F. R. Bennett

Total Bed-Material Discharge in Alluvial Channels

GEOLOGICAL SURVEY WATER-SUPPLY PAPER 1498-I

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By F. M. CHANG, D. B. SIMONS, *and* E. V. RICHARDSON

STUDIES OF FLOW IN ALLUVIAL CHANNELS

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SYMBOLS

Symbol	*Definition*	*Dimension*
A_1	Integration constant	-------
A_3	Volume constant of particle	-------
a	Thickness of contact-bed-material layer	L
b_e	Apparent thickness of contact-bed-material layer (only static force is considered)	L
C_a	Concentration at distance a from the bed	F/L^3
C_m	Mean concentration of contact-bed-material layer	F/L^3
c	Concentration in dry weight per unit volume at distance y from the bed	F/L^3
c_b	Weight of contact bed material in a unit volume of the water-bed-material mixture	F/L^3
c_s	Weight of suspended bed material in a unit volume of the water-bed-material mixture	F/L^3
D	Depth of flow	L
d	Bed-material diameter	L
d_e	Equivalent diameter (definition in the text, p. I 14)	L
d_{50}	Median diameter	L
e	Porosity of bed material	-------
$\bar{F}$	Vector notation of force acting on particle	F
g	Gravitational acceleration	L/T^2
I_1	Integral value	-------
I_2	Integral value	-------
j	Experimental coefficient, $j=a/b_c$	-------
K_b	$M \cdot K_{be}$	-------
K_{be}	Contact-bed-material coefficient	-------
K_T	Total bed-material discharge coefficient	-------
l	Mixing length	L
M	Coefficient $M=M_1 \cdot M_2$	-------
M_1	Sediment-discharge distribution coefficient	-------
M_2	Ratio of concentrations between contact-bed-material layer and bed material in stationary bed	-------
N	Number of particles	-------
n	Number of moving layers in contact-bed-material layer	-------
p	Percentage by weight of the bed material with diameter d	-------
p_t	Total percentage by weight of the bed material	-------
q_b	Contact-bed-material discharge per unit width of channel	F/TL
q_b'	Contact-bed-material discharge per unit area	F/TL^2
q_s	Suspended-bed-material discharge per unit width of channel	F/TL
q_T	Total bed-material discharge per unit width of channel	F/TL
R_s	Ratio of suspended-bed- to contact-bed-material discharge	-------
r_v	Ratio of the velocity at $\xi=\xi_a$ to the mean velocity of the fluid	-------
S	Slope of channel	-------
U	Mean velocity of flow	L/T
U_b	Mean velocity of contact-bed-material layer	L/T

Symbol	*Definition*	*Dimension*
U_m	Velocity at water surface	L/T
U_*	Shear velocity	L/T
u	Velocity at distance y from the bed	L/T
u'	Fluctuating component of velocity in the x direction	L/T
u_b	Actual velocity of the contact bed material	L/T
u_s	Actual velocity of the suspended bed material	L/T
$\overline{V}$	Vector notation of the relative velocity between particle and fluid	L/T
v'	Fluctuating component of velocity in the y direction	L/T
W	Work done by the relative motion between particle and fluid	FL
W_b	Work done by bed material	FL
W_f	Work done by fluid	FL
y	Distance from the bed surface	L
z	Exponent for suspended-bed-material distribution	-------
α	Slope of bed (angle)	-------
β	Ratio of the sediment transfer coefficient to the diffusion coefficient of fluid	-------
γ	Specific weight of fluid	F/L^3
γ_s	Specific weight of bed material	F/L^3
$\Delta\gamma$	Submerged specific weight of bed material	-------
ϵ	Efficiency	-------
ϵ_o	Diffusion coefficient of the fluid	-------
ϵ_s	Sediment transfer coefficient	-------
κ	von Karman's coefficient	-------
μ	Dynamic viscosity of fluid	FT/L^2
ν	Kinematic viscosity of fluid	L^2/T
ξ	Relative distance from bed surface $\xi=\frac{y}{D}$	-------
ρ	Density of fluid	FT^2/L^4
ρ_s	Density of bed material	FT^2/L^4
τ	Shear stress	F/L^2
τ_c	Critical tractive force	F/L^2
τ_o	Shear stress at the boundary (tractive force)	F/L^2
ϕ	Angle of internal friction of the submerged bed material	-------
ω	Fall velocity	L/T
ω_e	Equivalent fall velocity	L/T

STUDIES OF FLOW IN ALLUVIAL CHANNELS

TOTAL BED-MATERIAL DISCHARGE IN ALLUVIAL CHANNELS

By F. M. Chang, D. B. Simons, and E. V. Richardson

ABSTRACT

This is a study of total bed-material discharge (wash load excluded) in alluvial channels, based partly on existing theories.

The contact-bed-material discharge was obtained by utilizing the energy-work relation of the fluid and the bed material. An equation for velocity distribution was obtained by integrating the Reynolds equation through use of the Prandtl's hypothesis of mixing length.

Through application of the basic equation for the distribution of suspended bed material by M. P. O'Brien, the suspended-bed-material discharge was investigated in terms of contact-bed-material discharge. Then, the total bed-material discharge was obtained simply by adding the contact-bed-material discharge and the suspended-bed-material discharge.

The results were checked with available laboratory and field data and appeared to be mutually consistent and satisfactory.

INTRODUCTION

SCOPE

No purely theoretical approach to the general principles of sediment discharge in alluvial channels seems possible at present because of insufficient quantitative knowledge of fluid turbulence and its effect on sediment within the fluid. Hence, the approach used in this report is semitheoretical and is based on existing theory of turbulent flow and on general ideas derived from observation and reasonable speculation. This study concerns itself principally with—

1. The classification of the graded sands which comprise the bed material of natural alluvial channels.
2. The development of the velocity distribution, which includes consideration of the body force of the flow.
3. The subdivision of the total bed-material discharge into components based upon significant forces to which the particles are subjected.
4. Relations for estimating suspended-bed-material discharge, contact-bed-material discharge, and hence, total bed-material discharge.

5. Evaluation of the formula on total bed-material discharge by use of laboratory and field data.

The flume data (Simons, 1961b) used for the investigation were collected by the U.S. Geological Survey at Fort Collins, Colo., from a recirculating flume 8 feet wide, 150 feet long, and 2 feet deep, with a discharge capacity ranging from 0 to 22 cubic feet per second and an adjustable slope ranging from 0 to 1.5 percent. The flow investigations covered phenomena ranging from a plane bed without sediment movement to antidunes. The particle-size distributions for the bed material in the investigations are presented in figure 1. A sand bed 0.5 to 0.7 foot in depth was placed in the flume.

The kinds of data obtained for each equilibrium were: water-surface slope, water-sediment discharge, water temperature, depth, average velocity, velocity profiles, concentration of total bed-material discharge, concentration of fine-sediment discharge when fine sediment had been added to the flow, suspended-bed-material concentration, characteristics of the bed material, bed configuration, kinematic viscosity, apparent kinematic viscosity of the water–fine-sediment mixture, and descriptions and photographs of the water surface and corresponding bed configuration.

Field data given by Bishop (1961) for the following streams also were used for this investigation: (1) Colorado River—Taylor's Ferry

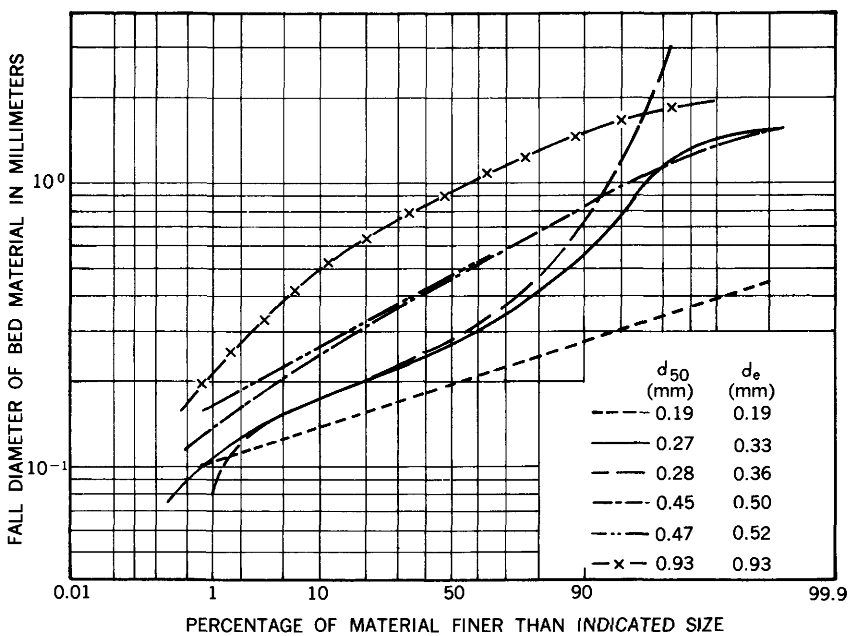

FIGURE 1.—Bed-material size distribution of sands for flume data.

sampling station, (2) Middle Loup River, and (3) Niobrara River (gaging section). These data were also tabulated by Chang (1962).

ACKNOWLEDGMENTS

The authors wish to express their appreciation for the cooperation afforded by the team of the U.S. Geological Survey and the faculty members of the Civil Engineering Department at Colorado State University. Many thanks are especially due Mr. C. F. Nordin, Jr., U.S. Geological Survey, and Dr. M. L. Albertson, Professor of Civil Engineering, Colorado State University, for their critical review and valuable comments.

ANALYSIS OF SEDIMENT SIZE

Natural alluvial bed-material particles have a wide range of physical properties, especially of size, shape, and density. Therefore, the determination of a representative mean diameter for a graded bed material is uncertain because ease of computation rather than theory have dictated procedure. Numerous methods have been used to define a representative mean diameter. These may be divided into two groups: (a) methods based on physical size and (b) methods based on fall velocity. The second method was developed because the physical size of a sediment particle is not an adequate measure of the behavior of the particle moving in a fluid (U.S. Inter-Agency Committee on Water Resources, 1958). The fall velocity of the individual particle in quiet water is a more fundamental measure of the sedimentation characteristics of the particle. However, fall velocity alone is not a complete measure of the size analysis for particles transported in a moving, turbulent fluid. Because the fluid is the main source of power for sediment transport, it is natural to consider that there must be a relation between energy dissipation of the fluid and sediment transport. Hence, the energy value of a particle is considered essential in the analysis of sediment size.

If a particle moves in a fluid with relative velocity $\overline{V}_1$ in the direction of flow against a resistant force $\overline{F}_1$, the rate of work done by the particle is $W_1=\overline{F}_1 \cdot \overline{V}_1$. If another particle moves in the fluid so that its rate of doing work is the same ($W_2=\overline{F}_2 \cdot \overline{V}_2=W_1$), then the second particle is assumed to be dynamically equivalent to the first one. From the standpoint of sediment transport, these two particles are considered equivalent in size. Consider the representative mean size for a group of particles of different sizes. The product of the number of the particle N and the rate of work done by a particle with representative size must be equal to the summation of the rate of work done by all particles. Mathematically, the equation may be written as

$$N(\overline{F} \cdot \overline{V})_{\text{representative}}=\sum_{i=1}^{N}(\overline{F}_i \cdot \overline{V}_i) \tag{1}$$

Unfortunately, the relative velocity, parallel to the bed, between a particle and the fluid in an open channel cannot be precisely determined at present because the velocity of the particle in the horizontal direction is not easy to measure. However, as Bagnold (1956) stated, the relative velocity parallel to the bed should be proportional to vertical relative velocity (fall velocity). Hence, the following analysis is based on fall velocity ω instead of relative velocity parallel to the bed. The force F acting on a particle is

$$F = A_3 d^3 (\rho_s - \rho) g \tag{2}$$

where A_3 is a volume constant of the particle, d is the diameter of particle, ρ_s and ρ are the densities of particle and fluid, respectively, and g is the gravitational acceleration. Further, the rate of work done W is

$$W = F\omega = A_3 d^3 (\rho_s - \rho) g \omega. \tag{3}$$

The equivalent diameter, d_e, is defined as the diameter of a particle which may represent a group of particles wherein the product of the number of the particles and the rate of work done by a particle having this equivalent diameter is equal to the summation of the rate of work done by all the particles.

The equivalent diameter for a group of particles having similar shapes but different sizes can be obtained as follows:

$$N A_3 d_e^3 (\rho_s - \rho) g \omega_e = \sum_{i=1}^{N} A_3 d_i^3 (\rho_s - \rho) g \omega_i \tag{4}$$

or

$$N d_e^3 \omega_e = \sum_{i=1}^{N} d_i^3 \omega_i. \tag{5}$$

For the size analysis of graded bed material, the sample is divided into subgroups by size, and each subgroup is expressed as a percent, p, by weight of the total sample. Then the equivalent fall velocity can be computed as follows:

$$\omega_e = \sum_{i=1}^{p_t} p_i \omega_i \Big/ \sum_{i=1}^{p_t} p_i. \tag{6}$$

The equivalent diameter is

$$d_e = f(\omega_e) \tag{7}$$

in which f denotes a function. These relations were experimentally obtained (U.S. Inter-Agency Committee on Water Resources, 1958). The following table compares the equivalent diameters, d_e, obtained by the method described above with the median standard fall diameters, d_{50}, for the basic flume data (Simons and Richardson, 1961b).

d_{50} (*mm*)	de (*mm*)	d_{50} (*mm*)	de (*mm*)
0.19	0. 19	0.45	0. 50
0.27	. 33	0.47	. 52
0.28	. 36	0.93	. 93

VELOCITY DISTRIBUTION IN ALLUVIAL CHANNELS

The Reynolds equation for the direction of the flow for a two-dimensional steady, uniform turbulent flow is expressed as

$$\rho g \sin \alpha + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) = \rho \frac{\partial(\overline{u'v'})}{\partial y} \tag{8}$$

in which α is bed slope, y is the distance from bed surface, μ is dynamic viscosity, u is the velocity at distance y from the bed, and u' and v' are the fluctuating components of velocity in the x and y directions.

Equation 8 may be integrated with respect to y, yielding

$$\rho g (\sin \alpha) y + \mu \frac{du}{dy} - \rho \overline{u'v'} = \text{constant}. \tag{9}$$

At the wall the first and third terms in the left side of the equation become zero, hence

$$\left(\mu \frac{du}{dy}\right)_{y=0} = \tau_o = \text{constant} \tag{10}$$

in which τ_0 is the shear stress at the boundary. Equation 9 becomes

$$g(\sin \alpha) y + \nu \frac{du}{dy} - \overline{u'v'} = \frac{\tau_o}{\rho} = U_*^2. \tag{11}$$

The term U_* is defined as shear velocity.

In most of the turbulent flow the Reynolds stress is expected to be much greater than the viscous stress except in the region near the boundary. Hence, excluding the thin layer near the boundary, equation 11 may be expressed as

$$g (\sin \alpha) y - \overline{u'v'} = U_*^2. \tag{12}$$

Through application of Prandtl's mixing length hypothesis

$$-\overline{u'v'} = l^2 \left(\frac{du}{dy}\right)^2 \tag{13}$$

in which l is the mixing length and is assumed to be

$$l = \kappa y. \tag{14}$$

Then equation 12 can be rewritten as

$$\kappa^2 y^2 \left(\frac{du}{dy}\right)^2 = U_*^2 \left(1-\frac{y}{D}\right). \tag{15}$$

Upon substitution of $\xi=\frac{y}{D}$, the following equation is obtained:

$$\frac{du}{d\xi}=\frac{U_*}{\kappa\xi}\sqrt{1-\xi} \tag{16}$$

in which κ is von Karman's coefficient.

Upon integration of equation 16 with respect to ξ with the boundary condition $u=U_m$ when $\xi=1$, in which U_m is the velocity at water surface, the following result is obtained:

$$\frac{U_m-u}{2U_*/\kappa}=\ln\left(\frac{\sqrt{\xi}}{1-\sqrt{1-\xi}}\right)-\sqrt{1-\xi} \tag{17}$$

and by substitution of the relation

$$U=\int_0^1 u d\xi = U_m - \frac{2}{3}\frac{U_*}{\kappa} \tag{18}$$

the following equation results:

$$\frac{U-u}{2U_*/\kappa}=\ln\left(\frac{\sqrt{\xi}}{1-\sqrt{1-\xi}}\right)-\sqrt{1-\xi}-\frac{1}{3}. \tag{19}$$

For clear water flow with a rigid boundary, von Karman's coefficient κ is a constant. However, for flow carrying suspended particles, Vanoni (1946), Vanoni and Brooks (1957), and Elata and Ippen (1961), among others, have shown that this value changes. The analysis of the data shows the relation between von Karman's coefficient computed from equation 18 and Reynolds number $\frac{U_* d_e}{\nu}$ as given in figure 2.

The velocity distributions of 16 runs from the experimental data and from two sets of field data of the Upper Bari Doab Canal in India (1954) are compared with the theoretical velocity distribution curves derived from equation 19 in figures 3 to 7. The measured velocities show irregularity due to turbulence and to the difficulty of measurement because of the suspended bed material. The deviation of the theoretical from the measured velocities is very severe when the relative depth y/D is less than 0.1 because the sediment particles tend to plug the Pitot tube. In addition, the effect of sand waves on the shape of the velocity distribution has not been considered. In spite

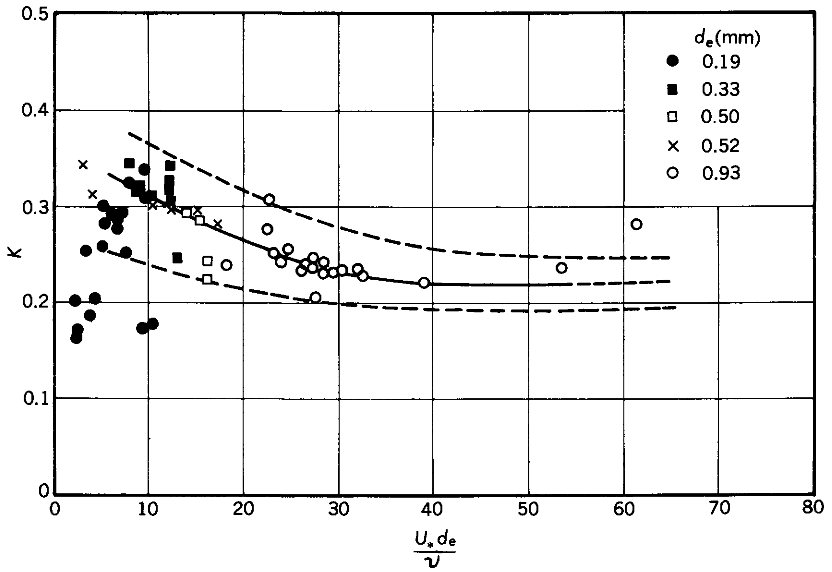

FIGURE 2.—Relation of sediment Reynolds number and von Karman's coefficient obtained from flume data.

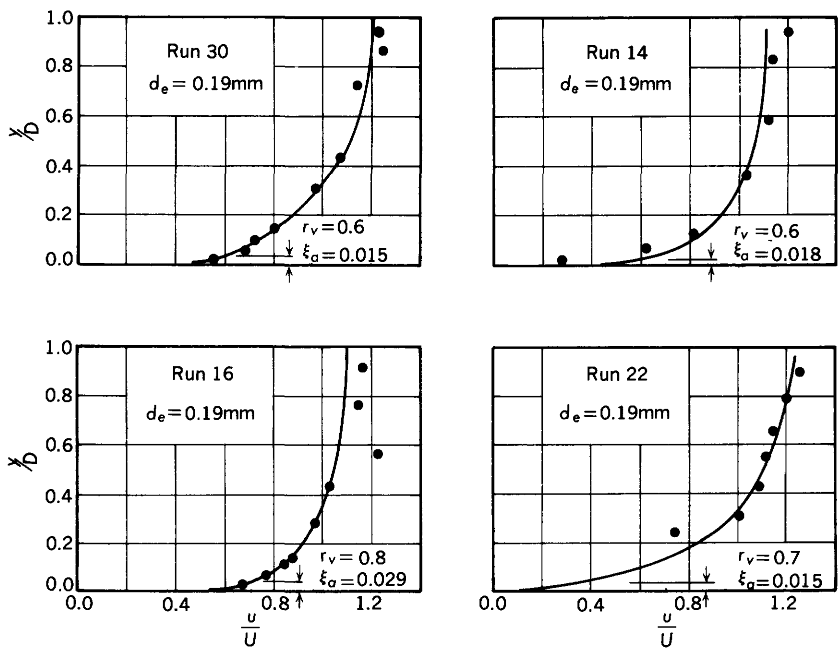

FIGURE 3.—Comparison of theoretical and measured velocity distribution for flume data for d_e=0.19-mm sand.

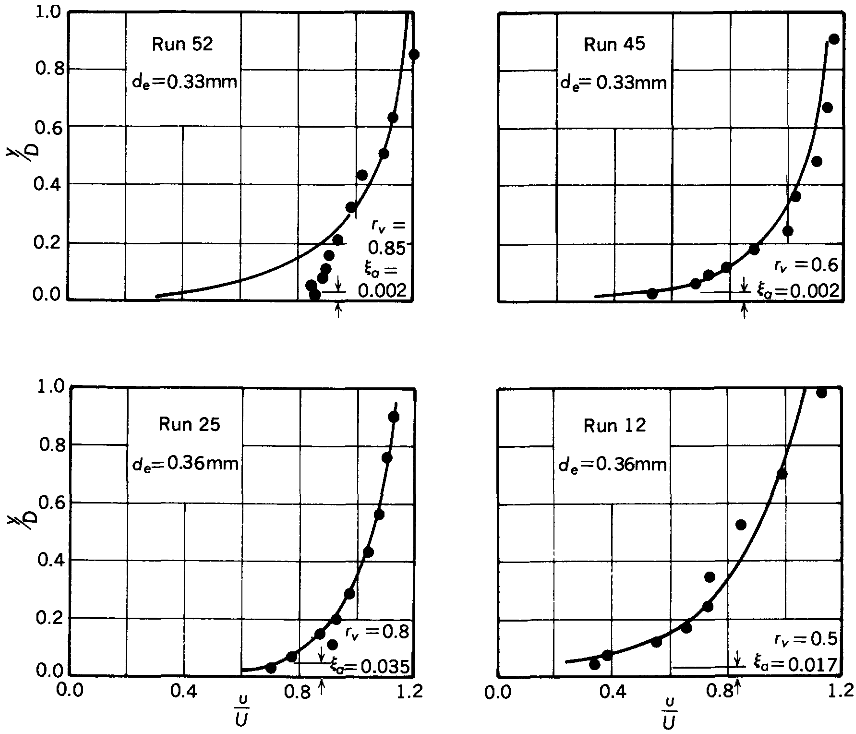

FIGURE 4.—Comparison of theoretical and measured velocity distribution for flume data for d_e=0-.33-, 0.35-mm sand.

of these sources of error, the result obtained seems satisfactory for evaluating bed-material discharge in alluvial channels.

BED-MATERIAL DISCHARGE

Total bed-material discharge is generally divided into contact-bed-material discharge and suspended-bed-material discharge. In two-dimensional uniform flow, the total bed-material discharge is expressed as

$$q_T = \int_0^a c_b u_b dy + \int_a^D c_s u_s dy. \tag{20}$$

The first term represents contact-bed-material discharge, and the second term represents suspended-bed-material discharge per unit width of channel. The terms c_b and c_s represent the weight of bed material in a unit volume of the water-bed-material mixture, and u_b and u_s are the actual velocities of the bed material. The limit a is the distance fromthe stationary boundary to the dividing line between bed and suspended bed materials.

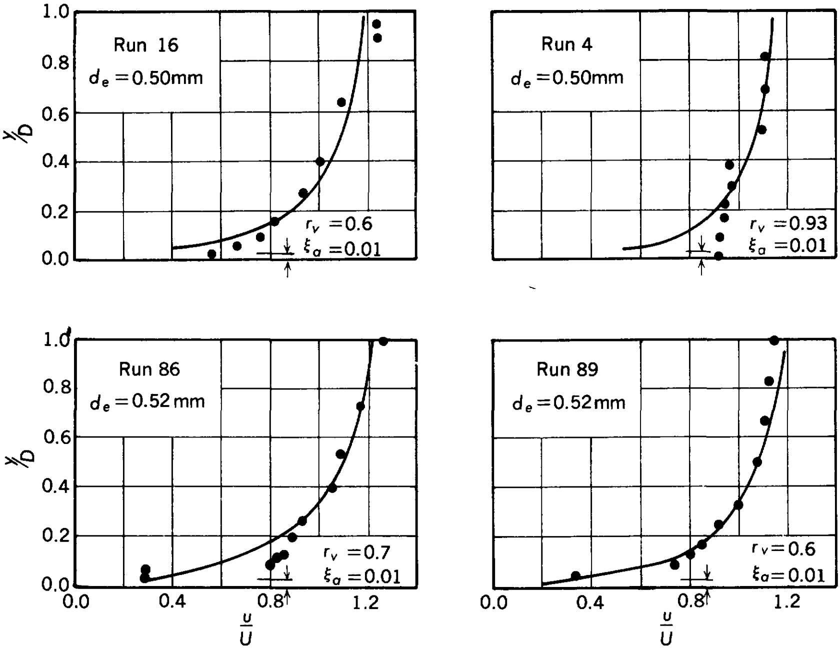

FIGURE 5.—Comparison of theoretical and measured velocity distribution for flume data for d_e=0.50-, 0.52-mm sand.

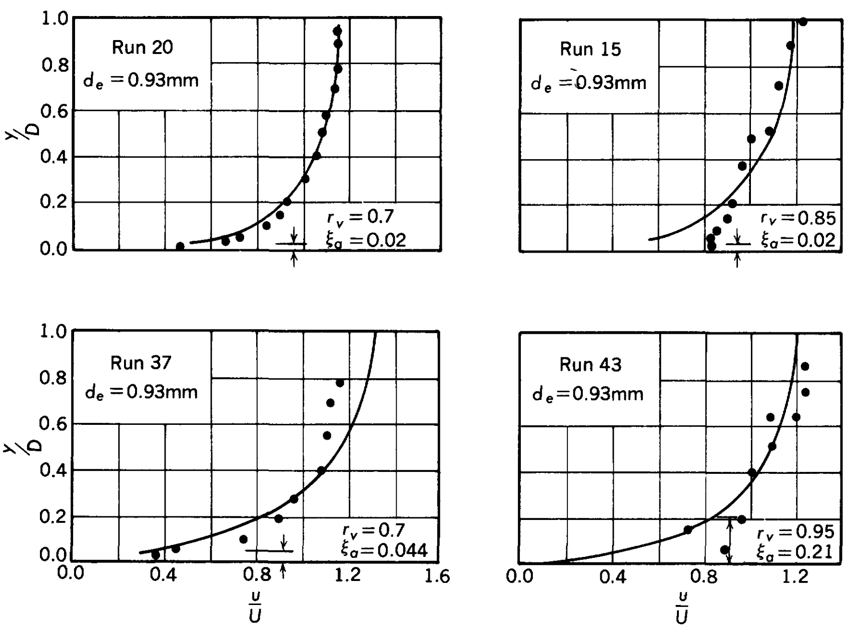

FIGURE 6.—Comparison of theoretical and measured velocity distribution for flume data for d_e=0.93-mm sand.

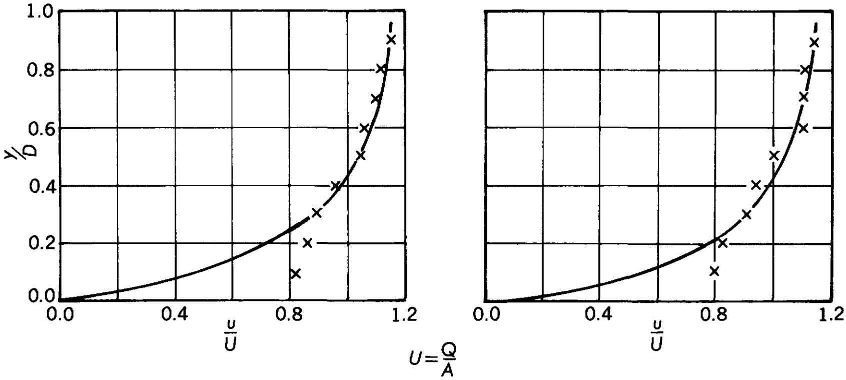

FIGURE 7.—Comparison of theoretical and measured velocity distribution for Upper Bari Doab Canal in India.

CONTACT-BED-MATERIAL DISCHARGE

Since the particles in the contact-bed-material layer move by rolling, sliding, or partly jumping, du Boys' assumption (Leliavsky, 1955) applies for the first approximation; that is, the propulsive movement of the granular bed material varies gradually and uniformly from a maximum at the surface to zero at some depth below this level.

If a plane alluvial bed is assumed to consist of several layers of bed materials for which the thickness of each layer is assumed to be one grain diameter, then the resisting force τ of the nth layers may be expressed as

$$\tau=(1-e)(\gamma_s-\gamma)nd \tan \phi \tag{21}$$

in which e is the porosity, γ_s and γ are the specific weights of bed material and fluid, respectively, n is the number of moving layers and ϕ is the friction angle of the bed material in the water.

The critical tractive force τ_c is defined as the force which causes motion of the bed material to begin and which may be approximately written as

$$\tau_c=(1-e)(\gamma_s-\gamma)d \tan \phi. \tag{22}$$

If the tractive force τ_o (applied at the upper surface of the first layer) is balanced by the resisting force at the nth layer (bottom), the following equation is established:

$$\tau_o=(1-e)(\gamma_s-\gamma)nd \tan \phi. \tag{23}$$

Since the tractive force is balanced at the nth layer, the movement of the nth layer is nil or very slight, and movement of $(n-1)$ layers above the nth layer can be assumed. The total thickness of the layer adjacent to the stationary bed in which the bed material moves either by rolling or sliding is then, from equations 22 and 23,

$$b_c=(n-1)d=\frac{\tau_o-\tau_c}{(1-e)(\gamma_s-\gamma)\tan\phi}. \tag{24}$$

However, in addition to the tractive force τ_o, the particles are subjected to an uplift force by the flowing fluid that will decrease the force toward the bed. Further, the concentration of bed material in the layers upward from the bottom will decrease, owing to the space increase created by jumping and elastic impact among the particles. If these points are considered, the thickness of the contact-bed-material layer, a, may be written as

$$a=j\cdot b_c=j\cdot\frac{\tau_o-\tau_c}{(1-e)(\gamma_s-\gamma)\tan\phi} \tag{25}$$

where j is an experimental coefficient. For the first approximation of the thickness of the layer, equation 25 is adequate since the principal consideration is within a very thin layer near the stationary boundary. This equation merely shows that the thickness of the contact-bed-material layer increases with the shear force.

By observing the thickness of the contact-bed-material layer through the plastic wall of the experimental flume and by computing j from equation 25, the value of j was found to be approximately 10. Einstein (1950) assumed that the thickness of the contact-bed-material layer was twice the geometric mean diameter of sediment and was independent of flow conditions. Danel, Durand, and Condolios (1954) stated that the thickness of the layer varies rapidly at low velocity and more slowly at higher velocity, depending on the settling velocity of the particles. The different viewpoints suggest that further study of the thickness of the contact-bed-material layer will be necessary.

The contact-bed-material discharge is expressed as

$$q_b=\int_0^a c_b u_b dy. \tag{26}$$

By use of mean values, equation 26 may be written as

$$q_b=M_1 C_m U_b a \tag{27}$$

where

$$M_1=\frac{\int_0^a c_b u_b dy}{C_m U_b a} \tag{28}$$

in which C_m is the mean concentration and U_b is the mean velocity of the contact-bed-material layer.

Since the contact-bed-material layer is usually very thin, it is natural to consider that C_m is closely related to the concentration of the stationary bed, which has a value of $(1-e)\gamma_s$. Hence,

$$C_m = M_2(1-e)\gamma_s \tag{29}$$

where $M_2 < 1.0$ and will approach 1.0 when all the bed material is sliding or rolling.

When equations 27 and 29 are combined, the contact-bed-material discharge per unit width of channel is

$$q_b = M(1-e)\gamma_s a U_b \tag{30}$$

where $M = M_1 \cdot M_2$.

The next problem is to find the mean velocity U_b of the contact-bed-material layer. If a contact-bed-material layer moves with velocity U_b against the resistant force τ, the rate of work done by the moving layer is

$$W_b = \tau U_b. \tag{31}$$

The resistant force is $\tau = \tau_o - \tau_c$, since the lowest layer of the contact bed material is considered to be stationary, as explained previously.

The energy required to do this work comes from the flow energy of the fluid in the stream; it may be written as

$$W_f = \tau_o U. \tag{32}$$

Bagnold (1960) defined this energy as stream power; he stated that "with respect to the movement of the bed load, the system must be regarded as a fluid-dynamic transporting machine with an efficiency where

$$\text{Efficiency} = \text{rate of work done/available power} = \epsilon. \tag{33}$$

The work rate is a certain proportion of the available power, and both work rate and power are related and measurable in the same units."

Therefore, the dynamic relation between equations 31 and 32 is

$$\tau U_b = \epsilon \tau_o U. \tag{34}$$

According to Bagnold, efficiency can be expressed as

$$\epsilon = K_{bc} \frac{\tau_o - \tau_c}{\tau_o} \tag{35}$$

where K_{bc} depends on grain size and possibly bed configuration.

When equations 30, 34, and 35 are combined, the contact-bed-material discharge per unit width of channel q_b is

$$q_b=K_b \frac{\gamma_s}{(\gamma_s-\gamma)\tan\phi}(\tau_o-\tau_c)U \tag{36}$$

where $K_b=M\cdot K_{bc}$ and is an experimental coefficient.

When the bed surface has an appreciable slope α, equations 25 and 36 become

$$a=j\cdot\frac{\tau_o-\tau_c}{(1-e)(\gamma_s-\gamma)(\cos\alpha\cdot\tan\phi-\sin\alpha)} \tag{37}$$

and

$$q_b=K_b\frac{\gamma_s}{(\gamma_s-\gamma)(\cos\alpha\cdot\tan\phi-\sin\alpha)}\cdot(\tau_o-\tau_c)\,\mathrm{U}. \tag{38}$$

SUSPENDED-BED-MATERIAL DISCHARGE

Most of the finer bed material moves downstream in suspension. The problem of bed-material suspension is best considered in the light of the turbulence mixing processes.

In steady two-dimensional open-channel flow, the basic equation for the distribution of suspended bed material by O'Brien (1933) is

$$\epsilon_s\frac{dc}{dy}+\omega_e c=0. \tag{39}$$

The sediment-transfer coefficient is assumed to be linearly proportional to the diffusion coefficient ϵ_o, which—by Prandtl's mixing length hypothesis—is expressed as

$$\epsilon_o=l^2\left|\frac{du}{dy}\right|. \tag{40}$$

Hence, the sediment transfer coefficient ϵ is given as

$$\epsilon_s=\beta\epsilon_o=\beta l^2\left|\frac{du}{dy}\right| \tag{41}$$

where β is a proportional constant. Upon substitution of equation 16 into equation 41, the following relation is obtained:

$$\epsilon_s=\beta\kappa DU_*\xi\sqrt{1-\xi}. \tag{42}$$

When this equation is substituted into equation 39, the basic equation for the distribution of suspended bed material in the vertical direction is

$$\beta U_*\kappa\xi\sqrt{1-\xi}\,\frac{dc}{d\xi}+\omega_e c=0. \tag{43}$$

By integration, the equation for the distribution of suspended bed material becomes

$$\frac{c}{C_a}=A_1\cdot\left(\frac{\sqrt{\xi}}{1-\sqrt{1-\xi}}\right)^z \tag{44}$$

where

$$A_1=\left(\frac{1-\sqrt{1-\xi_a}}{\sqrt{\xi_a}}\right)^z \tag{45}$$

$$z=\frac{2\omega_e}{\beta U_*\kappa} \tag{46}$$

and C_a is the concentration at $\xi=\xi_a$.

The value of β in equation 46 varies as the magnitude of the concentration of the suspended bed material changes. Rouse (1939) assumed that $\beta=1.0$; however, according to experimental data by Vanoni (1946), Ismail (1952), and Einstein and Chien (1954), the value of β is larger than 1.0. In view of these considerations, the value of β is assumed to be 1.5 for this analysis.

Since most suspended bed material is fine, the ratio of bed-material velocity to the surrounding-fluid velocity is assumed to be unity. Hence,

$$q_s=\int_a^D c_s u_s dy=D\int_{\xi_a}^1 cud\xi \tag{47}$$

which, by integration, becomes

$$q_s=DC_a\left(UI_1-\frac{2U_*}{\kappa}I_2\right) \tag{48}$$

where

$$I_1=A_1\int_{\xi_a}^1\left(\frac{\sqrt{\xi}}{1-\sqrt{1-\xi}}\right)^z d\xi \tag{49}$$

and

$$I_2=A_1\int_{\xi_a}^1\left(\frac{\sqrt{\xi}}{1-\sqrt{1-\xi}}\right)^z\left[\ln\left(\frac{\sqrt{\xi}}{1-\sqrt{1-\xi}}\right)-\sqrt{1-\xi}-\frac{1}{3}\right]d\xi. \tag{50}$$

The values of I_1 and I_2 were numerically integrated and are shown in figures 8 and 9.

The next problem is the determination of the concentration C_a at $\xi=\xi_a$. Since the contact-bed-material layer is thin, it is assumed, as Einstein (1950) did, that the rate of contact-bed-material discharge through a unit area $q_b'=\frac{q_b}{a}$ is a constant through the contact-bed-

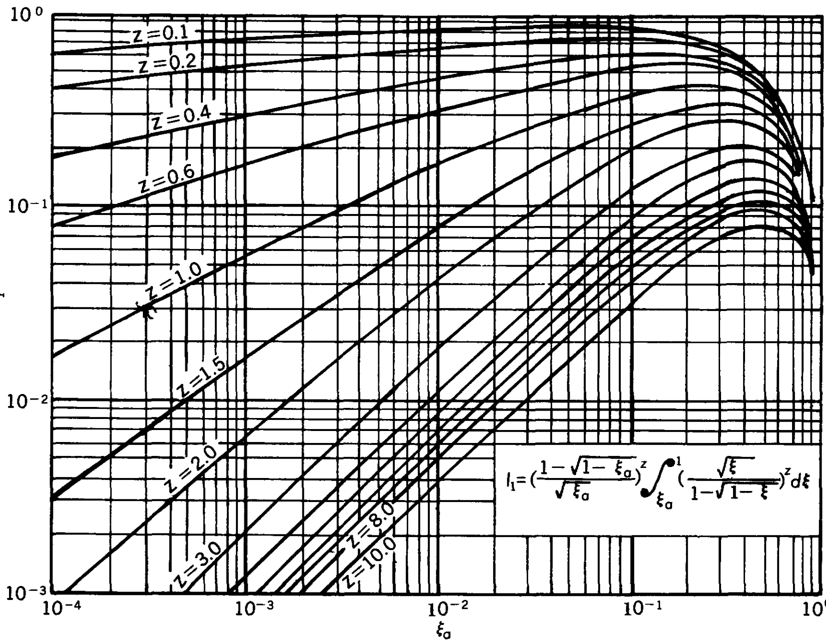

FIGURE 8.—Function I_1 in terms of relative contact-bed-material layer ξ_a for various values of exponent z.

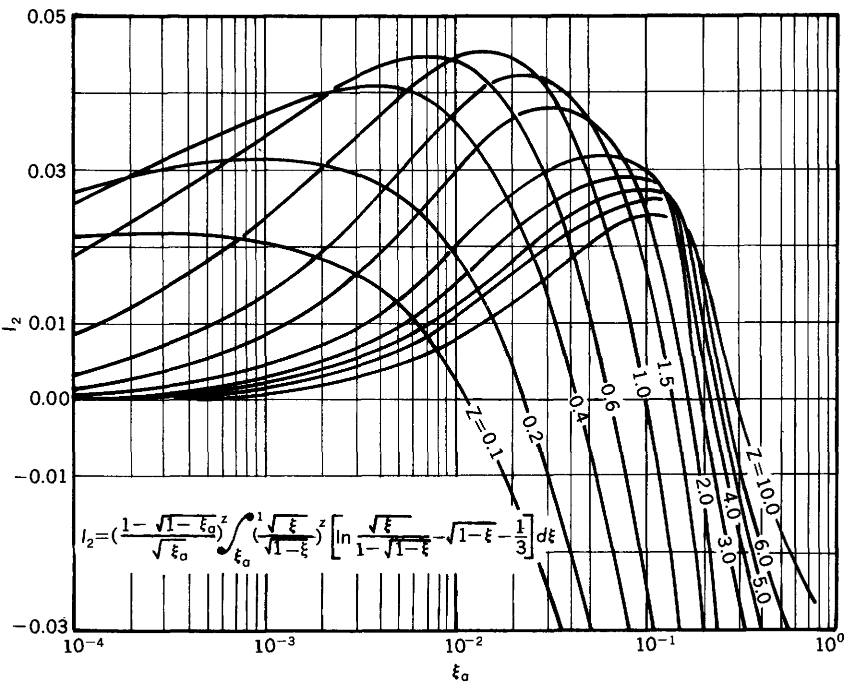

FIGURE 9.—Function I_2 in terms of relative contact-bed-material layer ξ_a for various values of exponent z.

material layer. Then the concentration at $\xi=\xi_a$ can be computed by dividing q_b' by the velocity u at $\xi=\xi_a$ as follows:

$$C_a=\frac{q_b}{au}. \tag{51}$$

The velocity u at $\xi=\xi_a$ may be obtained by equation 19; however, recall that equation 19 was derived on the assumption that the thin layer near the boundary was excluded. Therefore, it is necessary to assume that the velocity u at $\xi=\xi_a$ is proportional to the mean velocity U of the flow

$$u=r_v U. \tag{52}$$

This also parallels Einstein's assumption that u is proportional to U_*. Taking the average value of r_v from the measured velocity distributions given in figures 3 to 6 we found that $r_v=0.73$. These figures indicate typical lower regime conditions except for run 43 shown in figure 6. The value of r_v is larger for the upper flow regime (run 43 shows $r_v=0.95$). Figure 7 shows the measured velocity distributions for an alluvial river (Sehgal and Nayyar, 1954) from which r_v at $\xi=\xi_a$ has an observed value of about 0.8.

By use of $r_v=0.8$ and by combination of equations 51 and 52, the following relation for concentration at $\xi=\xi_a$ results:

$$C_a=\frac{q_b}{0.8aU}. \tag{53}$$

Therefore, the total suspended-bed-material discharge per unit width of channel q_s may be calculated from equations 48 and 53. That is,

$$q_s=\frac{Dq_b}{0.8aU}\left(UI_1-\frac{2U_*}{\kappa}I_2\right)=R_s q_b \tag{54}$$

where

$$R_s=\frac{D}{0.8aU}\left(UI_1-\frac{2U_*}{\kappa}I_2\right),$$

which is the ratio of suspended-bed- to contact-bed-material discharge.

TOTAL BED-MATERIAL DISCHARGE

The total bed-material discharge per unit width of channel q_T may now be calculated from

$$q_T=q_b+q_s=q_b(1+R_s). \tag{55}$$

By substitution of equation 38 into equation 55,

$$q_T = K_b \frac{\gamma_s}{(\gamma_s - \gamma)(\cos\alpha \cdot \tan\phi - \sin\alpha)} (\tau_o - \tau_c) U (R_s + 1) \tag{56}$$

or

$$q_T = K_T (\tau_o - \tau_c) U (R_s + 1) \tag{57}$$

where

$$K_T = K_b \frac{\gamma_s}{(\gamma_s - \gamma)(\cos\alpha \cdot \tan\phi - \sin\alpha)}. \tag{58}$$

The total bed-material discharge coefficient K_T is an experimental coefficient which is a function of the bed material, the bed configuration, and flow characteristics for the flume data. This coefficient was found to be a function of bed material and $\frac{U}{U_*} \cdot \frac{\tau_o}{\Delta\gamma d_e} \cdot s$, as shown in figure 10. These relations seem to undergo a radical change in slope when the flow regime changes. The condition between regimes has been defined by Simons and Richardson (1961a) as the transition zone. Computations of values of K_T for each of the three natural

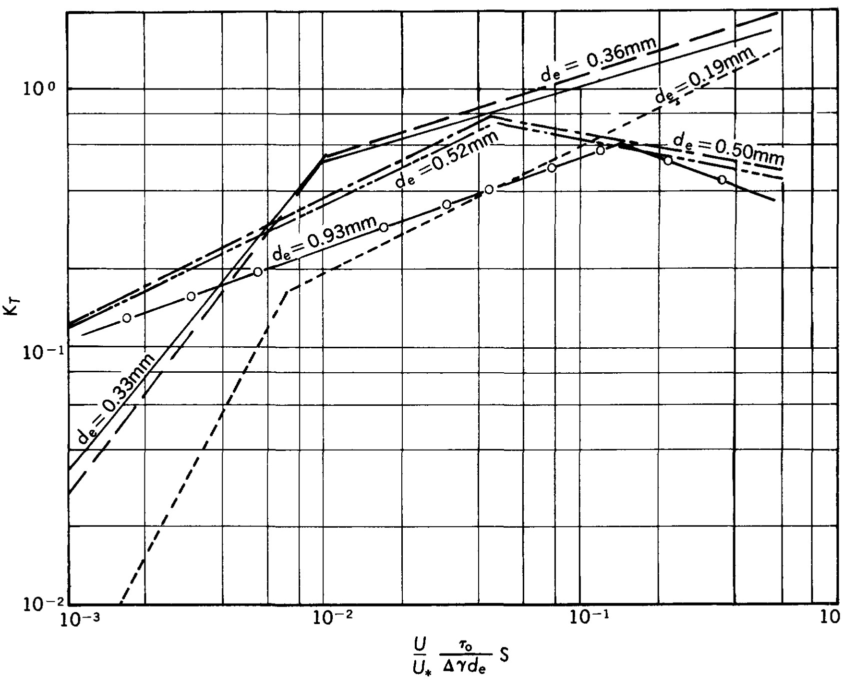

FIGURE 10.—Total bed-material discharge coefficient K_T versus $\frac{U}{U_*} \cdot \frac{\tau_o}{\Delta\gamma d_e} \cdot s$ relation for various sands from flume data.

rivers (Bishop, 1961) showed K_T to be approximately constant, as follows:

Name of river	d_{50} (*mm*)[1]	K_T
Colorado	0.34	1.10
Middle Loup	.36	.38
Niobrara	.28	.27

[1] Sufficient data were not available to compute d_e.

EVALUATION

The kinds of data required for computing total bed-material discharge per unit width are: (1) the flow depth, slope, and mean velocity, (2) the characteristics of bed material, or more specifically, the equivalent particle diameter (determined from fall velocity), particle density, and porosity, (3) the critical tractive force of the bed material, and (4) the temperature and density of the fluid.

From these data the total bed-material discharge can be computed by a procedure involving the following steps:

1. Calculate τ_o and U_* from

$$\tau_o=\gamma DS$$

and

$$U_*=\sqrt{\tau_o/\rho}.$$

2. Obtain von Karman's coefficient κ from figure 2 by first calculating

$$\frac{U_*d_e}{\nu}.$$

3. Determine the exponent z by equation 46.

4. Calculate the thickness of the contact-bed-material layer using equation 37 with $j=10$ and the relative thickness of the contact bed-material layer $\xi_a=a/D$.

5. Read I_1 and I_2 from figures 8 and 9, respectively, for corresponding values of z and ξ_a.

6. Calculate the ratio of suspended-bed- to contact-bed-material discharge by using equation 54 which is

$$R_s=\frac{D}{0.8aU}\left(UI_1-\frac{2U_*}{\kappa}I_2\right).$$

7. Read the total bed-material discharge coefficient K_T from figure 10 using $\frac{U}{U_*}\cdot\frac{\tau_o}{\Delta\gamma d_e}\cdot s$ and the size of bed material.

8. Calculate the total bed-material discharge per unit width of flow by using equation 57.

The comparison of the measured total bed-material discharge and the bed-material discharge estimated by this method for 184 flume runs and 57 sets of natural river data is given in figures 11 and 12. Only 4.3 percent of the estimated values deviates from the measured values by more than 100 percent.

Bishop (1961) computed the total bed-material discharge using Einstein's function for the same data. About 23 percent of the values estimated by the Einstein function deviates from the measured values by more than 100 percent. (See fig. 13.)

These results show the advantage of the method presented herein.

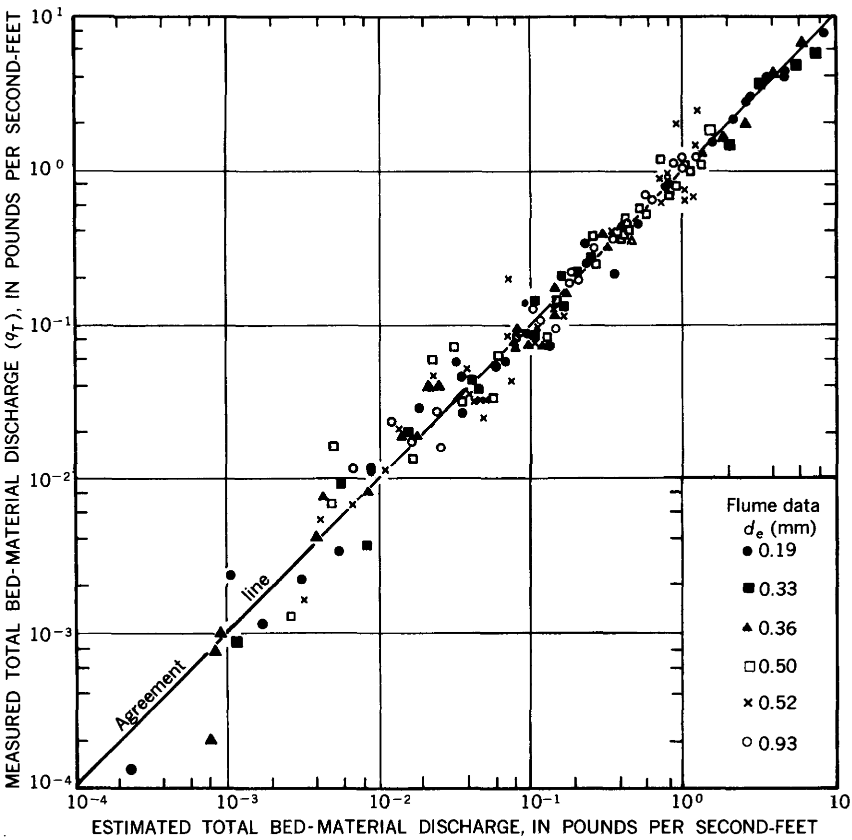

FIGURE 11.—Comparison of the estimated and measured total bed-material discharge for flume data.

$$q_T = K_T(\tau_o - \tau_c)\, U(R_s + 1)$$

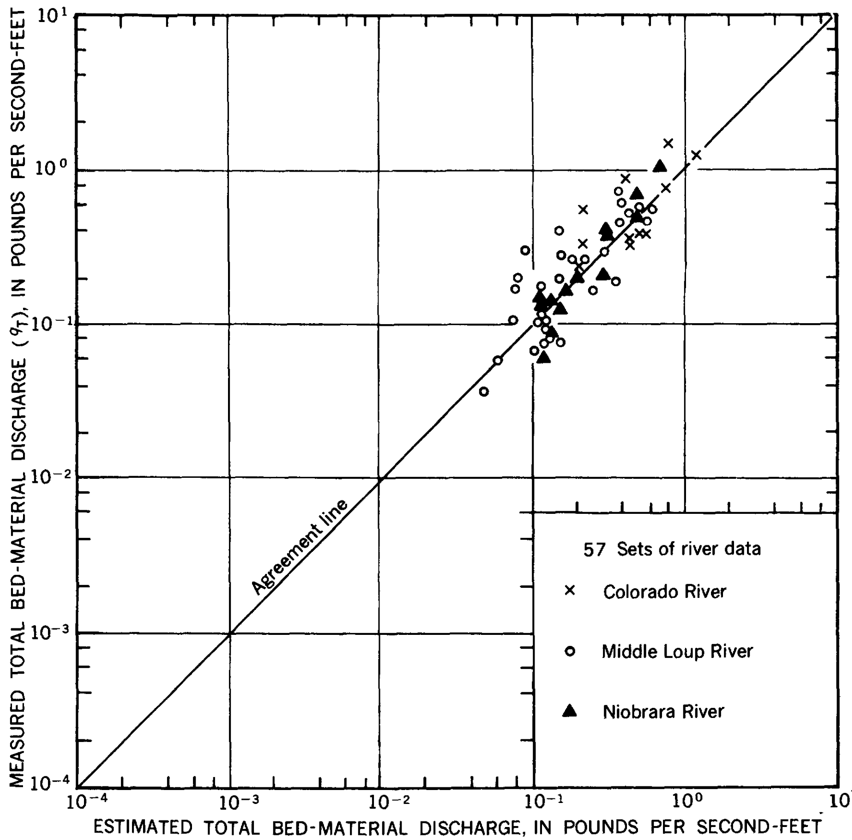

FIGURE 12.—Comparison of the estimated and measured total bed-material discharge for river data.

$$q_T = K_T(\tau_o - \tau_c)\,U(R_s + 1)$$

SUMMARY AND CONCLUSIONS

A size classification based upon the energy value of the particles was developed for graded bed material (sand). The classification considers the gradation of the bed material on the basis of the energy needed to suspend each size fraction. By use of this method of size classification, the need to subdivide the bed material into fractions in order to compute the total bed-material discharge is eliminated.

A velocity distribution equation was obtained by integrating the Reynolds equation by using the Prandtl's hypothesis of mixing length. In the derivation, the small region near the boundary was not considered because (1) this makes it possible to integrate the Reynolds equation, and (2) the velocity near the boundary cannot be easily expressed since the boundary itself is moving. The theoretical velocity distribution fitted the experimental data for the plane-bed configuration very well. If one considers the space and time variation

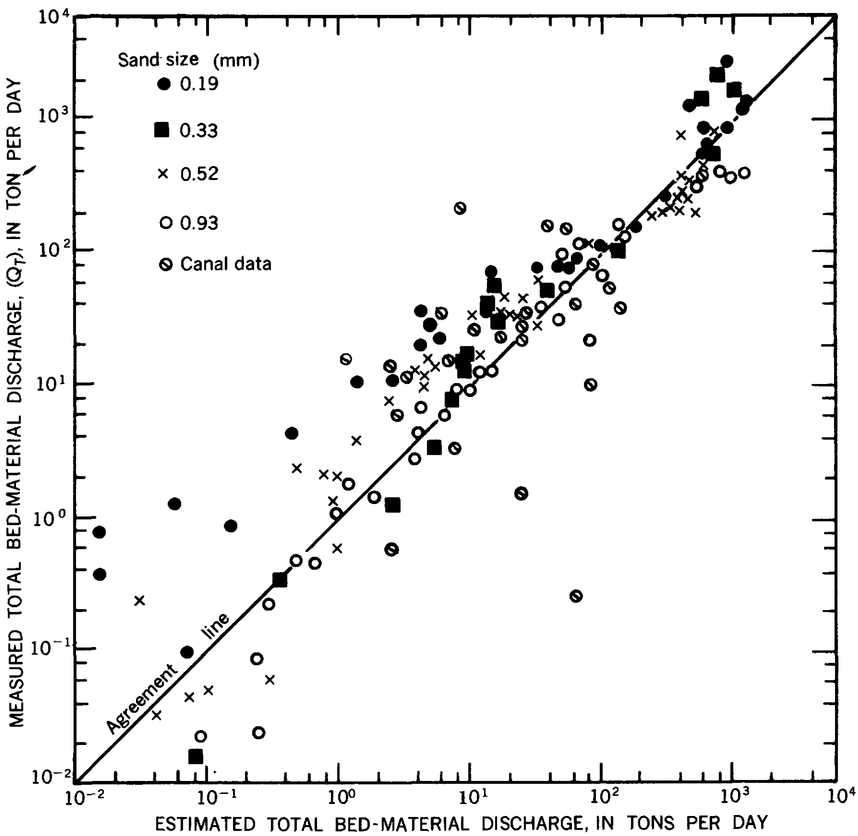

FIGURE 13.—Comparison of the estimated total bed-material discharge (Einstein's function) by Bishop (1961) and measured total bed-material discharge for both flume and river data.

of the point velocity when the dune bed configuration prevailed, the derived velocity distribution curve fits the data as well as could be expected.

An equation was developed to determine the total bed-material discharge by dividing it into two parts: the contact-bed- and suspended-bed-material discharges. The contact-bed-material discharge was obtained by applying the work-energy relation between the sediment and fluid. The fluid dissipates a part of its energy to transport bed material.

The suspended-bed-material discharge was obtained by integrating O'Brien's (1933) suspended-bed-material distribution equation, assuming that the horizontal velocity of suspended bed material is the same as that of the transporting fluid, and the sediment transfer coefficient of the fluid.

The integration limits used to determine the suspended-bed-material discharge were the distance from the stationary bed to the dividing line between contact and suspended bed material, a, and the distance from the bed to the water surface, D. The concentration at a was computed by assuming that the velocity at this point was 0.8 of the mean velocity of the fluid. This velocity was determined by an analysis of actual velocity distribution measured in the flume and in natural rivers.

The total bed-material discharge can be expressed as

$$q_T = K_T(\tau_o - \tau_c)U(R_s + 1).$$

The total bed-material discharge coefficient K_T is an experimental coefficient which was found to be a function of the bed material and $\frac{U}{U_*} \cdot \frac{\tau_o}{\Delta\gamma d_e} \cdot s$ for the flume data. The coefficient was found to be constant for each natural river studied. The results were checked with available laboratory and field data; they appear to be mutually consistent and satisfactory in the sense of sediment-transport theory for both flume and natural-river data.

The total bed-material discharge formula for alluvial channels that is developed in this paper has two principal advantages over formulas developed previously:

1. The computation procedure is not complicated.
2. The results are more accurate because the formula is based on the mean velocity of flow in the channel.

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