G. Allan Brown

Water Levels Near a Well Discharging From an Unconfined Aquifer

GEOLOGICAL SURVEY WATER-SUPPLY PAPER 1536-B

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By IRWIN REMSON, S. S. McNEARY, *and* J. R. RANDOLPH

GROUND-WATER HYDRAULICS

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A numerical solution of the differential equation for unsteady flow in an unconfined aquifer

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By Irwin Remson, S. S. McNeary, and J. R. Randolph

ABSTRACT

The differential equation describing unsteady radial flow to a well in an unconfined aquifer has no simple solution in terms of elementary functions because the transmissibility decreases as the aquifer is dewatered. A numerical solution and type curves are presented showing the head of water in a homogeneous aquifer as a function of radial distance from the well and time elapsed after the water level in the well is lowered to and maintained at a given level. This solution is similar to one presented by J. R. Philip in 1955 for linear diffusion in a flow system in which diffusivity is concentration dependent.

INTRODUCTION

The U.S. Geological Survey carried on studies of the occurrence and movement of subsurface water, particularly in the zone of aeration, at Seabrook, N.J., from 1950 to 1960. Because of similarities between unsaturated flow and unconfined saturated flow, these studies led to a numerical solution of the differential equation describing unsteady radial flow in an unconfined aquifer.

The theory and the solution are developed in this paper. The methods are demonstrated by applying them to a convenient hypothetical problem. Type curves are presented and are applied to practical problems in a report in preparation.

Acknowledgment is made of the generous collaboration of the Seabrook Farms Co. These studies were under direction of Henry C. Barksdale, area chief, Ground Water Branch, U.S. Geological Survey, Arlington, Va.

STATEMENT OF THE PROBLEM

The differential equation governing unsteady radial flow to a well in an aquifer (Jacob, 1950, p. 366) may be written

$$\frac{\partial h}{\partial t}=\frac{\partial}{\partial r}\left[\frac{T}{S}\frac{\partial h}{\partial r}\right]+\frac{T}{rS}\frac{\partial h}{\partial r}, \tag{1}$$

where

h=head of water at any point in the aquifer, in feet;

t=time, in days, since pumping started;

r=radial distance, in feet, from the discharging well;

T=coefficient of transmissibility of the aquifer, or rate of flow of water at the prevailing water temperature, through a vertical strip of aquifer 1 foot wide extending the full saturated height under a hydraulic gradient of 100 percent. T is normally expressed in gallons per day per foot of aquifer width (Theis, 1935). However, it is more convenient in the present development to express it in cubic feet of water per day per foot of aquifer width;

S=coefficient of storage of the aquifer, or volume of water released from or taken into storage by an aquifer per unit surface area of the aquifer per unit change in the component of head normal to that surface. It is a dimensionless quantity (Theis, 1935).

In the following application of equation 1, it is assumed that the aquifer is infinite in areal extent, homogeneous, and isotropic; that the discharging well penetrates and receives water from the entire thickness of the aquifer; and that the well has a negligibly small diameter.

In a confined aquifer which is not dewatered during the discharging period, the transmissibility may be assumed to be constant at all times and at all places. In such a confined aquifer, the water is released from or taken into storage by virtue of the elastic response of the aquifer and the water to pressure changes induced by the pumping. It is assumed that the water is removed from storage instantaneously as head declines. Solutions of equation 1 are available for such confined aquifers, where T and S may be taken as constant (Theis, 1935; Jacob, 1940).

An unconfined aquifer is dewatered as the head is lowered by a discharging well. Because the dewatered portion of the aquifer cannot transmit water, the transmissibility decreases with the decline in head. Furthermore, the water is released from or taken into storage in response to head changes, largely by gravity drainage or refilling of the zone through which the water table moves. Although the coefficient of storage is usually assumed to be constant and to be virtually equal to the specific yield of an unconfined aquifer, the release of water from some rocks is not instantaneous, and some variation in this coefficient does occur in both time and space during a period of discharge.

Even if a constant value of S is assumed, the ratio T/S is variable because T is a function of the head in an unconfined aquifer. An exact solution of equation 1 with variable T/S has not been achieved in terms of elementary functions. It is the purpose of this paper to present a numerical solution for equation 1 applied to an unconfined aquifer for an assumed relation between T and head.

RESTATEMENT OF THE DIFFERENTIAL EQUATION

A saturated soil will drain very rapidly to a critical range of moisture content within which values of capillary conductivity become small; thereafter, the soil drains very slowly. At Seabrook, N.J., for example, sediments above the capillary fringe and below the root zone rapidly drain to, and then remain approximately at, this critical moisture content. Subsequent drainage is very slow (Remson, Randolph, and Barksdale, 1960, p. 150–151). Therefore, as a reasonable approximation, S can be taken as constant in equation 1 if applied over a time period as long as a few days, as is commonly the case in analysis of pumping-test data.

For convenience, let

$$D=\frac{T}{S}. \tag{2}$$

Because S is taken as constant, the hydraulic diffusivity, D, varies with T , which in turn varies with hydraulic head.

Let h_n equal the constant hydraulic head throughout the aquifer before the well discharge begins, and let h_0 equal the head of water in a discharging well that commences to discharge at $t=0$. It is assumed that the head is instantaneously lowered to h_0 at t_0 and is maintained constant throughout the discharging period; in other words, the well is "pumped at constant drawdown." It is assumed also that the head of water immediately outside the pumped well is h_0. Furthermore, let

$$\theta=\frac{h-h_0}{h_n-h_0}. \tag{3}$$

Then θ is proportional to the difference between the hydraulic head at any point in the aquifer and the head at the discharging well, and is defined over the interval $0\leqq\theta\leqq1$. Thus, θ is a function of head, which in turn is a function of radial distance from the well and time. At the well, $\theta=0$; and at a remote point $(r\rightarrow\infty)$, $\theta=1$.

Substituting equation 2 and appropriate differentials of equation 3 in equation 1

$$\frac{\partial\theta}{\partial t}=\frac{\partial}{\partial r}\left[D\frac{\partial\theta}{\partial r}\right]+\frac{D}{r}\frac{\partial\theta}{\partial r}. \tag{4}$$

The boundary conditions for the case of a discharging well are

$$\theta=\theta_n \text{ at } t=0 \text{ and } r>0$$

and

$$\theta=\theta_0 \text{ at } r=0 \text{ and } t\geqq 0 \tag{5}$$

and $\theta_n>\theta_0$. It is assumed also that D is a single-valued function of θ.

In applying equation 4 to unconfined flow, we are essentially adopting the assumptions of Dupuit (1863) as described by Jacob (1950, p. 378).

In discussing the flow of ground water in a vertical plane above a horizontal sole or impermeable bottom, he [Dupuit] assumed the velocity along the water table to be proportional to the slope or tangent of its angle of inclination, instead of to the sine as it actually is. Furthermore, he assumed the flow to be horizontal at the water table and everywhere below and the velocity to be uniform from top to bottom.

As shown by R. W. Stallman of the U.S. Geological Survey (written communication, 1960), equation 4 can be derived from Jacob's general equation for unsteady unconfined flow, which also is based upon the Dupuit assumptions (Jacob, 1950, p. 384).

Equation 4 has the same form as the radial-diffusion equation where diffusivity is a function of concentration. In the diffusion case, D would be the diffusivity and θ would be the concentration. Philip (1955) has presented an iterative procedure for the numerical solution of the diffusion equation for one-dimensional flow, and his procedure may be adapted for solving equation 4. For clarity and convenience, much of Philip's development is reproduced below, most of his notation being retained.

DERIVATION OF THE ITERATIVE EQUATIONS

CONVERSION TO AN ORDINARY DIFFERENTIAL EQUATION

As a first step in the derivation of the iterative equations, the partial differential equation 4 is converted into an ordinary differential equation by means of the Boltzman transformation

$$\phi=rt^{-1/2}. \tag{6}$$

θ and D, both single-valued functions of r and t, are consequently single-valued functions of ϕ. Therefore, equation 4 can be rewritten

$$\frac{\partial\theta}{\partial\phi}\frac{\partial\phi}{\partial t}=\frac{\partial}{\partial\phi}\frac{\partial\phi}{\partial r}\left[D\frac{\partial\theta}{\partial\phi}\frac{\partial\phi}{\partial r}\right]+\frac{D}{r}\frac{\partial\theta}{\partial\phi}\frac{\partial\phi}{\partial r}. \tag{7}$$

When equation 6 and its derivatives are substituted into equation 7, the results can be simplified to the ordinary differential equation

$$-\frac{\phi}{2}\left(\frac{d\theta}{d\phi}\right)=\frac{d}{d\phi}\left[D\,\frac{d\theta}{d\phi}\right]+\frac{D}{\phi}\left(\frac{d\theta}{d\phi}\right). \tag{8}$$

This equation is subject to the boundary conditions

$$\theta=\theta_0=0,\text{ at }\phi=0,$$

and

$$\theta\rightarrow\theta_n=1,\text{ as }\phi\rightarrow\infty, \tag{9}$$

where $\theta_0\leqq\theta\leqq\theta_n$ for the case of a discharging well.

The boundary conditions state that, at any time, θ equals θ_0 at the pumped well, where $r=0$ and $\phi=0$. At any time, θ remains at its original value, θ_n, at an infinite distance from the well, where $r=\infty$ and $\phi=\infty$. The second condition states also that, when $t=0$, $\phi=\infty$ and heads are at their initial values throughout all parts of the system except at the well, where θ is θ_0.

For a homogeneous aquifer, it may be assumed that θ is a monotonically increasing function of ϕ, and equation 8 can be multiplied by $\frac{d\phi}{d\theta}$ and rearranged to give

$$\phi+\frac{2D}{\phi}=-2\,\frac{d}{d\theta}\left[D\,\frac{d\theta}{d\phi}\right]. \tag{10}$$

Because

$$\frac{d\theta}{d\phi}\rightarrow 0$$

as

$$\theta\rightarrow\theta_n=1,$$

the second boundary condition can be expressed in the form

$$\theta=\theta_n,\ \frac{d\theta}{d\phi}=0. \tag{11}$$

Equation 10 can then be written in integral form incorporating the second boundary condition:

$$\int_\theta^{\theta_n}\phi d\theta+\int_\theta^{\theta_n}\frac{2D}{\phi}\,d\theta=2D\,\frac{d\theta}{d\phi}. \tag{12}$$

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FINITE-DIFFERENCE FORM OF THE DIFFERENTIAL EQUATION

A finite-difference form of equation 12 can be developed by dividing the interval from θ_0 to θ_n into n equal steps of size $\delta\theta$. Let

$$\theta_p=\theta_0+p\delta\theta \tag{13}$$

where p is the number of equal steps of size $\delta\theta$ between θ_0 and θ_p.

The $\phi=f(\theta)$ curve can be replaced by a histogram with ϕ_p and θ_p as coordinates of the midpoint of step p. Then, by use of equation 13, the following finite-difference approximations are shown to hold:

$$\int_{\theta_p}^{\theta_{p+\frac{1}{2}}}\phi d\theta=\phi_p\frac{\delta\theta}{2} \tag{14}$$

and

$$\int_{\theta_{p-\frac{1}{2}}}^{\theta_p}\phi d\theta=\phi_p\frac{\delta\theta}{2}. \tag{15}$$

For $\theta=\theta_{p+1/2}$, the finite-difference form for the first term in equation 12 may now be written as

$$\int_{\theta_{p+\frac{1}{2}}}^{\theta_n}\phi d\theta=\int_{\theta_p}^{\theta_n}\phi d\theta-\int_{\theta_p}^{\theta_{p+\frac{1}{2}}}\phi d\theta=\int_{\theta_p}^{\theta_n}\phi d\theta-\phi_p\frac{\delta\theta}{2}. \tag{16}$$

Let $\overline{D}_{p+1/2}$ be the average value of D in the interval θ_p to θ_{p+1}, such that

$$\overline{D}_{p+\frac{1}{2}}=\frac{\int_{\theta_p}^{\theta_{p+1}}Dd\theta}{\int_{\theta_p}^{\theta_{p+1}}d\theta}. \tag{17}$$

Then, by analogy with equation 16, the finite-difference form for the second term in equation 12 is

$$\int_{\theta_{p+\frac{1}{2}}}^{\theta_n}\frac{2D}{\phi}d\theta=\int_{\theta_p}^{\theta_n}\frac{2D}{\phi}d\theta-\frac{\overline{D}_{p+\frac{1}{2}}\delta\theta}{\phi_p}. \tag{18}$$

The finite-difference form for the right side of equation 12 is

$$2D\frac{d\theta}{d\phi}=\frac{2\overline{D}_{p+1/2}\delta\theta}{\phi_{p+1}-\phi_p}. \tag{19}$$

By substitution of equations 16, 18, and 19 into 12 and rearrangement,

$$\phi_{p+1}-\phi_p=\frac{2\overline{D}_{p+1/2}}{\frac{1}{\delta\theta}\int_{\theta_p}^{\theta_n}\left[\phi+\frac{2D}{\phi}\right]d\theta-\left[\frac{\phi_p}{2}+\frac{\overline{D}_{p+1/2}}{\phi_p}\right]}. \tag{20}$$

Let

$$I_{p+1/2}=\frac{1}{\delta\theta}\int_{\theta_p}^{\theta_n}\left[\phi+\frac{2D}{\phi}\right]d\theta-\left[\frac{\phi_p}{2}+\frac{\overline{D}_{p+1/2}}{\phi_p}\right]. \tag{21}$$

Then

$$\phi_{p+1}-\phi_p=\frac{2\overline{D}_{p+1/2}}{I_{p+1/2}}. \tag{22}$$

In similar fashion, it can be shown that

$$\phi_p-\phi_{p-1}=\frac{2\overline{D}_{p-1/2}}{I_{p-1/2}} \tag{23}$$

where

$$I_{p-1/2}=\frac{1}{\delta\theta}\int_{\theta_p}^{\theta_n}\left[\phi+\frac{2D}{\phi}\right]d\theta+\left[\frac{\phi_p}{2}+\frac{\overline{D}_{p-1/2}}{\phi_p}\right]. \tag{24}$$

From equations 21 and 24,

$$I_{p+1/2}-I_{p-1/2}=-\phi_p-\frac{\overline{D}_{p+1/2}+\overline{D}_{p-1/2}}{\phi_p}. \tag{25}$$

When $\overline{D}_{p+1/2}+\overline{D}_{p-1/2}$ is replaced by $2D_p$,

$$I_{p+1/2}-I_{p-1/2}=-\phi_p-\frac{2D_p}{\phi_p}. \tag{26}$$

Equations 22 and 26 form the basis of the iterative procedure for solving equation 12, the transformed version of the basic flow equation 4.

EVALUATION OF I CLOSE TO $\theta=\theta_n$

The iterative solution involves the calculation of θ and ϕ values over the interval $\theta_0\leqq\theta\leqq\theta_n$. At $\theta=\theta_0$, $\phi=0$, and this value can be used directly. However, at $\theta=\theta_n$, $\phi_n=\infty$, and this value can only produce other values of infinity when used to determine $I_{n-1/2}$. The following provides a suitable method for determining $I_{n-1/2}$.

A range of values, $\theta=\theta_{n-\epsilon}$ to $\theta=\theta_n$, can be selected such that the variation of D is small over this range. In such an interval, it can be assumed that D is constant. This is valid for small values of ϵ because changes of water level are small in the vicinity of θ_n. If ϵ is taken as 1, $\theta_{n-\epsilon}$ is sufficiently remote from the well for D to be virtually constant. Therefore, $\overline{D}_{n-1/2}$ will be taken as the constant value of D throughout the interval θ_{n-1} to θ_n, and it will be used to derive $I_{n-1/2}$, which is equal to $I_{p+1/2}$ for $p=n-1$.

As indicated by equation 21, if

$$\int_{\theta_{n-1}}^{\theta_n}\left[\phi+\frac{2D}{\phi}\right]d\theta$$

can be evaluated, $I_{n-1/2}$ can be determined. Therefore, values of ϕ between θ_{n-1} and θ_n will be determined to evaluate this integral.

For constant diffusivity, equation 8 may be written

$$-\frac{\phi}{2}\frac{d\theta}{d\phi}=\overline{D}_{n-1/2}\frac{d^2\theta}{d\phi^2}+\frac{\overline{D}_{n-1/2}}{\phi}\frac{d\theta}{d\phi}. \tag{27}$$

In this case, the equation will be evaluated between the intervals

$$\theta=\theta_{n-1},\phi=\phi_{n-1} \text{ and } \theta=\theta_n,\phi=\phi_n. \tag{28}$$

Let

$$w=\frac{d\theta}{d\phi} \tag{29}$$

(Frank, 1950).

When (29) is substituted in equation 27, one obtains

$$\frac{dw}{w}=-\frac{\phi}{2\overline{D}_{n-1/2}}\,d\phi-\frac{d\phi}{\phi}. \tag{30}$$

Upon integration, equation 30 becomes

$$\ln w=-\frac{\phi^2}{4\overline{D}_{n-1/2}}-\ln\phi+\ln C_1, \tag{31}$$

where C_1 is a constant of integration. When terms are rearranged, equations 29 and 31 become

$$w=\frac{d\theta}{d\phi}=\frac{C_1}{\phi}\,e^{-4\frac{\phi^2}{\overline{D}_{n-1/2}}}. \tag{32}$$

Upon integration, equation 32 becomes

$$\theta=C_1\int\frac{e^{-4\frac{\phi^2}{4\overline{D}_{n-1/2}}}}{\phi}\,d\phi+C_2 \tag{33}$$

Let

$$u=\frac{\phi^2}{4\overline{D}_{n-1/2}} \tag{34}$$

When (34) is substituted in equation 33, one obtains

$$\theta=C_3\int\frac{e^{-u}}{u}\,du+C_2, \tag{35}$$

where the boundary conditions are

$$\theta=\theta_{n-1},u=u_{n-1}=\frac{\phi_{n-1}^2}{4\overline{D}_{n-1/2}} \text{ and } \theta=\theta_n,u=u_n=\frac{\phi_n^2}{4\overline{D}_{n-1/2}}. \tag{36}$$

If the second boundary condition is incorporated in equation 35, one obtains

$$\theta = C_3 \int_u^{u_n} \frac{e^{-u}}{u} \, du + \theta_n. \tag{37}$$

Because $\theta \rightarrow \theta_n = 1$ at a remote distance from the well, where $\phi \rightarrow \infty$, it may be assumed that $u_n = \frac{\phi_n^2}{4\overline{D}_{n-1/2}} = \infty$. Equation 37 can then be expressed in terms of the exponential integral function

$$\theta - \theta_n = C_3 Ei(u). \tag{38}$$

When the first boundary condition is substituted for the beginning of the "n"th interval, one obtains the equation

$$\theta_{n-1} - \theta_n = C_3 Ei\left[\frac{\phi_{n-1}^2}{4\overline{D}_{n-1/2}}\right] \tag{39}$$

or

$$C_3 = -\frac{\delta\theta}{Ei\left[\frac{\phi_{n-1}^2}{4\overline{D}_{n-1/2}}\right]}, \tag{40}$$

where $\delta\theta = \theta_n - \theta_{n-1}$. Thus, after both boundary conditions are satisfied, equation 38 becomes

$$\theta - \theta_n = -\frac{\delta\theta}{Ei\left[\frac{\phi_{n-1}^2}{4\overline{D}_{n-1/2}}\right]} Ei\left[\frac{\phi^2}{4\overline{D}_{n-1/2}}\right], \tag{41}$$

and, after terms are rearranged,

$$Ei\left[\frac{\phi^2}{4\overline{D}_{n-1/2}}\right] = \frac{\theta_n - \theta}{\delta\theta} Ei\left[\frac{\phi_{n-1}^2}{4\overline{D}_{n-1/2}}\right]. \tag{42}$$

With known values of θ_{n-1}, θ_n, ϕ_{n-1}, and $\overline{D}_{n-1/2}$, equation 42 permits the computation of values of ϕ corresponding to any values of θ in the range θ_{n-1} to θ_n.

The integral in equation 21 cannot be evaluated because $\phi \rightarrow \infty$ as $\theta \rightarrow \theta_n$. However, assume that $\phi_{n-3/4}$, $\phi_{n-1/2}$, and $\phi_{n-1/4}$ have been evaluated by means of equation 42. The integral in equation 21 can then be replaced by the following open-quadrature formula:

$$\int_{\theta_{n-1}}^{\theta_n} f(\theta) \, d\theta = \frac{\delta\theta}{3} [2f(\theta_{n-3/4}) - f(\theta_{n-1/2}) + 2f(\theta_{n-1/4})] \tag{43}$$

(Kunz, 1957, p. 139),

where
$$f(\theta) = \phi(\theta) + \frac{2D}{\phi(\theta)}. \tag{44}$$

APPLICATION TO A HYPOTHETICAL PROBLEM

Assume that the pumping level in a discharging well is at the bottom of the aquifer. Although $h_0=0$ is a poor choice so far as the Dupuit assumptions are concerned, it is convenient for illustrative purposes. If one lets $n=10$ and remembers that $\theta_0=0$ and $\theta_n=1$, values of θ_p can be tabulated as shown in table 1.

Assume that the aquifer has a transmissibility of 300 cubic feet per day per foot and its coefficient of storage is 0.30. From equation 2, $D_n=1{,}000$ cubic feet per day per foot. If D is assumed to be linearly related to θ, values of D_p and $\overline{D}_{p-1/2}$ can be listed as shown in table 1.

The computation was begun by assuming a trial value of ϕ_9. In this case, $27.11 \frac{\text{ft}}{\sqrt{\text{day}}}$ was chosen arbitrarily. By use of equation 42, tables of the exponential integral (Federal Works Agency, 1940), and the trial value of ϕ_9, it was found that $\phi_{9\frac{1}{4}}=33.05$, $\phi_{9\frac{1}{2}}=41.36$, and $\phi_{9\frac{3}{4}}=55.02$. These values were substituted into equation 43 and the result of that substitution into equation 21, giving $I_{9\frac{1}{2}}=42.36$.

TABLE 1.—*First iteration*

θ_p	D_p	$\overline{D}_{p-\frac{1}{2}}$	$I_{p-\frac{1}{2}}$	ϕ_p
$\theta_0=0.0$	0			1.84
$\theta_1=.1$	100	50	1,703	1.90
$\theta_2=.2$	200	150	1,596	2.09
$\theta_3=.3$	300	250	1,402	2.45
$\theta_4=.4$	400	350	1,155	3.06
$\theta_5=.5$	500	450	890.3	4.07
$\theta_6=.6$	600	550	640.5	5.79
$\theta_7=.7$	700	650	427.4	8.83
$\theta_8=.8$	800	750	260.1	14.60
$\theta_9=.9$	900	850	135.9	27.11
$\theta_{10}=1.0$	1,000	950	42.36	

Substituting the values of $I_{9\frac{1}{2}}$ and ϕ_9 into equation 26, one obtains $I_{8\frac{1}{2}}=135.9$, and substituting the values of ϕ_9 and $I_{8\frac{1}{2}}$ into equation 22, $\phi_8=14.60$. By alternate use of these equations, the values of $I_{p-1/2}$ and ϕ_p in table 1 were determined.

Although ϕ_0 in table 1 was computed as 1.84, it must equal 0 to satisfy the boundary requirements. Therefore, the original choice of ϕ_9 was too large, and the computation was repeated with a smaller value of ϕ_9. When this procedure was repeated five times, ϕ_0 was obtained very close to 0 as shown in table 2. It is noteworthy that $I_{9\frac{1}{2}}$ approaches 0 as ϕ_0 approaches 0. A mathematical discontinuity makes it impossible to derive a value of ϕ_0 exactly equal to 0.

The values of θ_p and the corresponding values of ϕ_p lead to the solution. If time is fixed in ϕ_p, the variation of ground-water head with distance can be determined at a given time. If distance is fixed in ϕ_p, the variation of ground-water head with time can be determined at a given distance from the discharging well.

As discussed above, a linear relation is assumed between D and θ. Therefore, the solution of equation 4 for $h_0=0$ and for any value of D_n $=1{,}000\ X\ \frac{\text{ft}^3}{\text{day ft}}$ can be determined by multiplying the respective values of ϕ_p in table 2 by $X^{1/2}$. The resulting curves relating θ and ϕ for $h_0=0$ and for different values of D_n are shown in figure 9. Values of ϕ are plotted on a logarithmic scale to improve the usefulness of the curves. However, this method of plotting makes apparent the error in the approximation of the boundary conditions.

Assume that one or more observation wells are available in the vicinity of a well discharging from an unconfined aquifer. If the pumping level in the discharging well is at the bottom of the aquifer, the field curves relating θ and ϕ can be compared with figure 9 and the value of D_n determined for the aquifer. Figure 10 is included to facilitate interpolation between the curves.

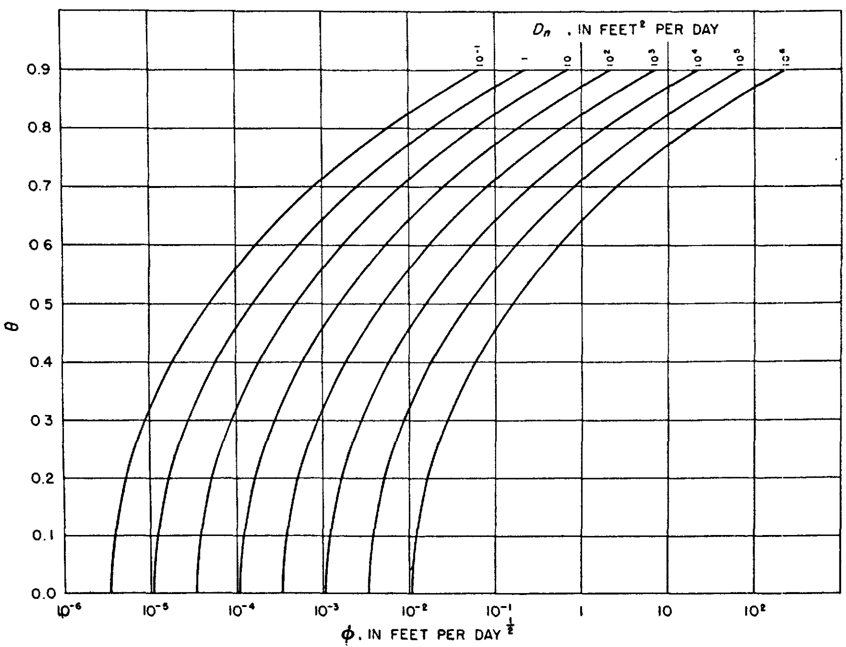

FIGURE 9.—Relation between θ and ϕ for different values of D_n.

TABLE 2.—*Final iteration*

θ_p	D_p	$\overline{D}_{p-1/2}$	$I_{p-1/2}$	ϕ_p
θ_0 =0.0	0			0.0004
θ_1 = .1	100	50	2, 634, 000	.0004
θ_2 = .2	200	150	2, 132, 000	.0005
θ_3 = .3	300	250	1, 390, 000	.0009
θ_4 = .4	400	350	722, 300	.0019
θ_5 = .5	500	450	294, 100	.0049
θ_6 = .6	600	550	91, 180	.0170
θ_7 = .7	700	650	20, 560	.0802
θ_8 = .8	800	750	3, 110	.5625
θ_9 = .9	900	850	265. 1	6. 9750
θ_{10}=1.0	1, 000	950	.07112	

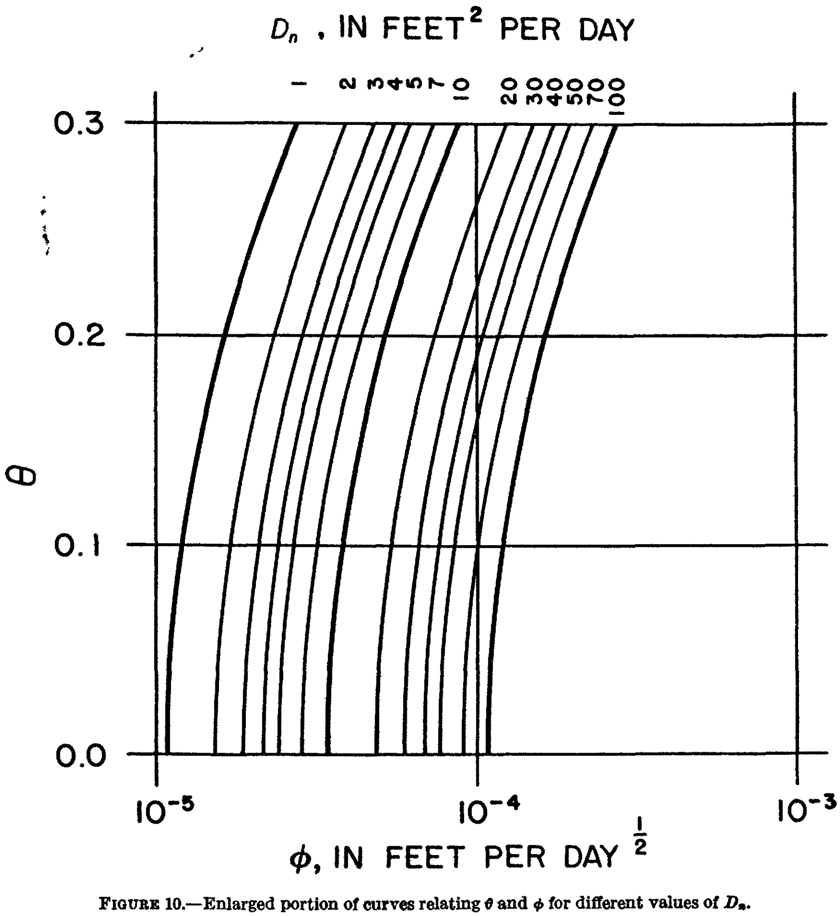

FIGURE 10.—Enlarged portion of curves relating θ and ϕ for different values of D_n.

The assumptions made in applying equation 4 to unsteady unconfined flow limit its range of applicability. It must be emphasized that figures 9 and 10 apply only when the pumping level in the discharging well is at the bottom of the aquifer, or $h_0=0$. This is a condition obviously difficult to attain and one that violates the Dupuit assumptions. However, similar curves can be developed for any values of h_0. In fact, a single set of curves can be developed for use with any value of h_0.

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