

# Calculation of Resistance and Error in an Electric Analog of Steady Flow Through Nonhomogeneous Aquifers

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# Calculation of Resistance and Error in an Electric Analog of Steady Flow Through Nonhomogeneous Aquifers

By ROBERT W. STALLMAN

GENERAL GROUND-WATER TECHNIQUES

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## GENERAL GROUND-WATER TECHNIQUES

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### CALCULATION OF RESISTANCE AND ERROR IN AN ELECTRIC ANALOG OF STEADY FLOW THROUGH NONHOMOGENEOUS AQUIFERS

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By ROBERT W. STALLMAN

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#### ABSTRACT

Several types of nonhomogeneous aquifers were modeled using resistance elements to represent finite sections of the aquifer. Various design equations were used to calculate model resistance in selected problems. Flow solutions obtained from the models qualitatively demonstrate the desirability of carefully selecting the design equation so as to minimize the model size for attaining a given accuracy of solution. Criteria are established which show the types of variations of transmissibility that can be modeled by making resistance inversely proportional to transmissibility without regard for spacing of the model grid. Areal trends of transmissibility are computed from the model data to illustrate numerical techniques. Head values computed by numerical techniques are compared with those observed on the analog to determine the accuracy of the analog solution.

#### INTRODUCTION

One type of electric model of ground-water flow is constructed as an assemblage of fixed resistance elements, each of which represents a large block of aquifer material. Each element in the model represents an analogous region in the aquifer system. The accuracy of the ground-water flow data derived from such a model is highly dependent on the accuracy with which the aquifer is represented by the model. The electric analog model, constructed as an assemblage of discrete resistance elements, is essentially equivalent to a set of finite-difference equations. Thus, errors in flow-problem solutions obtained from models of this type will be at least as great as the errors in a mathematical solution of the finite-difference equations representing the continuous aquifer.

Several sources of error must be considered in evaluating the accuracy of analog or finite-difference solutions. Those frequently cited are: (a) truncation or roundoff errors, (b) observational errors made in measurement of the model, (c) errors due to instability of model or analog control equipment, (d) errors due to representing the

continuous aquifer as a group of finite elements, and (e) inaccurate proportioning between the aquifer properties and model components. Mathematical investigations (Blanch, 1953; Douglas, 1956; Garza, 1956; Greenspan, 1957; Landau, 1956; Lawson and McGuire, 1953; Milne, 1949; Paschkis and Heisler, 1946; Walsh and Young, 1953; and Wasow, 1952) have defined errors due to the above sources (a) and (d) for certain finite-difference solutions. By proper design of the model, matrix error of the types (a) and (d) may be reduced to insignificance with certainty only if those factors producing error can be evaluated completely before the model is built. Unfortunately, this evaluation can be made in only a few problems in which errors have been or can be defined by mathematical studies. No general criteria as yet exist for exactly predicting errors of these types. This is because the errors are dependent on the fineness of the model grid representing the aquifer, on the manner in which the characteristics of the aquifer change in space, and on the curvature of the potential distribution being investigated. In nearly all flow-problem solutions performed as engineering studies, one or more of these factors is unknown. Thus, the model design initially can only be guided by intuitive reasoning from general knowledge of design versus error characteristics.

Although definition of the error inherent in an electric model solution is important, tolerance for error is ordinarily quite high in hydrologic studies. This is because the aquifer characteristics are seldom known with great accuracy, and therefore the model matrix itself can never be an exact replica of the aquifer. Because the hydraulic characteristics of aquifers are not likely to be known within  $\pm 10$  percent, it seems rather ludicrous to strive for analog accuracies on the order of 1 percent in general-purpose investigations of ground-water flow. Wide tolerances notwithstanding, no analog solution obtained may be considered a sufficiently accurate forecast or description of flow unless an error evaluation shows the solution accuracy to be within the tolerance demanded by final application of the solution. Error analysis has generally been made from a viewpoint that looks toward an unknown solution. For most ground-water flow problems this position is untenable. The viewpoint adopted here is that a solution should be obtained through model design guided by experienced intuition, and error evaluations should be made from the completed solution.

On the following pages, criteria are discussed for designing the model resistance elements which represent given blocks of the aquifer prototype. The interrelationship between the aquifer segment and the electric model component is discussed on the basis of a similitude between the electrical flow in the model and of ground-water motion

in the aquifer. A few features of model design, as related to aquifer characteristics, are studied in detail to indicate type problems for which caution in selecting the grid subdivision must be exercised. Finally, the use of finite-difference methods for evaluating the total error in the analog solution is proposed and discussed.

### EQUATIONS OF FLOW

At any given point in a nonhomogeneous aquifer, two-dimensional steady flow may be defined by the following differential equation:

$$T \left[ \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \right] + \frac{\partial h}{\partial x} \frac{\partial T}{\partial x} + \frac{\partial h}{\partial y} \frac{\partial T}{\partial y} = 0 \quad (1)$$

where  $T$  is the aquifer transmissibility,  $h$  is the height of the water level above an arbitrary horizontal reference plane, and  $x$  and  $y$  are the coordinates of the point at which  $h$  is defined. The differential terms of equation 1 may be written in finite-difference form (Southwell, 1946), whence

$$\begin{aligned} \frac{T_{p,n}(h_{p-1} + h_{p+1} + h_{n-1} + h_{n+1} - 4h_{p,n})}{\Delta x^2} + \frac{(h_{p+1} - h_{p-1})}{2\Delta x} \frac{(T_{p+1} - T_{p-1})}{2\Delta x} \\ + \frac{(h_{n+1} - h_{n-1})}{2\Delta y} \frac{(T_{n+1} - T_{n-1})}{2\Delta y} = 0 \quad (2) \end{aligned}$$

The subscript notation of equation 2 is identified on figure 1A, which shows a small segment of the aquifer subdivided by a rectilinear grid, with spacing  $\Delta x$  and  $\Delta y$ .

The model resistors connecting points of the analogous grid intersections, or nodes, of figure 1A are shown in figure 1B. To afford a correct analogic relation between the systems of 1A and 1B, the values of resistance of the elements in figure 1B must be compatible with the transmissibility distribution about the nodes of figure 1A. The equation of steady electrical flow to the junction  $(p, n)$  in figure 1B may be expressed in simplest form by the following:

$$\frac{e_{p-1} - e_{p,n}}{R_{p-1}} + \frac{e_{p+1} - e_{p,n}}{R_{p+1}} + \frac{e_{n-1} - e_{p,n}}{R_{n-1}} + \frac{e_{n+1} - e_{p,n}}{R_{n+1}} = 0 \quad (3)$$

in which  $e$  is the voltage at the junction indicated by the subscript, and  $R$  is the resistance of the element between junction  $(p, n)$  and the junction indicated by the subscript. Equation 3 may be rewritten in the following form:

$$\frac{e_{p-1}}{R_{p-1}} + \frac{e_{p+1}}{R_{p+1}} + \frac{e_{n-1}}{R_{n-1}} + \frac{e_{n+1}}{R_{n+1}} - e_{p,n} \left[ \frac{1}{R_{p-1}} + \frac{1}{R_{p+1}} + \frac{1}{R_{n-1}} + \frac{1}{R_{n+1}} \right] = 0 \quad (4)$$

To permit a more easily visualized comparison of the equations of



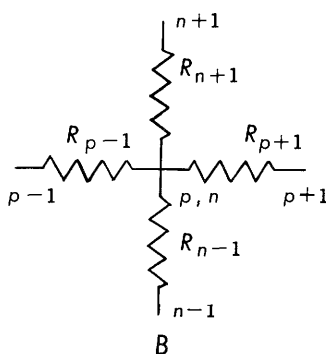
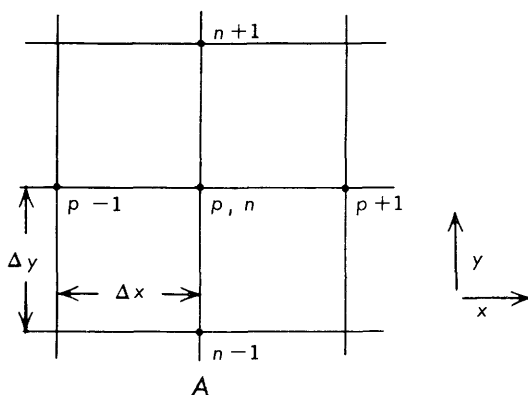


FIGURE 1.—Model relations at a point in a two-dimensional field of flow. *A*, grid reference in the aquifer; *B*, analogous resistance junctions of an electric model.

flow of ground water and electricity, let  $\Delta x = \Delta y$  and rewrite equation 2 as

$$h_{p-1} \left( T_{p,n} - \frac{T_{p+1} - T_{p-1}}{4} \right) + h_{p+1} \left( T_{p,n} + \frac{T_{p+1} - T_{p-1}}{4} \right) \\ + h_{n-1} \left( T_{p,n} - \frac{T_{n+1} - T_{n-1}}{4} \right) + h_{n+1} \left( T_{p,n} + \frac{T_{n+1} - T_{n-1}}{4} \right) - 4h_{p,n} T_{p,n} = 0 \quad (5)$$

Let

$$T_{p,n} - \frac{T_{p+1} - T_{p-1}}{4} = \frac{C}{R_{p-1}} \quad (6a)$$

$$T_{p,n} + \frac{T_{p+1} - T_{p-1}}{4} = \frac{C}{R_{p+1}} \quad (6b)$$

$$T_{p,n} - \frac{T_{n+1} - T_{n-1}}{4} = \frac{C}{R_{n-1}} \quad (6c)$$

$$T_{p,n} + \frac{T_{n+1} - T_{n-1}}{4} = \frac{C}{R_{n+1}} \quad (6d)$$

in which  $C$  is an arbitrarily selected constant relating model resistance to aquifer transmissibility. From equations 6a-d,

$$4T_{p,n} = C \left[ \frac{1}{R_{p-1}} + \frac{1}{R_{p+1}} + \frac{1}{R_{n-1}} + \frac{1}{R_{n+1}} \right] \quad (6e)$$

On the basis of the analogy between Ohm's and Darcy's laws,  $e$  may be taken to be proportional to  $h$ . Substitution of the latter proportionality and the relations 6a-e into equation 4 transforms equation 4 into equation 5. Thus, it is evident that equations 6 analogically relate the finite-difference equations of ground-water flow to the equations of electrical current flow for a junction like  $(p, n)$  in figure 1B.

#### RESISTANCE VALUES

Equations 6 may be used for calculating the resistance values around each junction of the model if the transmissibility distribution in the aquifer system is known. However, unless the transmissibility configuration and the node spacing are such that the value of a given resistance element, say,  $R_{p-1}$  of figure 1B, is the same regardless of whether it is calculated using node  $p, n$  or node  $p-1$  as the reference for equations 6, the model itself cannot be an exact finite-difference replica of the aquifer. The transmissibility distribution over the interval  $0 < x < 3$ , shown on figure 2, may be used to illustrate

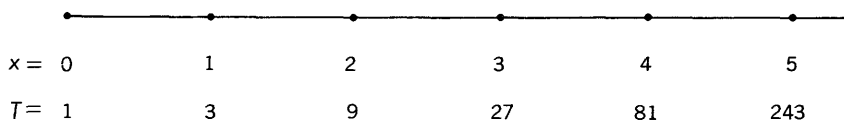


FIGURE 2.—Grid reference and corresponding values of  $T$  along  $y = \text{constant}$  for  $T = 3^x$ .

this point. There, it has been assumed that  $T = 3^x$  and  $\Delta x = 1$ . Thus,  $T$  is assumed constant along lines parallel to the  $y$  axis. Taking  $x = 2$  in figure 2 for the  $p, n$ -reference node of equations 6, the resistance of the element to be placed between  $x = 1$  and  $x = 2$  may be computed from equation 6a as follows:

$$\frac{C}{R_{1-2}} = 9 - \frac{27 - 3}{4} = 3$$

But the value of this same resistor is also defined by equation 6b if the  $p,n$ -reference node for equations 6 is taken at  $x=1$ . Thus, from equation 6b,

$$\frac{C}{R_{1-2}} = 3 + \frac{9-1}{4} = 5$$

a result markedly different from that found with the  $p,n$ -reference for equations 6 at  $x=2$ .

Rather than calculating  $R$  values by the finite-difference techniques, Karplus (1958, p. 181) has suggested making the values of the model resistors proportional to the integral of resistivity between nodes. Although the latter approach is justifiable as a much better approximation of the continuous aquifer media than equations like 6, error dependent on the flow regime will still arise. The latter error will be dependent mainly on the relative magnitude of the higher differentials of head and probably will be insignificant in many analog solutions.

Analysis of the form of  $T$  in space and integration of that form between all pairs of adjacent nodes according to Karplus' method for resistor design may be a laborious process. Study of equations 6 reveals what may be a less arduous means for calculating resistance

values. Note that terms like  $\frac{T_{p+1}-T_{p-1}}{4}$  are simply  $\frac{1}{4}$  of the  $\Delta T$  between two successive grid intervals. Thus, if  $T$  changes linearly, or almost linearly, over a span of three successive grid intervals, the resistance values between nodes, as given by equations 6, are simply inversely proportional to the arithmetic average of  $T$  between nodes. For all linear forms of  $T$ , this is true irrespective of which node is used as the  $p,n$ -reference in equations 6. It is also true for a host of other functional relations describing variations of  $T$  in space. The general nature of such functions can be discerned by defining  $T$  in the form of Taylor's series (see, for example, Scarborough, 1958, p. 338).

If equations 6 are to afford a single value of resistance between model nodes, regardless of which node is used for reference, then, from equations 6a and 6b for the nodes of figure 2,

$$4T_2 + T_3 - T_1 = 4T_3 - T_4 + T_2 = \frac{C}{R_{2-3}}$$

or

$$3(T_2 - T_3) + T_4 - T_1 = 0 \quad (7)$$

From Taylor's series, it can be shown that

$$T_2 - T_3 = -\frac{\partial T}{\partial x} \Delta x - \frac{\partial^3 T}{\partial x^3} \frac{\Delta x^3}{2^2 \cdot 3!} - \frac{\partial^5 T}{\partial x^5} \frac{\Delta x^5}{2^4 \cdot 5!} - \dots \quad (8a)$$

and

$$T_4 - T_1 = \frac{\partial T}{\partial x} \frac{\Delta x}{3} + \frac{\partial^3 T}{\partial x^3} \frac{9\Delta x^3}{2^2 \cdot 3!} + \frac{\partial^5 T}{\partial x^5} \frac{81}{2^3 \cdot 5!} \Delta x^5 + \dots \quad (8b)$$

From a substitution of equations 8a and 8b in equation 7,

$$0.1458 \frac{\partial^3 T}{\partial x^3} \Delta x^3 + 0.02070 \frac{\partial^5 T}{\partial x^5} \Delta x^5 + 0.001128 \frac{\partial^7 T}{\partial x^7} \Delta x^7 + \dots = 0 \quad (9)$$

without a loss of generality. An equation identical to the form of 9 may also be derived for the  $y$  axis. Equation 9 will always be satisfied if the differentials included are zero everywhere in the flow field. Thus if variations of  $T$  in space are defined by either linear or quadratic forms, equations 6 will always yield a single value of  $R$  for the space between a given pair of nodes irrespective of the reference position adopted for computing and irrespective of the magnitude of  $\Delta x$ . For other functional relations defining  $T$  in space, the series represented by equation 9 must be made insignificant by adopting an appropriately small value of  $\Delta x$  if equations 6 are used for computing values of resistance.

Equation 9 is helpful to the model designer only in that it specifies the type of variations of  $T$  for which resistance values can be computed as simple averages without concern for the magnitude of node spacing. However, for those situations in which the criterion of equation 9 is not satisfied along both the  $x$  and  $y$  axes, a model design based on equations 6 might be much less efficient than one based on Karplus' integration of hydraulic resistivity from one node to another. Under some circumstances, the latter approach may lead to a considerable amount of computation to define the resistor grid, but it may also lead to a greatly improved model efficiency by reducing the number of resistance elements required to obtain a given accuracy of the solution.

Consider, for one example, the case in which

$$T = a + bx \quad (10)$$

To represent  $T$  in the model we may use the fundamental model relation

$$R = \frac{T_o R_o}{T} \quad (11)$$

in which  $R_o$  is the resistivity of the model medium at the point where  $T = T_o$ . Following the form used by Karplus (1958, p. 181) with equations 10 and 11,  $dR = -\frac{bT_o R_o dx}{(a+bx)^2}$  and

$$R_{ix} = \int_{x=p\Delta x}^{x=(p+1)\Delta x} dR = -T_o R_o b \int_{x=p\Delta x}^{x=(p+1)\Delta x} \frac{dx}{(a+bx)^2} \quad (12)$$

in which  $R_{\xi x}$  is the resistor representing the aquifer between nodes  $p$  and  $p+1$ . From equation 12

$$R_{\xi x} = T_o R_o \Delta x \left[ \frac{1}{(a + bp\Delta x + b\Delta x)(a + bp\Delta x)} \right] \quad (13)$$

or

$$= \frac{T_o R_o \Delta x}{T_n^2}, \text{ for } b\Delta x \ll a + bp\Delta x \quad (14)$$

In a relation like  $T = a + bx$ , the finite-grid model cannot be designed by lumping resistivity characteristics between grid intersections of the prototype. This is evident from equations 13 and 14. Letting  $\Delta x$  approach zero results in the resistance element value being inversely proportional to the square of transmissibility, at variance with the basic model relation given by equation 11. Equations 13 and 14 illustrate that some model elements must be designed only as a direct function of the finite-difference equivalents of the lumped transmissibility relations between two specified nodes. The latter approach may be taken by utilizing a familiar modified form of Darcy's law. For  $T = a + bx$ , one-dimensional flow may be expressed as

$$v = -(a + bx) \frac{dh}{dx} \quad (15)$$

where  $v$  is the gross ground-water velocity. Integration of equation 15 between two points  $p$  and  $p+1$  along the  $x$  direction leads to

$$b \log \frac{a + bx_{p+1}}{a + bx_p} = \frac{h_p - h_{p+1}}{v} \quad (16)$$

Comparing equation 16 with Ohm's law, the left side of equation 16 is directly analogous to the resistance between points  $p$  and  $p+1$  if voltage is taken proportional to  $h$ , and if electrical current flow is proportional to  $v$ . Thus, for  $T = a + bx$ , the model resistance between nodes  $p$  and  $p+1$  could be made directly proportional to the value of the left side of equation 16.

### ONE-DIMENSIONAL FLOW SOLUTIONS

The solutions of a particular problem of one-dimensional flow obtained are compared here by using the various procedures already discussed for computing resistance values. From equations 6, the resistance values are independent of  $\Delta x$  for the relation  $T = a + bx$ , because equation 9 will always be satisfied. Thus, one set of values of model resistance might be computed simply by making them inversely proportional to the average  $T$  between two nodes. Another

set may be constructed from equation 16. Still another, but erroneous, set might be developed from equation 13. Further discussion of the latter set is included only to illustrate the error committed.

Analog solutions of the head distribution were obtained for

$$T=1 \times 10^4(1+x) \quad (17)$$

with a grid spacing of  $\Delta x=1.0$  assumed for the interval  $0 < x < 10.0$ , with  $h=0$  at  $x=0$ , and  $h=10.00$  at  $x=10.0$ . An analytical solution for the distribution of head is easily obtained as follows:  $v$ , the velocity at any point  $x$ , is obtained from Darcy's law and equation 17 as

$$v = -1 \times 10^4(1+x) \frac{dh}{dx} \quad (17a)$$

For steady flow  $v$  is constant with  $x$ . Equation 17a can be integrated between  $x_\alpha$  and  $x_\beta$ , from which

$$v \log_e \frac{1+x_\beta}{1+x_\alpha} = (h_\beta - h_\alpha) 10^4 \quad (17b)$$

Substituting the boundary conditions  $x_\alpha=0$ ,  $x_\beta=10$ ,  $h_\alpha=0$ , and  $h_\beta=10.00$  in equation 17b yields

$$v = \frac{10^5}{2.3979} \quad (17c)$$

Substituting equation 17c in equation 17b and letting  $x_\alpha=0$  shows

$$h = 4.170 \log_e (1+x) \quad (17d)$$

Equation 17d is the analytical solution of equation 17 for the boundary conditions stated. Head values calculated from equation 17d are given in the second column of table 1.

Equations 6, 13, and 16 were used individually to design three electric models of the aquifer defined by equation 17 over the interval  $0 < x < 10$ . Each model consisted of a series of variable resistors set to within 0.5 percent of the resistance indicated by the design equation. Voltage control and model readout were accurate to 0.1 percent or better. The theoretically correct head at any point  $x$  for each of the model designs is given by and was computed from the relation

$$h_x = 10.00 \frac{\sum_{x=10-\Delta x}^{x-\Delta x} R_{\xi x}}{\sum_{x=0} R_{\xi x}} \quad (18)$$

TABLE 1.—*Head distribution for one-dimensional flow through an aquifer with  $T=1 \times 10^4$  ( $1+x$ ), as determined from theory and model observations*

$x$	Analytical solution	Equation 6		Equation 16		Equation 13	
		Obs.	Theor.	Obs.	Theor.	Obs.	Theor.
0	0	0	0	0	0	0	0
1	2.891	2.81	2.817	2.89	2.891	5.51	5.500
2	4.582	4.50	4.507	4.57	4.582	7.34	7.333
3	5.781	5.70	5.714	5.77	5.781	8.26	8.250
4	6.712	6.64	6.652	6.70	6.712	8.82	8.800
5	7.472	7.41	7.421	7.46	7.472	9.19	9.167
6	8.115	8.06	8.071	8.10	8.115	9.46	9.429
7	8.672	8.63	8.634	8.66	8.672	9.66	9.625
8	9.163	9.12	9.131	9.15	9.163	9.79	9.778
9	9.604	9.59	9.598	9.60	9.604	9.92	9.900
10	10.000	10.00	10.000	10.00	10.000	10.00	10.000

Heads observed on the three models and the theoretically correct heads calculated by means of equation 18 are given in table 1 for comparison with the analytical solution. All the model observations agreed with the corresponding theoretical model analysis to within  $\pm 0.5$  percent. The electric model based on equation 16, as expected, produced the most accurate solution at a cost of greater time for computing resistance values. For this particular flow problem, the maximum error arising from the model based on equations 6 is only about 3.5 percent. This is not a significant error for most field applications.

Modeling  $T$  by the most fundamental finite-difference approach, as stated by equations 6, can sometimes lead to an unnecessarily large model to gain a given accuracy of the solution. The functional relation  $T=3^x$  was used earlier as an example in which resistance values for model design cannot be computed by equations 6 unless  $\Delta x$  is comparatively small. This is because the higher differentials of  $T=3^x$  are all larger than the first and second differentials, and therefore equation 9 cannot be satisfied, even approximately, unless  $\Delta x$  is very small. The effects of  $\Delta x$  on the values of  $R$ , calculated by means of equations 6, are illustrated in table 2. The resistance value was determined from equations 6 for the space between nodes 2 and 3 on figure 2, using both nodes alternately as reference, and letting  $\Delta x$  take several different values as node 2 was held at  $x=2$ . Where equation 6 is applicable, all the calculated resistance values in any one column of table 2 should be alike. Thus, for this particular functional relation, a net spacing of  $\Delta x=1$  is much too large to permit the simplified calculation of  $R$  by averaging  $T$  values between adjacent nodes. With  $\Delta x=0.10$ ,  $R$  calculated by equations 6b and 6c differs from the  $R$  calculated by averaging  $T$  values by only about 0.3 percent. The

TABLE 2.—R/C values calculated from equations 6a and 6b for the space between nodes 2 and 3 with  $T=3^x$  and the node configuration of figure 2, keeping node 2 at  $x=2$

$\Delta x =$	Calculated R/C values for indicated units of $\Delta x$			
	1.0	0.5	0.10	0.01
Calc. for node 2 by eq. 6b-----	0. 06667	0. 08622	0. 10531	0. 110505
Calc. for node 3 by eq. 6a.-----	0. 1111	0. 09019	0. 10532	0. 110505
Calc. $R_{2,3}/C=\frac{2}{T_3 + T_2}$ -----	0. 05556	0. 08134	0. 10501	0. 110501

error introduced in a solution will be dependent on these deviations to some unknown extent. If  $R$  is to be computed for  $T=3^x$  on the basis of average  $T$ , it would be judicial to select a  $\Delta x$  of less than 0.1 so as to gain a solution accuracy probably within one percent.

Further study of modeling relationships reveals a more simplified means for modeling  $T=3^x$  and increasing the solution accuracy obtained for a given number of resistance elements. Following much the same approach used in deriving equation 12 for calculating  $R_{tx}$ ,

$$R_{tx} = \int_{p\Delta x}^{(p+1)\Delta x} dR = 1.09861 R_0 T_0 \int_{p\Delta x}^{(p+1)\Delta x} 3^{-x} dx \quad (19)$$

for  $T=3^x$ . From equation 19

$$R_{tx} = R_0 T_0 [1 - 3^{-\Delta x}] 3^{-p\Delta x} \quad (20)$$

For  $\Delta x=1.0$  equation 20 is

$$R_{tx} = 0.6667 R_0 T_0 / T_p \quad (21)$$

Equation 21 is a relatively simple form for calculating  $R$ ; it eliminates the need for averaging  $T$  values at adjacent nodes, and is a more exact replica of conditions between nodes because of the integration performed. Contrary to the results from equations 13 and 14, equation 21 shows resistance to be inversely proportional to  $T$ . Therefore, it is in agreement with the basic analogy given by equation 11 and can be used directly for model design.

If resistors are designed using equations 6 and 21, an electric analog solution for one-dimensional flow over the interval  $0 < x < 4$ , with  $h=0$  at  $x=0$  and  $h=10$  at  $x=4$ , will produce the values of head given in table 3. The greatest error for the study reported in column 2 was about 2 percent for  $x$  smaller than 1.5. For  $x$  greater than 1.5, the observed values of head compared more favorably with the analytical solution. For example, five times the number of resistors were required for the solution given in column 2, where  $\Delta x=0.2$ , than



TABLE 3.—*Computed heads and heads observed on electric models for one-dimensional flow with  $T=3^x$* 

(1)	(2)	(3)	(4)
$x$	Model by eq. 6 $\Delta x=0.2$	Model by eq. 20 $\Delta x=1$	Analytical solution
0	0	0	0
1	6.62	6.75	6.75
2	9.00	9.00	9.00
3	9.74	9.76	9.75
4	10.00	10.00	10.00

were employed for the solution given in column 3. Yet, the solution accuracy appears to be considerably better in the data of column 3. Thus, for the function  $T=3^x$ , equation 21 yields a design approach much superior to the finite-difference design embodied in equations 6.

#### DESIGN OF MODEL OF TWO-DIMENSIONAL FLOW

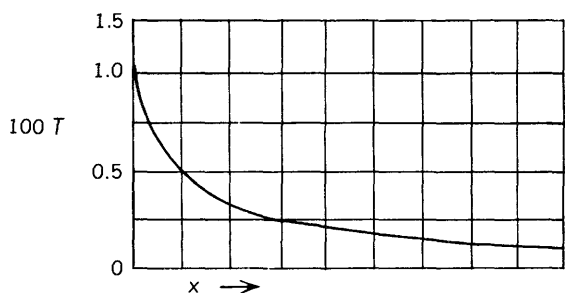
A mathematically rigorous description of model resistance for two-dimensional flow would require detailed knowledge of the flow characteristics about each point in the flow field. However, such knowledge is not ordinarily available and approximations to the more rigorous model design must be applied. One such approximation can be formed by considering the velocity components along the  $x$  and  $y$  directions separately for the purpose of calculating model resistance.

This approach is equivalent to assuming that  $\frac{\partial^2 h}{\partial x^2}$  and  $\frac{\partial^2 h}{\partial y^2}$  are very small compared with  $\frac{\partial h}{\partial x}$  and  $\frac{\partial h}{\partial y}$  at all points in the flow system, and that the higher spatial derivatives of  $T$  are much smaller than  $\frac{\partial T}{\partial x}$  or  $\frac{\partial T}{\partial y}$ . These conditions are satisfied in most aquifers. Thus, model resistance along the  $x$  and  $y$  axes can be computed independently of each other by integrating  $T$  along each axis in turn, as was done along the  $x$  axis in the derivation of equation 16.

Take as an example the aquifer wherein  $T$  is constant with  $y$ , and along the  $x$  axis

$$T = \frac{1}{100(1+x)} \quad (22)$$

over the rectangular region  $0 < x < 9$  and  $0 < y < 16$  shown in figure 3. Applying the same principle used in the development of



$R_{\xi x}$  150 250 350 450 550 650 750 850 950

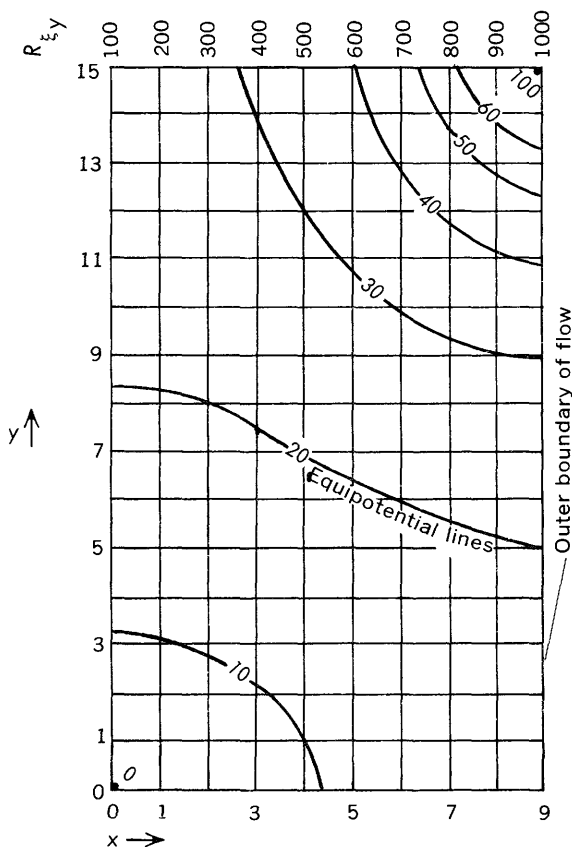


FIGURE 3.—Distribution of potential,  $T$ , and model resistance for two-dimensional flow in a rectangular region, with  $T = \frac{1}{100(1+x)}$ .

equation 16, the resistance values along the  $x$  direction should be

$$\frac{R_{\xi x}}{C} = 100(x_{p+1} - x_p) + 50(x_{p+1}^2 - x_p^2) \quad (23)$$

Similarly, the model resistors along the  $y$  axis have the following values:

$$\frac{R_{\xi y}}{C} = 100(1+x)(y_{n+1} - y_n) \quad (24)$$

in which  $R_{\xi y}$  represents the resistance between adjacent nodes  $n+1$  and  $n$ .

Resistors between nodes along the  $y$  direction represent an aquifer width of  $\Delta x$  between  $p - \frac{1}{2}$  and  $p + \frac{1}{2}$ . They must represent the hydraulic resistivity of the space in the aquifer between  $p - \frac{1}{2}$  and  $p + \frac{1}{2}$ ; and according to equation 22,  $T$  is nonlinear over this interval. Equation 24 is a first approximation in which nonlinearities in  $T$  have been ignored. Nonlinearities in  $T$  may be accounted for partly by defining the average velocity along the  $y$  axis as

$$v_y = - \left[ \frac{1}{\Delta x} \int_{p-\frac{1}{2}}^{p+\frac{1}{2}} T_x dx \right]_{n+\frac{1}{2}}^{n-\frac{1}{2}} \frac{dh}{dy} \quad (25)$$

in which the terms in brackets represent the effective  $T$  along the  $y$  axis for flow along that axis between  $(p, n+1)$  and  $(p, n)$  or between  $(p, n)$  and  $(p, n-1)$ ; and  $T_x$  is the functional relation between  $T$  and  $x$  only. From equation 25, following the technique used for obtaining equation 16, it can be shown that

$$\frac{R_{\xi y}}{C} = 100 \log_e \frac{1+x_{n+\frac{1}{2}}}{1+x_{n-\frac{1}{2}}} \quad (26)$$

for  $\Delta x = \Delta y = 1$ . Although equation 26 is a better approximation for  $R_{\xi y}$  than equation 24, it results in resistance values only 10 percent different from those specified by equation 24 along  $x=1$ . At larger values of  $x$ , the difference between equations 24 and 26 is negligible. Thus, equation 24, which is fundamentally equivalent to equations 6, was accepted for calculating  $R_{\xi y}$  in the model of figure 3. The technique for calculating  $R_{\xi x}$  and  $R_{\xi y}$  leading to equations 23 and 24 may be used for any functional variation of  $T$  in space. Resistance values computed by means of equations 23 and 24 are indicated on figure 3. These values were used in the construction of a model on which potentials, resulting from input and output through opposite corners of the system, were observed at each node. Only the equipotential lines developed from these observations are shown on figure 3.

# CALCULATING STATE OF THE MODEL FROM MEASUREMENTS OF POTENTIAL

Provided the model design and measurements were all correct, analysis of the observed potential distribution summarized on figure 3 should yield the original description of the  $T$  distribution, namely, equation 22. Such an analysis would provide a qualitative test of the accuracy of the model solution. To effect this test, a computing procedure previously outlined was followed (Stallman, 1956). Rather than use equation 22 for computing the spatial rate of change of  $T$ , it was assumed that the relation  $T=ax+by$  would approximate satisfactorily transmissibility over a small area of the flow field of figure 3. The latter approximation contains two unknowns,  $a$  and  $b$ . By algebraic manipulation (Stallman, 1956), the same form can be stated in terms of two other unknowns,  $\alpha$  and  $\frac{\Delta x}{T} \frac{\partial T}{\partial x}$ , which perhaps have a more easily visualized physical significance.  $\alpha$  is the angle between the  $x$  axis and the gradient of  $T$ , and  $\frac{\Delta x}{T} \frac{\partial T}{\partial x}$  is a nondimensional number whose magnitude is highly related to flow-field distortion caused by nonhomogeneity. A minimum of two equations is required to solve directly for the two unknowns,  $\alpha$  and  $\frac{\Delta x}{T} \frac{\partial T}{\partial x}$ .

Potentials observed on the model and used in the computations are given in table 4. The two nodes marked with code 1 in table 4 were used as centers for constructing two finite-difference expressions

TABLE 4.—Selected measurements of potential from a model of a rectangular flow field with  $T = \frac{1}{100(1+x)}$

$\begin{array}{c} \rightarrow x \\ \downarrow y \end{array}$	5	6	7	8	9
13	-----	-----	46. 16	-----	-----
12	-----	36. 70	*1, *5 40. 82	44. 84	-----
11	30. 48	*3, *5 33. 23	*1, *2, *3, *4, *5 36. 12	*4, *5 38. 72	40. 44
10	-----	30. 00	*2, *5 32. 05	33. 74	-----
9	-----	-----	28. 56	-----	-----

\*Code numbers identify sets of nodes used for finite-difference computations. See table 5.

TABLE 5.—*Aquifer characteristics calculated from selected potential measurements, assuming  $T = ax + by$* 

Computation code (see table 4)	Centers of node sets	$\frac{\Delta x}{T} \frac{\partial T}{\partial x}$ from eq. 22	$\frac{\Delta x}{T} \frac{\partial T}{\partial x}$ calculated from observed potentials	$\alpha$ calculated from observed potentials	Number of equations in set solved
1	$x = 7$ $y = 12.5$	-0.125	-0.110	+4°-45'	2
2	$x = 7$ $y = 11.5$	-0.125	-0.142	-4°-27'	2
3	$x = 6.5$ $y = 12$	-0.133	-0.391	-22°-27'	2
4	$x = 7.5$ $y = 12$	-0.105	-0.162	-8°-5'	2
5	$x = 7$ $y = 12$	-0.125	-0.150	-3°-42'	5

needed for calculating  $\alpha$  and  $\frac{\Delta x}{T} \frac{\partial T}{\partial x}$ . Under code 5, five finite-difference equations were constructed and normalized to solve for the two unknowns, the latter process being used to minimize the effects of random errors in observed potentials. Calculated results given in table 5 are keyed to the data of table 4 by code numbers 1 through 5. According to equation 22, all calculated values of  $\alpha$  should be zero. With the exception of code 3, the trend of  $T$  computed from the potential measurements agrees reasonably well with the trend computed from equation 22. The results given in table 5 indicate that under some conditions reasonably good estimates of  $T$  variations in aquifers could be obtained from water-level altitudes without other detailed knowledge of the aquifer characteristics.

Values of  $\alpha$  and  $\frac{\Delta x}{T} \frac{\partial T}{\partial x}$  calculated from the observed potentials are highly sensitive to errors in the potential distribution. A comparison of such calculated values with the correct values given by the equations defining the aquifer is useful for indicating areas of the model where even slight local errors have been made. However, calculated values of  $\alpha$  and  $\frac{\Delta x}{T} \frac{\partial T}{\partial x}$  cannot be used for directly obtaining a quantitative evaluation of total error in the observed potential.

#### COMPUTATION OF ERRORS IN POTENTIALS OBSERVED ON THE MODEL

In passing from equation 1 to equation 2, simple finite-difference approximations of the differentials in equation 1 were accepted as

adequate without qualification. Then the approximations inherent in equation 2 were applied as a basis for model design. Thus, errors due to the finite-difference approximations occur in the potential distribution observed on the model. A more accurate statement relating finite-differences to the differentials than was used in developing equation 2 may be obtained from Taylor's formula (Scarborough, 1958, p. 338) after letting  $\Delta x = \Delta y = \chi$ , whence

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \frac{h_{p-1} + h_{p+1} + h_{n-1} + h_{n+1} - 4h_{p,n}}{\chi^2} - \frac{2\chi^2}{4!} \left( \frac{\partial^4 h}{\partial x^4} + \frac{\partial^4 h}{\partial y^4} \right) - \frac{2\chi^4}{6!} \left( \frac{\partial^6 h}{\partial x^6} + \frac{\partial^6 h}{\partial y^6} \right) \dots \quad (27a)$$

$$\frac{\partial h}{\partial x} = \frac{h_{p+1} - h_{p-1}}{2\chi} - \frac{\chi^2}{3!} \frac{\partial^3 h}{\partial x^3} - \frac{\chi^4}{5!} \frac{\partial^5 h}{\partial x^5} - \dots \quad (27b)$$

$$\frac{\partial h}{\partial y} = \frac{h_{n+1} - h_{n-1}}{2\chi} - \frac{\chi^2}{3!} \frac{\partial^3 h}{\partial y^3} - \frac{\chi^4}{5!} \frac{\partial^5 h}{\partial y^5} - \dots \quad (27c)$$

The infinite series of differential terms on the right side of each of equations 27a-c has not been accounted for in the model design based on equation 2. Error due to this omission is called truncation error, which will be represented in the following by  $\epsilon_t$ . An expression for  $\epsilon_t$  at any point  $(p, n)$  can be obtained by substituting in equation 1 the equations 27a-c, and subtracting the following modified form of equation 2 from the result:

$$T \frac{(h_{p-1} + h_{p+1} + h_{n-1} + h_{n+1} - 4h_{p,n})}{\chi^2} + \frac{h_{p+1} - h_{p-1}}{2\chi} \frac{\partial T}{\partial x} + \frac{h_{n+1} - h_{n-1}}{2\chi} \frac{\partial T}{\partial y} = 0 \quad (2a)$$

Putting all the  $\epsilon_t$  into the potential at  $(p, n)$ , it can be seen that

$$\begin{aligned} \frac{4\epsilon_t}{\chi^2} = & \left[ \frac{2\chi^2}{4!} \left( \frac{\partial^4 h}{\partial x^4} + \frac{\partial^4 h}{\partial y^4} \right) + \frac{2\chi^4}{6!} \left( \frac{\partial^6 h}{\partial x^6} + \frac{\partial^6 h}{\partial y^6} \right) + \dots \right] \\ & + \left[ \frac{\chi^2}{3!} \left( \frac{\partial^3 h}{\partial x^3} \right) + \frac{\chi^4}{5!} \left( \frac{\partial^5 h}{\partial x^5} \right) + \dots \right] \left[ \left( \frac{\partial T}{\partial x} \right) T \right]_{p,n} \\ & + \left[ \frac{\chi^2}{3!} \left( \frac{\partial^3 h}{\partial y^3} \right) + \frac{\chi^4}{5!} \left( \frac{\partial^5 h}{\partial y^5} \right) + \dots \right] \left[ \left( \frac{\partial T}{\partial y} \right) T \right]_{p,n} \quad (28) \end{aligned}$$

As fixed by the assumptions leading to equation 28,  $\epsilon_t$  is the error in potential at  $(p, n)$  due to inadequate theoretical representation of potentials in the model region bounded by  $p-1$ ,  $p+1$ ,  $n-1$ , and  $n+1$ . It may be evaluated approximately by obtaining finite-difference estimates (see Southwell, 1946, p. 229-237) of all the derivatives of significance in equation 28 from the problem solution and values

of  $T$ ,  $\frac{\partial T}{\partial x}$ , and  $\frac{\partial T}{\partial y}$  from the model design criteria. Obviously from equation 28, the truncation error is dependent on both the degree of curvature of the potential surface and on the degree of non-homogeneity of the aquifer represented in the model.

In addition to the truncation errors, the observed potentials contain errors due to other sources such as faulty or inaccurate readout equipment, inaccurate resistance settings or selections, inadequate representation of the aquifer transmissibility by the model resistance, and others. The magnitude of local error from all these sources combined can be found by solving for  $h_{p,n}$  in equation 2a, after substituting in it the potentials observed on the model at points  $p+1, p-1, n+1$ , and  $n-1$ , and the values of  $T$ ,  $\frac{\partial T}{\partial x}$ , and  $\frac{\partial T}{\partial y}$  from the model design criteria. The difference between the value of  $h_{p,n}$  thus obtained,  $h_{p,n}$  (eq. 2a) and the value of  $h_{p,n}$  observed on the model,  $h_{p,n}$  (obs.), equals the sum of all local errors other than those arising from truncation of the finite-difference expressions for potential.

The total local error,  $\Delta_e h_{p,n}$ , at point  $(p, n)$  is comprised of only two parts, a component due to truncation and another component due to other omissions or inaccuracies, and can be expressed as

$$\Delta_e h_{p,n} = h_{p,n}(\text{eq. 2a}) - h_{p,n}(\text{obs.}) + \epsilon_t \quad (29)$$

The terms in equation 29 are illustrated schematically in profile through points  $(p-1, n)$ ,  $(p, n)$ , and  $(p+1, n)$  on figure 4.

The total error at any point in the model is dependent on the distribution and magnitude of  $\Delta_e h_{p,n}$  as just defined, on the spatial variations of  $T$ , and on the boundary conditions imposed on the region of flow. Thus, it is evident that the total error of the solution at any point will be a complicated function of all the variables of

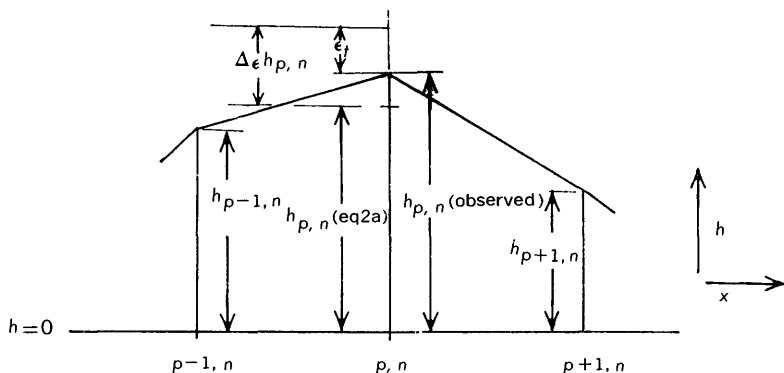


FIGURE 4.—Profile of potential distribution defining local errors.

flow and cannot be found from any simple relationship. However, equation 2 can be corrected to account approximately for local error as follows:

$$\frac{T_{p,n}[h_{p-1}+h_{p+1}+h_{n-1}+h_{n+1}-4(h_{p,n}+\Delta_\epsilon h_{p,n})]}{\Delta x^2} + \frac{(h_{p+1}-h_{p-1})}{2\Delta x} \frac{(T_{p+1}-T_{p-1})}{2\Delta x} + \frac{(h_{n+1}-h_{n-1})}{2\Delta y} \frac{(T_{n+1}-T_{n-1})}{2\Delta y} = 0 \quad (2b)$$

The addition of the term  $\Delta_\epsilon h_{p,n}$  in equation 2, as shown in equation 2b, could be simulated on the model of equation 2 by adding current from an external source at points  $(p,n)$  in proportion to  $T_{p,n}(\Delta_\epsilon h_{p,n})/\Delta x^{-2}$ . The resulting potential distribution in the model would be a solution of equation 2b, and it would be a better approximation than that obtained from the model based on equation 2. However, it would not be an exact solution because a corrected solution based on equation 2b would alter all the potential values, which in turn would alter the values of  $\Delta_\epsilon h_{p,n}$ . If the solution based on equation 2 is found to be relatively accurate, however, it is likely that only one correction will ordinarily be needed for an assessment of total error at any point. The total error may be obtained by subtracting the potentials observed on the model of equation 2 at all  $(p,n)$  from the potential observed on the model of equation 2b.

Such a procedure seems inordinately complicated and time consuming. However, for most analog solutions, brief studies of error, as defined by equations 28 and 29, will obviate the need for a complete and detailed error analysis. Using the data from table 4 for the point  $x=7, y=11$  of figure 3 as an example, the value of  $h_{(7,11)} \text{ (eq. 2a)}$  is found to be 36.12 from equation 2a. This equals exactly the value of  $h_{7,11} \text{ (obs.)}$ , and therefore, from equation 29, all the detectable local error arises from truncation. From an analysis for truncation error by equation 28, using the basic data of the solution presented in figure 3, it was found that  $\epsilon_t$  is of the order of  $-0.001$  at  $x=7, y=11$ . Unless  $\epsilon_t$  is larger than this over a significant area of the field of flow in that problem, it appears doubtful that the total error in potential at any point on the solution would be of importance in hydrologic studies.

### SUMMARY

Resistance elements of an electric model of two-dimensional steady ground-water flow through nonhomogeneous aquifers may be assigned values inversely proportional to the aquifer transmissibility if variations of transmissibility in space can be represented accurately by linear or quadratic equations. The solution error is independent of the analogic distance between resistor units, except for errors due to truncation of the finite-difference approximations of head, in models



of aquifers with variations of transmissibility of first or second degree. Where transmissibility changes in space are of third or higher degree, model size for a given solution accuracy can be made small by selecting resistance values inversely proportional to an integral form of transmissibility. For some types of aquifers, the latter design procedure may be preferred so as to yield solutions of adequate accuracy from a comparatively small number of components.

A procedure for computing local error at each point in the analog solution has been proposed, using finite-difference techniques. An approximation of the total error in the analog solution can be obtained by injecting currents proportional to the calculated local error at each point on the model.

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