Ground-Water Movements and Bank Storage Due to Flood Stages in Surface Streams

GEOLOGICAL SURVEY WATER-SUPPLY PAPER 1536-J

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By H. H. COOPER, Jr. *and* M. I. RORABAUGH

GROUND-WATER HYDRAULICS

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SYMBOLS

A Function defined in equation 11 (Dimensionless)
B Constant defined in equation 32 or 70 (Dimensionless)
a, b Constants defined in equation 14 or 20 (Dimensionless)
h Change in ground-water head at time t and distance x (L)
h_0 Maximum rise of stream stage (L)
l Distance from bank of stream to valley wall (L)
$N = 1/[e^{-\delta t_c}(1-\cos \omega t_c)]$ (Dimensionless)
Q Ground-water flow into stream per unit length at time t (L^2T^{-1})
S Coefficient of storage of aquifer (Dimensionless)
T Transmissibility of aquifer (L^2T^{-1})
t Time since beginning of stage oscillation (T)
t_c Time of flood crest
V Bank storage per unit length of stream at time t (L^2)
x Distance from bank of stream (L)
$\alpha = \pi^2\sigma/4l^2$ (T^{-1})
$\beta = \alpha/\omega = \pi\tau T/8l^2S$ (Dimensionless)
$\delta = \omega \cot (\omega t_c/2)$ Constant determining degree of asymmetry of curves ψ (T^{-1})
$\eta = \delta/\beta\omega$ (Dimensionless)
θ Function defined in equation (12) (Radians)
$\xi = (l-x)/l$ (Dimensionless)
$\sigma = T/S$ Hydraulic diffusivity of aquifer (L^2T^{-1})
τ Period or duration of stage oscillation (T)
φ Constant defined in equation 33 or 71 (Radians)
$\psi(t)$ Function representing stage hydrograph of stream (L)
$\omega = 2\pi/\tau$ Frequency of stage oscillation (T^{-1})

$\overline{F}(x,p) = \int_0^\infty e^{-pt}F(x,t)\,dt$ Laplace transformation of $F(x,t)$

$F_1 * F_2 = \int_0^t F_1(\lambda)F_2(t-\lambda)\,d\lambda$ Convolution (Faltung integral) of functions $F_1(t)$ and $F_2(t)$

GROUND-WATER HYDRAULICS

GROUND-WATER MOVEMENTS AND BANK STORAGE DUE TO FLOOD STAGES IN SURFACE STREAMS

By H. H. Cooper, Jr., and M. I. Rorabaugh

ABSTRACT

Solutions are derived for the changes in ground-water heads, ground-water flow, and bank storage that would occur as the result of a flood-wave stage oscillation described by the equation

$$\psi(t)=\begin{cases}Nh_0e^{-\delta t}(1-\cos\omega t), & \text{when } 0\leq t\leq\tau\\ 0, & \text{when } t\geq\tau\end{cases}$$

where τ is the duration or period of the flood wave, δ is a constant that imparts asymmetry, and N is a constant that causes the curve to peak at the height h_0. Numerical computations are presented for a variety of conditions. These show that in infinite aquifers the ground-water flow and bank storage decline very slowly after the flood wave has passed, the bank storage being as much as 14 percent of the maximum at $t=10\tau$ and 4 percent of the maximum at $t=100\tau$.

INTRODUCTION

The problem of evaluating changes in ground-water heads, ground-water flow, and bank storage caused by flood waves in surface streams and reservoirs arises in hydrologic problems. The shapes of the stage hydrographs that accompany flood waves vary widely and depend upon the characteristics of the drainage basin, the areal and temporal distribution of the storm or snowmelt, and the rating curve of the stream at a given section. However, the shapes of many flood-wave stage hydrographs may be approximated by one of the family of asymmetric curves defined by

$$\psi(t)=\begin{cases}Nh_0e^{-\delta t}(1-\cos\omega t), & \text{when } 0\leq t\leq\tau\\ 0, & \text{when } t\geq\tau\end{cases} \tag{1}$$

where h_0 is the maximum rise in stage, t is the time since the beginning of the flood wave, τ is the duration of the wave,

$$\omega=2\pi/\tau,\ \delta=\omega\ \cot(\omega t_c/2)$$

is a constant that determines the degree of asymmetry, t_c is the time of the flood crest, and

$$N=\frac{1}{e^{-\delta t_c}(1-\cos\omega t_c)}=\frac{\delta^2+\omega^2}{2\omega^2}e^{(2\delta/\omega)\arctan(\omega/\delta)} \qquad (1a)$$

is a constant that serves to make the curves of the family peak at the same height h_0. Curves corresponding to $\delta=\omega$ and $\delta=0$ are shown in the inset of figure 105. For $\delta=0$ the curve is sinusoidal.

Equations are derived for the changes in ground-water heads, ground-water flow to the stream, and bank storage caused by the stage oscillation described by equation 1. It will be assumed that the stream channel fully penetrates the aquifer and has vertical banks. For unconfined aquifers it will be assumed further that the slope and fluctuations of the water table are small in comparison to the thickness of the saturated zone, so that the transmissibility remains practically constant in space and time, and that the stream does not overflow its banks. An initially horizontal water table is assumed without loss of generality since, under the preceding assumptions, the results to be obtained may be superposed on the initial condition. Finally, it will be assumed that the changes in ground-water head are due solely to changes in the stage of the stream; this means that the storm or snowmelt is assumed to occur in some part of the drainage basin that is upstream from the section with which we are concerned.

ACKNOWLEDGMENTS

The authors are greatly indebted to H. R. Henry for his careful review of the mathematics. The computations from which figures 102 and 103 are constructed were performed by D. S. Handwerker of the Branch of Computations of the U.S. Geological Survey. The computations of bank storage for $t>10$ in the table and those used in the construction of the curves for $\delta=\omega$ in figures 105 and 106 were performed by George Ann Osborne.

ANALYSIS

AQUIFER BOUNDED BY VALLEY WALL

An aquifer of finite width l bounded on one side by the stream and on the other by a vertical valley wall (fig. 101) is first considered.

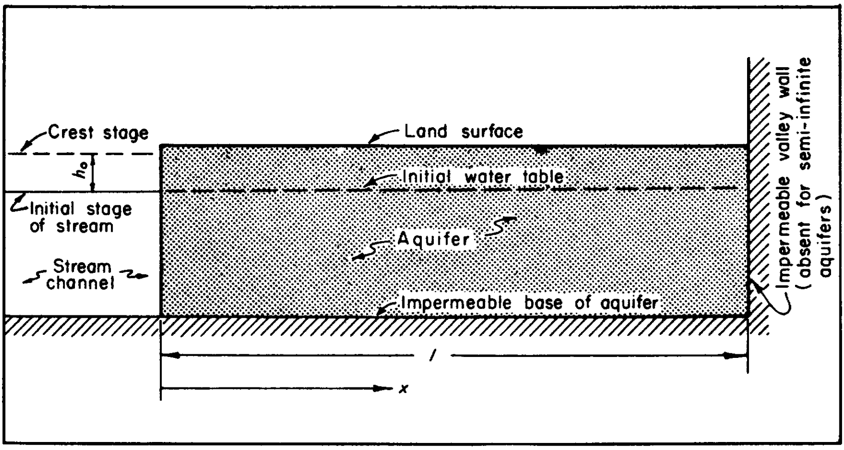

FIGURE 101.—Schematic sketch of aquifer showing assumed conditions.

CASE FOR $\delta>0$

GROUND-WATER HEADS

Two solutions are required, one for $t\leq\tau$ and one for $t\geq\tau$. The problem for $t\leq\tau$ is described by

$$\frac{\partial^2 h}{\partial x^2}-\frac{1}{\sigma}\frac{\partial h}{\partial t}=0 \tag{2}$$

$$h(x, 0)=0, \text{ when } 0\leq x\leq l \tag{2a}$$

$$\frac{\partial h(l,t)}{\partial x}=0, \text{ when } t\geq 0 \tag{2b}$$

$$h(0, t)=Nh_0e^{-\delta t}(1-\cos \omega t), \text{ when } 0\leq t\leq\tau. \tag{2c}$$

Equation 2 is the differential equation governing nonsteady flow of confined ground water. (See, for example, Jacob, 1949, p. 333.) It applies approximately to the flow of unconfined ground water provided the variations in the height of the water table are small in relation to the average depth of flow.

By applying the Laplace transformation with respect to time, the problem is reduced to

$$\frac{d^2\bar{h}}{dx^2}-\frac{p}{\sigma}\bar{h}=0 \tag{3}$$

$$\frac{d\bar{h}(l, p)}{dx}=0 \tag{3a}$$

$$\bar{h}(0, p)=Nh_0\left\{\frac{\omega^2}{(p+\delta)[(p+\delta)^2+\omega^2]}\right\} \tag{3b}$$

for which the solution is

$$\bar{h}(p)=Nh_0\left\{\frac{\omega^2}{(p+\delta)[(p+\delta)^2+\omega^2]}\right\}\frac{\cosh\,[(l-x)\sqrt{p/\sigma}]}{\cosh\,[l\sqrt{p/\sigma}]}. \tag{4}$$

The solution of the problem is the inverse transform of equation 4

$$h=\frac{Nh_0}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}\bar{h}(p)\,dp=\frac{Nh_0}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} f(p)\,dp \tag{5}$$

where γ is so large that $\bar{h}(p)$ is analytic in the half plane $R(p)\geq\gamma$ and where $f(p)=e^{pt}\bar{h}(p)$.

Because $\cosh\, l\sqrt{p/\sigma}$ is a single-valued function of p, $f(p)$ is also single-valued, and its singularities are poles. With this in mind, one can show by contour integration and Cauchy's residue theorem that the integral in equation 5 is $2\pi i$ times the sum of the residues of $f(p)$. Thus,

$$h=Nh_0\Sigma\ \mathrm{Res}\, f(p). \tag{6}$$

The function $f(p)$ has simple poles at $p=-\delta\pm i\omega$, $p=-\delta$, and those values of p for which $\cosh\, l\sqrt{p/\sigma}=0$. The latter values are given by

$$p=-\frac{(2n-1)^2\sigma\pi^2}{4l^2}=-(2n-1)^2\alpha, \qquad n=1,\,2,\,3,\,\ldots \tag{7}$$

The residue at $p=-\delta+i\omega$ is

$$\underset{p=-\delta+i\omega}{\mathrm{Res}}\, f(p)=\left[\frac{\omega^2e^{pt}}{(p+\delta)\,(d/dp)[(p+\delta)^2+\omega^2]}\,\frac{\cosh\,[(l-x)\sqrt{p/\sigma}]}{\cosh\,[l\sqrt{p/\sigma}]}\right]_{p=-\delta+i\omega}$$
$$=-\frac{e^{(-\delta+i\omega)t}}{2}\,\frac{\cosh\left[(l-x)\sqrt{(-\delta+i\omega)/\sigma}\right]}{\cosh\left[l\sqrt{(-\delta+i\omega)/\sigma}\right]} \tag{8}$$

and that at $p=-\delta-i\omega$ is

$$\underset{p=-\delta-i\omega}{\mathrm{Res}}\, f(p)=-\frac{e^{(-\delta-i\omega)t}}{2}\,\frac{\cosh\left[(l-x)\sqrt{(-\delta-i\omega)/\sigma}\right]}{\cosh\left[l\sqrt{(-\delta-i\omega)/\sigma}\right]} \tag{9}$$

By adding equations 8 and 9 and simplifying, it may be shown that

$$\underset{p=-\delta+i\omega}{\mathrm{Res}}\, f(p)+\underset{p=-\delta-i\omega}{\mathrm{Res}}\, f(p)=-e^{-\delta t}\mathrm{A}\cos\,(\omega t+\theta) \tag{10}$$

where

$$\mathrm{A}=\left[\frac{\cos^2 a\xi+\sinh^2 b\xi}{\cos^2 a+\sinh^2 b}\right]^{1/2}, \tag{11}$$

$$\theta=\arctan\left[\frac{f_s(a\xi,\,b\xi)f_c(a,\,b)-f_c(a\xi,\,b\xi)f_s(a,\,b)}{f_c(a\xi,\,b\xi)f_c(a,\,b)+f_s(a\xi,\,b\xi)f_s(a,\,b)}\right], \tag{12}$$

$$f_s(u,\,v)=\sin u\,\sinh v, \qquad f_c(u,\,v)=\cos u\,\cosh v, \tag{12a}$$

$$\xi=\frac{l-x}{l} \tag{13}$$

$$a=l\left[\frac{(\delta^2+\omega^2)^{1/2}+\delta}{2\sigma}\right]^{1/2}, \qquad b=l\left[\frac{(\delta^2+\omega^2)^{1/2}-\delta}{2\sigma}\right]^{1/2}. \tag{14}$$

The residue at $p=-\delta$ is

$$\operatorname*{Res}_{p=-\delta} f(p)=\left[\frac{\omega^2 e^{pt}}{[(p+\delta)^2+\omega^2](d/dp)(p+\delta)}\,\frac{\cosh[(l-x)\sqrt{p/\sigma}]}{\cosh[l\sqrt{p/\sigma}]}\right]_{p=-\delta}$$

$$=e^{-\delta t}\frac{\cosh[i(l-x)\sqrt{\delta/\sigma}]}{\cosh[il\sqrt{\delta/\sigma}]}$$

$$=e^{-\delta t}\frac{\cos[(l-x)\sqrt{\delta/\sigma}]}{\cos[l\sqrt{\delta/\sigma}]} \tag{15}$$

and the residue at $p=-(2n-1)^2\alpha$ is

$$\operatorname*{Res}_{p=-(2n-1)^2\alpha} f(p)=\left[\frac{\omega^2 e^{pt}}{(p+\delta)[(p+\delta)^2+\omega^2]}\,\frac{\cosh[(l-x)\sqrt{p/\sigma}]}{(d/dp)\cosh[l\sqrt{p/\sigma}]}\right]_{p=-(2n-1)^2\alpha}$$

$$=\frac{2\sqrt{\alpha\sigma}}{l}\,\frac{(2n-1)\omega^2 e^{-(2n-1)^2\alpha t}}{[\delta-(2n-1)^2\alpha]\{[\delta-(2n-1)^2]^2+\omega^2\}}$$

$$\times\frac{\cos[(2n-1)(l-x)\sqrt{\alpha/\sigma}]}{\sin[(2n-1)l\sqrt{\alpha/\sigma}]}. \tag{16}$$

With the substitution $\alpha=\pi^2\sigma/4l^2$ in the quantities under the square root radical, equation 16 reduces to

$$\operatorname*{Res}_{p=-(2n-1)^2\alpha} f(p)=\frac{\pi\sigma}{l^2}\sin[(2n-1)\pi x/2l]\,\frac{(2n-1)\omega^2 e^{-(2n-1)^2\alpha t}}{[\delta-(2n-1)^2\alpha]\{[\delta-(2n-1)^2\alpha]^2+\omega^2\}}. \tag{17}$$

Substituting from equations 10, 15, and 17 into equation 6, one finds the solution for $t\leq\tau$ to be

$$h_{t\leq\tau}=Nh_0\Bigg\{e^{-\delta t}\left[\frac{\cos[(l-x)\sqrt{\delta/\sigma}]}{\cos[l\sqrt{\delta/\sigma}]}-A\cos(\omega t+\theta)\right]$$

$$+\frac{\pi\sigma}{l^2}\sum_{n=1}^{\infty}\sin[(2n-1)\pi x/2l]\,\frac{(2n-1)\omega^2 e^{-(2n-1)^2\alpha t}}{[\delta-(2n-1)^2\alpha]\{[\delta-(2n-1)^2\alpha]^2+\omega^2\}}\Bigg\}. \tag{18}$$

Introducing the dimensionless parameters $\beta=\alpha/\omega=\pi\sigma\tau/8l^2$, $\eta=\delta/\beta\omega$, one can write solution 18 as

$$h_{t\leq\tau}=Nh_0\Bigg\{e^{-\eta\beta\omega t}\left[\frac{\cos[(l-x)\pi\sqrt{\eta}/2l]}{\cos[\pi\sqrt{\eta}/2]}-A\cos(\omega t+\theta)\right]$$

$$+\frac{4}{\pi}\sum_{n=1}^{\infty}\sin[(2n-1)\pi x/2l]\,\frac{(2n-1)e^{-(2n-1)^2\beta\omega t}}{[\eta-(2n-1)^2]+[\eta-(2n-1)^2]^3\beta^2}\Bigg\} \tag{19}$$

where A and θ are as defined in equations 11 to 14, except that equations 14 are now written

$$a=\pi\left[\frac{(\eta^2+1/\beta^2)^{1/2}+\eta}{8}\right]^{1/2},\qquad b=\pi\left[\frac{(\eta^2+1/\beta^2)^{1/2}-\eta}{8}\right]^{1/2}. \tag{20}$$

The solution for $t \geq \tau$ must satisfy equation 2 and the conditions

$$h(x, \tau) = h_{t \leq \tau}(x, \tau), \text{ when } 0 \leq x \leq l \tag{21a}$$

$$\frac{\partial h(l, t)}{\partial x} = 0, \text{ when } t \geq \tau \tag{21b}$$

$$h(0, t) = 0, \text{ when } t \geq \tau \tag{21c}$$

where the right side in equation 21*a* is the result obtained by substituting $t = \tau$ in solution 19. The solution for $t \geq \tau$ may be found from these conditions. However, it may be found more readily by using the principle of superposition. When applying this principle, one recognizes that the stage hydrograph described by equation 1 may be considered as consisting of two components, one beginning at $t = 0$ and continuing for all positive t, and another of equal magnitude but of opposite sign beginning at $t = \tau$. Thus, equation 1 may be written

$$\psi(t) = \psi_1(t) + \psi_2(t) \tag{22}$$

where

$$\psi_1(t) = Nh_0 e^{-\delta t}(1 - \cos \omega t), \text{ when } t \geq 0 \tag{23}$$

$$\psi_2(t) = \begin{cases} 0, \text{ when } t \leq \tau \\ -Nh_0 e^{-\delta t}(1 - \cos \omega t), \text{ when } t \geq \tau. \end{cases} \tag{24}$$

Equation 24 is equivalent to

$$\psi_2(t) = -e^{-\delta\tau} N h_0 e^{-\delta(t-\tau)}[1 - \cos \omega(t - \tau)], \text{ when } t - \tau \geq 0. \tag{25}$$

By the principle of superposition the solution h is the sum of the two solutions h_1 and h_2 corresponding, respectively, to the components $\psi_1(t)$ and $\psi_2(t)$ of the stage-hydrograph equation. The solution h_1 corresponding to $\psi_1(t)$ is equation 19 without the restriction $t \leq \tau$. The solution h_2 corresponding to equation 25 is equation 19 multiplied by the constant $-e^{-\delta\tau} = -e^{-2\pi\eta\beta}$ and with the substitution $t - \tau$ for t. After summing h_1 and h_2 and simplifying one finds the solution $h_{t \geq \tau}$ to be

$$h_{t \geq \tau} = [h_1 + h_2]_{t \geq \tau} = \frac{4Nh_0}{\pi} \sum_{n=1}^{\infty} \sin [(2n-1)\pi x/2l] \times \frac{(2n-1)\left[1 - e^{-[\eta - (2n-1)^2]2\pi\beta}\right] e^{-(2n-1)^2 \beta\omega t}}{[\eta - (2n-1)^2] + [\eta - (2n-1)^2]^3 \beta^2}. \tag{26}$$

It may be shown by substitution that equation 26 satisfies equations 2, 21*b*, and 21*c*. That it also satisfies equation 21*a* is shown by substituting $t = 0$ in equation 19 to obtain the equality

$$\frac{\cos [(l-x)\pi\sqrt{\eta}/2l]}{\cos [\pi\sqrt{\eta}/2l]} - A \cos \theta = -\frac{4}{\pi} \sum_{n=1}^{\infty} \sin [(2n-1)\pi x/2l] \frac{(2n-1)}{[\eta - (2n-1)^2] + [\eta - (2n-1)^2]^3 \beta^2}, \tag{27}$$

and by substituting $t=\tau$ and equality 27 in solution 26 to obtain

$$h_{t=\tau}=Nh_0\left\{e^{-2\pi\eta\beta}\left[\frac{\cos\left[(l-x)\pi\sqrt{\eta}/2l\right]}{\cos\left[\pi\sqrt{\eta}/2\right]}-A\cos\theta\right]\right.$$
$$\left.+\frac{4}{\pi}\sum_{n=1}^{\infty}\sin\left[(2n-1)\pi x/2l\right]\frac{(2n-1)e^{-(2n-1)^2 2\pi\beta}}{[\eta-(2n-1)^2]+[\eta-(2n-1)^2]^3\beta^2}\right\}. \quad (28)$$

This is identical with the result obtained by substituting $t=\tau$ in equation 19, and hence condition 21*a* is satisfied by solution 26.

GROUND-WATER FLOW INTO STREAM

Under the assumptions previously stated, the ground-water flow per unit length of stream in the direction of increasing x is, according to Darcy's law, $Q=-T(\partial h/\partial x)$. Hence the flow per unit length into the stream is

$$Q=T\frac{\partial h(0,t)}{\partial x}. \quad (29)$$

By differentiating equations 19 and 26 with respect to x, setting $x=0$, and substituting the results in equation 29, the equations for the flow per unit length of stream are found to be

$$Q_{t\leq\tau}=Nh_0\sqrt{\omega TS}\left\{e^{-\eta\beta\omega t}\left[\sqrt{\eta\beta}\tan\frac{\pi\sqrt{\eta}}{2}+B\cos(\omega t+\varphi)\right]\right.$$
$$\left.+\frac{4\sqrt{\beta}}{\pi}\sum_{n=1}^{\infty}\frac{(2n-1)^2e^{-(2n-1)^2\beta\omega t}}{[\eta-(2n-1)^2]+[\eta-(2n-1)^2]^3\beta^2}\right\} \quad (30)$$

$$Q_{t\geq\tau}=Nh_0\sqrt{\omega TS}\cdot\frac{4\sqrt{\beta}}{\pi}\sum_{n=1}^{\infty}\frac{(2n-1)^2[1-e^{-[\eta-(2n-1)^2]2\pi\beta}]e^{-(2n-1)^2\beta\omega t}}{[\eta-(2n-1)^2]+[\eta-(2n-1)^2]^3\beta^2} \quad (31)$$

where

$$B=(\eta^2\beta^2+1)^{1/4}\left[\frac{\cosh 2b-\cos 2a}{\cosh 2b+\cos 2a}\right]^{1/2} \quad (32)$$

$$\varphi=\arctan\left[\frac{a\sinh 2b+b\sin 2a}{b\sinh 2b-a\sin 2a}\right], \quad (33)$$

and a and b are as defined in equation 14 or 20.

BANK STORAGE

The bank storage per unit length of stream at time t is defined by

$$V=-\int_0^t Q\,dt. \quad (34)$$

For $t\leq\tau$ this is

$$V_{t\leq\tau}=-\int_0^t Q_{t\leq\tau}dt, \quad (35)$$

and for $t \geq \tau$ it is

$$V_{t \geq \tau} = -\int_0^{\tau} Q_{t \leq \tau} dt - \int_{\tau}^{t} Q_{t \geq \tau} dt. \tag{36}$$

Integrating equation 30 with respect to t and substituting the result in equation 35 yields the solution

$$V_{t \leq \tau} = Nh_0 \sqrt{\frac{TS}{\omega}} \left\{ e^{-\eta\beta\omega t} \left[\frac{1}{\sqrt{\eta\beta}} \tan \frac{\pi\sqrt{\eta}}{2} - \frac{B}{\eta^2\beta^2+1} (\sin (\omega t+\varphi) - \eta\beta \cos (\omega t+\varphi)) \right] - \left[\frac{1}{\sqrt{\eta\beta}} \tan \frac{\pi\sqrt{\eta}}{2} - \frac{B}{\eta^2\beta^2+1} (\sin \varphi - \eta\beta \cos \varphi) \right] + \frac{4\sqrt{\beta}}{\pi} \sum_{n=1}^{\infty} \frac{e^{-(2n-1)^2\beta\omega t} - 1}{[\eta-(2n-1)^2]\beta + [\eta-(2n-1)^2]^3\beta^3} \right\}. \tag{37}$$

This solution may be simplified as follows. By setting $t=0$ in equation 30 and in its derivative and equating the results to zero one obtains the two equalities

$$\eta\beta\sqrt{\eta\beta} \tan \frac{\pi\sqrt{\eta}}{2} + \eta\beta B \cos \varphi = -\frac{4\sqrt{\beta}}{\pi} \sum_{n=1}^{\infty} \frac{\eta\beta(2n-1)^2}{[\eta-(2n-1)^2] + [\eta-(2n-1)^2]^3\beta^2} \tag{38}$$

$$\eta\beta\sqrt{\eta\beta} \tan \frac{\pi\sqrt{\eta}}{2} + \eta\beta B \cos \varphi + B \sin \varphi = -\frac{4\sqrt{\beta}}{\pi} \sum_{n=1}^{\infty} \frac{\beta(2n-1)^4}{[\eta-(2n-1)^2] + [\eta-(2n-1)^2]^3\beta^2}. \tag{39}$$

From the series expansion

$$\tan x = \sum_{n=1}^{\infty} \frac{8x}{(2n-1)^2\pi^2 - 4x^2}$$

(Peirce, 1956, p. 104),

$$\sqrt{\beta\eta} \tan \frac{\pi\sqrt{\eta}}{2} = -\frac{4\sqrt{\beta}}{\pi} \sum_{n=1}^{\infty} \frac{\eta}{\eta-(2n-1)^2}. \tag{40}$$

By substituting equation 40 in equation 38 and then solving equal tions 38 and 39, we establish the equalities

$$B \cos \varphi = \frac{4\sqrt{\beta}}{\pi} \sum_{n=1}^{\infty} \frac{1+\eta[\eta-(2n-1)^2]\beta^2}{1+[\eta-(2n-1)^2]^2\beta^2} \tag{41}$$

$$B \sin \varphi = \frac{4\sqrt{\beta}}{\pi} \sum_{n=1}^{\infty} \frac{(2n-1)^2\beta}{1+[\eta-(2n-1)^2]^2\beta^2}. \tag{42}$$

Finally, a substitution of the equalities 40, 41, and 42 into equation 37 reduces the solution to

$$V_{t\leq\tau}=Nh_0\sqrt{\frac{TS}{\omega}}\left\{e^{-\eta\beta\omega t}\left[\frac{1}{\sqrt{\eta\beta}}\tan\frac{\pi\sqrt{\eta}}{2}-\frac{B}{\sqrt{\eta^2\beta^2+1}}\sin\left(\omega t+\varphi-\arctan\eta\beta\right)\right]\right.$$

$$\left.+\frac{4\sqrt{\beta}}{\pi}\sum_{n=1}^{\infty}\frac{e^{-(2n-1)^2\beta\omega t}}{[\eta-(2n-1)^2]\beta+[\eta-(2n-1)^2]^3\beta^3}\right\}. \quad (43)$$

To obtain the two terms on the right side in equation 36, equations 30 and 31 are integrated with respect to t. These results substituted in equation 36 yield

$$V_{t\geq\tau}=Nh_0\sqrt{\frac{TS}{\omega}}\left\{(e^{-2\pi\eta\beta}-1)\left[\frac{1}{\sqrt{\eta\beta}}\tan\frac{\pi\sqrt{\eta}}{2}-\frac{B}{\eta^2\beta^2+1}(\sin\varphi-\eta\beta\cos\varphi)\right]\right.$$

$$\left.-\frac{4\sqrt{\beta}}{\pi}\sum_{n=1}^{\infty}\frac{1-e^{-2\pi\eta\beta}+[e^{-[\eta-(2n-1)^2]2\pi\beta}-1]e^{-(2n-1)^2\beta\omega t}}{[\eta-(2n-1)^2]\beta+[\eta-(2n-1)^2]^3\beta^3}\right\}. \quad (44)$$

A substitution of the equalities 40, 41, and 42 into equation 44 reduces this result to

$$V_{t\geq\tau}=-Nh_0\sqrt{\frac{TS}{\omega}}\cdot\frac{4\sqrt{\beta}}{\pi}\sum_{n=1}^{\infty}\frac{[e^{-[\eta-(2n-1)^2]2\pi\beta}-1]e^{-(2n-1)^2\beta\omega t}}{[\eta-(2n-1)^2]\beta+[\eta-(2n-1)^2]^3\beta^3} \quad (45)$$

CASE FOR $\delta=\beta\omega(2n-1)^2$

In their present forms, the solutions 19, 26, 30, 31, 43, and 45 are not defined for values of $\delta=\beta\omega(2n-1)^2$, corresponding to $\eta=(2n-1)^2$, $n=1, 2, 3, \ldots$, because the terms that contain trigonometric functions of $(\pi\sqrt{\eta}/2)$ and the summed terms, which contain $[\eta-(2n-1)^2]$ as a factor in the denominator, become infinite for these values. To avoid this difficulty, alternate forms of the solutions will be obtained.

GROUND-WATER HEADS

Equation 4 may be written

$$\bar{h}(p)=\left\{\frac{Nh_0p\omega^2}{(p+\delta)[(p+\delta)^2+\omega^2]}\right\}\left\{\frac{\cosh\,[(l-x)\sqrt{p/\sigma}]}{p\cosh\,[l\sqrt{p/\sigma}]}\right\}. \quad (46)$$

Here, the first quantity in braces is the Laplace transform of the derivative of the function

$$\psi(t)=Nh_0e^{-\delta t}(1-\cos\omega t)$$

or the transform of

$$\psi'(t)=\frac{d\psi}{dt}=Nh_0e^{-\delta t}[\omega\sin\omega t-\delta(1-\cos\omega t)]$$

$$=Nh_0\omega e^{-\eta\beta\omega t}[\sin\omega t-\eta\beta(1-\cos\omega t)]. \quad (47)$$

The second quantity in braces is found by contour integration to be the Laplace transform of

$$g(x, t) = 1 - \frac{4}{\pi} \sum_{n=1}^{\infty} \sin\,[(2n-1)\pi x/2l]\, \frac{e^{-(2n-1)^2 \alpha t}}{(2n-1)}$$

$$= 1 - \frac{4}{\pi} \sum_{n=1}^{\infty} \sin\,[(2n-1)\pi x/2l]\, \frac{e^{-(2n-1)^2 \beta \omega t}}{(2n-1)}, \tag{48}$$

which is the solution of equation 2 with the boundary condition 2*c* changed to $h(0, t) = 1$ for $t > 0$. Equation 48, therefore, represents the position of the water table or piezometric surface after a sudden unit rise in river stage. It is analogous to a solution given by Churchill (1944, p. 196) for the temperature change in a wall whose face $x = l$ is insulated and whose face $x = 0$ is kept at unit temperature if the initial temperature is zero. Also, it is equivalent to the complement of a solution by Glover (see Dumm, 1954, eq. 2) for the position of the water table in an aquifer bounded by two parallel drains after a sudden rise of the water table due to recharge.

By using equations 46, 47, and 48, $h_{t\leq\tau}$ is obtained by convolution. Thus,

$$h_{t\leq\tau} = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt} \bar{\psi}' \bar{g}\, dp = \psi' * g = \int_0^t \psi'(\lambda) g(x, t-\lambda)\, d\lambda. \tag{49}$$

Performing the indicated integration in equation 49, one finds

$$h_{t\leq\tau} = Nh_0 \Bigg\{ (1 - \cos \omega t) e^{-\eta\beta\omega t} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin\,[(2n-1)\pi x/2l]$$
$$\times \Bigg[\frac{\eta[e^{-\eta\beta\omega t} - e^{-(2n-1)^2\beta\omega t}]}{\eta-(2n-1)^2} + \frac{1+\eta[\eta-(2n-1)^2]\beta^2 e^{-(2n-1)^2\beta\omega t}}{1+[\eta-(2n-1)^2]^2\beta^2}$$
$$+ e^{-\eta\beta\omega t} \left(\frac{(2n-1)^2\beta}{1+[\eta-(2n-1)^2]^2\beta^2} \sin \omega t - \frac{1+\eta[\eta-(2n-1)^2]\beta^2}{1+[\eta-(2n-1)^2]^2\beta^2} \cos \omega t \right) \Bigg] \Bigg\}. \tag{50}$$

Here, the only term that contains $\eta - (2n-1)^2$ as a factor in the denominator is the first term in the brackets. By l'Hospital's rule of limits

$$\lim_{\eta \to (2n-1)^2} \frac{\eta[e^{-\eta\beta\omega t} - e^{-(2n-1)^2\beta\omega t}]}{\eta - (2n-1)^2} = -\eta\beta\omega t e^{-\eta\beta\omega t}. \tag{51}$$

Therefore, in performing the summation indicated in equation 50, this term should be assigned the value indicated in equation 51 for the value of n such that $\eta - (2n-1)^2 = 0$.

It may be noted in passing that equations 49 and 50 are identical to the result obtained by using the first part of equation 1 and equation 48 in the formula of Duhamel (see Sneddon, 1951, p. 164). This formula gives a solution corresponding to a variable boundary condition in terms of a known solution corresponding to a constant boundary condition. As previously mentioned, equation 48 is obtainable from the complement of Glover's solution.

Equation 26 contains the factor

$$\gamma=\frac{1-e^{-[\eta-(2n-1)^2]2\pi\beta}}{[\eta-(2n-1)^2]+[\eta-(2n-1)^2]^3\beta^2} \tag{52}$$

which in turn contains $\eta-(2n-1)^2$ as a factor in the denominator. Applying l'Hospital's rule to this factor, one obtains

$$\lim_{\eta\to(2n-1)^2}\gamma=2\pi\beta \tag{53}$$

Therefore, when performing the summation in solution 26, one assigns to γ the value indicated in equation 53 whenever n is such that $\eta-(2n-1)^2=0$. With this substitution equation 26 is written

$$h_{t\geq\tau}=Nh_0\left\{8\beta\sqrt{\eta}e^{-\eta\beta\omega t}\sin\left[\pi x\sqrt{\eta}/2l\right]+\frac{4}{\pi}\left[\sum_{n=1}^{(\sqrt{\eta}-1)/2}+\sum_{n=(\sqrt{\eta}+3)/2}^{\infty}\right]\sin\left[(2n-1)\pi x/2l\right]\right.$$
$$\left.\times\frac{(2n-1)[1-e^{-[\eta-(2n-1)^2]2\pi\beta}]e^{-(2n-1)^2\beta\omega t}}{[\eta-(2n-1)^2]+[\eta-(2n-1)^2]^3\beta^2}\right\}. \tag{54}$$

Where $\frac{\sqrt{\eta}-1}{2}<1$, the first summation is empty and is interpreted as zero.

GROUND-WATER FLOW INTO STREAM

By substituting from equation 40 and by resolving the summed term into partial fractions, one can write equation 30 in the form

$$Q_{t\leq\tau}=Nh_0\sqrt{\omega TS}\left\{B\cos(\omega t+\varphi)e^{-\eta\beta\omega t}+\frac{4\sqrt{\beta}}{\pi}\sum_{n=1}^{\infty}\left[\frac{\eta[e^{-(2n-1)^2\beta\omega t}-e^{-\eta\beta\omega t}]}{\eta-(2n-1)^2}\right.\right.$$
$$\left.\left.-\frac{\{1+\eta[\eta-(2n-1)^2]\beta^2\}e^{-(2n-1)^2\beta\omega t}}{1+[\eta-(2n-1)^2]^2\beta^2}\right]\right\} \tag{55}$$

where the first summed term is given the value indicated in equation 51 for n such that $\eta-(2n-1)^2=0$. By substituting from equation 51 into equation 55 and re-collecting the partial fractions, one obtains

$$Q_{t\leq\tau}=Nh_0\sqrt{\omega TS}\left\{e^{-\eta\beta\omega t}\left[\frac{4\sqrt{\beta}}{\pi}(\eta\beta\omega t-1)+B\cos(\omega t+\varphi)\right]+\frac{4\sqrt{\beta}}{\pi}\left[\sum_{n=1}^{(\sqrt{\eta}-1)/2}\right.\right.$$
$$\left.\left.+\sum_{n=(\sqrt{\eta}+3)/2}^{\infty}\right]\left[\frac{(2n-1)^2e^{-(2n-1)^2\beta\omega t}}{[\eta-(2n-1)^2]+[\eta-(2n-1)^2]^3\beta^2}-\frac{\eta e^{-\eta\beta\omega t}}{\eta-(2n-1)^2}\right]\right\} \tag{56}$$

Substitution from equation 53 into equation 31 gives

$$Q_{t\geq\tau}=Nh_0\sqrt{\omega TS}\cdot\frac{4\sqrt{\beta}}{\pi}\Bigg\{2\pi\eta\beta e^{-\eta\beta\omega t}+\left[\sum_{n=1}^{(\sqrt{\eta}-1)/2}+\sum_{n=(\sqrt{\eta}+3)/2}^{\infty}\right]\left[\frac{(2n-1)^2[1-e^{-[\eta-(2n-1)^2]2\pi\beta}]e^{-(2n-1)^2\beta\omega t}}{[\eta-(2n-1)^2]+[\eta-(2n-1)^2]^3\beta^2}\right]\Bigg\}. \quad (57)$$

BANK STORAGE

Integrating equation 56 with respect to t and substituting the result in equation 35 one obtains

$$V_{t\leq\tau}=Nh_0\sqrt{\frac{TS}{\omega}}\Bigg\{e^{-\eta\beta\omega t}\left[\frac{4\sqrt{\beta}}{\pi}\,\omega t-\frac{B}{\sqrt{\eta^2\beta^2+1}}\sin\,(\omega t+\varphi-\arctan\,\eta\beta)\right]+\frac{4\sqrt{\beta}}{\pi}\left[\sum_{n=1}^{(\sqrt{\eta}-1)/2}+\sum_{n=(\sqrt{\eta}+3)/2}^{\infty}\right]\left[\frac{e^{-(2n-1)^2\beta\omega t}}{[\eta-(2n-1)^2]\beta+[\eta-(2n-1)^2]^3\beta^3}-\frac{e^{-\eta\beta\omega t}}{[\eta-(2n-1)^2]\beta}\right]\Bigg\}. \quad (58)$$

Finally, substituting from equation 53 into equation 45 one finds

$$V_{t\geq\tau}=Nh_0\sqrt{\frac{TS}{\omega}}\cdot\frac{4\sqrt{\beta}}{\pi}\Bigg\{2\pi e^{-\eta\beta\omega t}-\left[\sum_{n=1}^{(\sqrt{\eta}-1)/2}+\sum_{n=(\sqrt{\eta}+3)/2}^{\infty}\right]\frac{e^{-(2n-1)^2\beta\omega t}[e^{-[\eta-(2n-1)^2]2\pi\beta}-1]}{[\eta-(2n-1)^2]\beta+[\eta-(2n-1)^2]^3\beta^3}\Bigg\}. \quad (59)$$

CASE FOR $\delta=0$

For the special case in which $\delta=0$ and $\eta=0$, equation 1 reduces to

$$\psi(t)=\begin{cases}(h_0/2)\,(1-\cos\,\omega t), \text{ when } 0\leq t\leq\tau\\ 0, \text{ when } t\geq\tau\end{cases} \quad (60)$$

which defines a simple sinusoid, and the solutions 19, 26, 30, 31, 43, and 45 reduce to

$$h_{t\leq\tau}=\frac{h_0}{2}\left\{1-A\cos\,(\omega t+\theta)-\frac{4}{\pi}\sum_{n=1}^{\infty}\sin\,[(2n-1)\pi x/2l]\,\frac{e^{-(2n-1)^2\beta\omega t}}{(2n-1)+(2n-1)^5\beta^2}\right\} \quad (61)$$

$$h_{t\geq\tau}=-\frac{2h_0}{\pi}\sum_{n=1}^{\infty}\sin\,[(2n-1)\pi x/2l]\,\frac{[1-e^{(2n-1)^2 2\pi\beta}]e^{-(2n-1)^2\beta\omega t}}{(2n-1)+(2n-1)^5\beta^2} \quad (62)$$

$$Q_{t\leq\tau}=\frac{h_0}{2}\sqrt{\omega TS}\left\{B\cos\,(\omega t+\varphi)-\frac{4\sqrt{\beta}}{\pi}\sum_{n=1}^{\infty}\frac{e^{-(2n-1)^2\beta\omega t}}{1+(2n-1)^4\beta^2}\right\} \quad (63)$$

$$Q_{t\geq\tau}=\frac{h_0}{2}\sqrt{\omega TS}\cdot\frac{4\sqrt{\beta}}{\pi}\sum_{n=1}^{\infty}\frac{[e^{(2n-1)^2 2\pi\beta}-1]e^{-(2n-1)^2\beta\omega t}}{1+(2n-1)^4\beta^2} \quad (64)$$

$$V_{t\leq\tau}=\frac{h_0}{2}\sqrt{\frac{TS}{\omega}}\left\{\frac{\pi}{2\sqrt{\beta}}-B\sin\,(\omega t+\varphi)-\frac{4\sqrt{\beta}}{\pi}\sum_{n=1}^{\infty}\frac{e^{-(2n-1)^2\beta\omega t}}{(2n-1)^2\beta+(2n-1)^6\beta^3}\right\} \quad (65)$$

$$V_{t\geq\tau}=\frac{h_0}{2}\sqrt{\frac{TS}{\omega}}\cdot\frac{4\sqrt{\beta}}{\pi}\sum_{n=1}^{\infty}\frac{[e^{(2n-1)^2 2\pi\beta}-1]e^{-(2n-1)^2\beta\omega t}}{(2n-1)^2\beta+(2n-1)^6\beta^3} \quad (66)$$

where

$$A=\left[\frac{\cos^2(\pi\xi/\sqrt{8\beta})+\sinh^2(\pi\xi/\sqrt{8\beta})}{\cos^2(\pi/\sqrt{8\beta})+\sinh^2(\pi/\sqrt{8\beta})}\right]^{\frac{1}{2}} \tag{67}$$

$$\theta=\arctan\left[\frac{f_s(\pi\xi/\sqrt{8\beta})f_c(\pi/\sqrt{8\beta})-f_c(\pi\xi/\sqrt{8\beta})f_s(\pi/\sqrt{8\beta})}{f_c(\pi\xi/\sqrt{8\beta})f_c(\pi/\sqrt{8\beta})+f_s(\pi\xi/\sqrt{8\beta})f_s(\pi/\sqrt{8\beta})}\right] \tag{68}$$

$$f_s(u)=\sin u \sinh u, \qquad f_c(u)=\cos u \cosh u \tag{69}$$

$$B=\left[\frac{\cosh(\pi/\sqrt{2\beta})-\cos(\pi/\sqrt{2\beta})}{\cosh(\pi/\sqrt{2\beta})+\cos(\pi/\sqrt{2\beta})}\right]^{\frac{1}{2}} \tag{70}$$

$$\varphi=\arctan\left[\frac{\sinh(\pi/\sqrt{2\beta})+\sin(\pi/\sqrt{2\beta})}{\sinh(\pi/\sqrt{2\beta})-\sin(\pi/\sqrt{2\beta})}\right]. \tag{71}$$

If the stage of the stream were to continue to oscillate according to $\psi(t)=(h_0/2)(1-\cos\omega t)$ for $t>\tau$, equations 61, 63, and 65 would apply for all positive t. In this case the last term within braces in each equation—representing the transient effect produced by starting the oscillation at $t=0$—would die away as t became large. The remaining part of each equation is the steady-state periodic solution, which is an oscillation of frequency ω.

SEMI-INFINITE AQUIFERS, $\delta=0$

Where there is no valley wall, the geometry of the aquifer is that of the half plane $x\geq 0$. For this type of aquifer, we will obtain solutions only for the case of $\delta=0$ in equation 1, so that the stage hydrograph is the sinusoid defined by

$$\psi(t)=\begin{cases}(h_0/2)(1-\cos\omega t), & \text{when } 0\leq t\leq\tau\\ 0, & \text{when } t\geq\tau.\end{cases} \tag{71}$$

A laboratory investigation of the movements of ground water in the semi-infinite aquifer that would be caused by the stage oscillation described by equation 71 has been made by Todd (1955). Todd observed the changes in head in a Hele-Shaw channel caused by oscillations of various amplitudes and periods in a connecting reservoir and constructed from them curves representing the water table, ground-water flow, and bank storage.

GROUND-WATER HEADS

The problem is described by the conditions

$$\frac{\partial^2 h}{\partial x^2}-\frac{1}{\sigma}\frac{\partial h}{\partial t}=0 \tag{72}$$

$$h(x,0)=0, \text{ when } x\geq 0 \tag{72a}$$

$$\lim_{x\to\infty} h(x,t)=0, \text{ when } t\geq 0 \tag{72b}$$

$$h(0,t)=\begin{cases}\dfrac{h_0}{2}(1-\cos\omega t), & \text{when } t\leq\tau\\ 0, & \text{when } t\geq\tau\end{cases} \tag{72c}$$

for which the Laplace transforms for $t \leq \tau$ are

$$\frac{d^2\bar{h}}{dx^2} - \frac{p}{\sigma}\bar{h} = 0 \tag{73}$$

$$\lim_{x\to\infty} \bar{h}(x,p) = 0 \tag{73a}$$

$$\bar{h}(0,p) = \frac{h_0}{2}\left[\frac{1}{p} - \frac{p}{p^2+\omega^2}\right]. \tag{73b}$$

The solution of the transformed problem is

$$\bar{h} = \frac{h_0}{2}\left[\frac{e^{-x\sqrt{p/\sigma}}}{p} - \frac{pe^{-x\sqrt{p/\sigma}}}{p^2+\omega^2}\right]. \tag{74}$$

The inverse transforms of the two terms in brackets are listed by Erdélyi (1954, p. 245, 246). From the inverse transforms the solution for $t \leq \tau$ is found to be

$$h_{t\leq\tau} = \frac{h_0}{2}\left\{erfc\left[\frac{x}{2\sqrt{\sigma t}}\right] - e^{-x\sqrt{\omega/2\sigma}}\cos\left[\omega t - x\sqrt{\frac{\omega}{2\sigma}}\right] + \frac{1}{\pi}\int_0^\infty e^{-ut}\sin\left[x\sqrt{\frac{u}{\sigma}}\right]\frac{u}{u^2+\omega^2}\,du\right\}. \tag{75}$$

By resolving $\psi(t)$ into components as in equations 22, 23, and 25 and applying the principle of superposition, the solution for $t > \tau$ is found from equation 75 to be

$$h_{t>\tau} = \frac{h_0}{2}\left\{erfc\left[\frac{x}{2\sqrt{\sigma t}}\right] - erfc\left[\frac{x}{2\sqrt{\sigma(t-\tau)}}\right] + \frac{1}{\pi}\int_0^\infty [e^{-ut} - e^{-u(t-\tau)}]\sin\left[x\sqrt{\frac{u}{\sigma}}\right]\frac{u}{u^2+\omega^2}\,du\right\}. \tag{76}$$

GROUND-WATER FLOW INTO STREAM

When equation 75 is differentiated with respect to x, x is set equal to zero, and the result is substituted in equation 29, the result is

$$Q_{t\leq\tau} = \frac{h_0\sqrt{\omega TS}}{2}\left\{\cos(\omega t + \pi/4) - \frac{1}{\sqrt{\pi\omega t}} + \frac{1}{\pi\sqrt{\omega}}\int_0^\infty e^{-ut}\frac{u^{3/2}}{u^2+\omega^2}\,du\right\}. \tag{77}$$

By applying the principle of superposition one obtains from equation 77,

$$Q_{t>\tau} = \frac{h_0\sqrt{\omega TS}}{2}\left\{-\frac{1}{\sqrt{\pi\omega t}} + \frac{1}{\sqrt{\pi\omega t - 2\pi^2}} + \frac{1}{\pi\sqrt{\omega}}\int_0^\infty [e^{-ut} - e^{-u(t-\tau)}]\frac{u^{3/2}}{u^2+\omega^2}\,du\right\}. \tag{78}$$

Solutions for Q that are more convenient for computation are found as follows. The Laplace transform of equation 29 gives the Laplace transform of Q, which is

$$\bar{Q}=T\,\frac{d\bar{h}(0,p)}{dx} \tag{79}$$

and from equations 74 and 79,

$$\bar{Q}_{t\leq\tau}=-\frac{Th_0\omega^2}{2\sqrt{\sigma}}\left[\frac{1}{p^{1/2}(p^2+\omega^2)}\right]. \tag{80}$$

The inverse transform of the quantity in brackets is listed by Erdélyi (1954, p. 238, no. 10). From this and equation 80 one obtains

$$Q_{t\leq\tau}=\frac{h_0\sqrt{\omega TS}}{2}\cdot\sqrt{\frac{2}{\pi}}\sum_{n=1}^{\infty}\frac{(-1)^n(2\omega t)^{(4n-1)/2}}{\prod_{m=1}^{2n-1}(2m+1)}$$

$$=\frac{h_0\sqrt{\omega TS}}{2}\cdot\sqrt{\frac{2}{\pi}}\left[-\frac{(2\omega t)^{3/2}}{3}+\frac{(2\omega t)^{7/2}}{3\cdot5\cdot7}-\frac{(2\omega t)^{11/2}}{3\cdot5\cdot7\cdot9\cdot11}+\cdots\right]$$

$$=\frac{h_0\sqrt{\omega TS}}{2}\cdot\sqrt{\frac{2}{\pi}}\left\{\frac{1}{\sqrt{2\omega t}}\left[-\frac{(2\omega t)^2}{3}+\frac{(2\omega t)^4}{3\cdot5\cdot7}-\frac{(2\omega t)^6}{3\cdot5\cdot7\cdot9\cdot11}+\cdots\right]\right\} \tag{81}$$

This same result can be obtained by using the complement of Glover's solution [equation 48] in Duhamel's formula (Sneddon, 1951, p. 164). Thus, from equations 48 and 71, one could write

$$Q_{t\leq\tau}=T\,\frac{\partial h(0,x)}{\partial x}=T\left[\frac{\partial}{\partial x}\int_0^t\psi'(\lambda)g(x,t-\lambda)d\lambda\right]_{x=0}$$

$$=T\int_0^t\psi'(\lambda)\left[\frac{\partial}{\partial x}g(x,t-\lambda)\right]_{x=0}d\lambda$$

When the indicated operations are performed, a result identical to equation 81 is obtained.

By the principle of superposition the solution for $t\geq\tau$ is found from equation 81 to be

$$Q_{t\geq\tau}=\frac{h_0\sqrt{\omega TS}}{2}\cdot\sqrt{\frac{2}{\pi}}\sum_{n=1}^{\infty}\frac{(-1)^n[(2\omega t)^{(4n-1)/2}-(2\omega t-4\pi)^{(4n-1)/2}]}{\prod_{m=1}^{2n-1}(2m+1)}$$

$$=\frac{h_0\sqrt{\omega TS}}{2}\cdot\sqrt{\frac{2}{\pi}}\left\{\frac{1}{\sqrt{2\omega t}}\left[-\frac{(2\omega t)^2}{3}+\frac{(2\omega t)^4}{3\cdot5\cdot7}-\frac{(2\omega t)^6}{3\cdot5\cdot7\cdot9\cdot11}+\cdots\right]\right.$$

$$\left.-\frac{1}{\sqrt{2\omega t-4\pi}}\left[-\frac{(2\omega t-4\pi)^2}{3}+\frac{(2\omega t-4\pi)^4}{3\cdot5\cdot7}-\frac{(2\omega t-4\pi)^6}{3\cdot5\cdot7\cdot9\cdot11}+\cdots\right]\right\}. \tag{82}$$

BANK STORAGE

Integrating equation 77 with respect to t and substituting in equation 35 yields

$$V_{t\leq\tau}=\frac{h_0}{2}\sqrt{\frac{TS}{\omega}}\left\{\frac{1}{\sqrt{2}}-\sin\,(\omega t+\pi/4)+2\sqrt{\frac{\omega t}{\pi}}-\frac{\sqrt{\omega}}{\pi}\int_0^t dv\int_0^\infty\frac{e^{-uv}u^{3/2}}{u^2+\omega^2}\,du\right\}. \tag{83}$$

Since the improper integral converges uniformly in the interval $0\leq v\leq t$, we may change the order of integration and write

$$V_{t\leq\tau}=\frac{h_0}{2}\sqrt{\frac{TS}{\omega}}\left\{\frac{1}{\sqrt{2}}-\sin\,(\omega t+\pi/4)+2\sqrt{\frac{\omega t}{\pi}}-\frac{\sqrt{\omega}}{\pi}\int_0^\infty\frac{u^{3/2}}{u^2+\omega^2}\,du\int_0^t e^{-uv}dv\right\}$$

$$=\frac{h_0}{2}\sqrt{\frac{TS}{\omega}}\left\{\frac{1}{\sqrt{2}}-\sin\,(\omega t+\pi/4)+2\sqrt{\frac{\omega t}{\pi}}+\frac{\sqrt{\omega}}{\pi}\int_0^\infty(e^{-ut}-1)\,\frac{u^{1/2}}{u^2+\omega^2}\,du\right\}. \tag{84}$$

By the principle of superposition we find from equation 84

$$V_{t>\tau}=\frac{h_0}{2}\sqrt{\frac{TS}{\omega}}\left\{2\sqrt{\frac{\omega t}{\pi}}-2\sqrt{\frac{\omega t-2\pi}{\pi}}+\frac{\sqrt{\omega}}{\pi}\int_0^\infty[e^{-ut}-e^{-u(t-\tau)}]\,\frac{u^{1/2}}{u^2+\omega^2}\,du\right\}. \tag{85}$$

Solutions more convenient for computation are found by integrating equation 81 with respect to t and substituting in equation 35. This yields

$$V_{t\leq\tau}=\frac{h_0}{2}\sqrt{\frac{TS}{\omega}}\cdot\sqrt{\frac{2}{\pi}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2\omega t)^{(4n+1)/2}}{\prod\limits_{m=1}^{2n}(2m+1)}. \tag{86}$$

Then, by the principle of superposition we find from equation (86)

$$V_{t\geq\tau}=\frac{h_0}{2}\sqrt{\frac{TS}{\omega}}\cdot\sqrt{\frac{2}{\pi}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}[(2\omega t)^{(4n+1)/2}-(2\omega t-4\pi)^{(4n+1)/2}]}{\prod\limits_{m=1}^{2n}(2m+1)}. \tag{87}$$

ASYMPTOTIC SOLUTIONS

The solutions 82 and 87 for $Q_{t\geq\tau}$ and $V_{t\geq\tau}$ converge slowly for large values of t. Asymptotic expansions which will be more convenient for large values of t are developed as follows.

GROUND-WATER FLOW INTO STREAM

By synthetic division the algebraic fraction in the integrand of equation 78 may be expanded into the finite series

$$\frac{u^{3/2}}{u^2+\omega^2}=\frac{u^{3/2}}{\omega^2}-\frac{u^{7/2}}{\omega^4}+\frac{u^{11/2}}{\omega^6}-\ \ldots\ +(-1)^{n-1}\,\frac{u^{(4n-1)/2}}{\omega^{2n}}+(-1)^nR_{n+1} \tag{88}$$

where

$$R_{n+1}=\frac{u^{(4n+3)/2}}{(u^2+\omega^2)\,\omega^{2n}}.$$

Substituting from equation 88 into equation 78 and performing the indicated integration yields

$$Q_{t>\tau}=\frac{h_0\sqrt{\omega TS}}{2}\cdot\sqrt{\frac{2}{\pi}}\left\{\left[\frac{1}{(2\omega t-4\pi)^{1/2}}-\frac{1}{(2\omega t)^{1/2}}\right]-\left[\frac{3}{(2\omega t-4\pi)^{5/2}}-\frac{3}{(2\omega t)^{5/2}}\right]\right.$$

$$\left.+\left[\frac{3\cdot5\cdot7}{(2\omega t-4\pi)^{9/2}}-\frac{3\cdot5\cdot7}{(2\omega t)^{9/2}}\right]-\ \cdots\ +(-1)^{n-1}\left[\frac{\prod_{m=1}^{2n-3}(2m+1)}{(2\omega t-4\pi)^{(4n-3)/2}}-\frac{\prod_{m=1}^{2n-3}(2m+1)}{(2\omega t)^{(4n-3)/2}}\right]\right\}$$

$$-\frac{(-1)^n h_0\sqrt{TS}}{2\pi}\int_0^\infty[e^{-u(t-\tau)}-e^{-ut}]R_{n+1}du. \quad (89)$$

Here, the empty product, which occurs when $n=1$, is interpreted as unity. For values of ωt such that $(2\omega t-4\pi)>1$, the first term, which contains the finite sum, constitutes an asymptotic expansion of equation 78 by Poincaré's definition (Jeffreys, 1956, p. 499). Hence, by dropping the integral term from equation 89 and adding a finite number of the terms within the braces, one can compute Q to any desired degree of accuracy provided the value of ωt is sufficiently large. The asymptotic expansion can be written in the convenient form

$$Q_{t>\tau}\sim\frac{h_0\sqrt{\omega TS}}{2}\cdot\sqrt{\frac{2}{\pi}}\left\{\frac{1}{\sqrt{2\omega t-4\pi}}\left[1-\frac{3}{(2\omega t-4\pi)^2}+\frac{3\cdot5\cdot7}{(2\omega t-4\pi)^4}-\frac{3\cdot5\cdot7\cdot9\cdot11}{(2\omega t-4\pi)^6}+\ \cdots\right]\right.$$

$$\left.-\frac{1}{\sqrt{2\omega t}}\left[1-\frac{3}{(2\omega t)^2}+\frac{3\cdot5\cdot7}{(2\omega t)^4}-\frac{3\cdot5\cdot7\cdot9\cdot11}{(2\omega t)^6}+\ \cdots\right]\right\}. \quad (90)$$

For $(2\omega t-4\pi)>1$ the terms in each series will decrease to a minimum and then increase, and hence the two series are not convergent. However, because the series are alternating the magnitude of the error incurred in terminating the summations while the terms are still decreasing will be less than that of the first term neglected. For any given value of ωt the maximum accuracy will be attained by terminating the summation with the smallest but one of the terms, and from this fact one can determine the least value of ωt for which the asymptotic expansion gives the desired degree of accuracy.

BANK STORAGE

By a procedure similar to that used to obtain equation 90, the asymptotic solution for large values of ωt is obtained from equation 85.

$$V_{t>\tau}\sim\frac{h_0}{2}\sqrt{\frac{TS}{\omega}}\cdot\sqrt{\frac{2}{\pi}}\left\{\sqrt{2\omega t}\left[1+\frac{1}{(2\omega t)^2}-\frac{3\cdot5}{(2\omega t)^4}+\frac{3\cdot5\cdot7\cdot9}{(2\omega t)^6}-\ \cdots\right]\right.$$

$$\left.-\sqrt{2\omega t-4\pi}\left[1+\frac{1}{(2\omega t-4\pi)^2}-\frac{3\cdot5}{(2\omega t-4\pi)^4}+\frac{3\cdot5\cdot7\cdot9}{(2\omega t-4\pi)^6}-\ \cdots\right]\right\}. \quad (91)$$

COMPUTATIONS AND DISCUSSION

Shown in figures 102 and 103 are computed curves representing, respectively, the ground-water flow and the bank storage caused by the sinusoidal stage oscillation described by equation 1 when $\delta=0$. The curves for $\beta>0$, corresponding to aquifers of finite widths, were computed from equations 63 and 65 for $t\leq\tau$ and from equations 64 and 66 for $t\geq\tau$. Those for the semi-infinite aquifer ($\beta=0$) were computed from equations 81 and 86 for $t\leq\tau$ and from equations 82 and 87 for $t\geq\tau$. From these curves one may find the ground-water flow and bank storage for any combination of values of the parameters h_0, T, S, ω, and l that corresponds to 1 of the 7 values of β for which curves are given, including the limiting case of the semi-infinite aquifer for which $\beta=0$.

As figure 103 illustrates, the bank storage in finite aquifers declines very rapidly when β is large and declines slowly when β is small. For $\beta\geq 5$, almost all the bank storage will have returned to the stream after one flood-wave period ($t=\tau$); for $\beta\geq 0.5$, almost all of it will have returned after two flood-wave periods. (A value of $\beta=0.4$ would correspond, for example, to T=10,000 square feet per day, $S=0.10$, $\tau=10$ days, and l=1,000 feet.) On the other hand, in semi-infinite aquifers the bank storage declines very slowly, as shown on figure 103 and in the following table.

Bank storage in semi-infinite aquifers, $\delta=0$

t/τ	$\frac{V}{(h_0/2)\sqrt{TS/\omega}}$	Percent remaining	t/τ	$\frac{V}{(h_0/2)\sqrt{TS/\omega}}$	Percent remaining
1	2. 13793	64. 7	60	0. 18334	5. 6
2	1. 16116	35. 2	70	. 16964	5. 1
3	. 89620	27. 1	80	. 15861	4. 8
4	. 75669	22. 9	90	. 14949	4. 5
5	. 66707	20. 2	100	. 14178	4. 3
6	. 60327	18. 3	200	. 10013	3. 0
7	. 55486	16. 8	300	. 08172	2. 5
8	. 51651	15. 6	400	. 07075	2. 1
9	. 48515	14. 7	500	. 06328	1. 9
10	. 45889	13. 9	600	. 05776	1. 7
20	. 32027	9. 7	700	. 05347	1. 6
30	. 26038	7. 9	800	. 05002	1. 5
40	. 22502	6. 8	900	. 04715	1. 4
50	. 20101	6. 1	1,000	. 04473	1. 4

The remaining bank storage after 10 flood-wave periods is 13.9 percent of the maximum, and that after 100 periods is 4.3 percent. The values for large t in this table were computed from the asymptotic expansion 91.

In figure 104 the shape of the computed bank-storage curve for the semi-infinite aquifer is compared with that of the curve obtained by Todd (1955) from his model study. The two curves should not

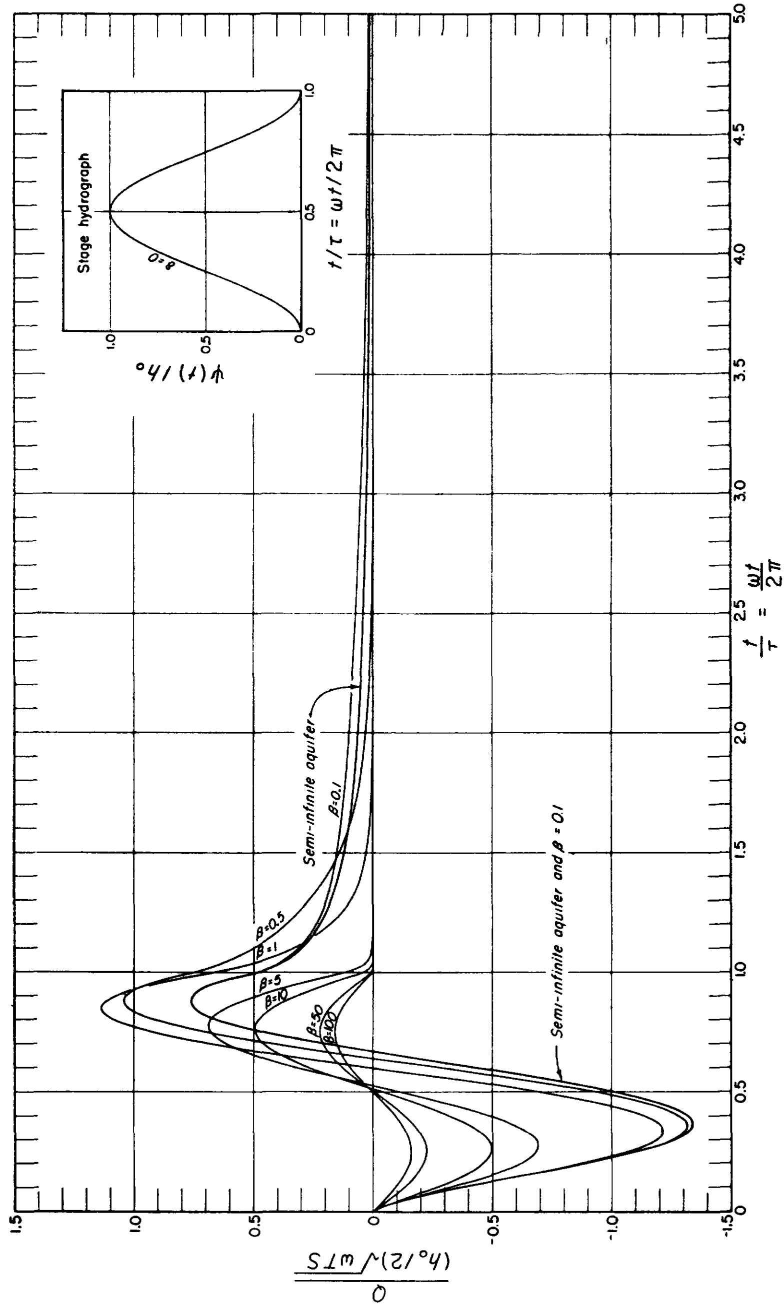

FIGURE 102.—Ground-water flow into stream resulting from stage oscillation defined by equation 1 when $\delta=0$.

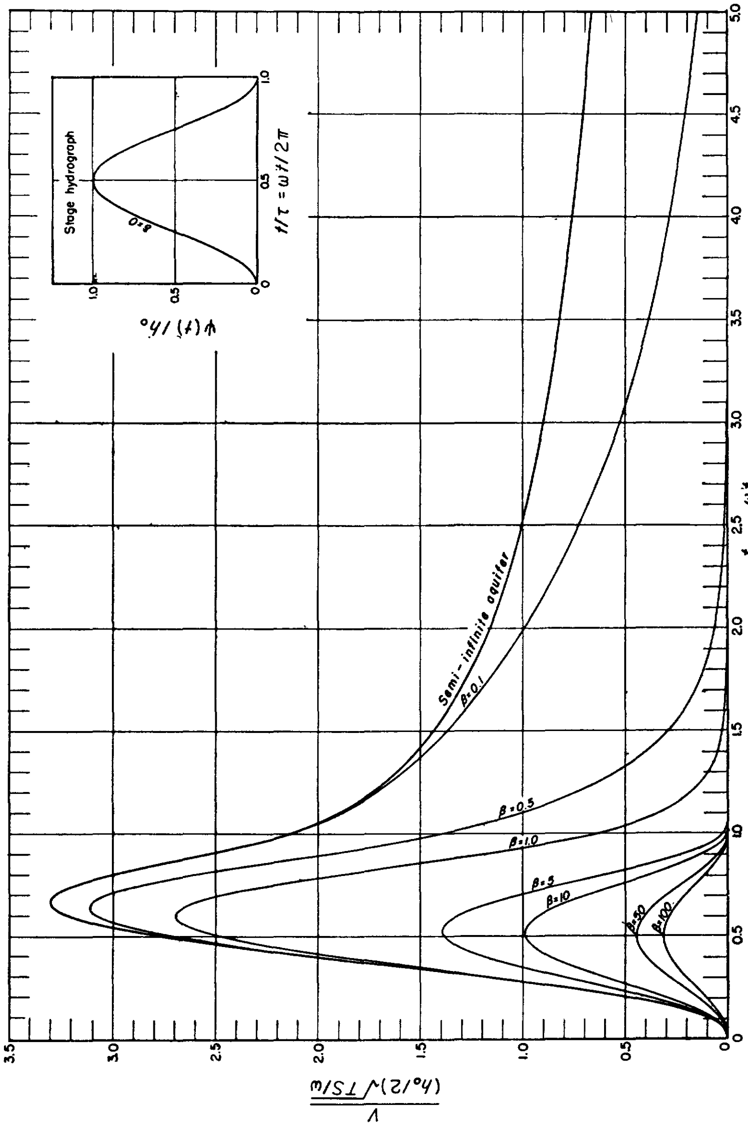

FIGURE 103.—Bank storage resulting from stage oscillation defined by equation 1 when $\delta=0$.

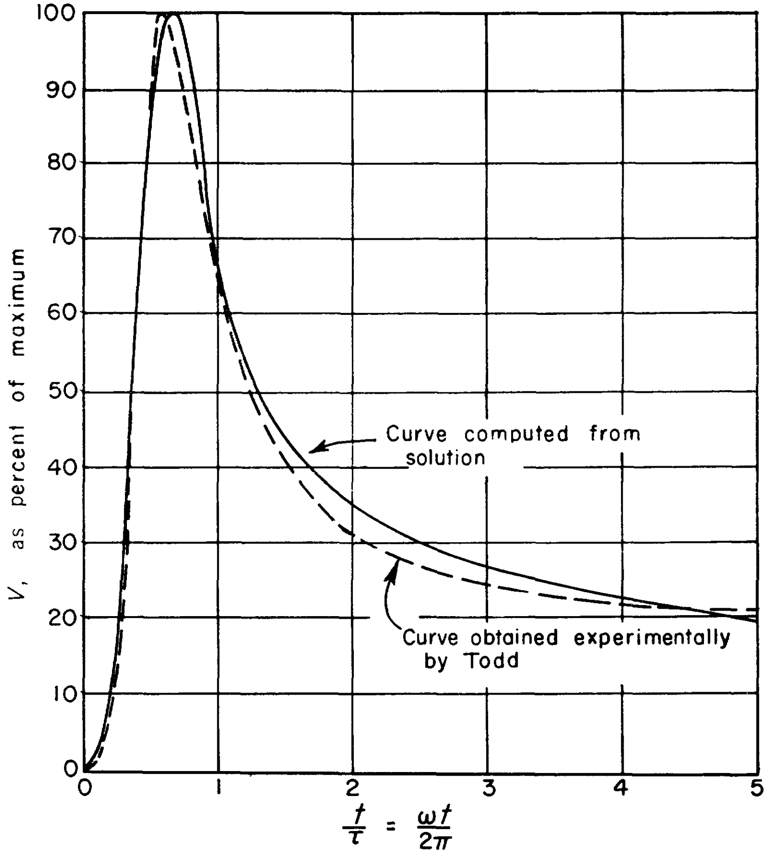

FIGURE 104.—Comparison of bank storage in semi-infinite aquifer computed from analytic solution with that obtained from model study by D. K. Todd.

agree exactly because the thickness of the saturated zone was allowed to vary in Todd's model, whereas this thickness is assumed to remain constant in the derivation of the analytic solution. Also, Todd's curve was constructed from the results of 14 experiments in which h_0 and τ were varied, and in combining the results into the composite curve, Todd assumed that V varies as the first power of τ, basing this assumption on figure 2 of his paper, which is a plot of V_{max} versus $h_0\tau$. Although the points of this plot are somewhat scattered, he considered that a linear relationship was reasonably well established. The analytic solution indicates, however, that V varies as the one-half power of τ, and the scattering of Todd's data is practically eliminated when one plots V_{max} versus $h_0\sqrt{\tau}$.

To illustrate the effect of asymmetry in the stage oscillation, curves representing the ground-water flow and bank storage for $\delta=\omega$, $\beta=0.1$ are compared with those for $\delta=0$, $\beta=0.1$ in figures 105 and 106.

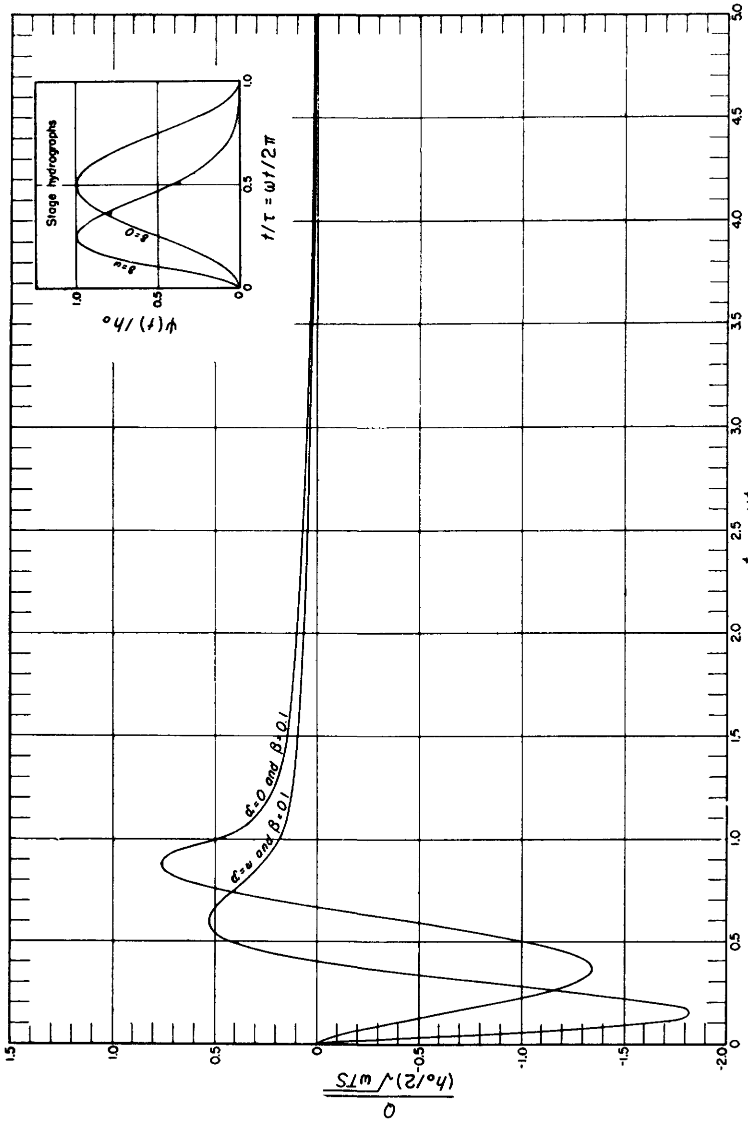

FIGURE 105.—Ground-water flow into stream resulting from stage oscillation defined by equation 1 when $\delta=0$ and $\delta=\omega$

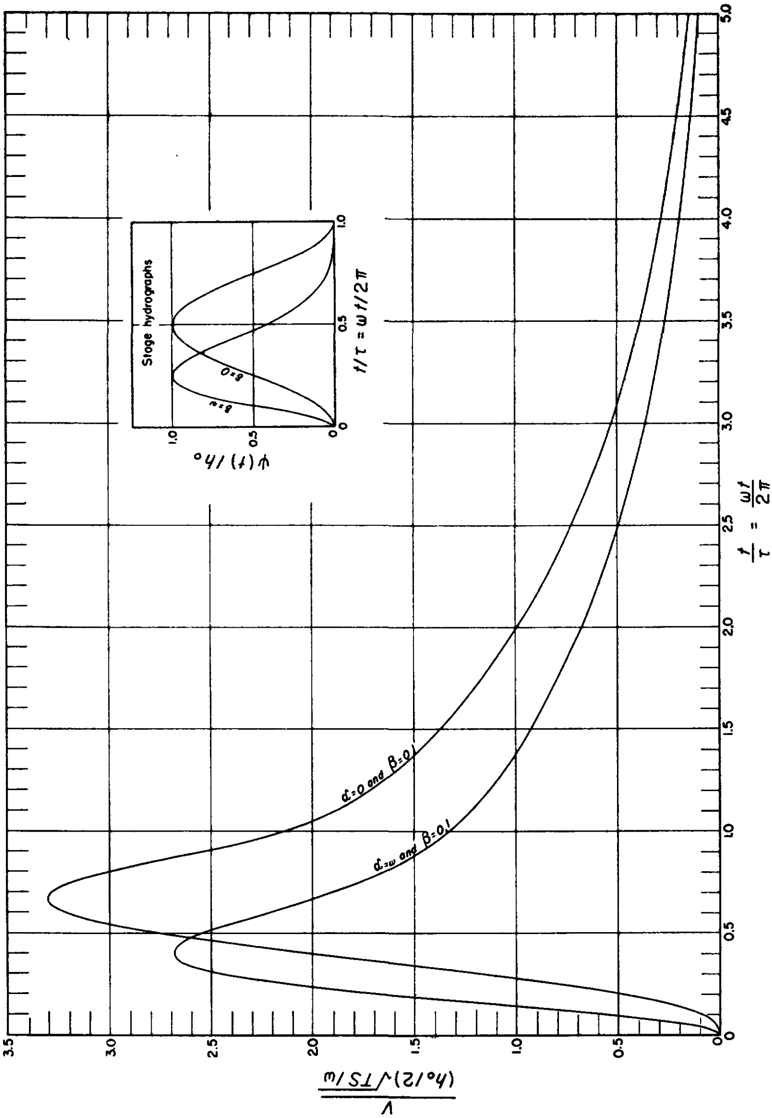

FIGURE 106.—Bank storage resulting from stage oscillation defined by equation 1 when δ=0 and δ=ω.

As might be expected, the asymmetry causes the rate of flow out of the stream to be larger during the early part of the flood wave owing to the more rapid rise in stage; but, because it also shortens the duration of this flow, it lessens the maximum bank storage.

Many investigators have concluded from studies of streamflow hydrographs (for example, Barnes, 1939) that the ground-water contribution—or base flow—of a stream declines according to $Q_t = Q_0 K^{-t}$, which is equivalent to

$$Q_t = Q_0 e^{-ct} \tag{92}$$

where Q_0 and Q_t are the base flows at the beginning and end of an interval of time, t, and c is an empirical constant that depends on the characteristics of the basin. For large values of t, equation 64, which applies to finite aquifers, approaches

$$Q_{t \geq \tau} = \frac{2h_0\sqrt{\beta\omega TS}(e^{2\pi\beta}-1)}{\pi(1+\beta^2)} e^{-\beta\omega t}$$

which has the form of equation 92. Likewise, equation 31 approaches this form. Thus, for finite aquifers equation 92 applies to the bank-storage component of base flow for large values of t where the stage hydrograph is described by equation 1. On the other hand, equation 78 for semi-infinite aquifers approaches

$$Q_{t > \tau} = \frac{h_0\sqrt{\omega TS}}{2} \left\{ \frac{1}{\sqrt{\pi\omega t - 2\pi^2}} - \frac{1}{\sqrt{\pi\omega t}} \right\}$$

which does not have the form of equation 92.

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