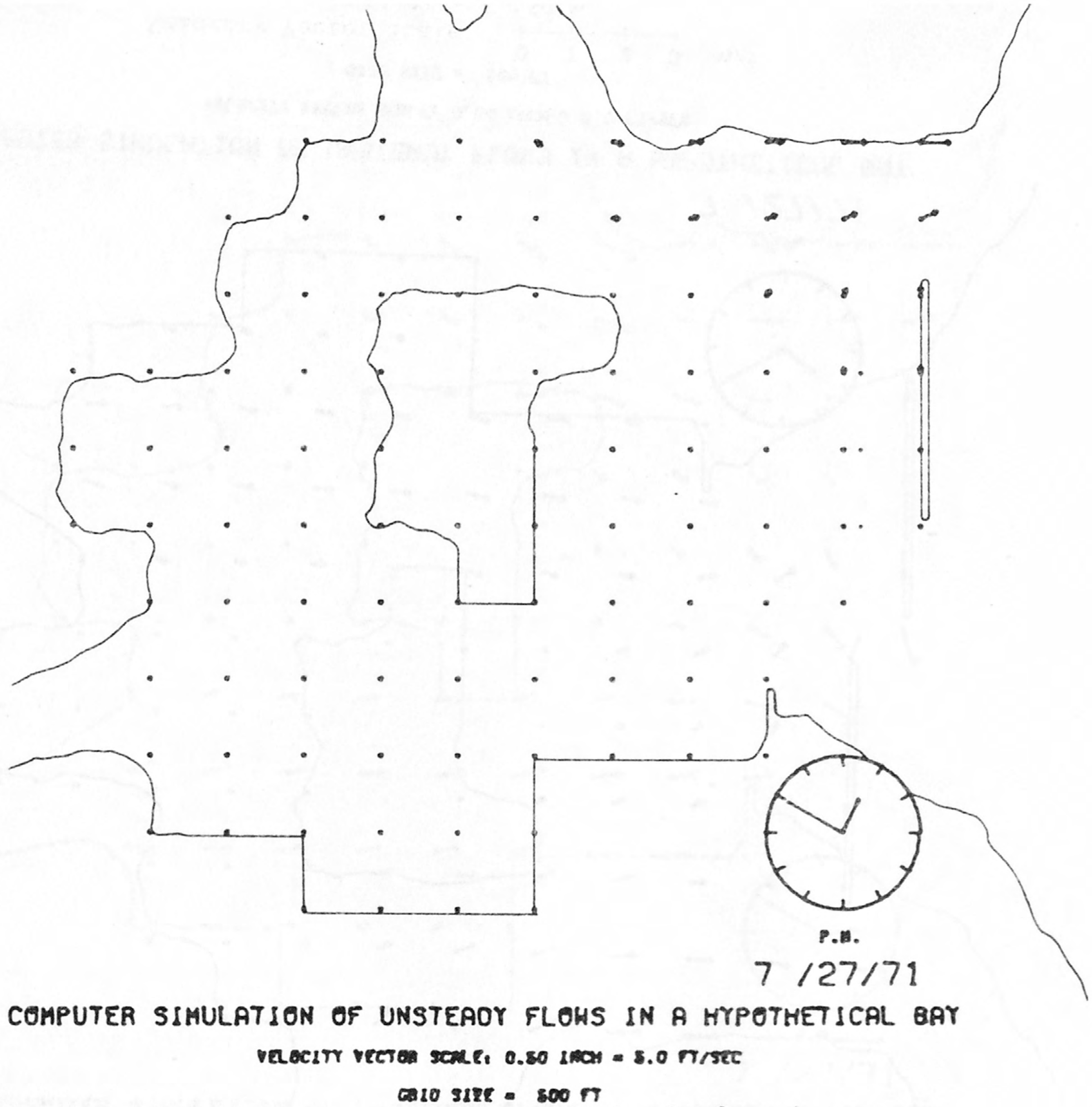
VELOCITY VECTOR SCALE: 0.50 INCH = 5.0 FT/SEC

GRID SIZE = 500 FT

Velocity Vector Scale: 0 1 2 3 m/s

Grid Size = 150 m

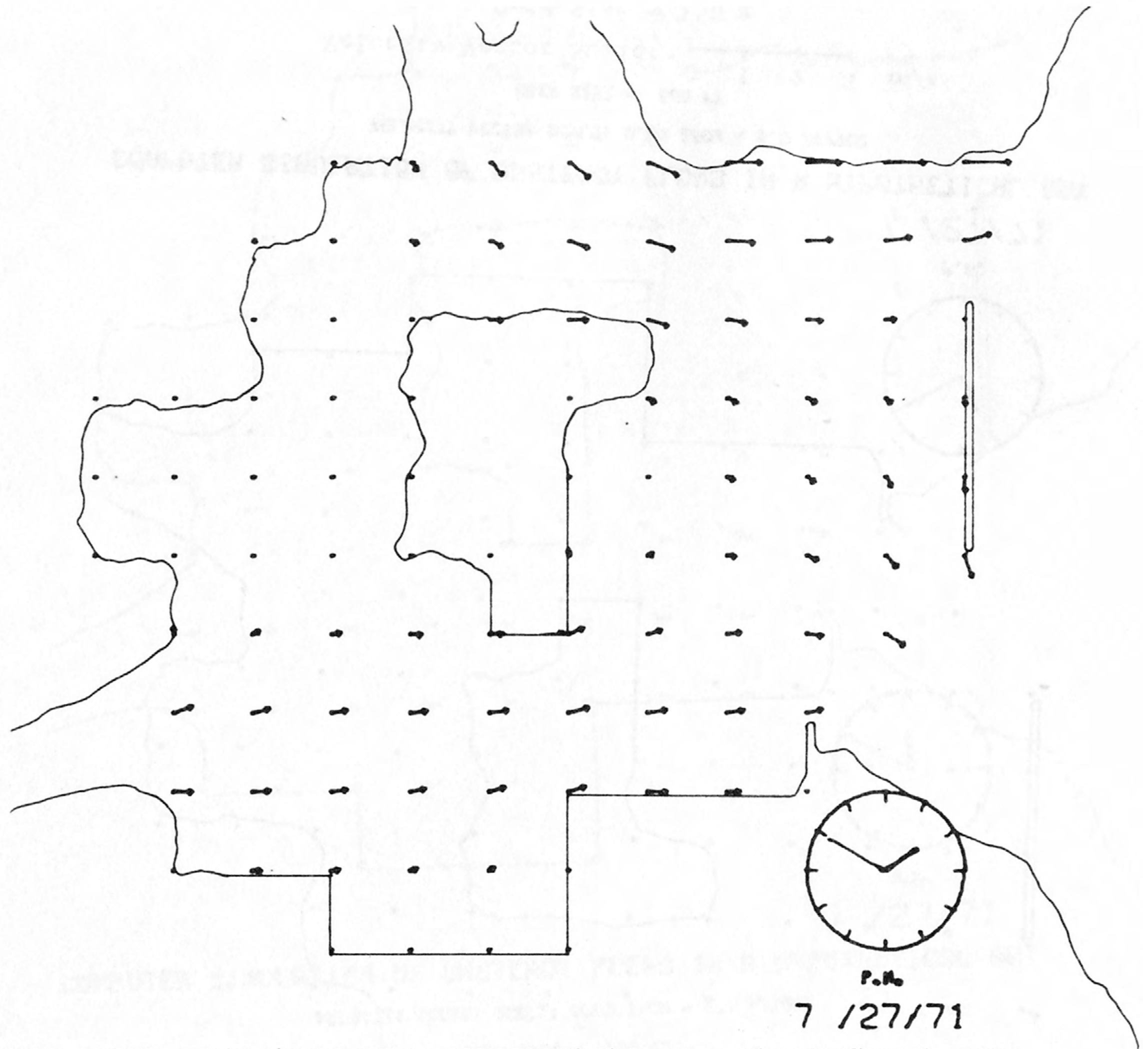
Figure 12c. - Sample computer outputs of flow simulation in a hypothetical bay.



Velocity Vector Scale: 0 1 2 3 m/s

Grid Size = 150 m

Figure 12d. - Sample computer outputs of flow simulation in a hypothetical bay.



COMPUTER SIMULATION OF UNSTEADY FLOWS IN A HYPOTHETICAL BAY

VELOCITY VECTOR SCALE: 0.50 INCH = 5.0 FT/SEC

GRID SIZE = 500 FT

Velocity Vector Scale: 0 1 2 3 m/s

Grid Size = 150 m

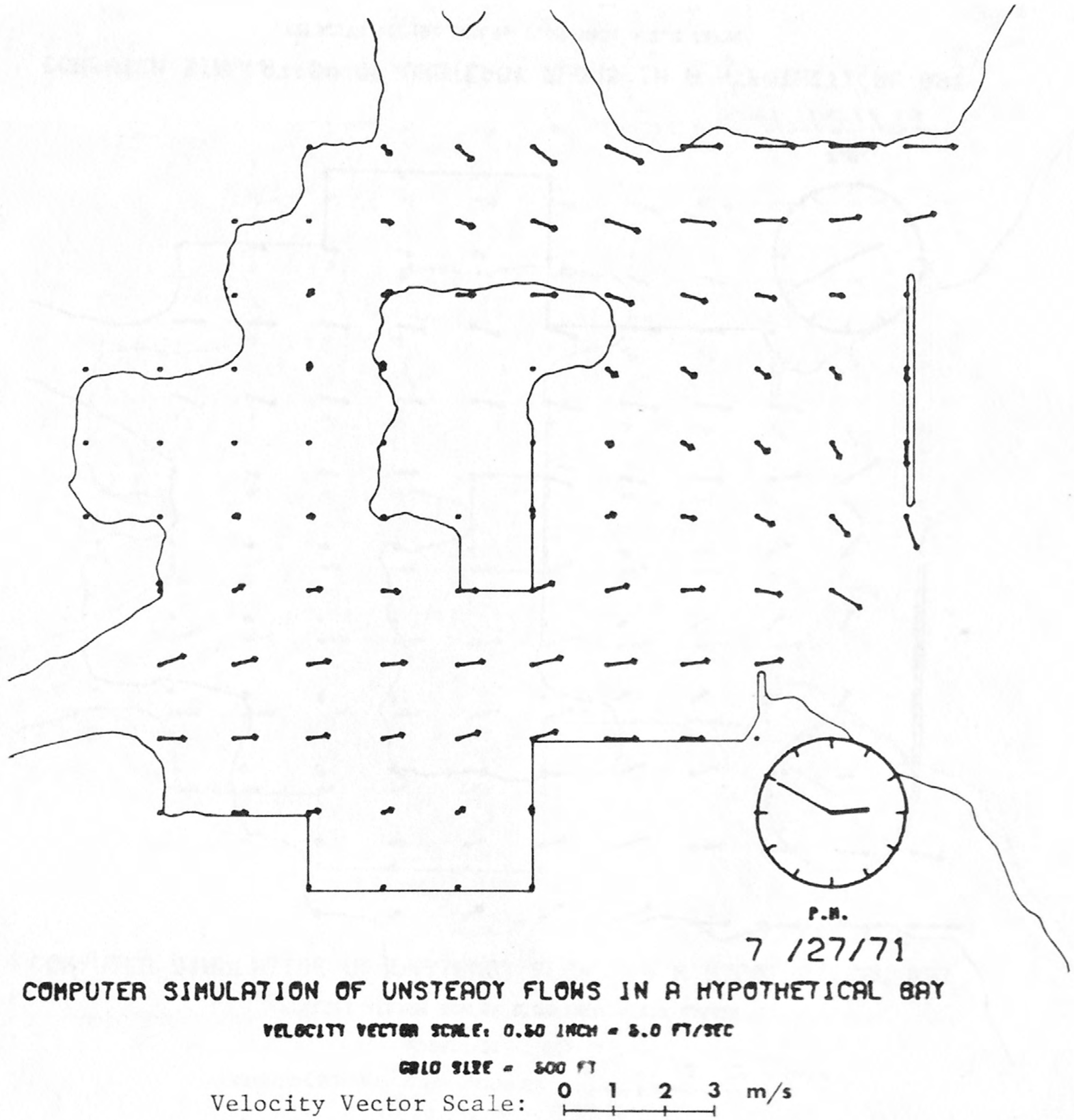
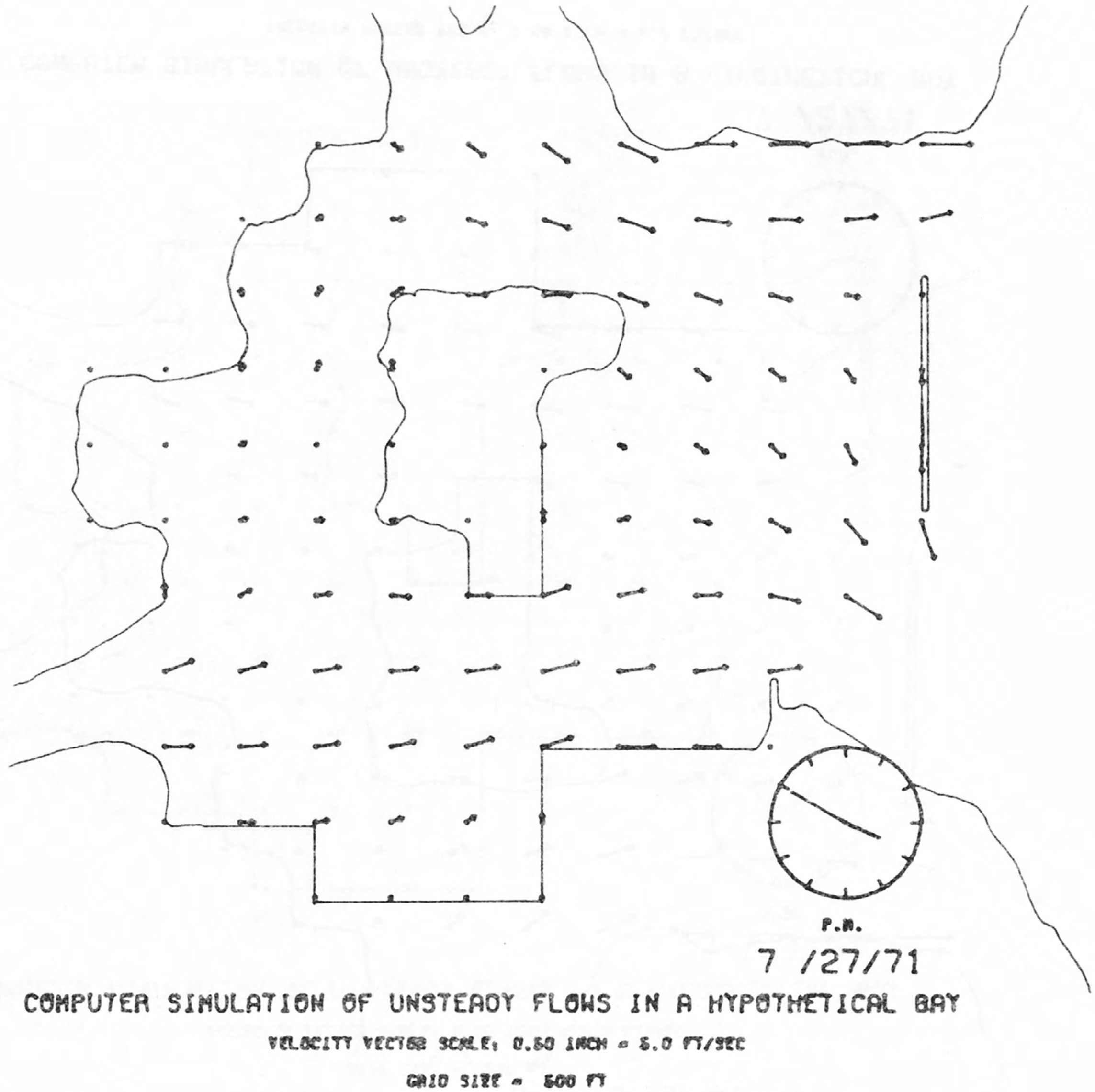


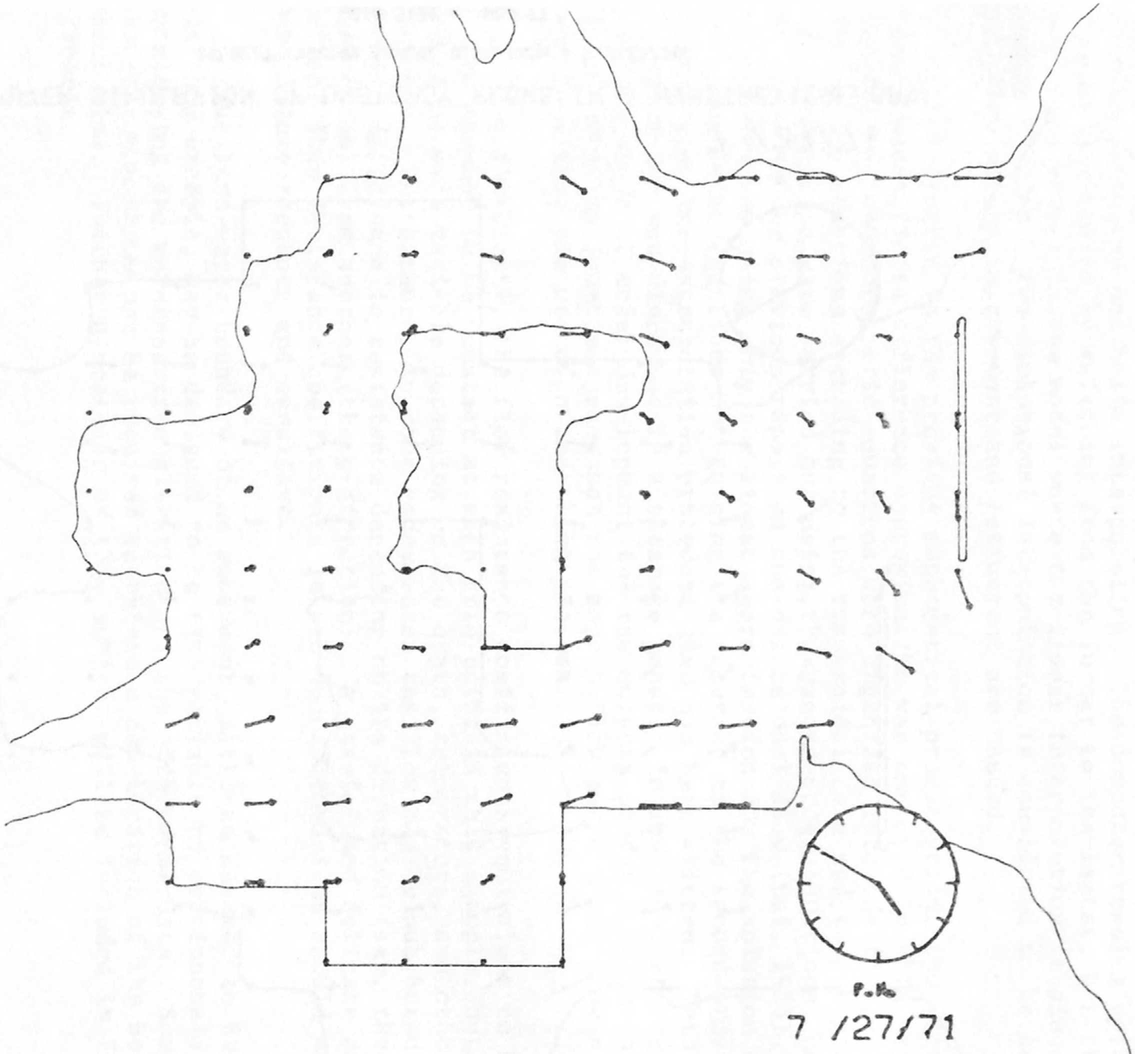
Figure 12f. - Sample computer outputs of flow simulation in a hypothetical bay.



Velocity Vector Scale: 0 1 2 3 m/s

Grid Size = 150 m

Figure 12g. - Sample computer outputs of flow simulation in a hypothetical bay.



COMPUTER SIMULATION OF UNSTEADY FLOWS IN A HYPOTHETICAL BAY

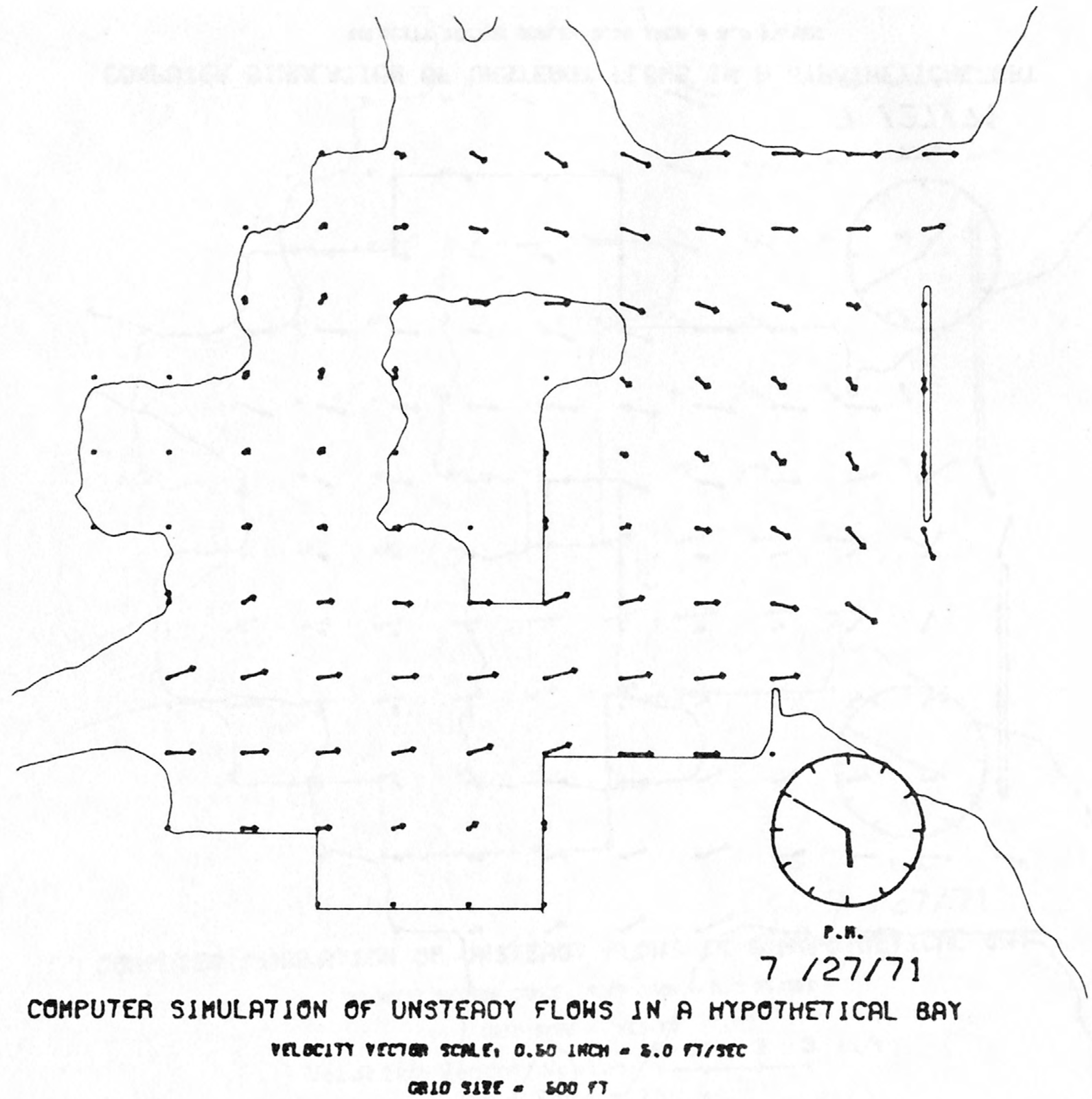
VELOCITY VECTOR SCALE: 0.50 INCH = 5.0 FT/SEC

GRID SIZE = 500 FT

Velocity Vector Scale: 0 1 2 3 m/s

Grid Size = 150 m

Figure 12h. - Sample computer outputs of flow simulation in a hypothetical bay.



Velocity Vector Scale: 0 1 2 3 m/s

Grid Size = 150 m

Figure 12i. - Sample computer outputs of flow simulation in a hypothetical bay.

In the earlier versions of this model, the variables and parameters at the point of intersection (x_i, y_i)[as defined in eq 3.44, eq 3.45, eq 3.60, and eq 3.61], where the bicharacteristics or the streamline meets the current time plane, were evaluated by linear interpolation from the corresponding values at the adjacent grid points, and in the latter versions by using parabolic interpolation. The computer results were generally improved by switching from the former to the latter, but there were many points in the model where the linear interpolation retained better results. Two-dimensional interpolation is considered to be an area in which future improvement and refinement are needed.

As indicated in the previous mathematical presentation, the direct second-order finite difference approximation was employed in this report; namely, the characteristic equations were organized into the finite difference equations according to the trapezoidal rule and the programming and computation were carried out using the second-order equations. However, as shown in the previous report on one-dimensional flow (Lai, 1965), a program system using only the linear approximation for the solution of the characteristic equations but raising its accuracy to the second-order by applying the extrapolation procedure also has been written. This will be presented and discussed in a separate paper. In any case, at the present stage of model development the raw outputs produced by any of the above versions sometimes show not too smooth flow patterns at some sites and requiring the use of a smoothing process.

For simplicity, the flow resistance coefficients equivalent to Manning n are assumed to be constant at each grid point in this example, but they could be made variable depending on the depth, temperature, and other factors. Furthermore, in some embayments, the flow may be subjected to marked difference in resistance depending on its direction (say, the x-direction) from another (the y-direction). A careful and delicate adjustment of the flow resistance coefficients in two-space dimensions should make the model more accurate and sensitive.

The land-water boundary of an embayment, although assumed to be fixed in this example, may be designed to be time variable by continuously monitoring the water-surface elevation and the bathymetry data. Some cut-and-try procedures may be required to obtain a new location of the boundary each time. Further discussion of this subject will be included in future reports.

SUMMARY AND CONCLUSIONS

The basic partial differential equations for two-dimensional unsteady flows in wide estuaries and embayments are of the hyperbolic type, and can be solved numerically by the method of characteristics with the aid of a digital computer. A set of computer programs to solve unsteady flows may be written and organized to form a mathematical model from which various complex two-dimensional tidal flows can be simulated.

For the solution of the two-dimensional unsteady flow by the method of characteristics, a numerical scheme was applied based on specified time intervals, similar to that used by Lai (1965) for one-dimensional flow solution. The basic partial differential equations, consisting of one equation of continuity and two equations of motion, are first transformed into characteristic equations along bicharacteristics, and then to corresponding difference equations using the trapezoidal rule.

Unlike the situation for the one-dimensional flow case, the characteristic equations and their finite difference forms in the two-dimensional flow do not entirely reduce to a total differential or difference form. Rather they consist of two parts -- the first representing totdal differentiation along the bicharacteristic and the second representing partial differential in space. By appropriate combination of the finite difference equations, however, it is possible to eliminate the partial derivative terms lying on the advanced time plane, thus reducing to a set amenable to numerical solution.

For the computation of each interior point, four bicharacteristics and one streamline, all passing through the same point on the advanced time plane, are used for the numerical solution. The treatment of boundary points are manifold, more difficult, and cumbersome, requiring special combinations of the bicharacteristics and the streamline. The boundary points can be classified into three types, points along the edges, points at salient-angle corners, and points at reentrant-angle corners. Not only does the type of boundary point necessitate different combinations of the bicharacteristics, but the variation of boundary conditions, such as closed-wall or open-water, the orientation of the boundary, and the combination of known (given or obtainable) and unknown boundary values, calls for a different assortment of boundary equations.

To carry out the numerical simulation of the two-dimensional unsteady flow in a water body, the study area is covered by an $x - y$ grid system, and the grid points are then classified into type, boundary conditions and orientation if applicable. The computation can be started by appropriate initial values and carried out to the desired length of time by suitable boundary-value data. The sample computer run shown in this report outlines the procedures for model operation, displays some of the obtainable output information, and indicates the workability of the mathematical method and the function of the model.

The results of flow simulation studies demonstrate the reasonableness of the physical principles and the mathematical method, the feasibility of the numerical procedures and computer algorithm, and the workability of the digital model as developed in this report. The usefulness of the model and its potential for complete implementation and full operation should be further investigated by more extensive numerical experiments and field applications.

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