Shortcuts and Special Problems in Aquifer Tests

GEOLOGICAL SURVEY WATER-SUPPLY PAPER 1545-C

Shortcuts and Special Problems in Aquifer Tests

Compiled by RAY BENTALL

METHODS OF AQUIFER TESTS

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A collection of reports prepared by W. C. Walton, W. J. Drescher, G. A. LaRocque, Jr., C. V. Theis, R. H. Brown, M. A. Warren, H. G. May, S. W. Lohman, C. S. Conover, H. O. Reeder, R. W. Stallman, H. H. Cooper, Jr., S. M. Lang, and E. A. Moulder, revised by Ray Bentall

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PREFACE

Certain fundamental principles, methods, and formulas for the analysis of ground-water problems from aquifer-test data were developed by Günter Thiem, C. V. Theis, H. H. Cooper, Jr., and C. E. Jacob. Special methods for solving the fundamental formulas, also for solving particular ground-water problems, have been developed by others. Some of these special methods and solutions have been described in short papers that heretofore have been available only to personnel of the U.S. Geological Survey or have been available to the general public as open-file reports at designated depositories.

Seventeen of these previously unpublished papers appear on the following pages. Some present shortcut methods—including the use of special charts, scales, or graphs—for the solution of the general nonequilibrium formula; some extend the equilibrium straight-line methods for purposes of obtaining more information with less work; some analyze specific boundary problems; and one discusses hydraulic and economic factors in the spacing of wells in a multiple-well system.

The original papers have been revised slightly and edited for purposes of clarity and conformity. Elementary information, including fundamental definitions and explanations of fundamental methods and formulas, generally has been deleted. In a few papers, clarifying statements have been added. Where necessary, the formulas have been changed so that the units used are those that have been adopted by the U.S. Geological Survey. The definitions for the commonly used symbols have been deleted from the individual papers; unless otherwise stated, the following symbols and units are used as defined below:

T=the coefficient of transmissibility, in gallons per day per foot (gpd per ft), of the aquifer;

S=the coefficient of storage of the aquifer, a dimensionless ratio;

Q=the rate of discharge, in gallons per minute (gpm), of the pumped well;

s=the water-level drawdown, in feet, in the pumped well, in an observation well, or at a point in the vicinity of the pumped well;

r=the distance, in feet, from the pumped well to an observation well or to a point for which the drawdown is to be determined;

t=the time, in days, since pumping began;

m=the thickness, in feet, of the aquifer;

x=the distance, in feet, from a drain to an observation well or to a point for which the drawdown is to be determined;

$$u=\frac{1.87\ r^2S}{Tt};\text{ and}$$

$$u'^2=\frac{1.87\ x^2S}{Tt}.$$

In some places, to avoid confusion and for purposes of consistency, generally accepted notation has been altered; for example, the Bessel function $K_0(x)$ has been changed to $K_0(x')$, where x' is related to the distance r, and the drain function $D(u)$ has been changed to $D(u')$, where u' is as defined above. Also, the term "log plot" has been adopted for use in referring to a graph having two logarithmic scales; "semilog plot" for a graph having one logarithmic and one arithmetic scale; and "arithmetic plot" for a graph having two arithmetic scales. If the type of graph is not designated, it is an arithmetic plot.

Unless otherwise stated, the following reports are based on the assumptions postulated by Theis: that the aquifer is homogeneous and isotropic; that the aquifer is of infinite areal extent and constant thickness; that the pumped well has an infinitesimal diameter and extends to the bottom of the aquifer; and that water taken from storage in the aquifer is discharged instantaneously with decline in head.

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COMPOSITE TYPE CURVE FOR ANALYZING AQUIFER-TEST DATA

By WILLIAM C. WALTON and WILLIAM J. DRESCHER

ABSTRACT

The composite type curve, a plane figure, is bounded by the three standard type curves that commonly are used in the analysis of aquifer-test data. It is recommended that the composite type curve be made of transparent plastic and that not only the axes for each type curve, but also convenient match points and the formulas in which the coordinates of the matched points are substituted, be indicated on it. The plastic composite type curve is easily and inexpensively made, is permanent, and saves time and insures greater accuracy than do type curves plotted on log paper.

THE USE OF TYPE CURVES IN ANALYZING AQUIFER-TEST DATA

The solution of certain simultaneous equations that are used in computing the coefficients of transmissibility and of storage involves the matching of a data curve, which is based on field measurements, to the type curve of one of the functions $W(u)$, $K_0(x')$, or $D(u')$. Each of these functions occurs in one of the pair of formulas that has been developed for the analysis of a specific hydraulic condition. The well function, $W(u)$, occurs in the Theis (1935) nonequilibrium formula, which is used if a water-table or artesian aquifer receives no recharge from the underlying or overlying beds. The Bessel function, $K_0(x')$, was defined by Jacob (1946), and the formula in which this function occurs is used if an elastic artesian aquifer receives recharge from an underlying or overlying semipermeable confining bed. The drain function, $D(u')$, which was derived by Ferris (1950), occurs in the formula that is used if a natural or artificial drain forms the boundary on one side of a water-table aquifer. The type curves result from the plotting, on log paper, of the functions $W(u)$, $K_0(x')$, and $D(u')$ against u, x', and u', respectively.

The data curve is plotted on log paper of the same scale as the type curve; it is then superposed on the type curve in such a way that a match of the curves is obtained when the respective axes of the two curves are parallel. A match point is selected at random, and the two pairs of coordinates are substituted in the pertinent formulas to solve for the unknowns.

THE COMPOSITE TYPE CURVE

The matching of a data curve to a type curve, if both are plotted on log paper, generally requires the use of a light table. Moreover, the type curve generally is broken into segments because it extends over more log cycles than are included on page-size sheets of log paper, and the consequent shift in scales often leads to confusion and error. The need for a light table and the inconvenience that results from the use of a segmented type curve can be eliminated if the type curve is cut from transparent plastic material of ample size. Furthermore, the three type curves can be cut from the same piece of plastic in such a way that each forms one side of a three-sided plane area. The three-sided plane area is referred to as a composite type curve.

To prepare a plastic composite type curve of convenient size, proceed as follows: From appropriate tables (Theis, 1935; Jacob, 1946; Ferris, 1950) plot the type curves related to the well function, $W(u)$, the Bessel function, $K_0(x')$, and the drain function, $D(u')$, on separate sheets of 3-inch-cycle log paper. Place the three curves so as to form a somewhat triangular-shaped plane figure with the end points of the curves, taken two at a time, forming the vertices; this figure is the composite type curve (fig. 1). Securely fasten a piece of transparent plastic, such as 0.020-inch clear acetate, over the curve. Transfer the three type curves and their major axes to the plastic with a scribe and cut away the excess material outside the curves with a pair of scissors. On the plastic, print in ink the information pertinent to each curve (the surface of the plastic must be roughened to take India ink, or regular acetate ink can be used); then cover the printing with transparent adhesive so that it will not smear or rub off. The axes of each curve should be inked in a different color to prevent confusion. Small holes should be punched through the intersections of the major axes so that match points can be transferred easily to the data-curve sheets.

In order to avoid inaccuracies due to variations in the graph paper produced by different companies, the field data should be plotted on the same brand of paper as that used in making the type curves.

CONCLUSION

The plastic composite type curves are easily made and are much more permanent than type curves constructed on graph paper. The cost of materials necessary for their construction is negligible and the curves are small enough to fit into most brief cases. Compared

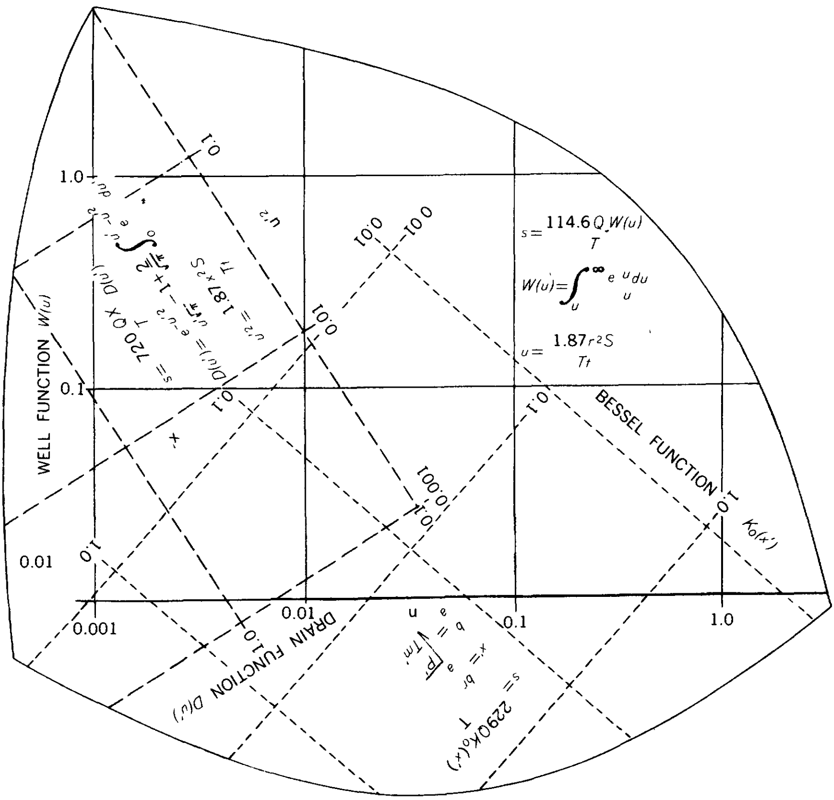

FIGURE 1.—The composite type curve.

to the use of type curves plotted on log paper, use of the plastic composite type curve consumes less time, is more convenient, and insures greater accuracy.

GRAPHIC METHOD FOR PLOTTING AQUIFER-TEST DATA

By George A. LaRocque, Jr.

ABSTRACT

Data for the Theis and Cooper-Jacob methods of analysis of an aquifer test can be plotted rapidly and accurately by graphic determination of the individual values of r^2/t.

The Theis method requires a log plot. Values of r^2/t are determined directly from the corresponding values of t_m (where t_m is the time, in minutes, since pumping began) by means of a series of parallel lines, the points of which represent all such corresponding values. The data curve is then obtained by plotting r^2/t against s with the s scale along the same axis as the t_m scale.

The Cooper-Jacob method requires a semilog plot. Values of r^2/t are determined by placing the data sheet in proper position over a basic reference sheet on which has been plotted corresponding values of r^2/t and t_m for r=100 feet. The data curve is then obtained by plotting r^2/t against s.

Analogous methods of graphing field data may prove useful in other methods of aquifer-test analysis.

THE PLOTTING OF FIELD DATA

One of the most time-consuming, tedious, and irksome jobs in certain methods of analyzing the data obtained during an aquifer test is the arithmetic computation of the value of r^2/t for each depth-to-water measurement. However, values of r^2/t can be found graphically and the data from an aquifer test can be plotted rapidly and accurately either when the measurements are made in the field or later in the office.

THE GRAPHIC DETERMINATION OF R^2/T FROM A LOG PLOT AND ITS APPLICATION

The Theis (1935) nonequilibrium method for determination of T and S requires the plotting of r^2/t against s on log paper. The abscissa, r^2/t, of any point can be determined graphically from the corresponding ordinate, t_m (where t_m is the time, in minutes, since pumping began), in the following manner.

For each observation well, determine the value of r^2/t for the arbitrarily selected times t_m=0.144, 1.44, 14.4, * * * 1.44×10^n minutes, where n is determined by the extent of the data. Any values of t_m could be used, but the selection of those that involve the conversion factor 1,440 simplifies the calculation. The suggested values of t_m give the following values of r^2/t:

t_m	r^2/t
0.144	$r^2 \times 10^4$
1.44	$r^2 \times 10^3$
14.4	$r^2 \times 10^2$
1.44×10^n	$r^2 \times 10^{(3-n)}$.

For each value of r, plot these values of r^2/t against t_m, with r^2/t as abscissa and t_m as ordinate, and draw a straight line, or lines, through the plotted points (fig. 2). The points on the resulting diagonal line or lines represent all values of t_m and the corresponding values of r^2/t for the observation well that is r feet from the pumped well. Lines drawn for various values of r result in a family of parallel curves.

To obtain the Theis data curve directly from the field data, a second vertical scale representing the drawdown, s, must be added to the log graph of r^2/t against t_m. This scale is entirely independent of the vertical time scale and may be placed on the graph in any convenient manner that will provide for adequate plotting of the data. The abscissa of any point on the data curve is the value of r^2/t that corresponds to a given t_m, and the ordinate is the appropriate value of s. Hence, to locate the point opposite the value of t_m on the vertical time scale, move horizontally to the appropriate diagonal line to determine r^2/t; then move vertically to a point opposite the observed value of s on the drawdown scale. Enough points to define the Theis curve for data for any observation well can be found in this manner (fig. 2).

THE GRAPHIC DETERMINATION OF r^2/t FROM A SEMILOG PLOT AND ITS APPLICATION

The Cooper-Jacob (1946) modified nonequilibrium method for determination of T and S requires the plotting of r^2/t against s on semilog paper if the data for more than one observation well for various times are to define the data curve. Here again, the abscissa, r^2/t, of any point can be determined graphically from the corresponding ordinate, t_m.

Select a convenient value of r, such as $r=100$ feet, and plot a series of parallel curves representing r^2/t against t_m, with r^2/t on the logarithmic scale and t_m on the arithmetic scale (fig. 3). The corresponding values for t_m and r^2/t can be obtained most readily from the log plot in figure 2. No matter what value of r is selected, the shape of the curves shown in figure 3 remains the same; a change in the r value merely displaces the curves along the logarithmic scale. For convenience, figure 3 will be referred to as the basic reference sheet.

Before plotting the data curve for a series of observation wells, set up a semilog data sheet with the r^2/t and s scales along the logarithmic and arithmetic scales, respectively. For one of the observation wells, determine the value of r^2/t for any convenient time, such as $t=1$ day.

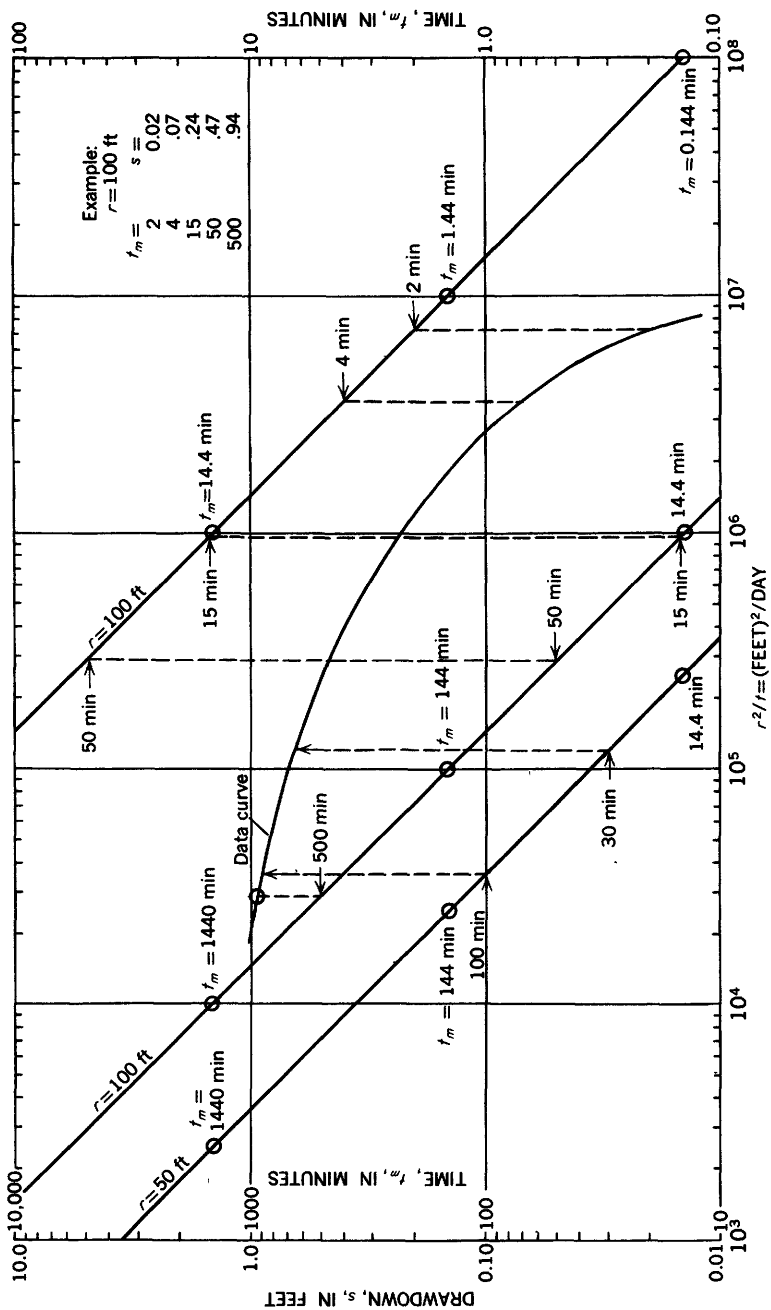

FIGURE 2.—Sketch of a data curve for the Theis nonequilibrium method of analysis of an aquifer test, illustrating the graphic determination (log method) of values of r^2/t from corresponding values of t_m.

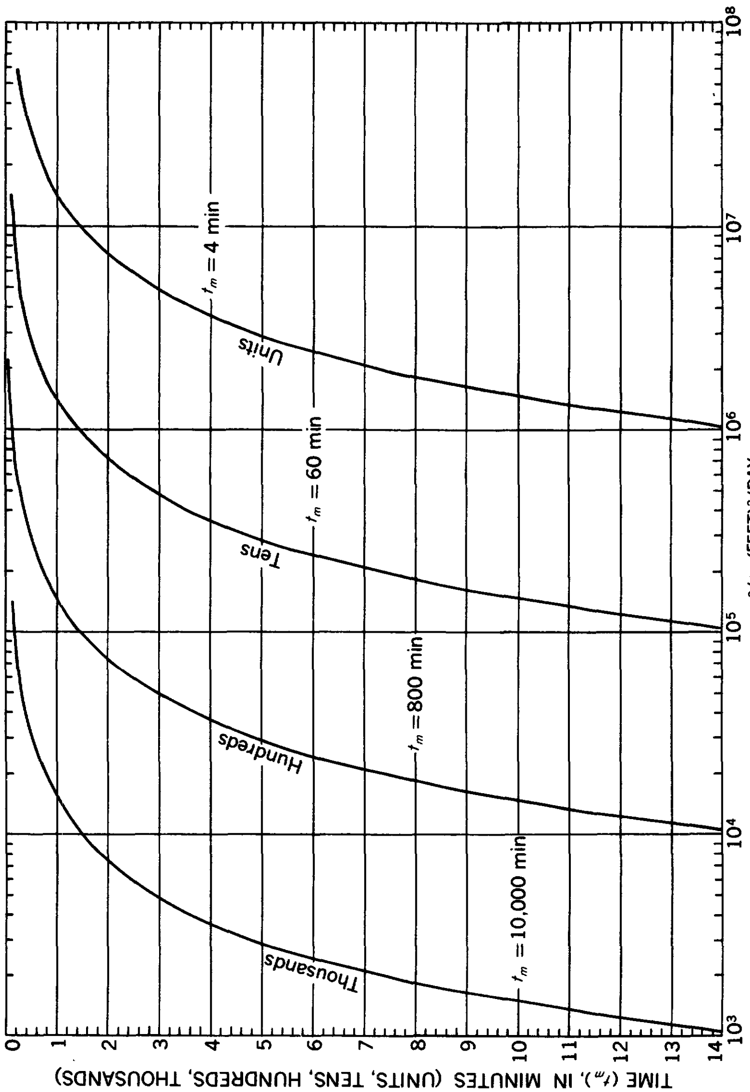

FIGURE 3.—Basic reference sheet: semilog plot of corresponding values of r^2/t and t_m (r=100 ft).

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Next, place the data sheet over the basic reference sheet in such a way that the arithmetic grids coincide; then shift the data sheet along the logarithmic scale of the basic reference sheet, to the left or right, until the value of r^2/t (t=1 day) falls upon $r^2/t=1\times10^4$ (r=100 feet, t=1 day) for the series of curves. For example, if it is desired to plot data for an observation well where r=200 feet (that is, r^2=40,000), shift the data sheet along the logarithmic scale of the basic reference sheet so that $r^2/t=4\times10^4$ is over $r^2/t=1\times10^4$ for the series of curves; similarly, if r=50 feet (that is, r^2=2,500), place the data sheet so that $r^2/t=2.5\times10^4$ is over $r^2/t=1\times10^4$ for the series of curves. With the data sheet thus in proper position over the series of curves on the basic reference sheet, points on the Cooper-Jacob data curve can be plotted directly from the field data. The abscissa of any point on the data curve is the value of r^2/t that corresponds to a given t_m, and the ordinate is the appropriate value of s. Hence, to locate the point opposite the value for t_m on the vertical time scale, move horizontally to the appropriate curve among the family of curves and locate the corresponding value of r^2/t on the data sheet; then move vertically to a point opposite the observed value of s on the drawdown scale. This process can be repeated for the various values of r and t_m, and the Cooper-Jacob plot of r^2/t against s is thus defined (fig. 4).

CONCLUSION

The graphic determination of values of r^2/t from corresponding values of t_m by means of a log plot for the Theis nonequilibrium method or a semilog plot for the Cooper-Jacob modified nonequilibrium method will save much time in plotting the data. The results will be as accurate as the scale of the paper will allow, and they will be more accurate than if a series of slide-rule computations had been used. Furthermore, the time that any drawdown occurred can be determined readily from the graph.

The log method of graphic determination of x^2/t, which is virtually the same as r^2/t, from corresponding values of t_m can also be used when the hydraulic properties of the aquifer are determined by the drain-function method.

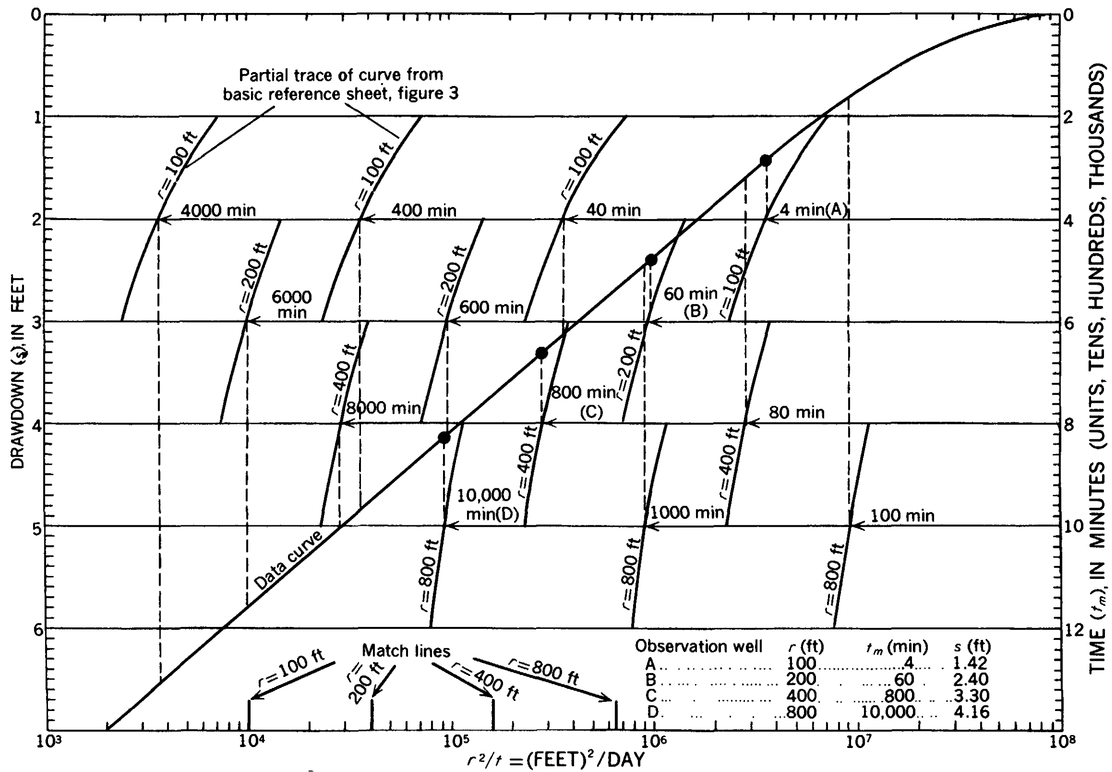

FIGURE 4.—Sketch of a data curve for the Cooper-Jacob modified nonequilibrium method of analysis of an aquifer test, illustrating the graphic determination of values of r^2/t from corresponding values of t_m by the use of the basic reference sheet (semilog plot).

CHART FOR THE COMPUTATION OF DRAWDOWNS IN THE VICINITY OF A DISCHARGING WELL

By Charles V. Theis

ABSTRACT

The exponential integral method used by Theis for calculating water-level drawdowns caused by the discharge of a well is laborious and time-consuming. A chart that has been devised to solve such problems by a combination of graphic and arithmetic methods greatly simplifies the computations, and the results are sufficiently accurate for practical purposes. The chart was made with multiple scales in order to give the complete range of possible values within the compass of a single diagram. The chart can be used to find the drawdown during pumping, or the residual drawdown after pumping has stopped, at any given time and place in unbounded aquifers, in aquifers that have either a line-source boundary or an impermeable boundary, or in aquifers that have multiple boundaries.

DESCRIPTION OF THE CHART

Because use of the exponential integral (Theis, 1935) for calculating water-level drawdowns near a discharging well is somewhat laborious and because the necessary tables are not always available, a graphic method of solution seems desirable. The chart in plate 1 is an effective means of streamlining and reducing the time-consuming computations for several drawdown problems. Computations from the chart represent drawdowns near a discharging well. Except F, which is the drawdown factor to be determined graphically, the terms and symbols necessary for use of the chart are those previously defined.

To give the complete range of possible values within the compass of a single diagram, the chart in plate 1 had to be made with multiple scales. On the top margin of the chart are two S/T scales marked A and B. On the left margin are seven t scales marked 1, 2, 3, 4, 5, 6, and 7. On the bottom margin are six r scales marked A–3 and B–1, A–4 and B–2, A–5 and B–3, A–6 and B–4, A–7 and B–5, and B–6 (that is, the A–3 scale is the same as the B–1 scale, etc.); the appropriate r scale is determined by the S/T and t scales used—hence, scale A–3 is to be used with scales A and 3, and scale B–2 with scales B and 2. The one F scale is on the right margin.

THE SOLUTION OF WATER-LEVEL DRAWDOWN PROBLEMS

The drawdown in a nonartesian aquifer is theoretically greater downgradient than upgradient from the discharging well. The difference becomes significant when the drawdown in the vicinity of the well is more than a small fraction of the full thickness of saturation; the computation of this difference is beyond the scope of this paper. For aquifers in which the drawdown is only a small fraction of the total thickness of saturation, the procedures outlined below are valid.

A. DRAWDOWNS IN AN UNBOUNDED AQUIFER

An aquifer is considered to be unbounded if the cone of depression that results when a well is discharging water does not extend to any of the boundaries of the aquifer. To find the drawdown, s, at a given time and place in an unbounded aquifer, proceed as follows:

1. Compute the value of S/T.
2. Compute the value of Q/T.
3. Find, on the chart, the point formed by the intersection of a vertical line from the value of S/T and a horizontal line from the given value of t. In the most elementary use of the chart, this point is on the corresponding profile of the cone of depression; this profile is represented by the diagonal line that passes through the point and is parallel to the diagonals shown.
4. Follow the diagonal through the point thus found (the position of the required diagonal may have to be interpolated between the diagonals shown) to its intersection with the vertical from the desired value of r. If this intersection does not lie on the chart, move horizontally from the intersection of the diagonal with the right edge of the chart to the left edge, and thence along the diagonal from that point to the desired value of r which is 100 times the value of r on the appropriate r scale.
5. Move horizontally to the right margin of the chart and read the value of the drawdown factor F.
6. Multiply this F by Q/T to obtain s, in feet.
7. To find the limit of the cone of depression, follow steps 1 through 3. The limit is the r at which the profile (diagonal) found in 3 meets the upper edge of the chart, that is, where s is, for practical purposes, 0 feet.

Example.—In an aquifer of wide areal extent, $S=0.2$ and $T=100{,}000$ gpd per ft. If a well is pumped continuously at a rate of 1,000 gpm, what is the drawdown at a distance of 0.4 mile (2,100 ft) from the well at the end of 0.5 year (180 days)?
From the given data: $S/T=2\times10^{-6}$ and $Q/T=10^{-2}$.
From the diagram: $F=215$.
Hence, $s=F(Q/T)=215\times10^{-2}=2.15$ feet.

B. RESIDUAL DRAWDOWNS IN AN UNBOUNDED AQUIFER

The drawdown that remains at any time after a well has been pumped is known as the residual drawdown. To find the residual drawdown for a given time and place in an unbounded aquifer, proceed as follows:

1. Follow the procedure outlined in *A* to obtain the factor F for the time since pumping started (t_1). Call this factor F_1.

2. Similarly, obtain the factor F_2 for the time since pumping stopped (t_2).

3. The difference, F_1-F_2, multiplied by Q/T is the residual drawdown, s_r.

Example.—In an aquifer of wide areal extent, $S=0.2$ and $T=100{,}000$ gpd per ft. If a well is pumped continuously at a rate of 1,000 gpm for 0.5 year (180 days), what is the residual drawdown at a distance of 0.4 mile (2,100 ft) from the well 0.5 year (180 days) after pumping is stopped?

From the given data: $S/T=2\times10^{-6}$; $Q/T=10^{-2}$; $t_1=360$ days; and $t_2=180$ days.

From the diagram: $F_1=290$ and $F_2=215$.

Hence, $s_r=(F_1-F_2)(Q/T)=(290-215)\times10^{-2}=0.75$ foot.

C. DRAWDOWNS IN AN AQUIFER HAVING A LINE-SOURCE BOUNDARY

If an aquifer is bounded along one edge by a surface- or ground-water body whose horizontal trace can be idealized as a long straight line along which the head cannot be lowered, the aquifer is said to have a line-source boundary. To find the drawdown at a given time and at a given place in an aquifer that has a line-source boundary, proceed as follows:

1. On a line perpendicular to the line source through the pumped well, assume the existence of an "image well" on the other side of the line source at a distance equal to that of the pumped well from the line source; this hypothetical well is assumed to recharge the aquifer at a rate equal to the discharge rate of the pumped well.

2. Compute the distances, r_a and r_i, from the point where the drawdown is to be determined to the discharging well and to the recharging image well, respectively.

3. Follow the procedure outlined in *A* to obtain the F factors, F_a and F_i, for the distances r_a and r_i, respectively.

4. The difference, F_a-F_i, multiplied by Q/T is s.

Example 1.—An areally extensive artesian aquifer, for which $T=$ 10,000 gpd per ft and $S=0.0003$, crops out for about 20 miles under the ocean in a manner that can be idealized as a straight line. A well that is 1 mile (5,300 ft) from the outcrop flows at the rate of 100 gpm.

a. What is the drawdown midway between the outcrop and the well after the well has flowed for 0.01 year (about 3½ days)?

From the given data: $S/T=3\times10^{-8}$; $Q/T=10^{-2}$; $r_a=2{,}600$ feet; and $r_i=7{,}900$ feet.

From the diagram: $F_a=200$ and $F_i=25$.

Hence, $s=(F_a-F_i)(Q/T)=(200-25)\times 10^{-2}=1.75$ feet.

b. What is the drawdown at a point that is both 1 mile (5,300 ft) from the well and 1 mile from the outcrop after the well has flowed 0.02 year (about 7¼ days)?

From the given data: $r_a=5{,}300$ feet and $r_i=\sqrt{4+1}$ $=2.24$ miles$=11{,}800$ feet.

From the diagram: $F_a=130$ and $F_i=25$.

Hence, $s=(F_a-F_i)(Q/T)=(130-25)\times 10^{-2}=1.05$ feet.

c. If the well flows for 0.01 year and then is sealed, what is the residual drawdown midway between the outcrop and the well 0.01 year later?

From the given data: $t_1=7\frac{1}{4}$ days and $t_2=3\frac{1}{2}$ days; $r_a=2{,}600$ feet and $r_i=7{,}900$ feet.

From the diagram: $F_{1a}=275$, $F_{1i}=65$, $F_{2a}=200$, and $F_{2i}=25$.

Hence, $s_r=[(F_{1a}-F_{1i})-(F_{2a}-F_{2i})](Q/T)$ $=[(275-65)-(200-25)]\times 10^{-2}=0.35$ foot.

Example 2.—An areally extensive nonartesian aquifer discharges into a stream whose course may be considered straight. The aquifer, for which $T=100{,}000$ gpd per ft and $S=0.2$, extends for a considerable distance along the stream. A well that is 2 miles from the stream pumps at the rate of 1,000 gpm.

a. What is the drawdown midway between the well and the outcrop 0.5 year (180 days) after pumping began?

From the given data: $S/T=2\times 10^{-6}$; $Q/T=10^{-2}$; $r_a=5{,}300$ feet; and $r_i=15{,}800$ feet.

From the diagram: $F_a=55$ and $F_i=0$ (as yet, the stream has had no effect).

Hence, $s=(F_a-F_i)(Q/T)=(55-0)\times 10^{-2}=0.55$ foot.

b. What is the drawdown midway between the well and the outcrop 5 years (1,800 days) after pumping began?

From the given data: the same as for (a).

From the diagram: $F_a=260$ and $F_i=50$.

Hence, $s=(F_a-F_i)(Q/T)=(260-50)\times10^{-2}=2.10$ feet.

D. DRAWDOWNS IN AN AQUIFER HAVING AN IMPERMEABLE BOUNDARY

If an aquifer is limited on one edge by a nearly vertical impermeable boundary whose horizontal trace can be idealized as a long straight line across which no water moves, the drawdown for a given time and place in the aquifer can be found as follows:

1. On a line perpendicular to the horizontal trace of the boundary, assume the existence of an "image well" on the other side of the boundary at a distance equal to that of the pumped well from the boundary; this hypothetical well is assumed to discharge at a rate equal to the discharge rate of the pumped well.
2. Determine the values of F_a and F_i by the method outlined in C.
3. The sum, F_a+F_i, multiplied by Q/T is s.

Example.—A well in a wide valley filled with alluvium is located 2 miles from a crystalline mountain mass that forms the valley wall. The well is pumped at the rate of 1,000 gpm, and the aquifer has $T=100{,}000$ gpd per ft and $S=0.2$.

a. What is the drawdown midway between the well and the outcrop 0.5 year (180 days) after pumping began?

From the given data: $S/T=2\times10^{-6}$; $Q/T=10^{-2}$; $r_a=5{,}300$ feet; and $r_i=15{,}800$ feet.

From the diagram: $F_a=55$ and $F_i=0$ (as yet, the boundary has had no effect).

Hence, $s=(F_a+F_i)(Q/T)=(55+0)\times10^{-2}=0.55$ foot.

b. What is the drawdown midway between the well and the outcrop 5 years (1,800 days) after pumping began?

From the given data: the same as for *a*.

From the diagram: $F_a=260$ and $F_i=50$.

Hence, $s=(F_a+F_i)(Q/T)=(260+50)\times10^{-2}=3.10$ feet.

E. DRAWDOWNS IN AN AQUIFER HAVING MULTIPLE BOUNDARIES

To find the drawdown in an aquifer that has more than one boundary, not only must the pumped well be reflected by an appropriate image well across each boundary but each such well must be similarly reflected across every other boundary. For parallel boundaries, the system of image wells becomes infinite—reflection of reflection ad infinitum. A few reflections probably will suffice for most drawdown problems because the effects of the distant image wells are negligible. For types of boundaries whose horizontal traces cannot be represented by a straight line of infinite length, see Muskat (1937, p. 621–676).

CONCLUSION

Use of the chart described in this paper greatly simplifies the computation of the water-level drawdown caused by pumping, or the residual drawdown after pumping has stopped, at any place in unbounded aquifers, in aquifers that have either a line-source boundary or an impermeable boundary, or in aquifers that have multiple boundaries.

USE OF A SLIDE RULE IN SOLVING GROUND-WATER PROBLEMS BY THE NONEQUILIBRIUM FORMULA

By Charles V. Theis and Russell H. Brown

ABSTRACT

The calculation, by the Theis nonequilibrium formula, of drawdowns caused by a well discharging from an aquifer is both tedious and time consuming. Such problems can be solved directly by means of a slide rule, which has special scales developed by the senior author. The special scales represent the values of the drawdown factor, F; the number of scales necessary to cover the range of F depends on the length, in log cycles, of the A scale of the slide rule. The appropriate special scale is attached over the D scale of the rule. If the value for t on the B scale is set opposite the value for S/T on the A scale, then the drawdown factor is the value on the special scale that is opposite the value for r on the C scale. The required drawdown is then Q/T multiplied by the drawdown factor.

THE NEED FOR A SIMPLIFIED METHOD OF QUANTITATIVE ANALYSIS

During many ground-water investigations, the quantitative phases of the work often require the application of the Theis nonequilibrium formula. The existing literature on ground-water hydraulics describes in detail the manner in which this formula is to be used in determination of the T and S of an aquifer (for example, Theis, 1935). However, where quantitative analyses involve the prediction of drawdowns or the study of multiple-well systems, the conventional computational processes become unduly burdensome and time consuming. A desire to eliminate repetitious routine in these processes, without sacrificing any accuracy in the final results, prompted the senior author to develop a simple modification of the slide rule which would permit direct solution of the nonequilibrium formula without recourse to charts or tables.

PREPARATION OF THE SPECIAL SLIDE-RULE SCALES

The basic Theis nonequilibrium formula is

$$s=\frac{114.6Q}{T}\int_u^\infty\left(\frac{e^{-u}}{u}\right)du=\frac{114.6Q}{T}W(u),\text{ where } u=\frac{1.87r^2S}{Tt}.$$

This can be rewritten as

$$s=(Q/T)F, \tag{1}$$

in which F, termed the "drawdown factor," is a function of u and is defined by

$$F=114.6W(u). \tag{2}$$

The value of u can be computed easily on the slide rule. The value of $W(u)$ is determined by u and can be found from the tables given by Wenzel (1942). Equation 2 shows that the value of F is determined by $W(u)$. The values of F can be affixed to special scales that coincide with the slide-rule scale; several special scales are necessary because the range of useful values of F is beyond the limits of the slide rule.

To prepare the special slide-rule scales:

1. Express S/T as 1 to 100 times an even power of 10, that is, $S/T=(S/T)'\times 10^{(2n)_{S/T}}$; for example, 0.000002 is 2×10^{-6} and 0.00002 is 20×10^{-6}. Set $(S/T)'$ on the A scale of the slide rule; the left half of the scale provides for numbers 1 to 10 and the right half for 10 to 100.
2. Express t, in days or years, as 1 to 100 times an even power of 10, that is, $t=t'\times 10^{(2n)_t}$; for example, 3 days or 3 years is 3×10^0 and 0.2 day or year is 20×10^{-2}. Set the value of t' on the B scale opposite the value for $(S/T)'$ on the A scale.
3. Express r, in feet if t is in days and in miles if t is in years, as 1 to 10 times any power of 10; that is, $r=r'\times 10^{n_r}$; for example, 2,000 is 2×10^3. Find r' on the C scale.
4. On the A scale, opposite r' on the C scale, is the value of $(r^2S/Tt)'$, where $r^2S/Tt=(r^2S/Tt)'\times 10^N$; the slide will have to be shifted to read the value on the A scale for some values of r'.
5. If the A scale were displaced to the left by a distance equal to that from the index to 1.87, on the A scale, opposite the value of r' on the C scale would be the value of $1.87(r^2S/Tt)'$, where $1.87(r^2S/Tt)=u=u'\times 10^{n_u}$.
6. For each u, determine the corresponding values of $W(u)$ and F.
7. Prepare special scales giving the F values that correspond to the values of u', and hence of r', in the positions corresponding to those of u' on the displaced A scale (pl. 2).

For convenience, the appropriate special scale can be attached over the D scale; the values of F can now be read directly on the new D scale from the values of r' on the C scale.

The necessary number of special scales depends on the number of log cycles on the A scale of the slide rule. A 2-cycle rule requires four scales, each of which can be designated by the number that represents the characteristic of the logarithm of r^2S/Tt. To find this number, N, if F can be determined when the slide is in its initial position: algebraically subtract the power of 10 used for t, $(2n)_t$, from the power of 10 used for S/T, $(2n)_{S/T}$, and, inasmuch as the r factor is squared, add twice the power of 10 used for r, n_r; hence,

$$N=(2n)_{S/T}-(2n)_t+2n_r.$$

However, if the slide must be shifted in order to read the value of F, r^2S/Tt is multiplied or divided by 10^2. Consequently, if the slide must be shifted to the right, the number of the scale, N_R, is two larger than $N(N_R=N+2)$, and if the slide must be shifted to the left, the number, N_L, is two less than N (that is, $N_L=N-2$).

USE OF THE SPECIAL SLIDE-RULE SCALES

The drawdown caused by a well discharging from an aquifer at a given time and place can be determined, as follows, by using a slide rule for the solution of the Theis nonequilibrium formula:

1. Set $(S/T)'$, defined above, on the A scale of the slide rule.
2. Set the value of t' on the B scale opposite the value of $(S/T)'$ on the A scale.
3. Find r' on the C scale.
4. Attach the appropriate special scale over the D scale.
5. Opposite the value of r' on the C scale, read the value of F on the special scale.
6. Find s, using equation 1.

CONCLUSION

The direct computation of drawdowns by the use of the slide rule eliminates tedious arithmetic calculations and is a great time saver. One advantage of the slide rule over charts or tables is that, with one setting of the scale, the drawdown factors that correspond to many distances from the pumped well for a given aquifer at a given time can be read quickly. Furthermore, the results are as accurate as those obtained by other methods.

GRAPHIC SHORTCUTS IN APPLYING THE NONEQUILIBRIUM FORMULA TO GROUND-WATER PROBLEMS

By Moultrie A. Warren

ABSTRACT

The coefficients of transmissibility and of storage generally are determined from aquifer-test data by the Theis nonequilibrium formula, which involves a matching of the data curve r^2/t versus s with the type curve $W(u)$ versus u. However, the solution is greatly facilitated if a distance-drawdown plot for a given time is matched with the type curve $W(u)$ versus $\sqrt{u}$ or if a time-drawdown plot for a given distance is matched with the type curve $W(u)$ versus $1/u$.

Likewise, determination of the T and S of an aquifer that is bounded by a straight-line ditch generally is accomplished by matching the data curve x^2/t versus s with the type curve $D(u')$ versus $(u')^2$. Again, the solution is more easily obtained if the time-drawdown curve is matched with the type curve $D(u')$ versus $1/(u')^2$ or if the data curve s/x versus x is matched with the type curve $D(u')$ versus u'.

If the hydraulic constants for an infinite aquifer or an aquifer bounded by a straight-line drain are known or assumed, distance-drawdown profiles for any desired time can readily be drawn by using the appropriate type curve, $W(u)$ versus $\sqrt{u}$ or $D(u')$ versus u', and time-drawdown profiles for any desired distance can readily be drawn by using the appropriate type curve, $W(u)$ versus $1/u$ or $D(u')$ versus $1/(u')^2$.

STANDARD TYPE CURVE $W(u)$ VERSUS u

The solution of hydrologic problems by means of the Theis nonequilibrium formula involves the use of the Theis type curve, which is a log plot of corresponding values of $W(u)$ and u (Theis, 1935). Such problems entail less arithmetic computation and are solved in less time if the appropriate data curvep are superposed over type curves that represent corresponding values of $W(u)$ and $\sqrt{u}$ $W(u)$ and $1/u$.

The Theis nonequilibrium formula is

$$s=\frac{114.6Q}{T}\int_u^\infty \frac{e^{-u}}{u}\,du=\frac{114.6Q}{T}W(u) \tag{1}$$

where

$$u=\frac{1.87r^2S}{Tt}. \tag{2}$$

From equation 1

$$\log s = \log \frac{114.6Q}{T} + \log W(u)$$

and from equation 2

$$\log \frac{r^2}{t} = \log \frac{T}{1.87S} + \log u.$$

Thus, if a log plot of the curve s versus r^2/t is superposed on a log plot of the type curve of $W(u)$ versus u, the coordinates of a match point give corresponding values of $W(u)$ and s and of r^2/t and u, which can be substituted in equations 1 and 2 to solve for T and S, respectively.

If corresponding values of s and r are given for some specific time since pumping started, the above method can be used in solving for T and S. However, because t is a known constant, equation 2 can be rewritten conveniently as follows:

$$r^2 = \frac{Tt}{1.87S} \times u,$$

from which

$$\log r^2 = \log \frac{Tt}{1.87S} + \log u.$$

In this case, the log plot of s versus r^2 is superposed on the curve of $W(u)$ versus u and the solution is obtained as described above. For this method, the values of r^2 are not divided by t; this is a slight saving in labor.

TYPE CURVE $W(u)$ VERSUS $\sqrt{u}$

The above solution for T and S from the distance-drawdown curve for a specified time can be simplified further. Equation 2 can be rewritten as follows:

$$r = \sqrt{\frac{Tt}{1.87S}} \cdot \sqrt{u},$$

from which

$$\log r = \log \sqrt{\frac{Tt}{1.87S}} + \log \sqrt{u}.$$

A log plot of s versus r can be superposed on a log type curve of $W(u)$ versus $\sqrt{u}$ to obtain values for T and S. This procedure avoids the necessity of obtaining values of r^2 or r^2/t.

Values of $W(u)$ for corresponding values of $\sqrt{u}$ are easily obtained. For example, if $\sqrt{u}$ is 0.9, u is 0.81; therefore, the value of $W(u)$

for u equal to 0.81 is plotted against the value of $\sqrt{u}$ equal to 0.9; table 1 gives enough data to plot the curve with sufficient accuracy for most purposes. A plot of the type curve for $W(u)$ versus $\sqrt{u}$ is shown in figure 5.

TABLE 1.—*Values of* W(u) *for values of* $\sqrt{u}$ *from* 10^{-4} *to 3.0*

N	$\sqrt{u}$				
	$N \times 10^{-4}$	$N \times 10^{-3}$	$N \times 10^{-2}$	$N \times 10^{-1}$	N
1.0	17.8435	13.2383	8.6332	4.0379	0.2194
1.5	17.0328	12.4276	7.8226	3.2396	.03484
2.0	16.4572	11.8520	7.2472	2.6813	.003779
2.5	16.0109	11.4058	6.8012	2.2569	.0002709
3.0	15.6462	11.0411	6.4368	1.9187	.00001245
3.5	15.3386	10.7334	6.1294	1.6418	
4.0	15.0709	10.4657	5.8621	1.4092	
5.0	14.6246	10.0194	5.4167	1.0443	
6.0	14.2599	9.6548	5.0532	.7745	
7.0	13.9516	9.3465	4.7462	.5721	
8.0	13.6846	9.0795	4.4806	.4197	
9.0	13.4490	8.8439	4.2468	.3050	

TYPE CURVE $W(u)$ VERSUS $1/u$

When the solution for T and S is to be made from the time-drawdown curve for an individual observation well, r is a known constant. The curve s versus $1/t$ can be superposed on the type curve $W(u)$ versus u to find the appropriate values for the solution of equations 1 and 2. However, equation 2 can be written

$$t=\frac{1.87r^2S}{T}\times\frac{1}{u},$$

from which

$$\log t=\log\frac{1.87r^2S}{T}+\log\frac{1}{u}.$$

A log plot of s versus t can be superposed on a log-type curve of $W(u)$ versus $1/u$ to obtain values for T and S. This method, which was suggested by N. J. Lusczynski of the U.S. Geological Survey (written communication, 1949), eliminates obtaining the reciprocals of t, as suggested by Wenzel (1942).

The type curve $W(u)$ versus $1/u$ is easily obtained. When u equals 1, $1/u$ is also 1, and the values of $W(u)$ are identical. If the type curve $W(u)$ versus u is turned over from right to left or from left to right (the corresponding axes being kept parallel) and if the point where u equals 1 is matched to the point where $1/u$ equals 1, the type curve $W(u)$ versus $1/u$ can be drawn by simply tracing the original Theis type curve. Or values of $W(u)$ that correspond to values of $1/u$ can be obtained directly from Wenzel's table of $W(u)$ values (Wenzel, 1942.

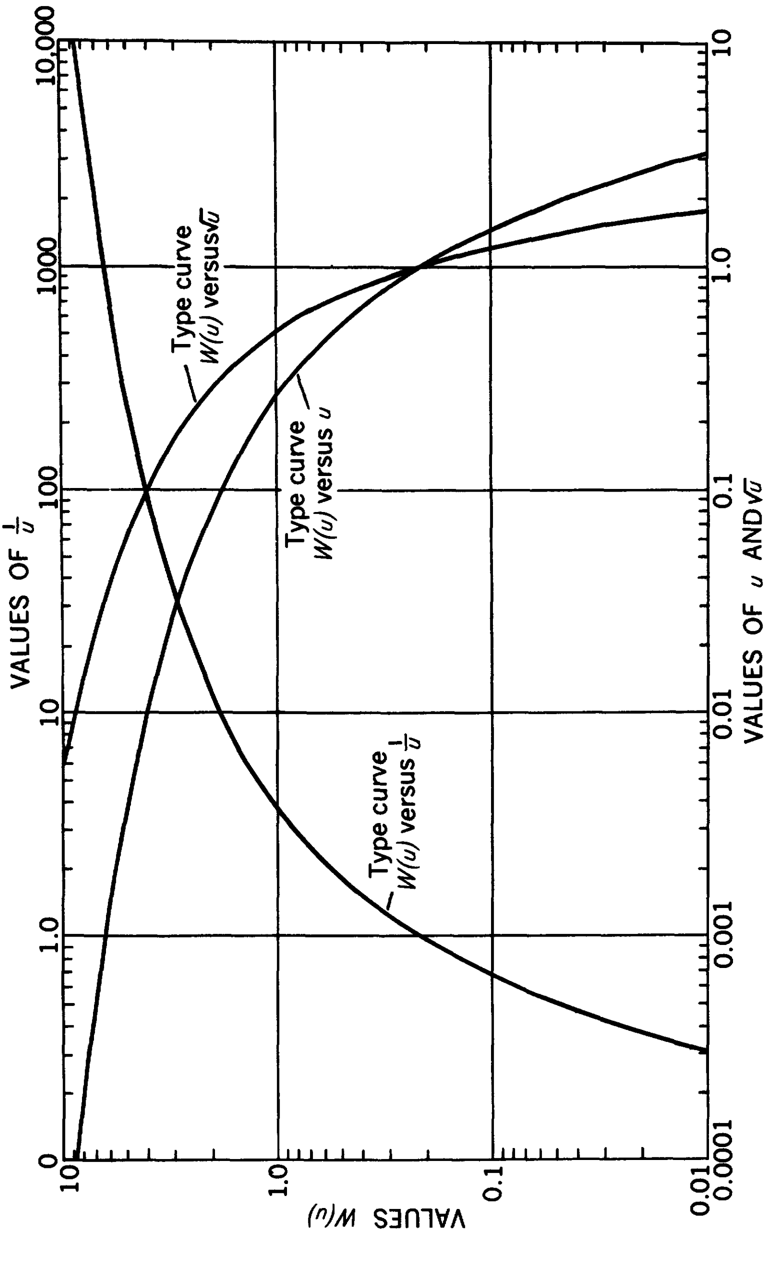

FIGURE 5.—Type curves $W(u)$ versus u, $W(u)$ versus $\sqrt{u}$, $W(u)$ versus $1/u$.

table facing p. 89). For example, when $u=0.04$, $1/u=25$, and the value of $W(u)$ for $u=0.04$ is plotted against the value of $1/u=25$; table 2 gives enough data to plot the curve with sufficient accuracy for most purposes. A plot of the type curve for $W(u)$ versus $1/u$ is shown in figure 5.

TABLE 2.—*Values of* W(u) *for values of* 1/u *from 0.1 to* 9.0×10^5

N	$1/u$						
	$N\times10^{-1}$	N	$N\times10$	$N\times10^2$	$N\times10^3$	$N\times10^4$	$N\times10^5$
1.0	----------	0.2194	1.8229	4.0379	6.3315	8.6332	10.9357
1.5	0.0001686	.3985	2.1964	4.4401	6.7367	9.0387	11.3412
2.0	.001148	.5598	2.4679	4.7261	7.0242	9.3263	11.6289
2.5	.003779	.7024	2.6813	4.9482	7.2472	9.5495	11.8520
3.0	.008591	.8290	2.8572	5.1301	7.4296	9.7319	12.0344
3.5	.01570	.9422	3.0066	5.2837	7.5838	9.8861	12.1887
4.0	.02491	1.0443	3.1365	5.4167	7.7172	10.0194	12.3220
5.0	.04890	1.2227	3.3547	5.6394	7.9402	10.2426	12.5451
6.0	.07854	1.3749	3.5342	5.8218	8.1229	10.4253	12.7279
7.0	.1146	1.5071	3.6859	5.9758	8.2771	10.5796	12.8821
8.0	.1469	1.6242	3.8180	6.1094	8.4109	10.7134	13.0159
9.0	.1829	1.7285	3.9340	6.2266	8.5282	10.8307	13.1333

THE CALCULATION OF DRAWDOWNS FROM THE *W(u)* TYPE CURVES

To decrease the time and labor involved in computing values of the drawdown caused by a pumped well or well field for known or assumed values of Q, r, t, T, and S, several nomographs, special slide-rule scales, and charts have been developed. However, drawdown profiles can be obtained readily without any of these special aids by means of the type curves described above.

If a distance-drawdown curve is desired for a given time, a value of s is computed for a convenient value of r. This point is plotted on a sheet of log paper, which is superposed on the type curve $W(u)$ versus $\sqrt{u}$ so that the point matches the corresponding $\sqrt{u}$ value. The desired curve is obtained by tracing the type curve through the point on the upper sheet, keeping the corresponding axes parallel.

If a time-drawdown curve is desired for a given distance, a value of s is computed for a convenient value of t. This point is plotted on a sheet of log paper, which is superposed on the type curve $W(u)$ versus $1/u$ so that the point matches the corresponding $1/u$ value. The desired curve is obtained by tracing the type curve through the point on the upper sheet and keeping the corresponding axes parallel.

If desired, two points on each curve can be calculated to check the work and increase the accuracy of matching. The points can be taken one log cycle apart and should include the range where greatest accuracy is desired. The method is illustrated by the following example.

A pumped well discharges 2,000 gpm from an aquifer having T=200,000 gpd per ft and S=0.20. Radial profiles of the drawdown due to pumping are desired for 1, 2, 5, and 20 days, and time-drawdown curves are desired for distances of 100, 300, and 1,000 feet.

For this problem Q, T, and S are constants for which values have been given. The expression for u is

$$u=\frac{1.87r^2S}{Tt}=\frac{1.87(0.20)r^2}{200{,}000t}=1.87\times10^{-6}\frac{r^2}{t}$$

and for s is

$$s=\frac{114.6Q}{T}W(u)=\frac{114.6\,(2{,}000)}{200{,}000}W(u)=1.146\ W(u).$$

To obtain the required distance-drawdown curves, the convenient value of 100 feet is chosen for r and corresponding values of u and $\sqrt{u}$ are computed for the times 1, 2, 5, and 20 days. Values of $W(u)$ can then be obtained from the type curve or table and the corresponding values of s can be calculated. Table 3 summarizes these computations and shows the resulting match points. Each s versus r curve is obtained by finding point n' on the new graph, superposing the graph on the type curve $W(u)$ versus $\sqrt{u}$ with point n' matched to point n and the axes of the two graphs alined, and tracing the type curve through n'. Figure 6 shows the required distance-drawdown curves and, with points 4 and 4′ matched, illustrates the method.

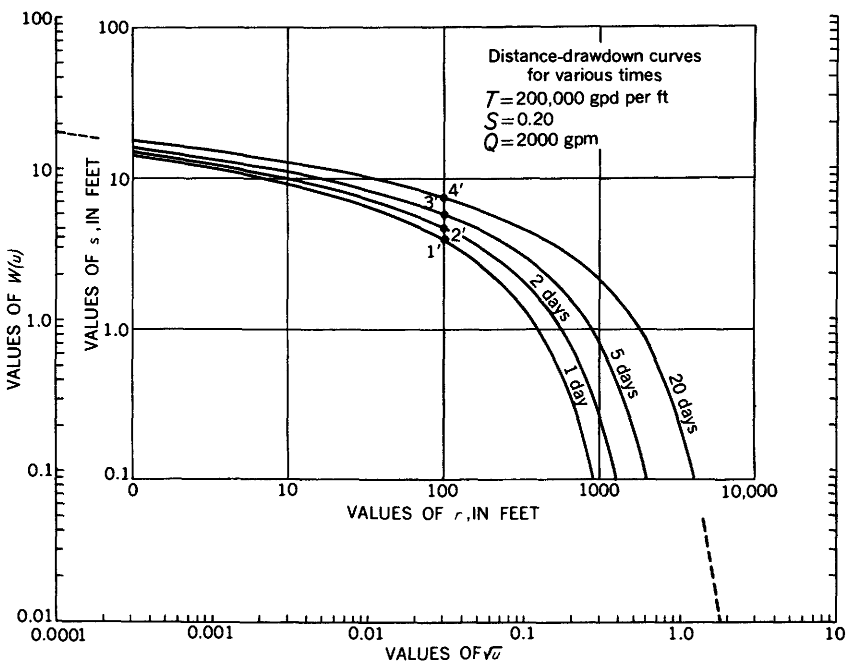

FIGURE 6.—Illustration of the method for constructing distance-drawdown curves.

TABLE 3.—*Computed data for distance-drawdown curves*

Point n	t (days)	u	$\sqrt{u}$	$W(u)$
Plotted on type curve				
1	1	1.87×10^{-2}	0.137	3.42
2	2	9.35×10^{-3}	.0967	4.10
3	5	3.74×10^{-3}	.0612	5.02
4	20	9.35×10^{-4}	.0306	6.40

Point n'	t (days)	r (feet)	s (feet)
Plotted on superposed graph			
1′	1	100	3.92
2′	2	100	4.70
3′	5	100	5.75
4′	20	100	7.33

To obtain the required time-drawdown curves, the convenient value of 10 days is chosen for t and corresponding values of $1/u$ are computed for the distances 100, 300, and 1,000 feet. Values of $W(u)$ can then be obtained from the type curve $W(u)$ versus $1/u$ and the corresponding values of s can be calculated. Table 4 summarizes these computations and shows the resulting match points. Each s versus t curve is obtained by finding point n' on the new graph, superposing the graph on the type curve $W(u)$ versus $1/u$ with point n' matched to point n and the axes of the two graphs alined, and tracing the type curve through n'. Figure 7 shows the required time-drawdown curves and, with points 7 and 7′ matched, illustrates the method.

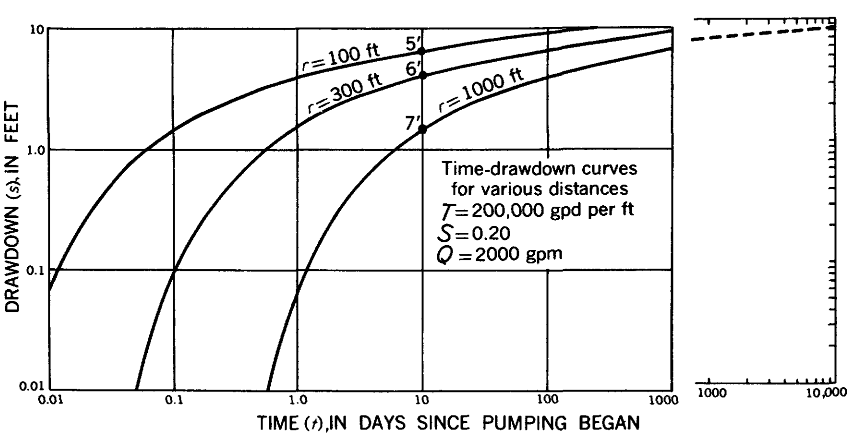

FIGURE 7.—Illustration of the method for constructing time-drawdown curves.

TABLE 4.—*Computed data for time-drawdown curves*

Point n	r (feet)	$1/u$	$W(u)$
Plotted on type curve			
5	100	535	5.707
6	300	59.5	3.526
7	1,000	5.35	1.28
Point n'	5 (feet)	t (days)	s
Plotted on superposed graph			
5′	100	10	6.54
6′	300	10	4.04
7′	1,000	10	1.47

DRAIN-FUNCTION TYPE CURVES

A formula for the analysis of the behavior of the water level near a straight-line drain or ditch of infinite length in an ideal aquifer of infinite areal extent under nonequilibrium conditions was presented by Ferris (1950) as follows:

$$s=\frac{Qx}{2T}\left[\frac{e^{-(u')^2}}{u'\sqrt{\pi}}-1+\frac{2}{\sqrt{\pi}}\int_0^{u'} e^{-(u')^2}du'\right]=\frac{Qx}{2T}D(u') \tag{3}$$

and

$$(u')^2=\frac{1.87x^2S}{Tt}. \tag{4}$$

For these equations, Q is the discharge into the drain in gallons per day per foot length of drain. From equation 3

$$\log s=\log\frac{Qx}{2T}+\log D(u')$$

and from equation 4

$$\log x^2/t=\log\frac{T}{1.87S}+\log (u')^2.$$

Hence, T and S can be determined from the time-drawdown curve, where x is a constant, by matching a log plot of s versus x^2/t to a log plot of the drain-function type curve, $D(u')$ versus $(u')^2$ (fig. 8). However, equation 4 may be written

$$t=\frac{1.87x^2S}{T}\cdot\frac{1}{(u')^2},$$

from which

$$\log t=\log\frac{1.87x^2S}{T}+\log\frac{1}{(u')^2}.$$

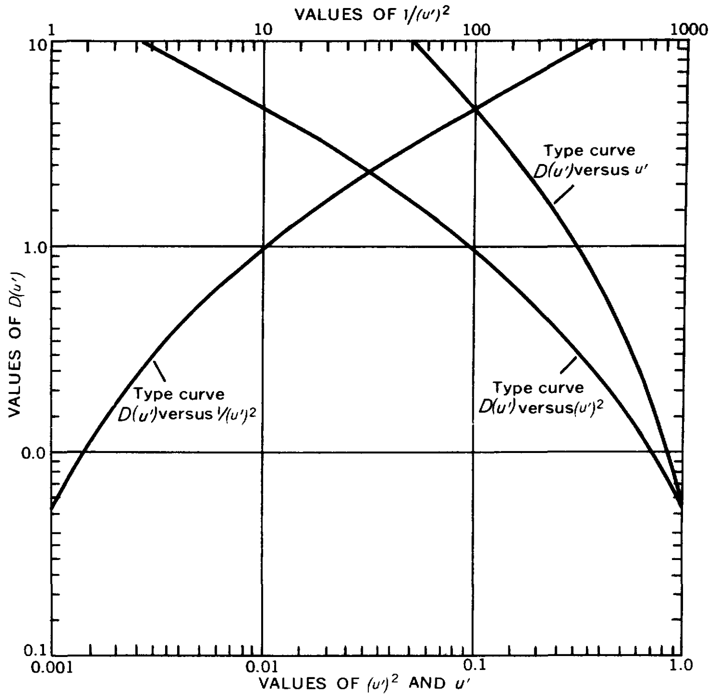

FIGURE 8.—Type curves $D(u')$ versus $(u')^2$, $D(u')$ versus u', and $D(u')$ versus $1/(u')^2$.

Thus, T and S can be determined by matching a log plot of s versus t and the type curve $D(u')$ versus $1/(u')^2$ (fig. 8). Also, for given values of Q, S, T, and x, a time-drawdown curve can be drawn by computing the value of s for some convenient time, t, plotting this point (t, s), on the time-drawdown graph overlying the corresponding values of $D(u')$ and $1/(u')^2$ on the type curve (the axes of the two graphs being alined), and tracing the underlying type curve.

If T and S are to be determined from the distance-drawdown curve for a given time, t is a known constant and equations 3 and 4 can be written as follows:

$$\frac{s}{x}=\frac{Q}{2T}D(u')$$

and

$$x=\sqrt{\frac{Tt}{1.87S}}\cdot u'.$$

Therefore

$$\log \frac{s}{x} = \log \frac{Q}{2T} + \log D(u')$$

and

$$\log x = \log \sqrt{\frac{Tt}{1.87S}} + \log u'.$$

T and S can be determined by matching the log plot of s/x versus x and the type curve $D(u')$ versus u' (fig. 8). A solution can, of course, also be obtained by matching the curve s/x versus x^2 with the type curve $D(u')$ versus $(u')^2$.

To obtain the distance-drawdown curve, s versus x, for given values of Q, S, T, and t, the curve s/x versus x must first be plotted. The value of s/x can be computed for some convenient value of x. The point (x, s/x) is then plotted on the graph, which is superposed over the type curve $D(u')$ versus u' so that the point overlies the corresponding values of $D(u')$ and u'. With the axes of the two graphs alined, the underlying type curve can be traced onto the upper graph. From this curve, a value of s/x can be obtained for every value of x; s can then be computed and the curve s versus x can be plotted.

CONCLUSION

By means of the described processes, distance-drawdown and time-drawdown curves can be obtained quickly. If reasonable care is used in constructing the type curves, in plotting and matching the computed points, and in alining the axes of the two graphs, and if the scales on the two sheets of graph paper do not differ by more than 0.25 percent, the results should be accurate within 2 percent for that part of the curve having a slope of less than 45°—provided that the points for matching are in that part of the curve. For the part of the curve where the slope is greater than 45°, an error of 5 or 10 percent is not considered serious.

The accuracy of this method probably is greater than that of most solutions by nomographs. Although slide-rule calculations may be arithmetically accurate to three significant figures, some accuracy is lost in plotting the curves. Moreover, the physical properties of most aquifers are seldom sufficiently uniform and accurately defined that drawdown predictions can be made with an error of less than 2 percent. Therefore, the increased time and labor necessary to obtain more accurate values of s for plotting curves is not justified.

For time-drawdown curves, further time and labor can be saved in plotting data for an aquifer test of short duration, if the time in minutes is used rather than the time in days. If this is done, the shape of the type curve remains unchanged but the right side of the formula for the drawdown must be divided by 1,440.

A SIMPLIFIED TIME- AND DISTANCE-DRAWDOWN GRAPH

By Harold G. May

ABSTRACT

Semilog plots of the time-drawdown relation for a given distance from the pumped well, or of the distance-drawdown relation for a given time since pumping started, cannot be used to determine the drawdowns for other distances, or times. However, if a time-drawdown curve is plotted on the same graph as a distance-drawdown curve for the same pumped well, drawdowns for various times and distances can be found with the aid of an index curve that is related to the two initial curves. Time-drawdown curves for various distances are parallel to the initial time-drawdown curve, and distance-drawdown curves for various times are parallel to the initial distance-drawdown curve. Therefore, the drawdown for any time and place can be found either from the appropriate time-drawdown curve or the appropriate distance-drawdown curve.

THE USE OF SEMILOG PLOTS IN THE SOLUTION OF DRAWDOWN PROBLEMS

One of the principal uses of aquifer-test results is the determination of the effects of pumping on water levels in the vicinity of the discharging well. These effects at a given place for various times can be illustrated by a time-drawdown curve (Cooper and Jacob, 1946) and for a given time and at various places by a distance-drawdown curve (Thiem, 1906). Each of the curves for a given time since pumping started or for a given distance from the pumped well is a semilog plot. To determine the effects for other constant values of time or distance, additional plots must be made. Because of space limitations on the graph paper, only a few of these curves are plotted on one graph. If interrelated time- and distance-drawdown curves for a well that has a given rate of discharge from an aquifer with predetermined hydraulic characteristics are plotted on the same graph, drawdowns for different times and distances can be found from the graph without plotting additional data.

THE TIME- AND DISTANCE-DRAWDOWN RELATION

If time- and distance-drawdown curves are plotted on the same graph (fig 9), the relation between the distance from the pumped well and the time since pumping started can be expressed as

$$r=10^{n}t \qquad (1)$$

where n is an integer. For the semilog plots, let the upper horizontal log scale represent the values of t, the lower horizontal log scale rep-

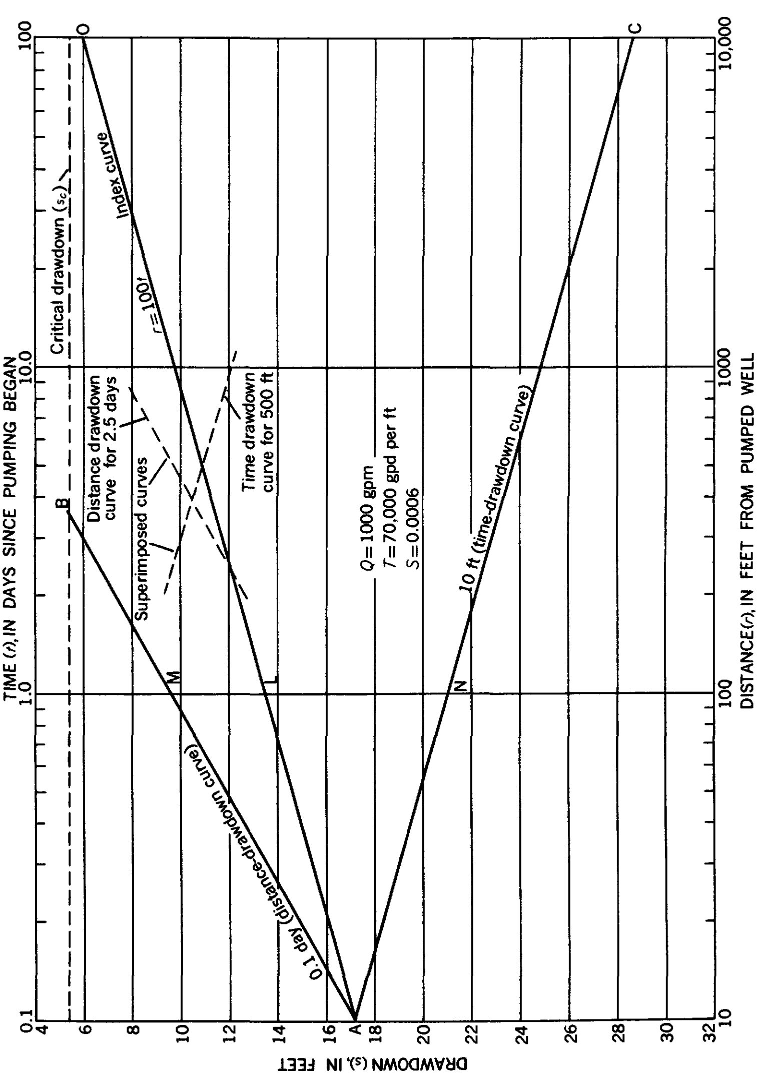

FIGURE 9.—Theoretical time- and distance-drawdown relation for an infinite aquifer having the indicated hydraulic characteristics.

resent the values of r, and the vertical arithmetic scale represent the values of s. In figure 9, $n=2$ for corresponding r and t values.

The semilog drawdown curve is a straight line if the value of u is less than 0.02 (Cooper and Jacob, 1946). The critical drawdown, s_c, which is the drawdown that must be reached for the straight-line plot to be valid, can be obtained by substituting the value of $W(u)$ that corresponds to u equal to 0.02 in the Theis nonequilibrium formula. The resulting formula for the critical drawdown is

$$s_c=\frac{384Q}{T}. \tag{2}$$

After determining the desired distance-time ratio and establishing the scales on the graph, calculate the critical drawdown and plot the horizontal line representing $s=s_c$. If an aquifer has a transmissibility of 70,000 gpd per foot and a well is discharging from the aquifer at a rate of 1,000 gpm, the critical drawdown, computed from equation 2, is 5.49 feet (fig. 9).

After determining the critical drawdown, plot, from the aquifer-test data, a distance-drawdown curve, AB, and a time-drawdown curve, AC; for the sake of accuracy, at least three points for which s is greater than s_c should be used to locate each line. In figure 9, AB represents the drawdowns at various distances from the pumped well when $t=0.1$ day, and AC represents the drawdowns at various times since pumping started where $r=10$ feet.

The drawdowns for several times and distances can be determined from the time- and distance-drawdown plot with the aid of a third line, called the index curve. The vertical distance between the time-drawdown curve and the index curve is n times the distance, measured along the same vertical line, between the distance-drawdown curve and the index curve. Consequently, the index curve, AO, can be plotted through points that divide the vertical line bounded by the two initial curves into segments whose lengths are in the ratio n to 1. In figure 9, any point L on AO divides the vertical line MN, where M is on AB and N is on AC, so that $NL=2LM$. That AO is the required curve is shown by equation 1.

USE OF THE TIME- AND DISTANCE-DRAWDOWN RELATION

The drawdowns for several times and distances can be determined from the time- and distance-drawdown graph. For the given conditions, the curves in figure 9 can be used to predict graphically the drawdowns at distances ranging from 10 to 10,000 feet and for pumping periods ranging from 0.1 day to 100 days.

The distance-drawdown curve for any time, t_1, is a line parallel to the distance-drawdown curve AB through the point where the vertical line $t=t_1$ intersects AO. In the problem in figure 9, for example, the drawdowns at the end of 2.5 days of continuous pumping can be determined by drawing a line parallel to AB through the point where the vertical line $t=2.5$ intersects AO. Then, drawdown values can be read from the graph for distances from 10 to about 1,800 feet, where the line intersects the critical-drawdown line. This distance-drawdown curve shows that, at a point 500 feet from a well discharging 1,000 gpm continuously for 2.5 days, the theoretical drawdown is about 9.6 feet (fig. 9).

The time-drawdown curve for any distance, r_1, is a line parallel to the time-drawdown curve AC through the point where the vertical line $r=r_1$ intersects AO. In the problem in figure 9, for example, the drawdowns at a distance of 500 feet for various times can be determined by drawing a line parallel to AC through the point where the vertical line $r=500$ intersects AO. Then, drawdown values for any time between 0.2 and 100 days can be determined from the graph. At the end of 2.5 days, the theoretical drawdown at a point 500 feet from the discharging well is 9.6 feet, which is the same as that obtained from the distance-drawdown curve (fig. 9).

CONCLUSION

The time- and distance-drawdown relation for a well discharging at a given rate from an aquifer that has given hydraulic properties can be represented graphically. From the semilog plot of the time-drawdown, the distance-drawdown, and the index curves, drawdowns can be determined for any time and place where the drawdown is less than the critical drawdown; the drawdown can be found for the given time from the time-drawdown curve for the given distance; or, the drawdown can be found for the given distance from the distance-drawdown curve for the given time. The two methods will give identical results.

METHOD FOR DETERMINATION OF THE COEFFICIENT OF STORAGE FROM STRAIGHT-LINE PLOTS WITHOUT EXTRAPOLATION

By Stanley W. Lohman

ABSTRACT

Each of the Cooper and Jacob, Jacob and Lohman, and Thiem methods for analysis of aquifer-test data makes use of a straight-line semilog plot. To date, the values to be substituted in the formulas used in computing the coefficient of storage for the point representing zero drawdown have been found by extrapolation; the author has developed new formulas for which the necessary values can be determined simply and directly from the data region of the straight-line plot. The formulas for the coefficient of storage for the various methods are similar, are easy to use, and require only values found directly from the graphs without extrapolation.

PRESENT METHOD FOR DETERMINATION OF THE COEFFICIENT OF STORAGE FROM STRAIGHT-LINE PLOTS

In straight-line methods of analyzing aquifer-test data for the determination of the hydraulic properties of an aquifer, the values substituted in the formulas now used to obtain S are the coordinates of the point representing zero drawdown, which must be found by extrapolation. The results obtained by extrapolation are satisfactory if the slope of the straight line is such that the point of zero drawdown falls within the confines of the semilogarithmic plot; otherwise, graphic extrapolation onto an adjoining sheet of graph paper for the required point may be difficult and inconvenient and may result in error. Although an arithmetic rather than a graphic extrapolation can be used, S can be determined simply and directly, without any extrapolation, from values that are obtained from the data region of the straight-line plot.

PROPOSED METHOD FOR DETERMINATION OF THE COEFFICIENT OF STORAGE FROM STRAIGHT-LINE PLOTS WITHOUT EXTRAPOLATION

For straight-line methods of analysis of aquifer-test data, Cooper and Jacob (1946) showed that only the first two terms of the infinite series resulting from the solution of the Theis (1935) exponential integral are necessary. The Theis equation can then be simplified as follows:

$$s=\frac{114.6Q}{T}\left(-0.5772-\log_e\frac{1.87r^2S}{Tt}-\ .\ .\ .\right)$$

$$=\frac{114.6Q}{T}\left(\log 0.562+\log_e\frac{Tt}{1.87r^2S}\right)$$

$$=\frac{2.3(114.6)Q}{T}\log_{10}\frac{0.3Tt}{r^2S}$$

$$=\frac{264Q}{T}\log_{10}\frac{0.3Tt}{r^2S}. \qquad (1)$$

The straight line for the Cooper and Jacob (1946) method of solution is the semilog plot of the corresponding values of s (or s/Q) and t or t/r^2; the slope of the line can be obtained from equation 1 by differentiating s (or s/Q) with respect to $\log_{10}t$ or $\log_{10}t/r^2$. Similarly, the straight line for the Jacob and Lohman (1952) method of solution for the nonsteady flow of an artesian well of constant drawdown is the semilog plot of corresponding values of s_w/Q and $t/r_w{}^2$ (where s_w is the drawdown and r_w is the radius of the well); the slope can be obtained from equation 1 by differentiating s_w/Q with respect to $\log_{10}t/r_w{}^2$. Also, the straight line for the Thiem (1906) method of solution for steady-state flow is the semilog plot of corresponding values of s and r; the slope can be obtained by differentiating s with respect to $\log_{10}r$ in equation 1.

In each of the straight-line methods, S generally has been determined from values found by extrapolating the straight-line part of the semilog plot to the point where s, s/Q, or s_w/Q is zero. Then, from equation 1,

$$s=\frac{264Q}{T}\log_{10}\frac{0.3Tt}{r^2S}=0.$$

Therefore,

$$\log_{10}\frac{0.3Tt}{r^2S}=0=\log_{10} 1.0$$

or

$$\frac{0.3Tt}{r^2S}=1$$

and

$$S=\frac{0.3Tt}{r^2}\left(\text{or }\frac{0.3Tt}{r_w{}^2}\right) \qquad (2)$$

or

$$S=2.1\times10^{-4}\frac{Tt_m}{r^2}\left(\text{or } 2.1\times10^{-4}\frac{Tt_m}{r_w{}^2}\right), \qquad (3)$$

where

t_m = the time, in minutes, since pumping began.

For the solution that does not involve extrapolation, equation 1 can be solved for S, for the Jacob and Lohman (1952) method, to give

$$S=\frac{0.3Tt/r_w^2}{\log_{10}^{-1}\left(\frac{Ts_w/Q}{264}\right)}. \tag{4}$$

Differentiating s_w/Q in equation 1 with respect to $\log_{10}t/r_w^2$ yields

$$\frac{d(s_w/Q)}{d[\log_{10}(t/r_w^2)]}=\frac{\Delta(s_w/Q)}{\Delta\log_{10}(t/r_w^2)}=\frac{264}{T}$$

or

$$T=\frac{264}{\Delta(s_w/Q)}, \tag{5}$$

where $\Delta(s_w/Q)$=the change over one log cycle of the ratio t/r_w^2. Equations 4 and 5 give

$$S=\frac{0.3Tt/r_w^2}{\log_{10}^{-1}\left(\frac{s_w/Q}{\Delta(s_w/Q)}\right)}=\frac{2.1\times10^{-4}Tt_m/r_w^2}{\log_{10}^{-1}\left(\frac{s_w/Q}{\Delta(s_w/Q)}\right)}. \tag{6}$$

Thus, to compute the value of S, select any convenient point on the semilog plot and substitute its coordinates, s_w/Q and t/r_w^2, in equation 6. The value of T can be computed from equation 5, and $\Delta(s_w/Q)$ is the value defined for that equation.

Similarly, for solutions involving semilog plots of s versus t or t/r^2, the corresponding form of equation 6 becomes

$$S=\frac{0.3Tt/r^2}{\log_{10}^{-1}(s/\Delta s)}=\frac{2.1\times10^{-4}Tt_m/r^2}{\log_{10}^{-1}(s/\Delta s)}, \tag{7}$$

where Δs=the change in drawdown over one log cycle of t or t/r.

For the analysis of a semilog plot of s versus r, equation 6 becomes

$$S=\frac{0.3Tt/r^2}{\log_{10}^{-1}(-2s/\Delta s)}=\frac{2.1\times10^{-4}Tt_m/r^2}{\log_{10}^{-1}(-2s/\Delta s)}, \tag{8}$$

where Δs=the change in drawdown over one log cycle of r and the negative sign in the denominator reflects the fact that the drawdown decreases as the distance from the discharging well increases. When substituting for Δs in equation 8, the numerical value should be prefixed with a minus sign, thus indicating that the slope of the straight line in the semilog plot is negative. The two negative signs then combine to make $(-2s/\Delta s)$ positive.

Equations 7 and 8 are applied in a manner similar to that described for equation 6. The method is illustrated by the example given in figure 10, which is a semilog plot of corresponding values of s and t_m/r^2 from data for an aquifer test in which the well is discharging

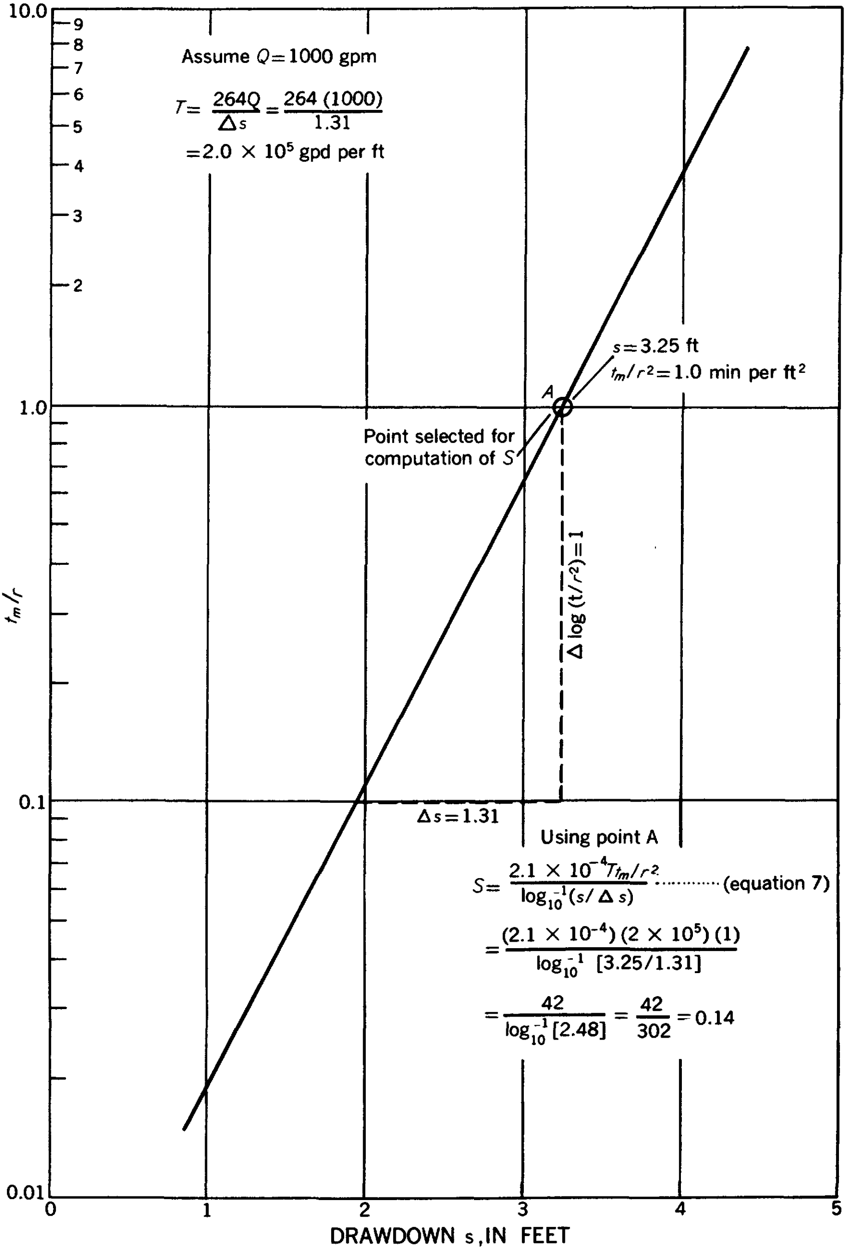

FIGURE 10.—Semilog plot of aquifer-test data and sample computations of the coefficients of transmissibility and storage.

at the rate of 1,000 gpm. Point A ($s=3.25$, $t_m/r^2=1$) was conveniently chosen to give an even value of t_m/r^2; $s=1.31$ is the change in drawdown over one log cycle measured from point A. Substituting these values in the appropriate formulas gives $T=200,000$ gpd per ft and $S=0.14$.

CONCLUSION

In the determination of the hydraulic properties of an aquifer by use of straight-line semilog plots, S can be found directly from values that are obtained from the data region of the straight-line plot without first extrapolating for the point where the drawdown would be zero. For the graphs of the corresponding values of s_w/Q and $\log_{10}t/r_w^2$, s and $\log_{10}t$ or $\log_{10}t/r^2$, and s and $\log_{10}r$, S can be found from equations 6, 7, and 8, respectively. The formulas are easy to use and the results are likely to be more accurate than those found in methods that require extrapolation.

SPECIAL DRAWDOWN SCALES FOR PREDICTING WATER-LEVEL CHANGES THROUGHOUT HEAVILY PUMPED AREAS

By Clyde S. Conover and Harold O. Reeder

ABSTRACT

Water-level declines for periods of heavy withdrawal can be predicted if the hydraulic properties of the aquifer are known. Such predictions generally are made from the Theis nonequilibrium formula. The computations for this method can be greatly simplified by use of a special drawdown scale that is based on a semilog drawdown-distance plot which reflects the hydraulic properties of the aquifer. The drawdown caused by any discharging well or the effect caused by any boundary can be determined from the scale for any point in the area; the net drawdown at that point is the sum of all such drawdown or buildup effects. If the net drawdown is determined for a sufficient number of points, lines showing the predicted lowering of the water table for the given period can be drawn on a map.

THE PROBLEM AND PROPOSED METHOD OF SOLUTION

The Theis nonequilibrium formula often is used in the quantitative analyses that are part of many ground-water investigations. The computations associated therewith may become very involved and tedious, especially if they deal with predictions of the decline of water levels throughout large areas in which there are many discharging wells. The process of predicting future water-level declines in a given area can be greatly simplified and shortened through the use of a special drawdown scale that reflects the hydrologic properties of the aquifer. Through the use of such a device, much of the computation can be reduced to scaling the pertinent values from a map that shows the location of the pumped wells. The net drawdown effect, which is the sum of the water-level declines caused by the individual pumped wells, can be determined readily for any desired point in the area. If the net drawdown effect at a number of points is desired, the process can be repeated for each point. From the net drawdown at a number of points at a given time, lines of predicted water-level changes for the multiple-well system can be drawn on a map.

GENERAL DESCRIPTION OF THE DRAWDOWN SCALE

The graduations on the finished special drawdown scale represent conveniently selected values of drawdown, in feet. The numbers on the scale are placed, in descending order of magnitude, at the proper distances from the reference point, which represents the drawdown at the pumped well. That this is the correct order is obvious from the fact that the drawdown is greatest at the pumped well and decreases with increasing distance from the pumped well. Each special scale should be inscribed with the map scale for which it is designed and with the values of the coefficients of transmissibility and of storage, the discharge rate of the well, and the time used to determine the drawdown.

PREPARATION OF THE DRAWDOWN SCALE

If the drawdown, s, caused by a pumped well, is to be determined, the following must be known: the coefficient of transmissibility, T; the coefficient of storage, S; the time, t, since pumping began—expressed in years if the distance, r, from the pumped well is in miles or in days if the distance is in feet; and the uniform rate of pumping, Q, of an individual well—expressed in gallons per minute or acre-feet per year. The discharge of a well pumped for irrigation generally is given in acre-feet per year. If the prediction of drawdown is for a time expressed in years, the value chosen for Q should be one that can be conveniently expressed in acre-feet per year; for example, a uniform discharge rate of 62 gpm is equivalent to 100 acre-feet per year. The pumpage from individual wells then can be expressed in multiples of hundreds of acre-feet per year and the respective drawdown effects can be modified proportionately to take into account the many discharge rates that may characterize multiple-well systems.

To make a special drawdown scale that accords with given information, proceed as follows:

1. Compute, with the aid of the previously described Theis chart or slide-rule scales, the drawdown factor, F, and then the drawdown, s, for various distances, r. Arrange the results in a form similar to that shown in table 5.

2. Plot on semilog paper the drawdown, s, on the log scale versus the distance, r, on the arithmetic scale, as shown in figure 11. The distance must be plotted to the same scale as that of the map on which the special drawdown scale is to be used. This semilog distance-drawdown plot differs from that used in analyzing aquifer tests because the desired result here is an arithmetic scale of distance on which selected values of drawdown will be correctly positioned; thus, the distance is plotted on the arithmetic scale and the drawdown is plotted on the log scale so that the plot will be of reasonable size.

3. Draw a smooth curve through the points plotted on the graph.

4. Calibrate the special scale by marking off selected values for the drawdown at the proper distances on the scale. This is accomplished by holding the special drawdown scale parallel to and exactly opposite

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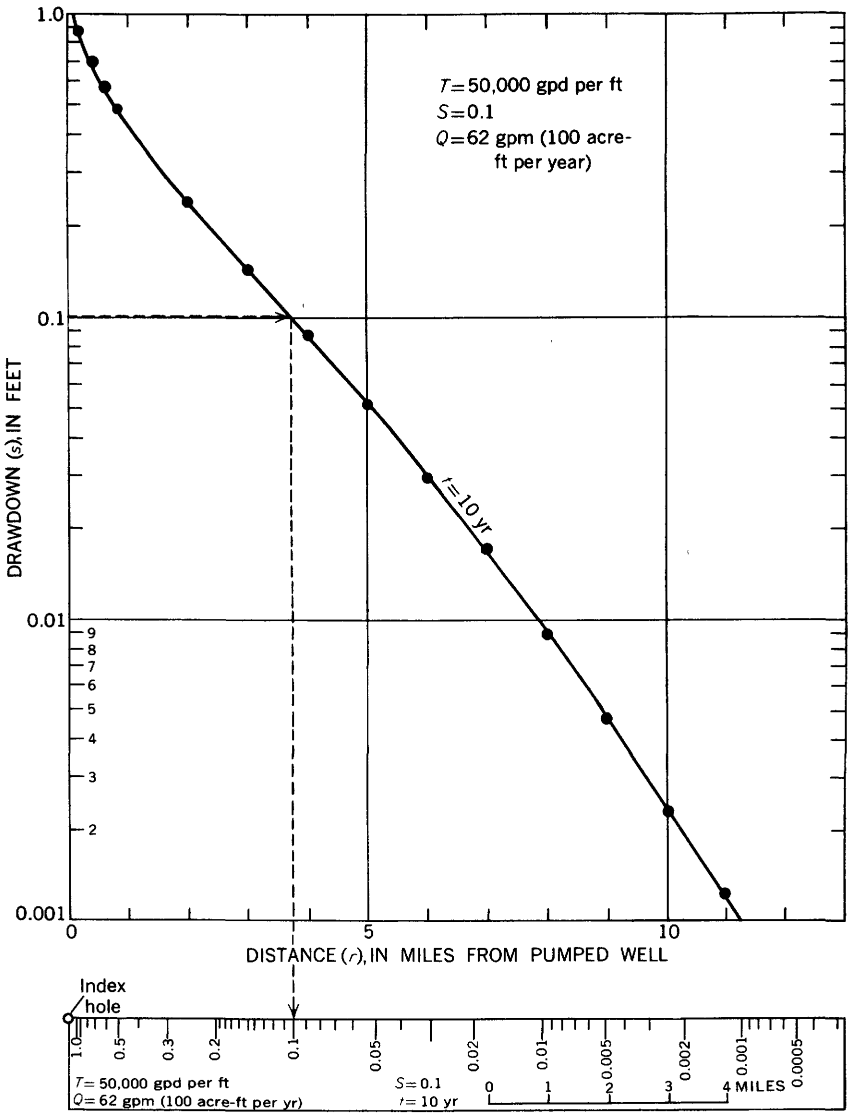

FIGURE 11.—Diagram of a special drawdown scale constructed from the graph of drawdowns under given conditions.

the distance scale of the graph. To place a drawdown value of 0.1 foot on the special scale, start at this value on the log scale of the graph, and move horizontally to the plotted curve and vertically downward to the special map scale, labelling the point thus found 0.1. This is the drawdown expected at the distance indicated on the graph in response to the pumping of one well under the known conditions. The distance in miles (or feet) from the pumped well to any point for which the drawdown has been determined need not be shown on the special drawdown scale.

5. Make a small hole in the special drawdown scale, at the index or reference point, through which a tack or other pointer can be inserted to secure the scale at the point on the map where the net drawdown effect of all pumping in the area is desired.

TABLE 5.—*Drawdown factors and drawdowns at various distances under the indicated conditions*

T=50,000 gpd per ft
S=0.1
Q=62 gpm (100 acre-ft per year)
t=10 years

$\frac{S}{T}=2\times10^{-6}$

$\frac{Q}{T}=12.4\times10^{-4}$

$s=\frac{Q}{T}F$

Distance, r (miles)	Drawdown factor, F	Drawdown, s (feet)
0. 2	711	0. 882
. 4	552	. 690
. 6	460	. 570
. 8	396	. 492
1. 0	346	. 430
2	196	. 244
3	118	. 1464
4	71	. 088
5	41	. 0508
6	24	. 0298
7	14	. 0174
8	7. 25	. 0090
9	3. 8	. 0047
10	1. 9	. 00236
11	1. 0	. 00124

The drawdown scale also can be constructed directly by use of the Theis slide-rule scales. (See p. C16.) The procedure requires computing the drawdown factor, F, for each value of drawdown desired to be inscribed on the special drawdown scale, and then solving for the particular distance at which the drawdown occurs. This procedure is convenient if the Theis slide-rule scales are available but requires a separate determination for each desired drawdown value. Furthermore, because of the wide spacing of graduations on the Theis slide-rule scales, the computed distances may differ slightly from those determined by the graphic method, which tends to smooth out irregularities inherent in the calculations.

USE OF THE DRAWDOWN SCALE

Place the reference point of the scale over the first point on the map at which a net drawdown prediction is desired and secure the scale to the map by inserting a tack or pin through the reference point. Rotate the scale around the point and tabulate the drawdown caused by each pumped well, as determined by the point on the scale that coincides with the well position. Multiply each of these values by the rate, in hundreds of acre-feet per year, at which each well is pumped or is expected to be pumped. The sum of all these adjusted values is the net drawdown prediction desired, based on the given or assumed conditions. The foregoing process can be repeated for each point on the map at which a drawdown prediction is desired.

The points selected for the drawdown predictions should not coincide with any pumped well or its immediate vicinity in order to insure that the results will be as realistic as possible. If the number of points is adequate, the predicted drawdown after a designated period of time under the given or assumed conditions may be shown conveniently on a map by lines of equal predicted drawdown.

If a boundary or boundaries are present, the appropriate system of image wells can be plotted on the map and the foregoing procedures followed. The effect of each image well will be added or subtracted, depending upon the type of boundary, in determining the net drawdown effect.

A PRACTICAL APPLICATION OF THE SCALE

A map of the predicted lowering of the water table over a 10-year period in the lower Animas Valley, Hidalgo County, N. Mex., is shown in figure 12. The lines show the decline of the water level that can be expected throughout the area if all the irrigation wells for which permits have been granted are pumped at the rates indicated in the respective permits throughout the period. The map indicates the location of those irrigation wells and also the arbitrarily selected points, shown as triangles, for which the net drawdown was determined for the end of a 10-year period during which normal water consumption, or crop requirements, was assumed. A special drawdown scale was made by use of the predetermined aquifer constants and a steady discharge rate of 62 gpm (100 acre-ft a year). As the scale was rotated around the index point, which was secured to one of the arbitrarily chosen points, to each pumped well, the respective drawdown effects were tabulated. Each drawdown was then adjusted by multiplying the number of hundreds of acre-feet the pumped

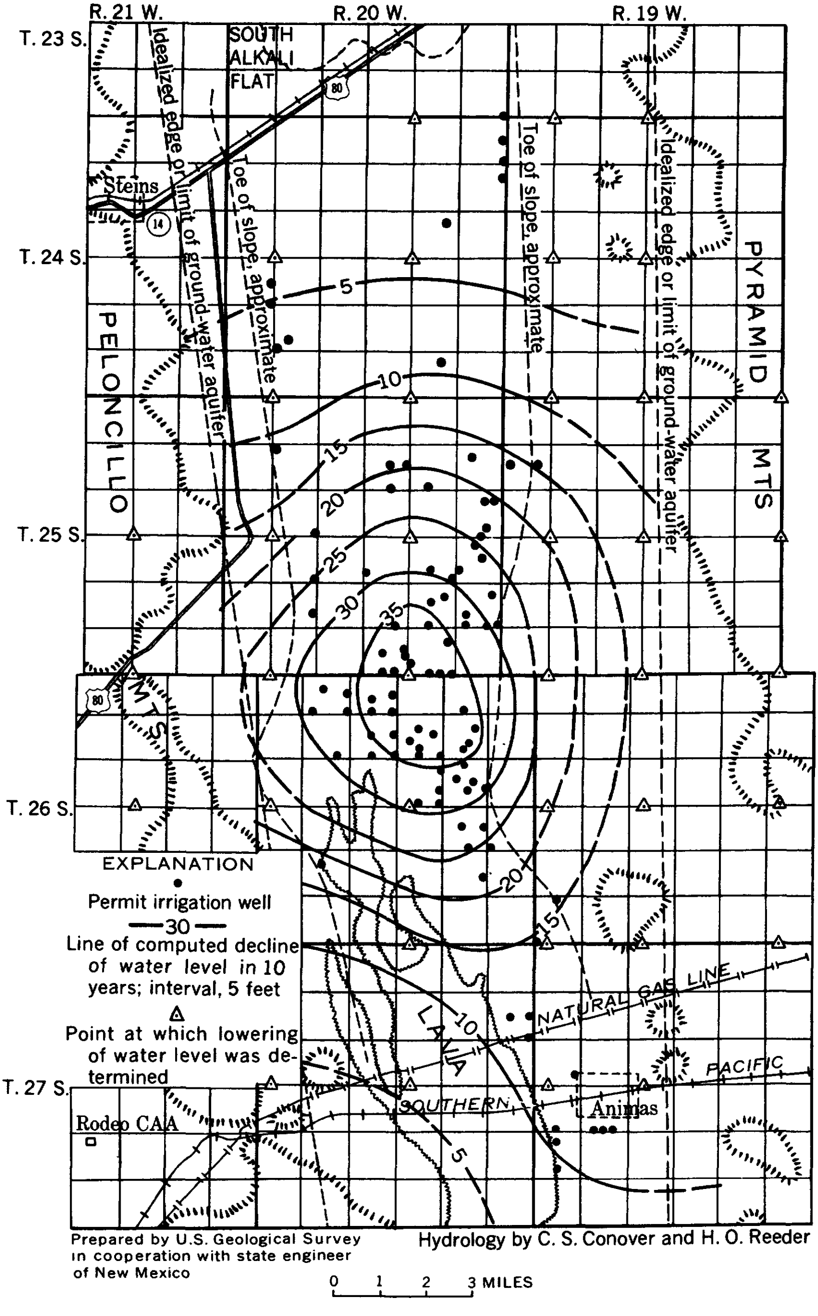

FIGURE 12.—Computed lowering of the water table in 10 years (1948–58) in the lower Animas Valley, Hidalgo County, N.Mex.

well was expected to deliver annually. For each selected point, the individual drawdown effects were totaled, resulting in the declines that would occur in an areally extensive aquifer. However, because the aquifer has two boundaries across which no recharge occurs, the real pumped wells were reflected across each boundary to establish two systems of image well. This was done by folding over at each boundary a transparent print of figure 12, thus eliminating the need for replotting all the wells. The drawdown effects from the image wells were determined for each of the selected points, using the method previously described. These were added to the previously totaled values, and the net drawdown was plotted for each point. Drawdown lines then were drawn to show graphically the predicted water-level changes throughout the area. The successive reflection of the image wells, beyond the first set of images, was not necessary in this example although in other problems it might be mandatory. If one of the boundaries had been a recharge boundary, the water-level changes caused by the image wells opposite that boundary would have been subtracted from those of the real wells.

Possibly a simpler method of dealing with boundaries, especially if the position of the boundary is uncertain, is to consider the aquifer as areally extensive. The drawdown effects can be determined beyond the boundary as if no boundary were present. After the drawdowns due to the real wells are plotted on the map and the lines of equal drawdown are constructed, the map can be folded at the best estimated position of the boundary or boundaries. The drawdown lines on the folded part or parts of the map will then be superposed on the drawdown lines of the remaining part of the map. By algebraically adding the several sets of drawdown lines, in accord with the recognized nature of the boundary or boundaries, a new set of lines can be drawn to show the resultant water-level changes. The primary advantage of this method is that the position of the boundary can be shifted, if necessary, after the computations are made and the water-level changes caused by the new positions of the image wells need not be recomputed; the sets of drawdown lines, properly superposed by refolding the map, need only to be added (or subtracted) again and a new resultant set of lines to be drawn.

CONCLUSION

Although only one practical application of the special drawdown scale has been described, its use would prove helpful in the solution of many field problems. The special scale offers a shortcut method for summing the several drawdown or buildup effects, especially in boundary problems that are to be analyzed by systems of multiple image wells.

TYPE CURVES FOR THE SOLUTION OF SINGLE-BOUNDARY PROBLEMS

By Robert W. Stallman

ABSTRACT

An aquifer that is laterally contiguous with saturated material that has a vastly different transmissibility is considered to have a boundary. If the horizontal trace of the boundary can be represented by a straight line, the hydraulic properties of the aquifer can be determined by the use of the image-well theory. For aid in the solution of single-boundary problems, the Theis nonequilibrium formula has been modified to include the boundary effect and a family of type curves has been developed from the original Theis type curve. Together with the family of type curves, the modified Theis formula is used to determine not only the coefficients of transmissibility and of storage of the aquifer but also the location of the boundary.

SINGLE BOUNDARIES AND THE IMAGE-WELL THEORY

If aquifer A is laterally contiguous with relatively impermeable material B, an insignificant amount of water flows from B to A when a well that taps A is pumped. This situation can be duplicated hydraulically by assuming that the aquifer is infinite and by postulating a discharging image well to give the required boundary effect. For example, if a pumped well is a distance a from the assumed boundary, the corresponding discharging image well is located on a line perpendicular to the boundary through the pumped well at a distance a from the boundary on the opposite side from the pumped well. If the image well is assumed to begin discharging at the same time and to discharge at the same rate as the pumped well, a theoretical ground-water divide is developed along the assumed boundary between A and B.

Similarly, if material B has a transmissibility vastly greater than that of aquifer A, B can be assumed to be an unlimited source of water and no drawdown occurs along the assumed boundary between A and B. This situation can be duplicated hydraulically by the above method except that the image well is assumed to recharge (rather than discharge) the infinite aquifer at the same rate as the pumped well is discharging from aquifer A.

MODIFIED THEIS FORMULA

The net drawdown in an observation well in an aquifer that has a single boundary is the algebraic sum of the drawdown caused by the pumped well and the theoretical drawdown caused by the image well. If s_o is the drawdown in an observation well and if s_p and s_i are the components of that drawdown caused, respectively, by the pumped well and the discharging or recharging image well, then

$$s_o = s_p \pm s_i. \tag{1}$$

The basic Theis formula is

$$s = \frac{114.6Q}{T} W(u), \text{ where } u = \frac{1.87r^2S}{Tt}. \tag{2}$$

From equations 1 and 2

$$s_o = \frac{114.6Q}{T}\left[W(u)_p \pm W(u)_i\right] = \frac{114.6Q}{T} \sum W(u), \tag{3}$$

where $W(u)_p$ and $W(u)_i$ are the values of $W(u)$ for the pumped well and the image well, respectively. The values of u_p and u_i corresponding to the values of $W(u)_p$ and $W(u)_i$ are

$$u_p = \frac{1.87r_p{}^2S}{Tt} \text{ and } u_i = \frac{1.87r_i{}^2S}{Tt}, \tag{4}$$

where r_p is the distance from the pumped well to the observation well and r_i is the distance from the image well to the observation well. From equations 4, u_p and u_i are seen to be related as follows:

$$u_i = (r_i/r_p)^2 u_p \text{ or } u_i = K^2 u_p, \tag{5}$$

where

$$K = r_i/r_p. \tag{6}$$

This procedure is similar to the one followed by Kazmann (1946).

MODIFIED TYPE CURVES AND THEIR APPLICATION

Just as a Theis type curve is used to solve for S and T in equations 2, one of a family of modified type curves can be used to solve for these values in equations 3 and 4. The shape of each of the modified curves depends on the relative positions of the observation well, the pumped well, and the image well—that is, on the ratio r_i/r_p, which is equal to K. Each modified Theis curve is a log plot of corresponding values of $\sum W(u)$ and $1/u_p$ (for convenience, $1/u_p$ can be used in place of u_p). Plate 3 shows the modified type curves for various values of K and u_p; the value of $\sum W(u)$ were found by first finding the appropriate u_i from equation 5 and then finding the algebraic sum of

$W(u)_p$ and $W(u)_i$, the values of which are found in the table of values of $W(u)$ and u (Wenzel, 1942, table facing p. 89).

A log plot of the corresponding values of the drawdowns in an observation well and the times since pumping started is a curve that belongs to the family of type curves. If the aquifer is of wide areal extent, all the plotted data fall on the Theis type curve, which is shown as the "parent curve" on the graph of the family of type curves. However, if the aquifer is laterally contiguous with saturated material of a vastly different transmissibility than that of the given aquifer, the observed data plot along the parent curve until such time that the effect of the boundary is first felt at the observation well and then will deviate from the parent curve along one of the modified curves.

For the solution of a single-boundary problem, the plot of the observed data is superposed on the family of type curves, keeping the $\sum W(u)$ axis parallel with the s axis and the $1/u_p$ axis parallel with the t axis. When the plotted data are matched to one of the modified type curves, a point common to both can be selected to obtain values of $\sum W(u)$ and $1/u_p$ corresponding to values of t and s, respectively. Equation 3 can then be solved for T and equations 4 can be solved for S. From the value of K for the particular modified curve that matched the data curve, the value of r_i can be computed from equation 6.

The position of the straight line that represents the assumed boundary between the aquifer and the more or less permeable material can be determined by use of data from three observation wells, provided the ratio K is sufficiently large and the observation wells are not too close together. For best results, K should be greater than 2.5 but less than 100. The distances, r_{ai}, r_{bi}, and r_{ci}, of the image well from the observation wells, a, b, and c, respectively, can be determined by use of the data curves and the appropriate modified type curves. The common intersection of arcs of circles drawn with radii r_{ai}, r_{bi}, and r_{ci}, and centers a, b, and c, respectively, determines the position of the image well. The assumed boundary is midway between the image and discharging wells and is oriented in a direction normal to the line through these two wells.

CONCLUSION

A log plot of the time-drawdown data from an aquifer test may match one of the family-type curves that have been modified from the Theis-type curve. If so, the aquifer has a single boundary. The coefficients of transmissibility and storage and the location of the straight line that represents the assumed boundary can be found by the use of the modified Theis formula and the appropriate type curve.

TYPE CURVES FOR NONSTEADY RADIAL FLOW IN AN INFINITE LEAKY ARTESIAN AQUIFER

By HILTON H. COOPER, JR.

ABSTRACT

The coefficients of transmissibility and storage of a leaky artesian aquifer and the leakance, or rate of leakage through its confining bed, can be determined by superposing a plot of nonsteady-state drawdowns on a family of type curves. Also, if the coefficients of the aquifer and of its confining bed are known, the family of type curves can be used for the prediction of drawdowns.

THE EFFECT OF LEAKANCE ON DRAWDOWNS IN AN ARTESIAN AQUIFER

The prediction of the ultimate water-level drawdown that will result from pumping is a common problem of economic importance. Mathematically, the problem is one of computing drawdowns for the steady-state condition, which occurs when the rate of withdrawal has been balanced entirely by the capture of water from sources outside the aquifer—that is, when water is no longer being withdrawn from storage within the aquifer. The capture may consist of an increase in the rate of recharge to the aquifer, a decrease in the rate of discharge from the aquifer, or, more probably, a combination of both (Theis, 1940, p. 277).

When water is being withdrawn from an artesian aquifer, the piezometric surface of the water in the aquifer is lowered throughout a large circular area that has the well at its center. Because all confining beds probably are permeable to some degree, the lowering of the piezometric surface results in a change in the rate of leakage through the confining bed. The change may consist of a decrease in the rate of leakage out of the aquifer or an increase in the rate of leakage into the aquifer, but in either case the change results in a net increase to the supply of water to the aquifer and, therefore, constitutes capture.

As the permeability of an effective confining bed is small, the change in the rate of leakage through a confining bed ordinarily is only a small fraction of a gallon per day per square foot. However, because a cone of depression that has been created by heavy pumping ordinarily encompasses many millions and even billions of square feet, leakage through the confining bed may result in the capture, by the aquifer, of a considerable quantity of water. Situations wherein practically all the water being discharged from wells is balanced by such capture probably are not uncommon.

An equation for the steady-state drawdown near a well discharging at a constant rate from an infinite leaky artesian aquifer was derived by Jacob (1946) after deGlee (1930), Steggewentz and Van Nes (1939), and others had analyzed the same problem. Jacob's equation, with an alteration of the terms in the argument, is as follows:

$$s=\frac{Q}{2\pi T}K_0(2v), \tag{1}$$

where Q and T are as previously defined,

K_0=the modified Bessel function of the second kind of zero order, and

$$v=\frac{r}{2}\sqrt{\frac{P'}{m'T}},$$

in which

P'=the coefficient of permeability of the confining bed in the vertical direction,

m'=the thickness of the confining bed,

and r is as previously defined.

Equations for the steady-state drawdowns near wells that tap variously bounded leaky artesian aquifers are given by Hantush and Jacob (1954). The development of these equations is based on the assumption that the change in the rate of leakage is proportional to the drawdown or, in other words, that the water level in the material that overlies the confining bed is not lowered and that the hydraulic gradient through the confining bed has adjusted completely to the new conditions. That situation is approached if the drawdown in the material that overlies the confining bed is small in comparison with that in the aquifer and if sufficient time has elapsed for the gradient through the confining bed to adjust to the drawdown.

If the T of the aquifer and the ratio between the P' and m' of the confining bed are known, the steady-state drawdowns can be computed from equation 1 or from the equations given by Hantush and Jacob (1954) for bounded leaky artesian aquifers. The ratio P'/m', which has been referred to by Hantush and Jacob (1955a) as the "leakance" of the confining bed, can be determined by a graphic method suggested by Jacob (1946, p. 204), as follows:

> From data on the steady-state distribution of drawdown in the vicinity of a well or a center of pumping, if leakage is known to occur, values of the parameters T and a/b [$a/b=P'/m'$] may be determined by a modification of the Theis (Jacob, 1940, p. 582) graphical method, in which the "type curve" is a plot of $K_0(x)$ [$K_0(x')$] against x [x'] on logarithmic paper and the observational data are plotted on logarithmic paper of the same scale with r as abscissa and s as ordinate.

This method, which has been developed and described in detail by Ferris and others (1962, p. 110), has advantages in both accuracy and ease of application, but its usefulness is limited in that it requires steady-state drawdowns, which rarely are obtainable. Other methods for determination of P'/m' from observed or extrapolated steady-state drawdowns and from nonsteady-state drawdowns in two or more wells have been devised by Hantush (1956), and a method for determination of P'/m' from nonsteady-state drawdowns by progressive approximations and use of a chart of curves was devised by R. E. Glover (U.S. Bur. Reclamation, 1960).

This paper describes a method whereby the T and S of a leaky artesian aquifer and the leakance, P'/m', of the confining bed are determined from nonsteady-state drawdowns by superposing a plot of the drawdowns on a family of type curves. The method is an extension of the Theis graphic method. If the hydraulic constants of both the aquifer and the confining bed are already known, the type curves may be used also for the prediction of drawdowns.

The author is indebted to M. I. Rorabaugh of the Geological Survey for his insistence on the worth of the type curves and for prompting the writing of this explanatory text.

DERIVATION OF THE TYPE CURVES

The type curves in plate 4A are based on data that were computed from the following equation, derived by Hantush and Jacob (1955a), for nonsteady radial flow in an infinite leaky artesian aquifer:

$$\frac{s}{Q/4\pi T}=2K_0(2v)-\int_{v^2/u}^{\infty}(1/y)\exp\left(-y-v^2/y\right)dy, \tag{2}$$

where

$u=r^2S/4Tt$

and the other terms are as previously defined.

Hantush and Jacob (1955a) gave two series expressions for the formal solution of the equation. One, which is convenient to use if t is large, is

$$\frac{s}{Q/4\pi T}=2K_0(2v)-I_0(2v)W(v^2/u)$$
$$+[\exp(-v^2/u)]\left\{W(u)+0.5772+\log_e u-u+(u/v^2)[I_0(2v)-1]\right.$$
$$\left.-u^2\sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{(-1)^{n+m}(n-m+1)!(v^2/u)^m u^n}{(n+2)!}\right\}, \tag{3}$$

and the other, which is convenient to use if t is small, is

$$\frac{s}{Q/4\pi T}=I_0(2v)W(u)-e^{-u}\{0.5772+\log_e(v^2/u)+W(v^2/u)-v^2/u$$

$$+[I_0(2v)-1]/u\}+(e^{-u}/u^2)\sum_{n=1}^{\infty}\sum_{m=1}^{n}\frac{(-1)^{n+m}(n-m+1)!(v^2/u)^n u^m}{(n+2)!^2}, \quad (4)$$

where

I_0=the modified Bessel function of the first kind of zero order,

$$W(x')=-Ei(-x')=\int_{x'}^{\infty}(e^{-x'}/x')dx',$$

and the other terms are as previously defined.

If the right-hand side of equations 2, 3, or 4 is represented by the term $L(u,v)$, the solution then may be written

$$s=\frac{Q}{4\pi T}L(u,v).$$

In the limiting case wherein P' and, hence, v approach zero, $L(u,v)$ approaches $W(u)$ and the solution becomes

$$s=\frac{Q}{4\pi T}W(u), \quad (5)$$

which is the familiar Theis nonequilibrium formula (Wenzel, 1942, p. 88).

The type curves, which are plots of $L(u,v)$ versus $1/u$, consist of two families, in one of which v is the parameter and in the other of which v^2/u is the parameter. The parent curve of both families, labeled $W(u)$, is the Theis nonequilibrium type curve. In plate 4*A*, the family of solid-line type curves, for which v is the parameter, corresponds to a plot of s versus t at some constant r, whereas the family of dashed-line type curves, for which v^2/u is the parameter, corresponds to a plot of s versus $1/r^2$ at some constant t. Either family of type curves may be used to compute T, S, and P'/m'.

METHOD FOR USE OF THE TYPE CURVES

To compute T, S, and P'/m' by use of the solid-line type curves, proceed as follows:

1. Plot s versus t/r^2 for each observation well on logarithmic graph paper having the same scale as the graph of the type curves.
2. Superpose this time-drawdown plot on the solid-line type curves

and, keeping the coordinate axes of the two graphs parallel, translate the data plat to the position where the earliest data approach the limiting curve labeled $W(u)$ and all the remaining data fall either between one pair of the curves labeled $v=2.2$, $v=2.0$, etc., or along one of them.

3. Select a convenient match point and note its coordinates (s, t/r^2, $L(u,v)$, and $1/u$).

4. Determine the value of v that corresponds to the value of r for each observation well. If the later data do not lie along one of the v-curves, estimate the value of v by interpolation.

5. Compute the hydraulic constants of the aquifer by making appropriate substitutions in the following equations:

$$T=\frac{Q}{4\pi}\cdot\frac{L(u,v)}{s}, \tag{6}$$

$$S=4T\frac{t/r^2}{1/u}, \tag{7}$$

and

$$\frac{P'}{m'}=4T\frac{v^2}{r^2}. \tag{8}$$

Expressed in the customary Geological Survey units, equation 6 becomes

$$T=114.6Q\frac{L(u,v)}{s}, \tag{9}$$

equation 7 becomes

$$S=\frac{T(t/r^2)}{1.87(1/u)}, \tag{10}$$

and equation 8 remains unchanged.

To compute T, S, and P'/m' by use of the dashed-line type curves, proceed as follows:

1. Plot values of s, each from a different observation well but for identical values of t, versus t/r^2 on logarithmic graph paper having the same scale as the graph of the type curves.

2. Superpose this distance-drawdown plot on the dashed-line type curves and, keeping the coordinate axes of the two graphs parallel, translate the data plot to the position where all the data fall between one pair of the type curves or along one of them.

3. Select a convenient match point and note its coordinates (s, t/r^2, $L(u,v)$, and $1/u$).

4. Determine the value of v^2/u that corresponds to the value of t at which the drawdowns occurred. If the data do not lie along one of the type curves, estimate the value of v^2/u by interpolation.

5. Compute the values of T and S from equations 6 and 7 or from equations 9 and 10. If the value of P'/m' is to be expressed in units consistent with those for T and S in equations 6 and 7, use

$$\frac{P'}{m'}=S\,\frac{v^2/u}{t}, \tag{11}$$

but if it is to be expressed in the customary Geological Survey units, use

$$\frac{P'}{m'}=7.48S\,\frac{v^2/u}{t}. \tag{12}$$

If, when superposed on the dashed-line type curves, the plotted data fall in the region $v^2/u \geq 8$ and $L(u,v) \geq 10^{-2}$, steady-state conditions have been reached and the method of analysis suggested by Jacob (1946) and described by Ferris and others (1962, p. 112–115) is applicable.

AN APPLICATION OF THE TYPE CURVES

The method of analyzing a test in which steady-state conditions are not reached is illustrated by using postulated measurements of water-level drawdown at distances of 100, 500, and 1,000 feet from a well that discharges 1,000 gpm for 1,000 minutes from a leaky artesian aquifer. (See table 6.) The values of s are plotted against corresponding values of t/r^2 on logarithmic graph paper, and the plotted data are superposed on and matched to the solid-line type curves, as shown in plate 4*B*. For convenience, the match point is selected where $L(u,v)=1.0$ and $1/u=1.0$; then the corresponding coordinate values on the data sheet are $s=1.15$ and $t/r^2=1.87\times10^{-9}$. Substitution of these values in equations 9 and 10 gives

$$T=114.6Q\,\frac{L(u,\,v)}{s}=114.6\times1000\times\frac{1.0}{1.15}=100{,}000 \text{ gpd per ft}$$

and

$$S=\frac{T(t/r^2)}{1.87\,(1/u)}=\frac{100{,}000\times1.87\times10^{-9}}{1.87\times1.0}=0.0001.$$

TABLE 6.—*Postulated water-level drawdowns in three observation wells during a hypothetical test of an infinite leaky artesian aquifer*

[Pumped well began discharging 1,000 gpm at t=0 minutes]

Time since pumping began, t		Observation well 1 (r=100 ft)		Observation well 2 (r=500 ft)		Observation well 3 (r=1,000 ft)	
Minutes	Day	t/r^2 (day/ft²)	Drawdown, s (ft)	t/r^2 (day/ft²)	Drawdown, s (ft)	t/r^2 (day/ft²)	Drawdown, s (ft)
0.2	0.000139	1.39×10^{-8}	1.76	5.56×10^{-10}	0.01	1.39×10^{-10}	0.00
0.5	.000347	3.47×10^{-8}	2.75	1.39×10^{-9}	.14	3.47×10^{-10}	.00
1	.000694	6.94×10^{-8}	3.59	2.78×10^{-9}	.45	6.94×10^{-10}	.02
2	.00139	1.39×10^{-7}	4.26	5.56×10^{-9}	.93	1.39×10^{-9}	.14
5	.00347	3.47×10^{-7}	5.28	1.39×10^{-8}	1.76	3.47×10^{-9}	.55
10	.00694	6.94×10^{-7}	5.90	2.78×10^{-8}	2.34	6.94×10^{-9}	.99
20	.0139	1.39×10^{-6}	6.47	5.56×10^{-8}	2.85	1.39×10^{-8}	1.46
50	.0347	3.47×10^{-6}	6.92	1.39×10^{-7}	3.31	3.47×10^{-8}	1.95
100	.0694	6.94×10^{-6}	7.11	2.78×10^{-7}	3.50	6.94×10^{-8}	2.10
200	.139	1.39×10^{-5}	7.20	5.56×10^{-7}	3.51	1.39×10^{-7}	2.11
500	.347	3.47×10^{-5}	7.21	1.39×10^{-6}	3.52	3.47×10^{-7}	2.11
1,000	.694	6.94×10^{-5}	7.21	2.78×10^{-6}	3.52	6.94×10^{-7}	2.11

In the matched position, the time-drawdown plot for observation well 1 falls slightly below the solid-line curve for v=0.02, or at about v=0.025. Substitution of r=100 and v=0.025 in equation 8 gives the following value for leakance:

$$\frac{P'}{m'}=4T\frac{v^2}{r^2}=4\times100{,}000\times\frac{(0.025)^2}{(100)^2}=0.025 \text{ gpd per ft}^3.$$

The same value for leakance would be obtained from the time-drawdown plot for observation well 2, which falls at v=0.125, and from that for observation well 3, which falls at v=0.25. The precise agreement in the values obtained from the three time-drawdown plots is due, of course, to the fact that the postulated data that were used for this illustration represent idealized conditions.

The value of s for each of the three observation wells for t=100 minutes (0.0694 day) falls slightly below the dashed-line type curve for v^2/u=2.0, or at about v^2/u=2.1. Substitution of this value for v^2/u in equation 12 gives

$$\frac{P'}{m'}=7.48S\frac{v^2/u}{t}=7.48\times0.0001\times\frac{2.1}{0.0694}=0.023 \text{ gpd per ft}^3.$$

LIMITATIONS OF THE METHOD

Equation 2, from which the data for the type curves were computed, is based on the assumption that the change in the rate of leakage through the confining bed is proportional to the decline in head in the aquifer. However, when the flow is unsteady, such proportionality is possible only if the hydraulic gradient through the confining bed adjusts instantaneously to the decline in head in the aquifer, a condition that could be approached if the confining bed is sufficiently thin, if the rate of decline in head in the aquifer is sufficiently slow, and (or) if the diffusivity of the confining bed is sufficiently large. The term "diffusivity" as used here, is the change in the volume of water in a unit volume of material per unit change in head and is expressed as the ratio of the permeability of the confining bed to the "specific storage coefficient," where the specific storage coefficient is the storage coefficient of the confining bed divided by the thickness of the confining bed. The permeability and, hence, the diffusivity of all confining beds is small in comparison with those of aquifers. Furthermore, in materials that compact substantially when drained, the specific storage coefficient is large and the diffusivity, therefore, is very small indeed.

CONCLUSION

Because the adjustment of the hydraulic gradient through a confining bed generally lags considerably behind the decline in head, the water yielded by an artesian aquifer is derived largely, if not entirely, from storage in the confining bed. For this reason, most time-drawdown plots deviate from the Theis curve to a greater degree than if leakage alone were involved. The method for determining leakance is presented with reservation because, if applied under the mistaken assumption that the deviations are due to leakage, it yields erroneously large values. However, whenever the results of an aquifer test indicate that leakage occurs, the determination of T and S by use of the family of type curves described in this paper has advantages over that by use of the Theis type curve alone.

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DRAWDOWN PATTERNS IN AQUIFERS HAVING A STRAIGHT-LINE BOUNDARY

By Solomon M. Lang

ABSTRACT

If two laterally contiguous saturated materials have vastly different transmissibilities, the drawdown pattern caused by a well discharging from one of the materials is either that caused by a well discharging from an aquifer that has a recharge boundary or that caused by a well discharging from an aquifer that has an impermeable boundary. In the preceding paper, Stallman modified the Theis nonequilibrium formula for use in simulated straight-line boundary problems. In this paper it is demonstrated that for times after the initial pumping period—that is, where the value of u is equal to or less than 0.02—the drawdowns can be found from a simplification of Stallman's modification of the Theis formula.

For each type of boundary, the simplification of the modified formula yields certain distance relations that provide a convenient method for locating the loci of points of equal net drawdown. The loci of equal net drawdowns caused by a well discharging near a recharge boundary are circles that are centered on the line through the pumped well and the postulated image well. The drawdowns caused by a well discharging near an impermeable boundary do not attain equilibrium but each locus for a given drawdown moves farther from the well with time; the loci near the well are oval, farther from the well they are teardrop shaped, and still farther from the well they intersect the boundary at right angles.

STRAIGHT-LINE BOUNDARY CONDITIONS AND THE IMAGE-WELL THEORY

The Theis (1935) nonequilibrium formula has been modified by Stallman, in the preceding paper in this series, for use in situations where an aquifer from which water is pumped is laterally contiguous with saturated material of vastly different transmissibility and the boundary between the two may be represented by a straight line. If the adjacent material has a much higher transmissibility than the pumped aquifer, virtually no drawdown will occur along the boundary; if the adjacent material has a much lower transmissibility than the pumped aquifer, virtually no water will flow through the boundary. The modified formula was developed from the image-well theory. The postulated image well is a recharging well if the adjacent material is more permeable and is a discharging well if the adjacent material is less permeable than the pumped aquifer.

The solution given by Stallman can be extended to provide a more convenient method for computing the drawdowns in the vicinity of a well discharging from an aquifer that has a boundary and also for determining the time at which the drawdowns approximate stabilization for a well discharging near a recharge boundary.

DRAWDOWN PATTERNS CAUSED BY A WELL DISCHARGING NEAR A RECHARGE BOUNDARY

In an aquifer that has a recharge boundary, the net drawdown at any point is given by the following equation:

$$s_o = s_p - s_i = \frac{114.6Q}{T} [W(u)_p - W(u)_i], \tag{1}$$

where s_o is the net drawdown at any point, the subscript $_p$ relates the given symbol to the pumped well, and the subscript $_i$ relates the given symbol to the image well. For two points that have the same drawdown, $s_{o_1} = s_{o_2}$, and

$$\frac{114.6Q}{T} [W(u)_{p_1} - W(u)_{i_1}] = \frac{114.6Q}{T} [W(u)_{p_2} - W(u)_{i_2}]$$

or

$$W(u)_{p_1} - W(u)_{i_1} = W(u)_{p_2} - W(u)_{i_2}. \tag{2}$$

According to Cooper and Jacob (1946), where t becomes sufficiently large so that u is equal to or less than 0.02, the value of $W(u)$ can be approximated as $(-0.5772 - \log_e u)$ and equation 2 can be written in the form

$$-0.5772 - \log_e u_{p_1} + 0.5772 + \log_e u_{i_1}$$
$$= -0.5772 - \log_e u_{p_2} + 0.5772 + \log_e u_{i_2}$$

or

$$\log_e u_{i_1} - \log_e u_{p_1} = \log_e u_{i_2} - \log_e u_{p_2},$$

from which

$$\frac{u_{i_1}}{u_{p_1}} = \frac{u_{i_2}}{u_{p_2}}. \tag{3}$$

By definition

$$u = \frac{1.87 r^2 S}{Tt}.$$

If t is the same for both drawdown points, the only variable in u is r^2 and

$$\frac{r_{i_1}^2}{r_{p_1}^2} = \frac{r_{i_2}^2}{r_{p_2}^2}$$

or

$$\frac{r_{i_1}}{r_{p_1}} = \frac{r_{i_2}}{r_{p_2}} = \frac{r_i}{r_p} = c, \tag{4}$$

where c is a constant.

Under equilibrium conditions, the locus of points of equal net drawdown at a given time around a well near a recharge boundary is a circle (Muskat, 1937, p. 175–177) that is centered on the line through the image and pumped wells on the side of the pumped well away from the boundary (fig. 13). Such a circle can be located easily by deter-

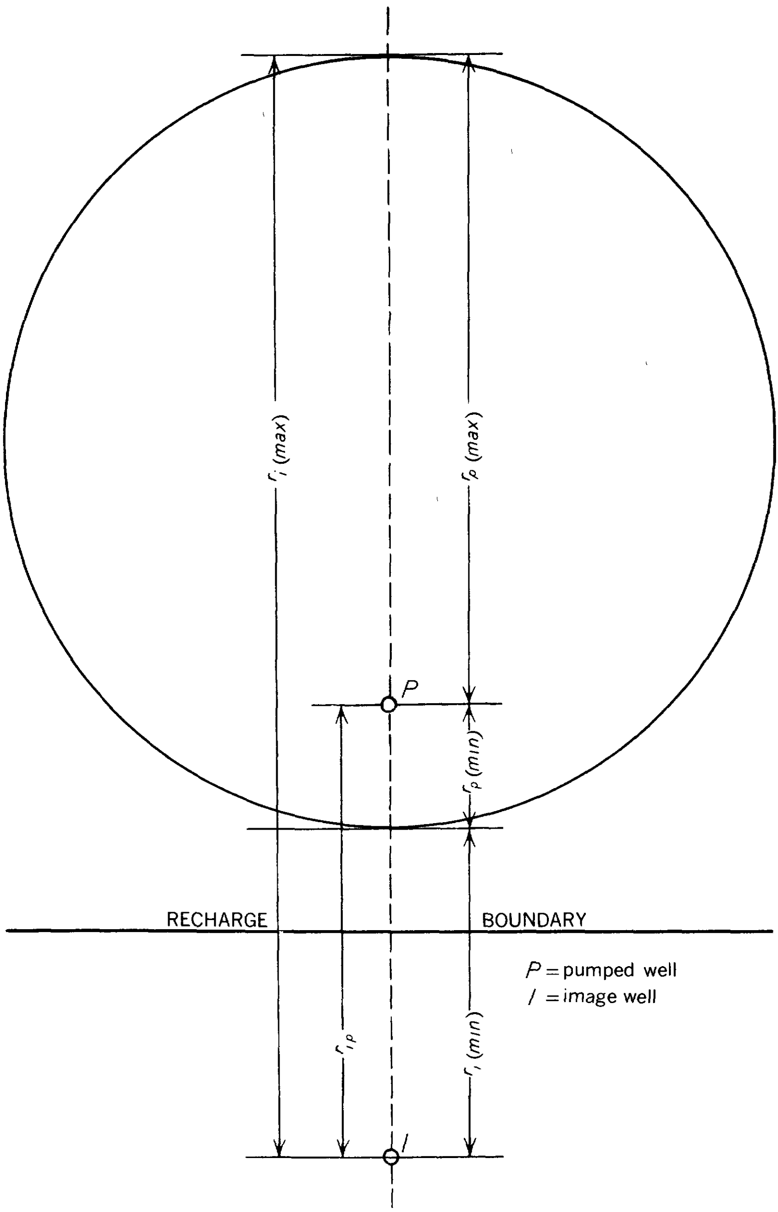

FIGURE 13.—Sketch showing the locus of points of equal net drawdown under equilibrium conditions in an aquifer that has a recharge boundary.

mining the two points where it intersects the line through the image and pumped wells. The point of intersection between the pumped well and the boundary represents the place where a specific drawdown is at a minimum distance from both the image and pumped wells, $[r_{i\,(min)}, r_{p(min)}]$. The other point of intersection represents the place where that drawdown is at a maximum distance from both the image and pumped wells $[r_{i\,(max)}, r_{p(max)}]$.

Therefore,

$$\frac{r_{i\,(min)}}{r_{p\,(min)}}=\frac{r_{i\,(max)}}{r_{p\,(max)}}=\frac{r_i}{r_p}=C.$$

From figure 13,

$$r_{i\,(min)}=r_{ip}-r_{p(min)} \tag{5}$$

and

$$r_{i\,(max)}=r_{ip}+r_{p\,(max)} \tag{6}$$

where

r_{ip}=the distance from the pumped well to the image well.

From these equations,

$$r_{p(min)}=\frac{r_{ip}}{(r_i/r_p)+1} \tag{7}$$

and

$$r_{p\,(max)}=\frac{r_{ip}}{(r_i/r_p)-1}. \tag{8}$$

One-half the distance between the two points of intersection locates the center of the circle and is the radius of the circle.

If the time necessary for u to become as small as 0.02 is determined for the greatest distance ($r_{i\,(max)}$) on a given circle of net drawdown, then u will be less than 0.02 for any other point on that circle because the value of u varies with the square of the distance r. In other words, the drawdown for all the points on the circle can be considered to have reached approximate stabilization at the time necessary for u at $r_{i\,(max)}$ to equal 0.02.

For diagrammatic representation, lines of net drawdown generally are determined for specified magnitudes and uniform intervals, as shown in figure 14. By letting u equal 0.02 for the distance $r_{i\,(max)}$ for each circle, the buildup due to the recharging image well can be computed. If the buildup is added to the specified net drawdown, the result is the drawdown effect of the pumped well. The ratio u_i/u_p, which equals the ratio r_i^2/r_p^2, can then be determined. Equations 5, 6, 7, and 8 then can be used to calculate the maximum and minimum values of r_i and r_p. Finally, the time necessary for u to equal 0.02 for the distance $r_{i\,(max)}$ can be determined. The method is illustrated by the following example.

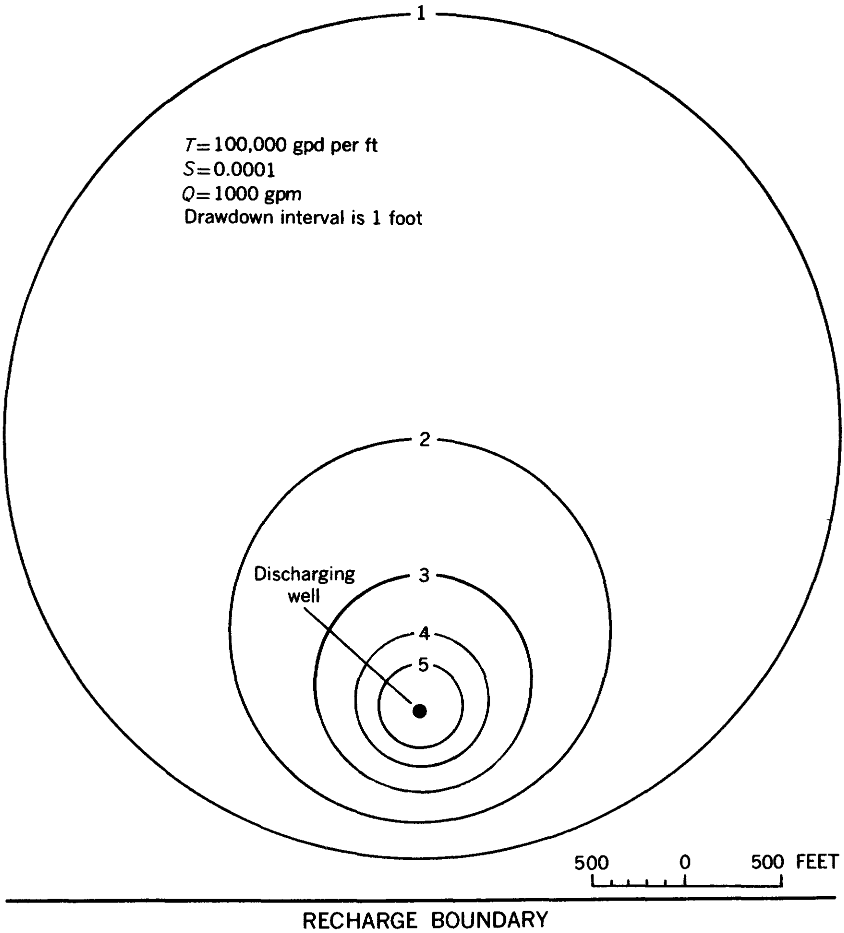

FIGURE 14.—Equilibrium positions of lines of equal net drawdown around a well discharging near a recharge boundary.

Example.—In a certain ground-water investigation an artesian aquifer bounded by a perennial stream was found to have $T=1\times10^5$ gpd per ft and $S=1\times10^{-4}$. Test drilling demonstrated the feasibility of constructing a supply well, capable of being pumped continuously at 1,000 gpm (Q), at a location 1,000 feet from the stream. The net drawdown pattern that will ultimately result in the vicinity of the pumped well and the pumping time required for each drawdown locus to become stabilized in position as a circle are desired.

For the analysis of the problem, a recharging image well is assumed to be located directly across the stream at a distance of 2,000 feet (r_{ip}) from the pumped well. The location of the 5-foot net drawdown circle ($s_o=5$ ft) then can be determined by the following computations.

Let u_i for the distance $r_{i(max)}$ of this circle equal 0.02, then $W(u)_i$ is 3.35. The buidup effect of the image well is

$$s_i=\frac{114.6Q}{T}W(u)_i=\frac{114.6\times 1{,}000\times 3.35}{1\times 10^5}=3.84 \text{ feet}$$

and, hence, the drawdown due to the pumped well is

$$s_p=s_o+s_i=5.00+3.84=8.84 \text{ feet.}$$

To determine the ratio r_i/r_p for the 5-foot drawdown circle:

$$W(u)_p=\frac{Ts_p}{114.6Q}=\frac{1\times 10^5\times 8.84}{114.6\times 1{,}000}=7.71;$$

therefore

$$u_p=2.5\times 10^{-4}.$$

Then

$$\frac{u_i}{u_p}=\frac{2\times 10^{-2}}{2.5\times 10^{-4}}=80=\frac{r_i^2}{r_p^2};$$

therefore

$$\frac{r_i}{r_p}=8.94.$$

The desired maximum and minimum distances then are

$$r_{p(max)}=\frac{r_{ip}}{(r_i/r_p)-1}=\frac{2{,}000}{8.94-1}=252 \text{ feet,}$$

$$r_{i(max)}=r_{ip}+r_{p(max)}=2{,}000+252=2{,}252 \text{ feet,}$$

$$r_{p(min)}=\frac{r_{ip}}{(r_i/r_p)+1}=\frac{2{,}000}{8.94+1}=201 \text{ feet,}$$

and

$$r_{i(min)}=r_{ip}-r_{p(min)}=2{,}000-201=1{,}799 \text{ feet.}$$

The radius of the 5-foot drawdown circle is

$$\frac{r_{i(max)}-r_{i(min)}}{2}=\frac{2{,}252-1{,}799}{2}=226 \text{ feet.}$$

The distance of the center of the circle from the image well is

$$r_{i(min)}+\text{radius}=1{,}799+226 \text{ feet}=2{,}025 \text{ feet.}$$

The time at which the locus of the 5-foot drawdown becomes a circle is

$$t=\frac{1.87r_{i(max)}^2S}{Tu_i}=\frac{(1.87)(2{,}252)^2(1\times 10^{-4})}{(1\times 10^5)(2\times 10^{-2})}=0.475 \text{ day.}$$

The results of computations for the drawdown circles for net drawdowns of 1, 2, 3, 4, and 5 feet are given in table 7 and the relative positions of the circles of equal net drawdown are shown in figure 14.

TABLE 7.—*Data for determining the drawdown pattern caused by a well discharging near a recharge boundary*

s_o (feet)	$r_{p(max)}$ (feet)	$r_{i(max)}$ (feet)	$r_{p(min)}$ (feet)	$r_{i(min)}$ (feet)	Drawdown circle: Radius (feet)	Drawdown circle: Distance of center from image well	t (days)
1	3, 640	5, 640	785	1, 215	2, 212	3, 427	2. 97
2	1, 440	3, 440	590	1, 410	1, 015	2, 425	1. 11
3	719	2, 719	419	1, 581	569	2, 150	. 69
4	419	2, 419	296	1, 704	358	2, 062	. 55
5	252	2, 252	201	1, 799	226	2, 025	. 48

DRAWDOWN PATTERNS CAUSED BY A WELL DISCHARGING NEAR AN IMPERMEABLE BOUNDARY

For an aquifer that has an impermeable boundary, the modified Theis equation for the net drawdown at any point is

$$s_o = s_p + s_i = \frac{114.6Q}{T}[W(u)_p + W(u)_i]. \tag{9}$$

For two points of equal drawdown, $S_{o_1} = S_{o_2}$, and

$$W(u)_{p_1} + W(u)_{i_1} = W(u)_{p_2} + W(u)_{i_2}.$$

When u is equal to or less than 0.02, the approximate value of $W(u)$ is $(-0.5772 - \log_e u)$. Therefore

$$-0.5772 - \log_e u_{p_1} - 0.5772 - \log_e u_{i_1} - 0.5772 - \log_e u_{p_2} - 0.5772 - \log_e u_{i_2}$$

or

$$\log_e u_{p_1} + \log_e u_{i_1} = \log_e u_{p_2} + \log_e u_{i_2},$$

from which

$$u_{p_1} u_{i_1} = u_{p_2} u_{i_2}. \tag{10}$$

If t is the same for both points,

$$r_{p_1} r_{i_1} = r_{p_2} r_{i_2} = r_p r_i = c. \tag{11}$$

The loci of points of equal drawdown around a well near an impermeable boundary are not circles, and therefore their positions at a given time cannot be fixed completely by determining their points of intersection with the line through the pumped and image wells. The loci nearest the pumped well are oval shaped; those farther away from the pumped well become progressively more elongated toward the boundary until the last curve that closes, before the boundary is reached, is distinctly teardrop shaped. More distant loci do not close upon themselves but intersect the boundary at right angles. However, all drawdown lines are symmetrical about the line through the pumped and image wells.

To obtain the locus of equal net drawdown for a given elapsed pumping time, various values of r_p must be assumed for substitution in equation 11 so that corresponding values or r_i can be computed. These values of r_p and r_i are then plotted to obtain points on the required net drawdown curve.

As an aid in assuming proper values of r_p, the points of intersection of the net drawdown locus with the line through the pumped and image wells can be determined. Figure 15 shows that the point of intersection on the side of the pumped well away from the boundary is at a minimum distance from the pumped well but at a maximum distance from the image well; if it exists, the intersection on the boundary side of the pumped well is at a maximum distance from the pumped well but at a minimum distance from the image well. Therefore, from equation 11,

$$r_{i(max)}r_{p(min)}=r_{i(min)}r_{p(max)}=r_ir_p=c. \tag{12}$$

From figure 15,

$$r_{i(max)}{}^2=r_{ip}+r_{p(min)} \tag{13}$$

and

$$r_{i(min)}{}^2=r_{ip}-r_{p(max)}. \tag{14}$$

From these equations,

$$r_{p(min)}{}^2+r_{p(min)}r_{ip}-r_ir_p=0 \tag{15}$$

and

$$r_{p(max)}{}^2-r_{p(max)}r_{ip}+r_ir_p=0. \tag{16}$$

The points of intersection of a net drawdown locus and the line through the pumped and image wells can be determined from equations 15 and 16.

However, as for a well discharging near a recharge boundary, net drawdown curves for specified magnitudes and uniform intervals

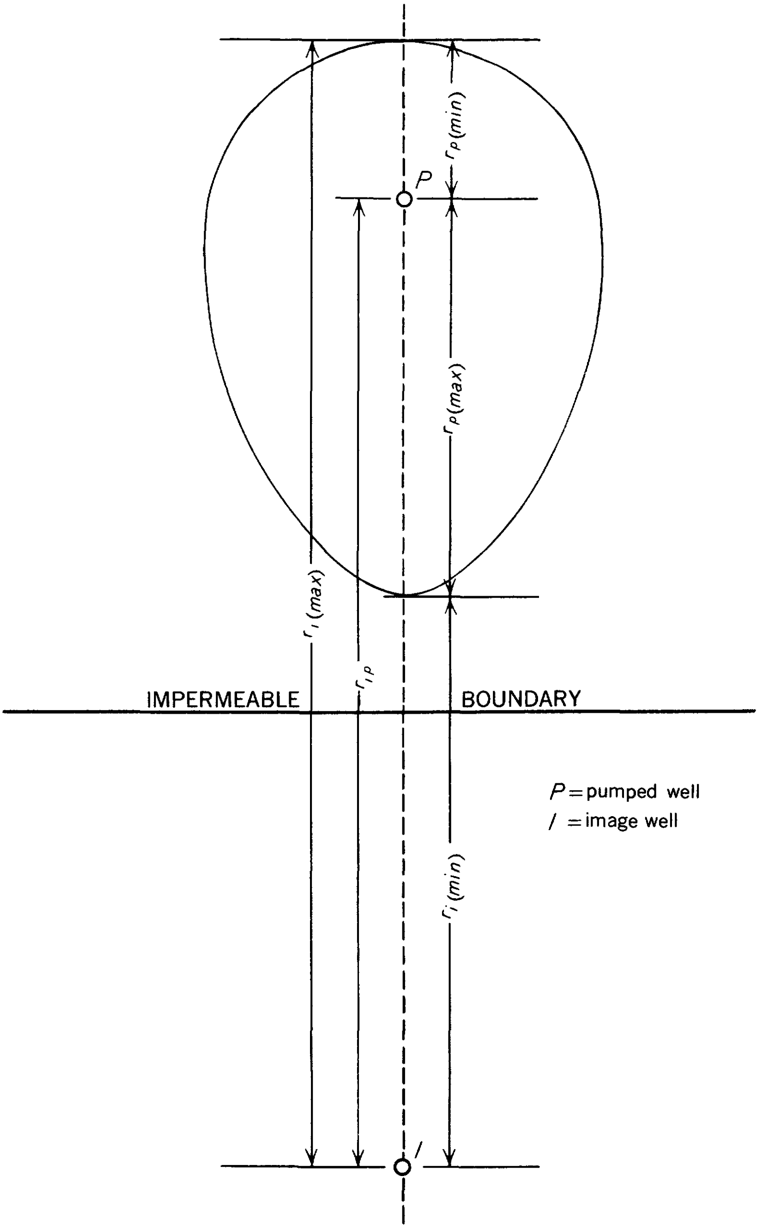

FIGURE 15.—Sketch showing the locus of points of equal net drawdown in an aquifer that has an impermeable boundary.

generally are desirable for use in diagrams. To obtain such curves, the product of u_i and u_p must be determined for each locus. Where u_i and u_p are each less than 0.02, equation 9 can be written as follows:

$$s_o=\frac{114.6Q}{T}[-2(0.5772)-\log_e u_iu_p], \tag{17}$$

from which

$$\log_e u_iu_p=-\left(\frac{s_oT}{114.6Q}+1.154\right) \tag{18}$$

If the proper values for s_o, Q, and T are substituted in equation 18, the product u_iu_p can be computed directly. From the relation

$$u_iu_p=\left(\frac{Kr_ir_p}{t}\right)^2, \text{ where } K=\frac{1.87S}{T}, \tag{19}$$

it follows that

$$r_ir_p=\frac{t}{K}\sqrt{u_iu_p}. \tag{20}$$

From equation 20, the product r_ir_p can be computed. Equations 15 and 16 can then be used to determine the points of intersection of the desired net drawdown locus and the line through the pumped and image wells. After these two points are located, other points on the same locus can be found by assuming intermediate values of r_p and computing corresponding values of r_i from equation 11. The method is illustrated by the following example.

Example.—For the same field situation described in the previous example (for which T is 1×10^5 gpd per ft and S is 1×10^{-4}), assume that the artesian aquifer is bounded by an impermeable barrier. Determine the net drawdown pattern in the vicinity of the pumped well, assuming that the pumping rate of 1,000 gpm (Q) has been maintained for 20 days (t).

For the analysis of this problem, a discharging image well is assumed to be located across the stream at a distance of 2,000 (r_{ip}) from the pumped well. The location of the 20-foot net drawdown locus (s_o=20 ft) is determined by the following computations.

From equation 18

$$\log_e u_iu_p=-\left(\frac{s_oT}{114.6Q}+1.154\right)=-\left(\frac{20\times1\times10^5}{114.6\times1{,}000}+1.154\right)=-18.60.$$

Therefore,

$$u_iu_p=8.35\times10^{-9} \text{ and } \sqrt{u_iu_p}=9.14\times10^{-5}.$$

From equation 20

$$K=\frac{1.87S}{T}=1.87\times10^{-9},$$

and hence

$$r_ir_p=\frac{t}{K}\sqrt{u_iu_p}=\frac{20\times9.14\times10^{-5}}{1.87\times10^{-9}}=9.78\times10^{5}.$$

The desired maximum and minimum distances are then found as follows:

$$r_{p(min)}{}^2+r_{p(min)}r_{ip}-r_ir_p=r_{p(min)}{}^2-2{,}000r_{p(min)}-9.78\times10^{5}=0;$$

therefore

$$r_{p(min)}=405 \text{ feet};$$

$$r_{i(max)}=r_{ip}+r_{p(min)}=2{,}000+405=2{,}405 \text{ feet};$$

$$r_{p(max)}{}^2-r_{p(max)}r_{ip}+r_ir_p=r_{p(max)}{}^2-2{,}000r_{p(max)}+9.78\times10^{5}=0;$$

therefore

$$r_{p(max)}=850 \text{ feet}$$

and

$$r_{i(min)}=r_{ip}-r_{p(max)}=2{,}000-850=1{,}150 \text{ feet}.$$

The above computations determine the points of intersection of the 20-foot net drawdown locus with the line through the pumped and image wells and give the limits to be used in assuming values of r_p for finding corresponding values of r_i.

Computations to determine the position of the loci for net drawdowns of 15, 16, 17, 18, and 19 feet indicate that the curves intersect the boundary. For these loci, after $r_{p(min)}$ and $r_{i(max)}$ are determined, values for r_p are assumed and the corresponding values of r_i are computed. The results for the above calculations are given in table 8 and the corresponding loci are illustrated in figure 16.

The position of any net drawdown locus about a well discharging from an aquifer that has an impermeable boundary changes with time. A diagram that would show such changes can be drawn, if desired. The product u_iu_p remains constant for a specified drawdown curve. By substituting various values for t in equation 20, corresponding values for r_ir_p can be determined. The positions of the locus at different times are then found by the procedures previously described.

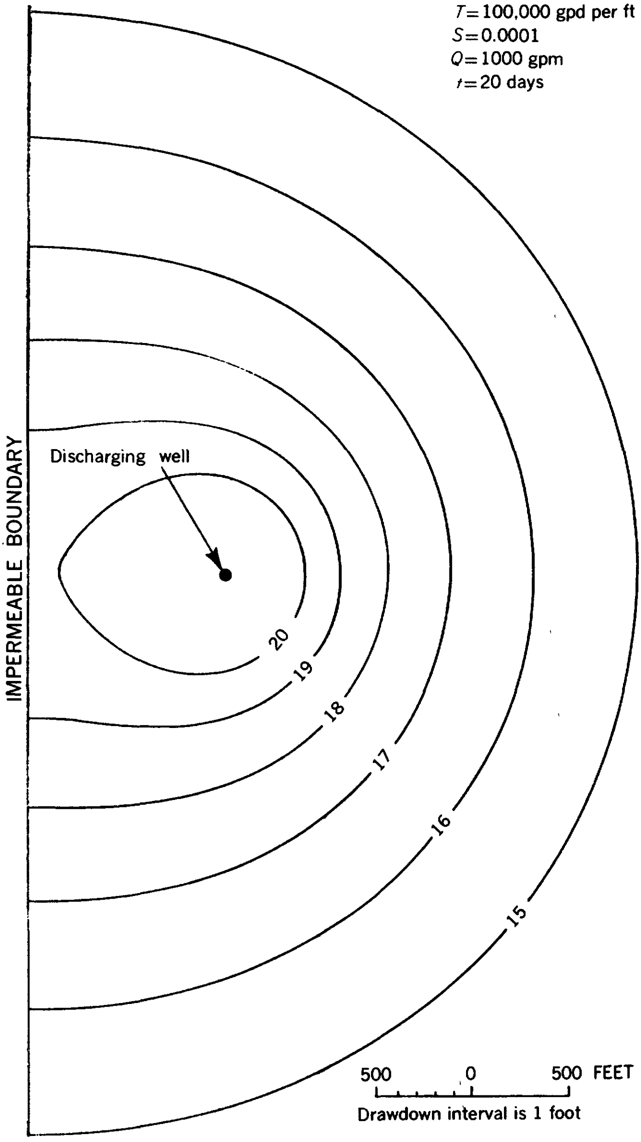

FIGURE 16.—Positions of net drawdown lines around a well discharging near an impermeable boundary.

TABLE 8.—*Data for determining the drawdown pattern caused by a well discharging near an impermeable boundary*

[All units are in feet]

$s_o=15$ $r_{p(min)}=2{,}105$ $r_{i(max)}=4{,}105$		$s_o=16$ $r_{p(min)}=1{,}570$ $r_{i(max)}=3{,}570$		$s_o=17$ $r_{p(min)}=1{,}150$ $r_{i(max)}=3{,}150$		$s_o=18$ $r_{p(min)}=828$ $r_{i(max)}=2{,}828$		$s_o=19$ $r_{p(min)}=585$ $r_{i(max)}=2{,}585$		$s_o=20$ $r_{p(min)}=405$ $r_{i(max)}=2{,}405$	
r_p	r_i	r_p	r_i	r_p	r_i	r_p	r_i	r_p	r_i	r_p	r_i
2,200	3,930	1,600	3,510	1,200	3,020	850	2,750	600	2,520	440	2,220
2,300	3,760	1,680	3,340	1,270	2,850	900	2,600	650	2,320	480	2,040
2,400	3,600	1,760	3,190	1,340	2,700	950	2,460	700	2,160	520	1,880
2,500	3,460	1,840	3,050	1,410	2,570	1,000	2,340	750	2,010	560	1,750
2,600	3,330	1,920	2,930	1,480	2,450	1,050	2,230	800	1,890	600	1,630
2,700	3,200	2,000	2,810	1,550	2,340	1,100	2,130	850	1,780	640	1,530
2,800	3,090	2,080	2,700	1,620	2,240	1,150	2,030	900	1,680	680	1,440
2,900	2,980	2,160	2,600	1,690	2,140	1,200	1,950	950	1,590	720	1,360
--------	--------	2,240	2,510	1,760	2,060	1,250	1,870	1,000	1,510	760	1,290
--------	--------	2,320	2,420	1,830	1,980	1,300	1,800	1,050	1,440	800	1,220
--------	--------	--------	--------	1,900	1,910	1,350	1,730	1,100	1,370	850	1,150
--------	--------	--------	--------	--------	--------	1,400	1,670	1,150	1,310	--------	--------
--------	--------	--------	--------	--------	--------	1,450	1,610	1,200	1,260	--------	--------
--------	--------	--------	--------	--------	--------	1,500	1,560	--------	--------	--------	--------

CONCLUSION

If a pumped well taps an aquifer that has a recharge boundary, the drawdown pattern soon becomes stabilized in the vicinity of the well. Under equilibrium conditions, the locus of points having the same net drawdown is a circle. As the well is pumped for longer periods of time, the area for which the loci of equal net drawdowns are circles becomes larger. However, the locus of the nearest points of zero drawdown does not become stabilized until t becomes infinite. Hence, the cone of depression continues to expand until t equals infinity, at which time the locus of zero drawdown becomes a circle of infinite radius—that is, the straight line that represents the recharge source.

If a pumped well taps an aquifer that has an impermeable boundary, the drawdown pattern does not become stabilized but the position and shape of a specified net drawdown locus changes with time. As each locus for a given drawdown moves farther from the well, it changes from its initial oval shape to a teardrop shape and eventually intersects the boundary at right angles.

THE CONE OF DEPRESSION AND THE AREA OF DIVERSION AROUND A DISCHARGING WELL IN AN INFINITE STRIP AQUIFER SUBJECT TO UNIFORM RECHARGE

By Russell H. Brown

ABSTRACT

The expansion of the cone of depression around a discharging well is limited only by the boundaries of the aquifer. However, the expansion of the area from which water is diverted by the discharging well continues only until the recharge to the area of diversion is equal to the discharge from the well. This is best illustrated by the analysis of a specific hydrologic system.

The flow field in an aquifer that is bounded by two parallel streams is static only so long as the natural recharge to the aquifer balances the natural discharge. A well discharging from the aquifer causes a new flow field that can be considered to be the resultant of the initial flow field and the flow field that would be caused by a well discharging from an aquifer that has two parallel boundaries across which no recharge occurs.

Contour maps and profiles of the water table, prepared from values found by appropriate formulas, show the resultant flow fields for various times since pumping started in an infinite strip aquifer. These maps show that even though the cone of depression continues to expand, the position of the ground-water divide, and hence the area of diversion, becomes virtually static after a certain period of time.

THE PROBLEM

A complete understanding of the source of water derived from wells continues to be elusive. The fallacious idea that the cone of depression around a discharging well expands only until its outer limit encompasses an area in which the recharge balances the discharge of the well persists despite Theis' (1938, 1940) able discussion of the factors that control the response of an aquifer to withdrawals by pumping. The purpose of this paper is to show, by numerical example and related discussion, how a particular hydraulic system, in which the natural recharge balances the natural discharge, responds to the introduction of a discharging well and ultimately reaches a new state of equilibrium.

Perhaps some misunderstanding arises from loose or imperfect interpretations of the term "cone of depression." As stated by Theis (1938, p. 891), the cone of depression is

> * * * the geometric solid included between the water table or other piezometric surface after a well has begun discharging and the hypothetical position the water table or other piezometric surface would have had if there had been no discharge by the well.

This definition is equally apt for the synonymous, and possibly more descriptive, term "cone of influence."

HYDRAULIC CONDITIONS IN AN UNDISTURBED INFINITE STRIP AQUIFER

The hydraulic system devised for illustrative analysis is shown in part by the cross section and oblique view in figure 17; it is described as a water-table aquifer that is of width $2a$, of thickness m, and of infinite length, and that is bounded by two parallel perennial streams which are hydraulically continuous with the full thickness of the aquifer and in which the stage remains constant. The thickness of the aquifer is sufficiently great that the drawdowns to be created in it are negligible proportions of the original saturated thickness. The aquifer is recharged uniformly, with respect to space and time, by precipitation at the rate of W inches per year.

The profile of the water table for the cross section in figure 17 can be plotted from values obtained from the formula which was derived by J. G. Ferris of the U.S. Geological Survey (oral communication) from an equation given by Jacob (1943, p. 566) for use in determining steady-state profiles. Ferris' formula is

$$h_o=\frac{1.71\times10^{-3}W}{2T}\,(2aX-X'^2),$$

where

h_o=the elevation of the water table above stream stage;

X=the distance of the observation point from the centerline of the aquifer; and

$X'=a-X$=the distance of the observation point from the nearest stream.

The above equation becomes

$$h_o=6.4041\times10^{-2}(31.68\underline{X}-\underline{X}'^2),$$

where

$$\underline{X}=X/1{,}000 \text{ and } \underline{X}'=X'/1{,}000,$$

if the following dimensions and hydraulic constants for the system are assumed:

$2a=6$ miles, or 31,680 feet;
$W=6$ inches per year;
$T=80{,}000$ gpd per ft; and
$S=0.20$.

The calculated values of h_o and the corresponding values of X' are given in table 9; these values located the position of the water table in the aquifer.

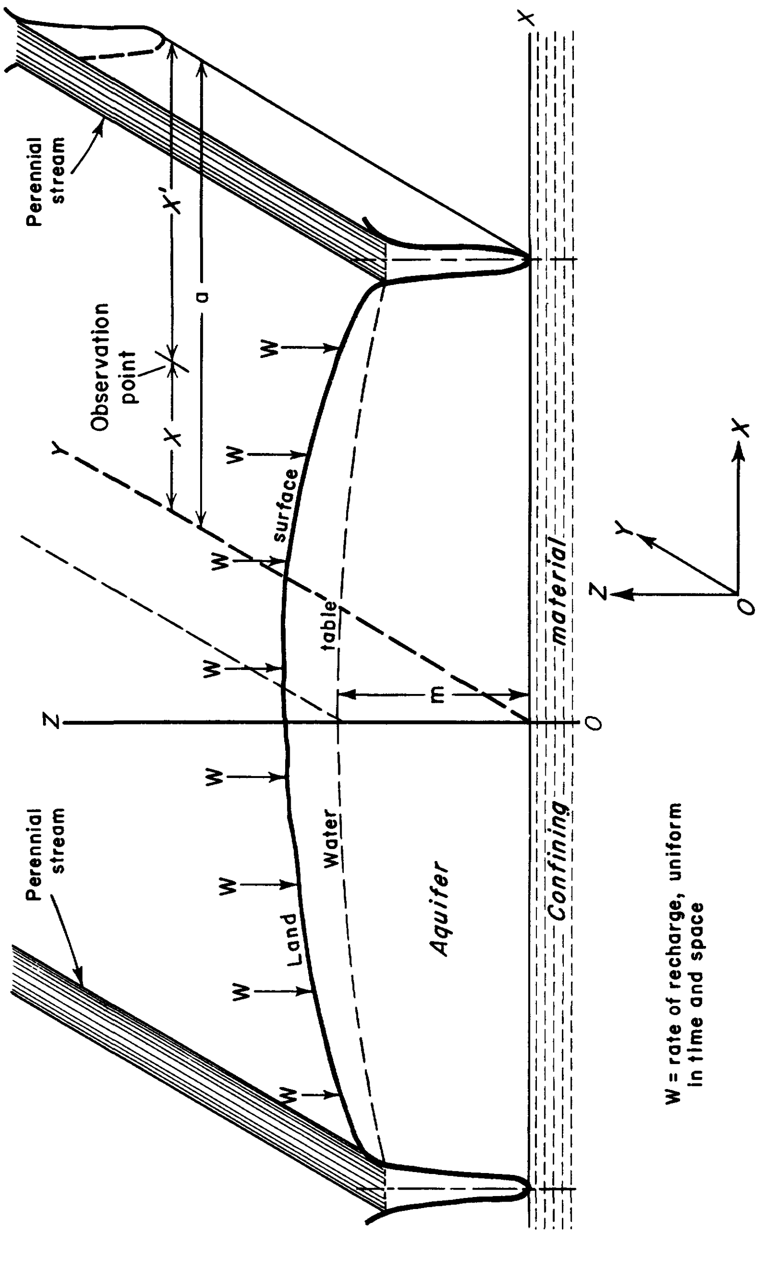

FIGURE 17.—Cross section and oblique view of part of an infinite strip aquifer bounded by two parallel streams.

TABLE 9.—*Data for plotting the water-table profile for a cross section of the aquifer oriented in the X-direction.*

[All units are in feet]

X'	X	h	X'	X	h
15, 840	0	16. 07	7, 840	8, 000	11. 97
15, 340	500	16. 05	6, 840	9, 000	10. 88
14, 840	1, 000	16. 00	5, 840	10, 000	9. 66
13, 840	2, 000	15. 81	4, 840	11, 000	8. 32
12, 840	3, 000	15. 49	3, 840	12, 000	6. 85
11, 840	4, 000	15. 04	2, 840	13, 000	5. 25
10, 840	5, 000	14. 47	1, 840	14, 000	3. 52
9, 840	6, 000	13. 76	840	15, 000	1. 66
8, 840	7, 000	12. 93	0	15, 840	0

The water-table profile is shown schematically in figure 20 and is plotted to scale in figure 18. If the gradient of the streams and the water table is small and in the direction normal to the plane of the cross section, the water-table contour map (fig. 18) can be drawn as a family of straight lines paralleling the streams. The ground-water divide, which is the line along which the water table is highest, is midway between the streams. Because the aquifer is symmetrical with respect to the X– and Y–axes (fig. 17), water-table profiles need to be drawn for only one quadrant of the XZ– or YZ–plane and the contour lines need to be drawn in only one quardrant of the XY–plane.

The map and profile of the water table, as shown in figure 18, depict the initial state of balance between natural recharge to and discharge from the aquifer for this particular hydraulic system. If the rate of recharge does not vary, if the stream level does not change, and if no other recharge to or discharge from the system occurs, then the shape and position of the water table are unvarying with time and remain as shown. Owing to symmetry, the aquifer obviously discharges equally into the two streams. The natural discharge, D, into either stream, for the assumed dimensions and constants of this hydraulic system, expressed in gallons per day per foot of stream channel, is

$$D=\frac{Wa(7.48)}{12\times 365}=\frac{6\times 15,840\times 7.48}{12\times 365}=162 \text{ gpd per ft.}$$

FLOW FIELDS IN AN INFINITE STRIP AQUIFER

Before analyzing the changes that will take place in the hydraulic system through the introduction of a discharging well, consider briefly the nature of the initial flow field, the flow field that will be introduced, and the resultant flow field. As may be seen from the orientation of the three coordinate axes in figure 17, the initial conditions in the hydraulic system are such that the prevalent two-dimensional flow can be described entirely by reference to the XZ–plane; no flow components exist in the third, or "Y–," direction. If the drawdowns to be created in the aquifer are a very small proportion of the saturated thickness, the flow field related only to the discharging well is satisfactorily approximated as two dimensional (radial flow) and can be described entirely by reference to the XY–plane. When the two flow fields are combined, therefore, the resultant field will be three dimensional, requiring reference to all three coordinate axes for proper description.

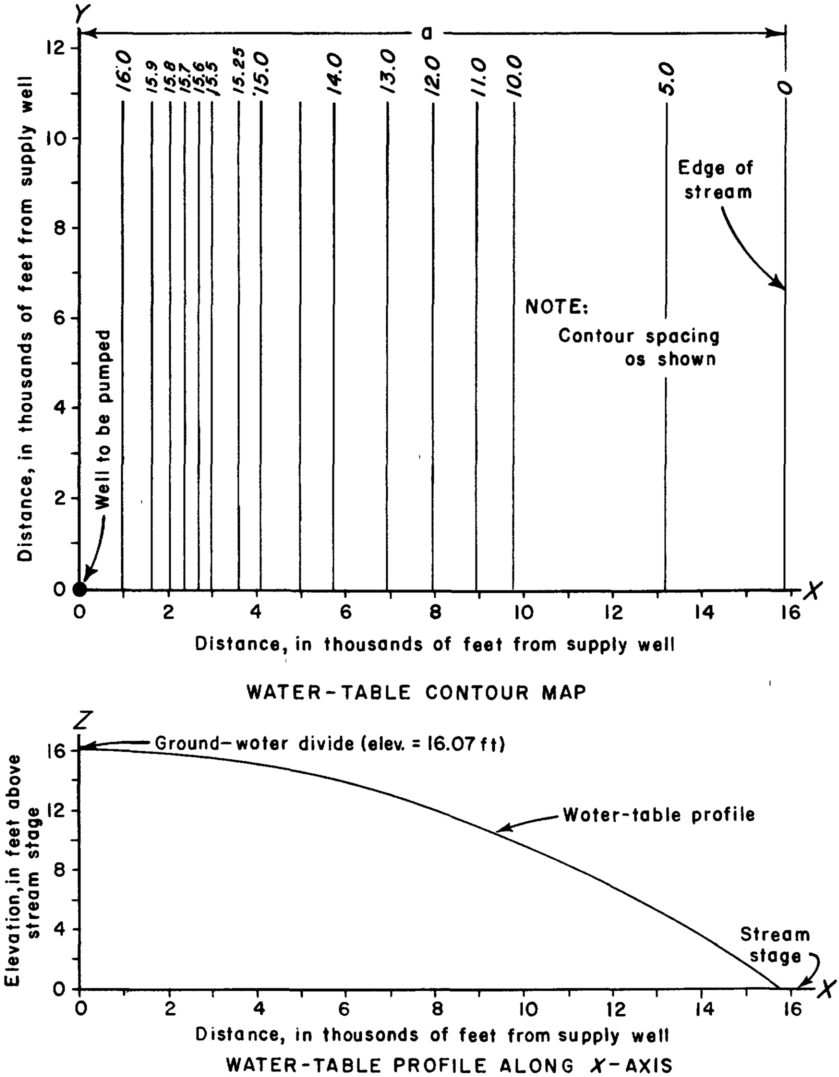

FIGURE 18.—Contour map and profile of the water table before pumping begins.

As for certain problems of mathematical physics, the analysis of the flow of fluids through porous media employs the principle of superimposition. Superimposition is especially helpful in studying a three-dimensional problem, such as the one developed herein, which is recognizable as the resultant of two or more component flow fields, each of which involves only one- or two-dimensional flow. The pattern of the distribution of head throughout one flow field is superimposed over the correctly oriented pattern of the distribution of head throughout the component flow field; the distribution of head throughout the resultant field is obtained by algebraic summation. The prime criterion for the applicability of this analytical technique is that the differential equations describing the component flow field be linear (Jacob, 1950, p. 361). A differential equation is said to be linear if it is of the first degree with respect to the dependent variable and its derivatives (Ingersoll and Zobel, 1948, p. 11–12; Phillips, 1934, p. 38, 49). The differential equations that describe the initial steady-state flow field and the unsteady-state field to be superimposed with the introduction of the discharging well are, respectively, of the form

$$Wx = -Ph\frac{dh}{dx} \text{ (Jacob, 1950, p. 381)}$$

and

$$\frac{\partial^2 h}{\partial r^2}+\frac{1}{r}\frac{\partial h}{\partial r}=\frac{S}{T}\frac{\partial h}{\partial t} \text{ (Jacob, 1950, p. 366).}$$

In both equations, h is the dependent variable and nowhere does it, or any of its derivatives, appear with an exponent higher than one. Both equations are therefore of the first degree, or linear; hence, the application of the principle of superimposition is justified.

HYDRAULIC CONDITIONS RESULTING FROM A WELL DISCHARGING FROM AN INFINITE STRIP AQUIFER

Assume that a well is drilled to the base of the aquifer at a point midway between the two boundary streams. This location, which is on the ground-water divide, is chosen primarily to simplify the ensuing computations; the choice does not alter the significance of the discussion. Assume also that the well is pumped continuously at a steady rate of 200 gpm. Of interest, now, is the resultant flow field at selected times after pumping begins and at the ultimate new steady state. With the propriety of using the technique of superimposition established, the problem is restated, in terms of the two previously recognized constituent flow fields, as follows:

1. The initial flow field reflects uniform recharge, over the entire infinite strip aquifer, in balance with the aquifer discharge to the two

parallel boundary streams. Steady-state conditions prevail—that is, the flow field does not vary with time. Thus, if this flow field is to be added algebraically to some other, the time or times, selected for determining the resultant field is immaterial; the configuration of the initial flow field is that depicted, in plan and profile, in figure 18.

2. The flow field introduced by the discharging well develops in the manner predictable for a discharging well at the center of an infinite strip aquifer that is subject to no recharge and which is bounded by two parallel streams whose stages are held constant. This is an unsteady-flow field that expands with elapsed pumping time. Thus, the configuration and extent of the field depends upon the time since pumping began; the delineation of its ultimate steady state (and some intermediate states) involves recourse to the image-well theory in order to satisfy the boundary conditions of no drawdown at the streams.

Inasmuch as the initial flow field has been identified for any and all times, the problem of defining the resultant of the two described constituent flow fields at selected times becomes primarily an exercise in describing the unsteady field developed by the discharging well. Fortunately, the mathematical theory required for such an exercise has been presented by Theis (1935) and by Hantush and Jacob (1955b, p. 106–107) and need not be redeveloped here. However, a limited discussion of the mechanics of the exercise, employed in the above-cited works, is appropriate.

To meet the stipulation that no drawdown can occur along the aquifer boundaries, image wells are postulated as shown in figure 19. Each image well is assumed to recharge or discharge, as appropriate, at 200 gpm simultaneously and continuously with the real discharging well. Thus, along either boundary the theoretical effects of all wells on one side are seen to annul exactly the effects of the wells correspondingly placed on the other side, and the condition of no drawdown is fulfilled. Because the system of image wells extends to infinity in both directions, an infinite period of time is required for the superimposed flow field, and hence the resultant flow field, to become steady. In other words, infinite time is needed for the effects from the most distant image wells to reach the real aquifer. However, the degree to which different parts of the flow field approach stabilization in finite times can be judged by examining the illustrations that follow.

The times (since pumping started) arbitrarily selected for determining the composite or resultant flow field are 30 days, 500 days, and when the new steady-state conditions are realized, that is, at time $t=\infty$. Water-table profiles and water-table contour maps similar to those shown in figure 18 will illustrate adequately the resultant flow fields for each of the given times.

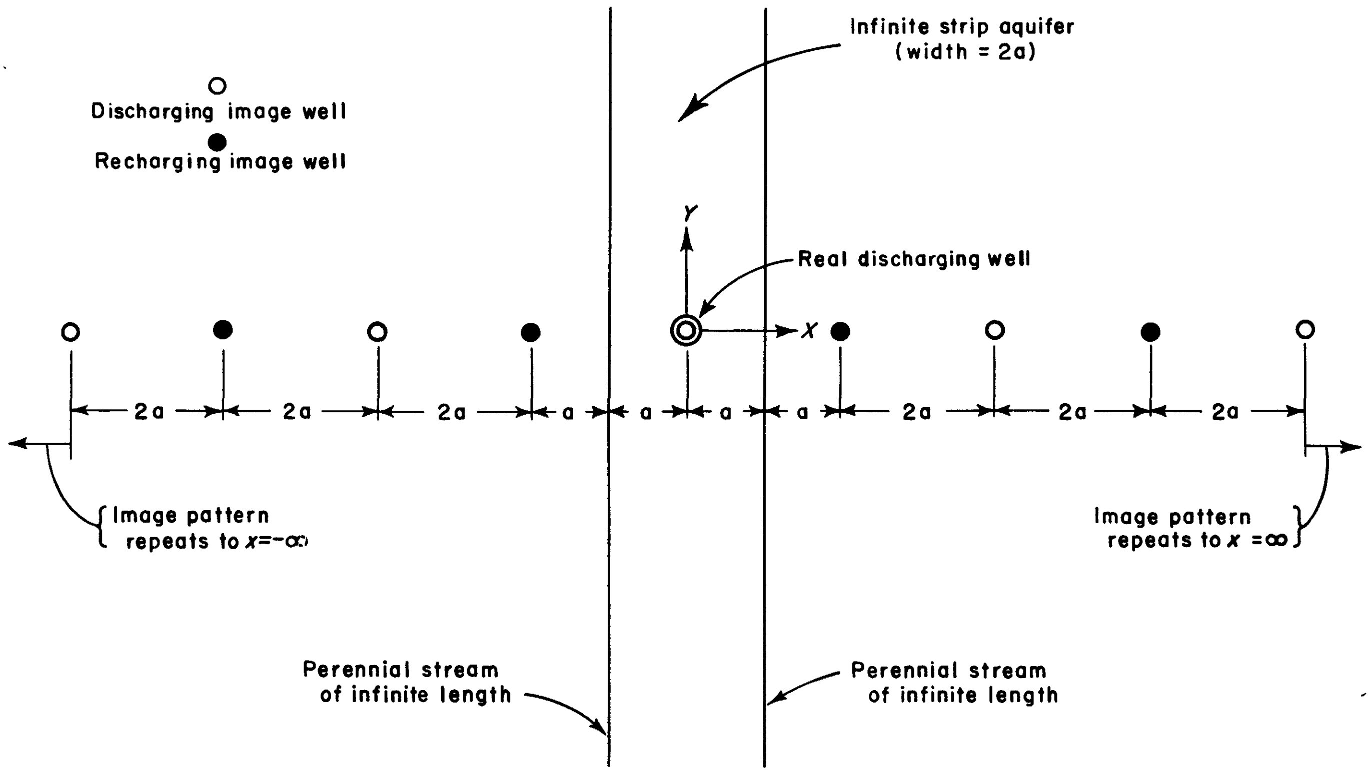

FIGURE 19.—Image-well array for a single discharging well at the center of an infinite strip aquifer bounded by perennial streams.

In the early stages of development, the flow field caused by the discharging well is the cone of depression; its profile can be drawn from values obtained by using the familiar Theis (1935) nonequilibrium equation. When enough time has elapsed for the cone of depression to reach the boundary streams, the effects of the image wells must be considered and Hantush and Jacob's equation, which sums all image- and real-well effects, must be used to determine the drawdowns.

The drawdowns for selected distances from the discharging well at the end of 30 days' pumping are given in table 10. The limit of the cone of depression is arbitrarily considered to be where the drawdown is 0.01 foot. At the end of the 30-day period, this drawdown occurs at a distance of 3,800 feet from the discharging well. Thus, each line of equal drawdown at the end of 30 days is one of a family of concentric circles whose common center is the discharging well. The lines of equal drawdown are indicated, for one quadrant only, by light short dashes on the map in figure 20. The heavy solid lines on the same map are water-table contours depicting the resultant flow field. Each point used in constructing the water-table contours of the resultant flow field was determined by subtracting the computed or interpolated drawdown for that point from the initial water-table elevation of the same point. The initial water-table elevation is shown by the straight parallel lines, which coincide in part with the water-table contours of the resultant flow field and which are represented by light long dashes where they do not so coincide. The shaded area in figure 20 is enclosed by the shifted ground-water divide, which is the division between that part of the flow field contributory to the well and that part contributory to the stream.

TABLE 10.—*Drawdown data for an infinite aquifer tapped by a single discharging well*

[All units are in feet]

Drawdown, s	Distance from discharging well, r		Drawdown, s	Distance from discharging well, r	
	At end of 30-day pumping period	At end of 500-day pumping period		At end of 30-day pumping period	At end of 500-day pumping period
0.01	3,800	15,400	0.30	1,260	5,150
.02	3,340	13,700	.40	1,020	4,180
.04	2,860	11,700	.60	690	2,820
.05	2,700	11,000	.80	480	1,950
.07	2,450	10,000	1.00	330	1,370
.10	2,170	8,880	1.50	140	570
.15	1,840	7,540	2.00	60	240
.20	1,600	6,570	3.00	----------	40
.25	1,420	5,790			

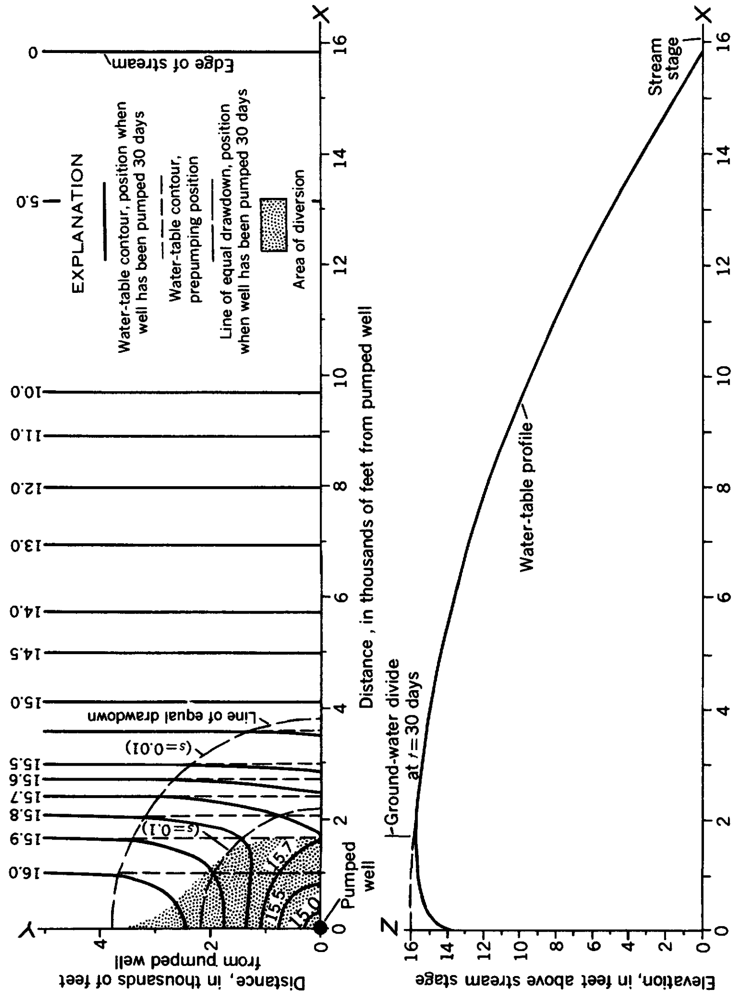

FIGURE 20.—Contour map and profile of the water table when the period of pumping equals 30 days.

Analysis of the flow field, caused only by the discharging well, by the end of a 500-day pumping period reveals that the cone of depression can be represented satisfactorily by lines of equal drawdown, which again are a family of concentric circles. Table 10 shows the drawdown values computed by use of the Theis nonequilbrium equation; the practical limit of the cone of depression, as marked by the position of the 0.01 foot drawdown value, is about 15,400 feet from the discharging well. Thus, the cone has expanded so that measurable drawdowns are produced at distances from the real discharging well that are almost equal to half the width of the aquifer; that is, the cone has almost reached the streams. Therefore, if pumping continues beyond 500 days, the buildup effects of the first pair of recharging image wells begin to offset the drawdown effect from the real well, and more than the simple application of the Theis equation is required for proper analysis of the cone of depression. Figure 21 was prepared for $t=500$ days by following the same procedures and conventions described in figure 23. The shaded area bounded by the ground-water divide is seen to be larger than in figure 20; hence, the configuration of the cone of depression had not become static at $t=30$ days.

The ultimate steady-state configuration of the cone of depression, which occurs after the lapse of an infinite period of pumping, is found from computations using the formula developed by Hantush and Jacob (1955b):

$$s=\frac{1{,}440Q}{\pi T}\sum_{n=1}^{\infty}\frac{1}{n}\left\{\sin\left[\frac{n\pi(x-a)}{2a}\right]\sin\left(-\frac{n\pi}{2}\right)\right\}e^{-\pi ny/2a}, \tag{1}$$

which is the steady-state drawdown (when $t\rightarrow\infty$) at any point (x, y)—where the coordinate axes are oriented as shown in figure 17—in an infinite strip aquifer tapped by a single discharging well at the coordinate origin. Inspection of equation 1 shows that where n is an even number the term "sin $(-n\pi/2)$" is zero. Therefore only odd numbers need to be considered, and, as successive odd numbers are assigned to n in evaluation of the series summation, the value of the term "sin $(-n\pi/2)$" changes alternately from -1 to $+1$.

Two simplified forms of equation 1 are helpful in performing the desired drawdown computations. For a transverse profile coincident with the X–axis (fig. 17), any point on the cone of depression has the general space coordinates $(x, 0)$. For $y=0$, the exponential function in equation 1 equals 1. Thus the equation is shortened to the form

$$s=\frac{1{,}440Q}{\pi T}\sum_{n=1}^{\infty}\frac{1}{n}\left\{\sin\left[\frac{n\pi(x-a)}{2a}\right]\sin\left(-\frac{n\pi}{2}\right)\right\}. \tag{2}$$

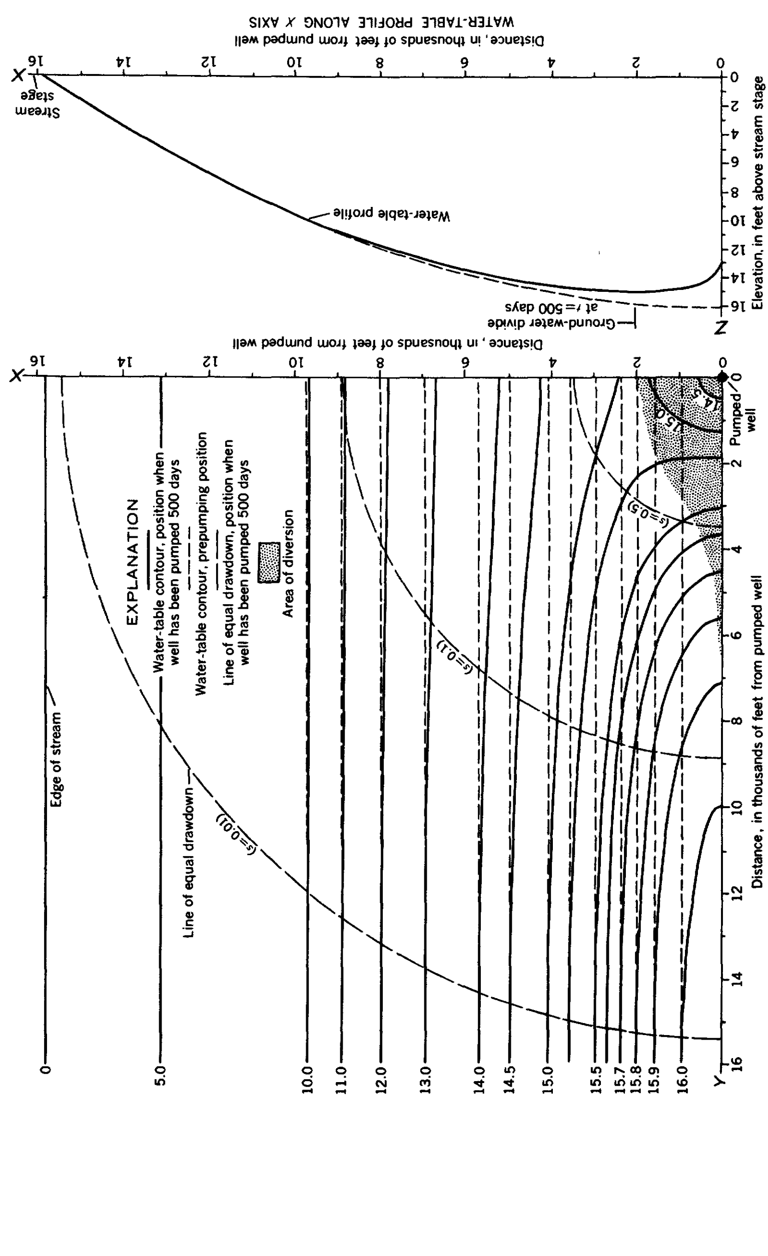

Figure 21.—Contour map and profile of the water table when the period of pumping equals 500 days.

Substituting the values specified in this problem for Q, T, and a, equation 2 becomes

$$s=\frac{200(1{,}440)}{\pi(80{,}000)}\sum_{n=1}^{\infty}\frac{1}{n}\left\{\sin\left[n\pi\left(\frac{x}{31{,}680}-\frac{1}{2}\right)\right]\sin\left(-\frac{n\pi}{2}\right)\right\}$$

or

$$s=1.146\sum_{n=1}^{\infty}\frac{1}{n}\left\{\sin\left[n\pi\left(\frac{x}{31{,}680}-\frac{1}{2}\right)\right]\sin\left(-\frac{n\pi}{2}\right)\right\}. \tag{2a}$$

Similarly, for a longitudinal profile coincident with the Y–axis (fig. 17), any point on the cone of depression has the general space coordinates $(0, y)$. Where $x=0$, the first sine term in equation 1 becomes identical to the second, and, even though their respective values change alternately from -1 to $+1$, their product is always $+1$. Equation 1 is, therefore, shortened to the form

$$s=\frac{1{,}440Q}{\pi T}\sum_{n=1}^{\infty}\frac{1}{n}\,e^{-\pi ny/2a}. \tag{3}$$

Appropriate substitutions for Q, T, and a in equation 3 give

$$s=1.146\sum_{n=1}^{\infty}\frac{1}{n}\,e^{-\pi ny/31,680}. \tag{3a}$$

Equations 2a and 3a are used to compute the steady-state drawdowns for selected points along the transverse profile (X–axis) and the longitudinal profile (Y–axis), respectively; sample detailed computations for a selected point on each profile appear in table 11, and results for several points along each profile appear in table 12. These results offer a ready means for determining the drawdown at any point whose coordinates comprise any combination of the x and y values chosen for points on the transverse and longitudinal profiles. The solutions for points on the two coordinate axes can be combined to yield solutions for points between those axes because equations 2 and 3 recombine to form the more general formula, equation 1. To find the drawdown at point (x, y) : multiply, term by term, the series expressions developed for the appropriate X– and Y–direction solutions; find the summation of the products; and multiply by the factor 1.146 (table 11, last column). The results, for selected points, appear in table 12.

The steady-state drawdowns computed for the selected points whose coordinates are indicated in table 12 are subtracted from the initial steady-state water-table elevations of the same set of points. The resultant elevations are the basis for the water-table profiles and contour map in figure 22, which indicates the steady-state flow field (when $t=\infty$).

A comparison of the contours and profiles of water-table positions resulting after pumping periods of 30 days (fig. 20), 500 days (fig.

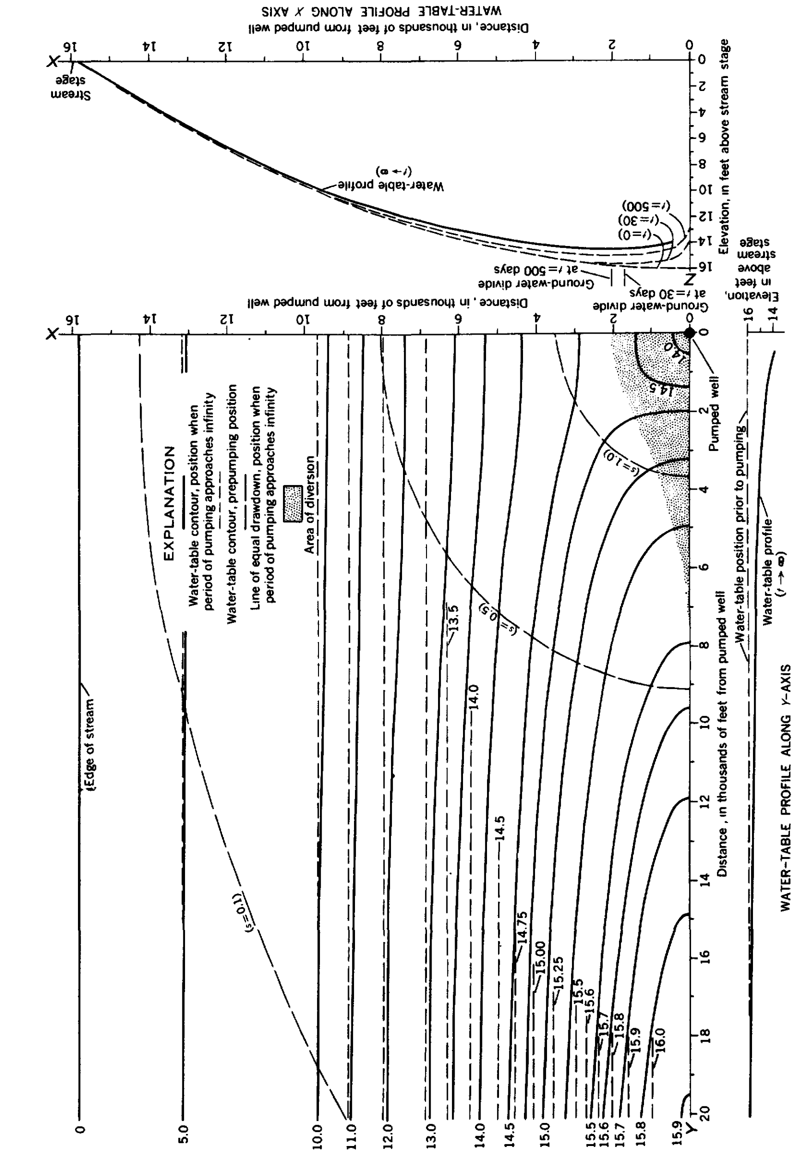

FIGURE 22.—Contour map and profiles of the water table when the period of pumping approaches infinity.

TABLE 11.—*Sample computations of steady-state drawdowns at the indicated coordinate points in an infinite strip aquifer*

[The three drawdown values computed below are entered in table 12]

n	$x=1{,}000$, $y=0$				$x=0$, $y=1{,}000$			$x=1{,}000$, $y=1{,}000$
	nA	$\sin (nA)$	$\sin\left(-\frac{n\pi}{2}\right)$	$\frac{1}{n}\sin (nA) \sin\left(-\frac{n\pi}{2}\right)$	nA'	$e^{-nA'}$	$\frac{1}{n}e^{-nA'}$	
1	2	3	4	5	6	7	8	5 x 7
1	−1.472	−0.995	−1	+0.995	0.0992	0.906	0.906	+0.901
3	−4.416	+.956	+1	+.319	.2975	.743	.248	+.237
5	−7.360	−.880	−1	+.176	.4958	.609	.122	+.107
7	−10.304	+.770	+1	+.110	.6942	.499	.071	+.055
9	−13.248	−.630	−1	+.070	.8925	.410	.046	+.029
11	−16.192	+.466	+1	+.042	1.0908	.336	.031	+.014
13	−19.136	−.282	−1	+.022	1.2892	.275	.021	+.006
15	−22.080	+.089	+1	+.006	1.4875	.226	.015	+.001
17	−25.024	+.109	−1	−.006	1.6858	.185	.011	−.001
19	−27.968	−.301	+1	−.016	1.8842	.152	.008	−.002
21	−30.912	+.483	−1	−.023	2.0825	.125	.006	−.003
23	−33.856	−.646	+1	−.028	2.2808	.102	.004	−.003
25	−36.800	+.782	−1	−.031	2.4792	.084	.003	−.003
27	−39.744	−.897	+1	−.033	2.6775	.069	.003	−.002
29	−42.688	+.962	−1	−.032	2.8758	.056	.002	−.002
31	−45.632	−.997	+1	−.032	3.0742	.046	.001	−.001
33	−48.576	+.993	−1	−.030	3.2725	.038	.001	−.001
35	−51.520	−.950	+1	−.027	3.4708	.031	.001	−.001
37	−54.464	+.870	−1	−.024	3.6692	.025	.001	−.001
39	−57.408	−.758	+1	−.019			Σ=1.501	.000
41	−60.352	+.615	−1	−.015				Σ=1.330
43	−63.296	−.448	+1	−.010				
45	−66.240	+.263	−1	−.006				
47	−69.184	−.069	+1	−.001				
49	−72.128	−.129	−1	+.003				
				Σ=1.439				
	$A=\pi\left(\frac{x}{2a}-\frac{1}{2}\right)$ $A=\pi\left(\frac{1{,}000}{31{,}680}-\frac{1}{2}\right)=-1.4716$ $s=(1.146)(1.439)=1.7$ (See eq 2a.)				$A'=\frac{\pi y}{2a}=\frac{1{,}000\pi}{31{,}680}=0.099166$ $s=(1.146)(1.501)=1.7$ (See eq 3a.)			$s=(1.146)(1.330)=1.5$

21), and when $t\to\infty$ (fig. 22) shows plainly how this hydraulic system responds to newly imposed discharge, by pumping, and reveals the degree and promptness with which different parts of the system approach new stabilization. The migration of the ground-water divide, from its initial position (fig. 18) coincident with the longitudinal centerline of the aquifer to its final position enclosing the shaded area of figure 22, is evident.

The introduction of the term "area of diversion" is appropriate in labeling and discussing the shaded areas shown in figures 20, 21, and 22. Inasmuch as the outer limit of the shading is the ground-water divide, recharge from precipitation anywhere on the shaded areas must eventually reappear as well discharge. In other words, each of the three shaded areas represents, for the indicated time since pumping began, the area in which the recharge is being diverted to the discharging well instead of escaping from the aquifer entirely as discharge into the two boundary streams. Table 13 shows the size of the three areas as well as the quantity of recharge being received

TABLE 12.—*Steady-state drawdown data for an infinite strip aquifer tapped by a single discharging well*

[Well is on centerline of aquifer. Computations involve use of equations 1, 2a, or 3; sample computations are given in table 11]

X-coordinate distance from well (feet)	Y-coordinate distance from well (feet)															
	0	500	1,000	2,000	3,000	4,000	5,000	6,000	7,000	8,000	9,000	10,000	12,000	14,000	16,000	20,000
0	---	2.1	1.7	1.3	1.1	0.9	0.8	0.7	0.6	0.6	0.5	0.5	0.4	0.3	0.2	0.2
500	2.0	1.9	1.7	1.3	1.1	.9	---	---	---	---	---	---	---	---	---	---
1,000	1.7	1.7	1.5	1.3	1.1	.9	.8	.7	.6	.6	.5	---	---	---	---	---
2,000	1.3	1.3	1.3	1.1	1.0	.9	.8	.7	.6	.5	.5	.4	---	---	---	---
3,000	1.1	1.1	1.1	1.0	.9	.8	.7	.7	.6	.5	.5	.4	.3	---	---	---
4,000	.9	.9	.9	.9	.8	.7	.7	.6	.5	.5	.4	.4	.3	.3	---	---
5,000	.8	.8	.8	.8	.7	.7	.6	.6	.5	.5	.4	.4	.3	.3	.2	.1
6,000	.7	.7	.7	.7	.6	.6	.5	.5	.5	.4	.4	.4	.3	.2	.2	.1
7,000	.6	.6	.6	.6	.5	.5	.5	.4	.4	.4	.4	.3	.3	.2	.2	.1
8,000	.5	---	.5	.5	.5	.4	.4	.4	.4	.3	.3	.3	---	---	---	---
9,000	.4	---	---	.4	.4	.4	.4	.3	.3	.3	.3	.3	.2	.2	.1	1
10,000	.4	---	---	.3	.3	.3	.3	.3	.3	.3	.2	.2	.2	.2	.1	1
11,000	.3	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---
12,000	.2	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---
13,000	.2	---	---	.2	---	.2	---	.1	---	.1	---	.1	.1	.1	.1	.0
14,000	.1	---	---	.1	---	.1	---	.1	---	.1	---	.1	.1	.1	.0	.0
15,000	.1	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---
15,840	.0	---	---	---	---	---	---	---	---	---	---	---	---	---	---	---

by each area. Thus, if the well has been pumped for 30 days, its area of diversion has expanded to 15.2×10^6 square feet. Uniform recharge of 6 inches per year on this area is equivalent to pouring water into the aquifer at the rate of 110 gpm. Obviously, this is not enough to offset the steady rate at which water is being removed through the well (200 gpm). The area of diversion continues to expand until, at $t=\infty$, the intercepted recharge exactly balances the rate at which the well is being pumped.

TABLE 13.—*Extent of the area of diversion and computed recharge for that area*

Pumping period (days)	Area of diversion (millions of square feet)		Total recharge (gpm)
	In one quadrant	In four quadrants	
30	3.80	15.2	110
500	6.15	24.6	175
∞	6.95	27.8	200

While the area of diversion is still expanding, part of the well discharge is intercepted recharge and the balance is withdrawal from storage in the aquifer. Although theoretically the withdrawal from storage does not cease before $t=\infty$, figures 21 and 22 and table 13 show that virtual stabilization of the entire flow system, accompanied by the cessation of significant draft upon storage, occurs at some finite time not greatly in excess of 500 days.

CONCLUSION

The demonstrated response of a given hydraulic system to imposed well discharge illustrates that a steady pumping rate is satisfied only by a decrease at the same rate in the discharge from the aquifer to the streams. Although some ground water is taken from storage in developing the flow field needed to meet the demands of the discharging well, in the final analysis the system reaches its new steady state by effecting a decrease in the natural discharge to the streams; the decrease is exactly equal to the newly imposed discharge.

The fallacious idea that the cone of depression around a discharging well expands only until its outer limit encompasses an area in which the recharge balances the discharge is perhaps most easily controverted, at least for the particular hydrologic system described in this paper, through substitution of "area of diversion" for "cone of depression." The area of diversion must expand until it has captured enough recharge to counterbalance the newly imposed discharge. The only limits to the expansion of the cone of depression are the physical boundaries of the system.

GROUND-WATER MOVEMENT IN A RECTANGULAR AQUIFER BOUNDED BY FOUR CANALS

By RUSSELL H. BROWN

ABSTRACT

Most rectangular aquifers are in areas that are dissected by networks of irrigation canals or drainage ditches. If the water level is initially the same both throughout the aquifer and in the peripheral canals and if an instantaneous change of water level occurs either within the aquifer or in the canals, the formula from which the resultant ground-water movement can be determined is analogous to that used in certain heat-flow problems.

Water-level data from recording-gage charts can be used to determine the hydraulic characteristics of the rectangular aquifer. The formula is most easily solved by the use of two log plots, one of which is superposed on the other. Computations for one of the log plots are facilitated by the use of a semilog plot of curves representing one-dimensional flow in an infinite strip aquifer.

If the hydraulic properties of the rectangular aquifer are known, the formula can be used for predicting water-level recession curves for any point in the aquifer. Again, the semilog plot is used for one-dimensional flow.

THE OCCURRENCE OF RECTANGULAR AQUIFERS

An aquifer of rectangular shape is not as rare as might be assumed. For example, in the greater Miami area in southeastern Florida, the principal aquifer is dissected into many rectangles by a network of interconnected drainage ditches or canals, most of which are oriented parallel to the section lines of the "Federal system of rectangular surveys"; the aquifer is highly permeable limestone in which water occurs under water-table conditions. Other examples of rectangular aquifers can be found in irrigated areas, especially in the western and southwestern parts of the United States, and in nursery plots where the position of the water table is carefully controlled by networks of irrigation and drainage ditches.

The writer is indebted to R. W. Stallman of the U.S. Geological Survey for prompting the preparation of this paper. Mr. Stallman used the fundamental theory and techniques described in this paper to analyze data that were collected for, but which were not included in, a special project report (Stallman, 1956).

FORMULAS FOR THE SOLUTION OF HYDRAULIC PROBLEMS RELATING TO A RECTANGULAR AQUIFER

If a cloudburst occurs over a rectangular aquifer at a time when the water level in the bordering canals or ditches is closely regulated, information from water-level recorder charts can be used to solve relevant hydraulic problems by procedures that are analogous to those used in certain heat-flow problems.

Consider a water-table aquifer underlying an area of rectangular shape, the four sides of which are bounded by drainage ditches or canals that are hydraulically continuous with the full thickness of the aquifer. Assume that, initially, the water table is everywhere at the same elevation as the controlled (fixed) water surface in the ditches. If a heavy rain causes a sudden and general rise of the water table to an elevation substantially higher than the fixed water level in the ditches, the subsequent positions of the water table within the aquifer as the ground water drains to the ditches can be determined from a formula. No recharge is assumed for the period after the rain, and the maximum vertical movement of the water table is assumed to be a small proportion of the initial saturated thickness. The last specification characterizes the ground-water flow as two-dimensional—that is, the flow is virtually in a horizontal plane and has no significant vertical components.

Ingersoll and Zobel (1948, p. 183–185) have analyzed the analogous problem of heat flow in a rectangular rod of infinite length (or of finite length with perfectly insulated ends). Initially, the rod has a uniform temperature; then the four plane surfaces are subjected to an abrupt temperature change and are kept at the new temperature. The solution is equally applicable if the temperature of the four plane surfaces remains constant and if the temperature throughout the entire rod changes abruptly. The heat flows only in the plane of the rectangular cross section—that is, the flow will be two-dimensional and the length of the rod is not a variable in the flow equation.

The relation given by Ingersoll and Zobel for heat conduction at the center of an infinite rectangular rod can be written in abbreviated form as follows:

$$\frac{T_c-T_s}{T_0-T_s}=\Big[S(Z)\Big]\Big[S(Z')\Big], \tag{1}$$

where

$$Z=\frac{\alpha t}{l^2} \text{ and } Z'=\frac{\alpha t}{w^2},$$

and where

t = the elapsed time since the temperature change occurred (and heat flow began),

T_0 = the initial temperature throughout the rod ($t \leq 0$),

T_s = the new temperature of the rod surfaces. T_s is instantaneously established and is held constant thereafter ($t \geq 0$),

T_c = the temperature at any point on the centerline (axis) of the rod at time $t>0$,

α = the thermal diffusivity of the rod,

l = the length of the rectangular cross section of the rod,

w = the width of the rectangular cross section of the rod,

$S(Z)$ or $S(Z')$ = the symbol for a series of exponential and trigonometric terms that is evaluated, for this special case (the center of the rectangle), by Ingersoll and Zobel (1948, p. 255–256).

Each bracketed term in equation 1 represents the solution for heat conduction in one dimension; in other words, the bracketed terms represent heat flow in infinite slabs of thickness l and w, respectively. That the product of the two bracketed terms gives the solution sought for this two-dimensional flow problem is the result of a principle commonly employed in the theory of heat conduction. This principle recognizes the validity of developing the heat-conduction equations for problems involving two- or three-dimensional flow by determining the product of the simpler individual flow equations pertaining to each of the respective dimensions. Thus the proper combination of elemental solutions, each involving flow in one or two dimensions, yields solutions for flow in a variety of simple solids.

The temperature factor, T, in a heat-flow system is analogous to the head, h, in a ground-water system. The reference datum for all the values of head is arbitrarily chosen as the fixed position of the water level in the drainage ditches. Thus, the left side of equation 1 becomes h_c/h_o, where h_c is the head at the center point of the rectangular aquifer at any time $t>0$, and h_o is the change in stage that is "instantaneously" impresesd upon the water table at $t=0$ and which initiates the drainage toward the ditches.

The only heat-flow term implicit in the right side of equation 1 is the thermal diffusivity, α, which Theis (1935, p. 520) identified as being analogous to the quotient of T/S. Equation 1 could now be rewritten in ground-water notation but, because in its present form the equation is restricted to the special case for a point at the center of the rectangular aquifer, it is more desirable to find the general equation that is applicable to any point in the rectangle. For convenience of reference, the origin of coordinates is chosen as one corner of the aquifer rectangle, so that the dimension l is oriented in the X-direction and the dimension w in the Y-direction. The solution given by Ingersoll and Zobel (1948, p. 125) for one-dimensional flow in an infinite slab can be rewritten directly in ground-water notation as follows:

$$\frac{h}{h_0}=\frac{2}{l}\sum_{n=1}^{\infty}\left[\left(e^{-n^2\pi^2Tt/l^2S}\right)\frac{l}{n\pi}(1-\cos n\pi)\sin\frac{n\pi x}{l}\right]\frac{2}{w}$$
$$\sum_{n=1}^{\infty}\left[e^{-n^2\pi^2Tt/w^2S}\right)\frac{w}{n\pi}(1-\cos n\pi)\sin\frac{n\pi y}{w}\right] \quad (2)$$

where

h=the head at the point (x, y) in the rectangular aquifer at any time $t>0$,
n=a positive integer,
l=the length of the rectangular aquifer, and
w=the width of the rectangular aquifer.

As in equation 1, each pair of brackets with its preceding coefficient identifies the solution that pertains to flow in one dimension; the indicated product provides the solution sought for the two-dimensional flow system of this problem.

If n in equation 2 is an even number, the parenthetical expression containing the cosine term is zero. Therefore, only odd numbers for n need be considered; this means that the cosine parenthetical expression always equals 2.

Let

$$Z=\frac{Tt}{l^2S} \quad (2a)$$

and

$$Z'=\frac{Tt}{w^2S}. \quad (2b)$$

For any time t, Z and Z' are seen to be related as follows:

$$Z'=Z\left(\frac{l}{w}\right)^2. \quad (3)$$

Substituting Z and Z' as appropriate in equation 2, factoring $2l/\pi$ and $2w/\pi$ from the bracketed expressions, expanding the indicated summations into series, and substituting 1, 3, 5, etc. for successive n's gives the following equation:

$$\frac{h}{h_0}=\frac{4}{\pi}\left[e^{-\pi^2Z}\sin\frac{\pi x}{l}+\frac{1}{3}e^{-9\pi^2Z}\sin\frac{3\pi x}{l}+\frac{1}{5}e^{-25\pi^2Z}\sin\frac{5\pi x}{l}+\ldots\right]$$
$$\frac{4}{\pi}\left[e^{-\pi^2Z'}\sin\frac{\pi y}{w}+\frac{1}{3}e^{-9\pi^2Z'}\sin\frac{3\pi y}{w}+\frac{1}{5}e^{-25\pi^2Z'}\sin\frac{5\pi y}{w}+\ldots\right] \quad (4)$$

SEMILOG PLOTS FOR ONE-DIMENSIONAL FLOW IN AN INFINITE STRIP AQUIFER

Equation 4 offers no clue to any solution as direct as the superposition of a data plot on a single-type curve. Because the combined evaluation of the two series of terms involves the factors l, w, x, and y, a separate curve exists for each point in the aquifer rectangle for each value of the ratio l/w. Obviously, an infinite number of curves would be required to satisfy the possible combinations of x and y coordinates and ratios of l/w. The practical use of equation 4, therefore, presupposes knowledge of the dimensions l and w, or their ratio, as well as the coordinates x and y of the point or points at which the water-table movement is being observed or is to be predicted. According to the nature of the specific-field problem and the kind of additional information available, these data are used in solving equation 4 either for the ratio T/S or for predicting the position of the water table at specified points in the aquifer. The solution of either problem is expedited, however, by evaluating in advance one of the series in equation 4, using assumed values for Z (or Z') and x/l (or y/w). Sample computations for the assumed ratio $x/l=0.1$ appear in table 14. The results, together with the results of computations for other ratios of x/l, are assembled in table 15 and are plotted on semilog graph paper as shown in figure 23.

The family of curves in figure 23 is similar to a family of water-level recession curves because, for each selected point in the aquifer, values of Z, which contain t as the only variable (see eq 2a), are plotted against the ratios of the residual head to the initial change in head. Because the curves are developed by solving only one of the series in equation 4, they represent one-dimensional flow. Although they depict, therefore, only half the solution required for two-dimensional flow in a rectangular aquifer, they depict the complete solution for flow under similar hydraulic conditions in a strip aquifer of finite width and infinite length and bounded by two parallel ditches of infinite length.

The sample computations in table 14 and similar computations for the results given in table 15 show that, for any assumed value of the ratio x/l or y/w (which needs to range only from 0 to 0.5), either series expression in equation 4 is satisfactorily approximated by using only its first term, provided Z or Z' is greater than about 0.08. Thus, for this special condition, the following relation may be written:

$$\left(\frac{h}{h_0}\right)_x=\frac{4}{\pi}\left(e^{-\pi^2 Z}\sin\frac{\pi x}{l}\right), \qquad (5)$$

where the subscript x signifies that the ratio h/h_0 is expressed only for one-dimensional flow in the X-direction. The nature of equation

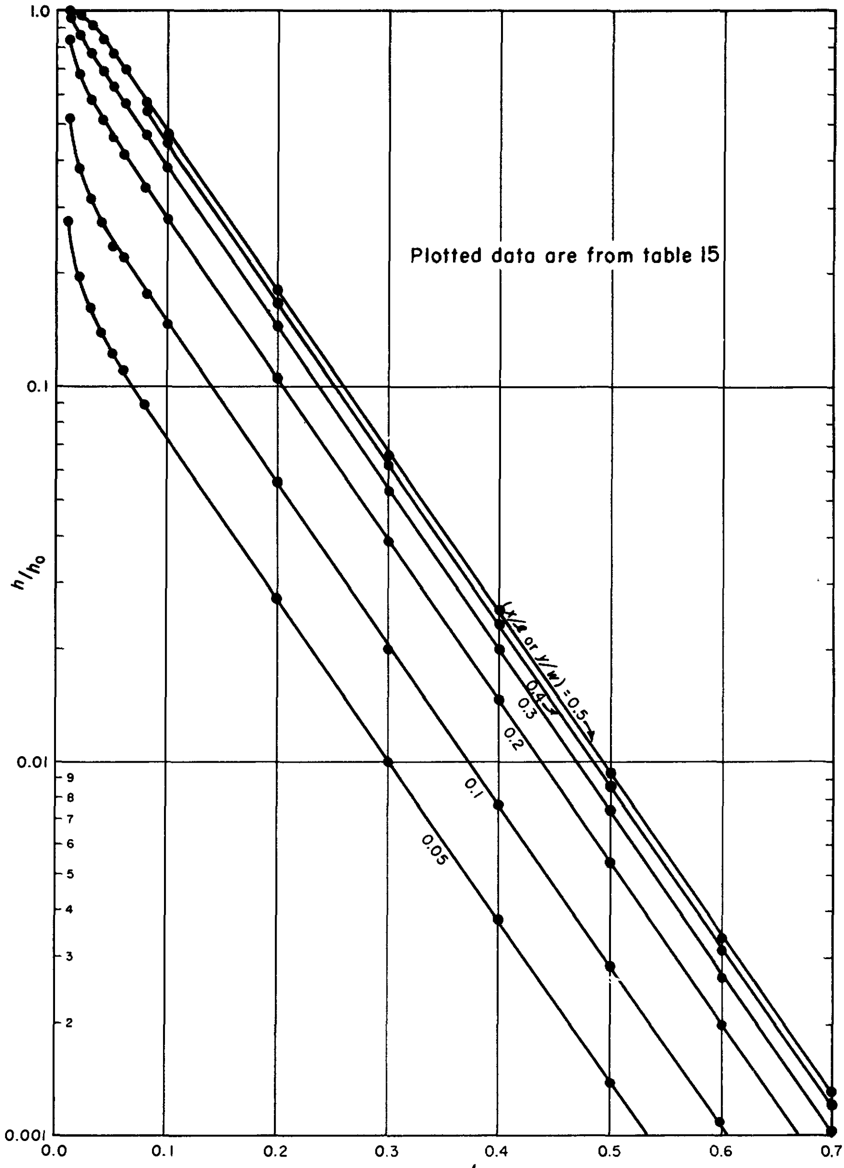

FIGURE 23.—Semilog plot of curves representing one-dimensional flow in an infinite strip aquifer of width l or w.

TABLE 14.—*Sample computations for values of* h/h$_0$ *when* x/l (*or* y/w) *is 0.1*

[Data in first and last columns constitute part of table 15 and are plotted in fig. 23 as curve $x/l=0.1$. Values of the exponential function are from U.S. Dept. Commerce, 1951, Tables of the exponential function e^x: Natl. Bur. Standards, Washington, D.C., U.S. Govt. Printing Office]

Z or Z'	$\pi^2 Z$	$e^{-\pi^2 Z}$	$e^{-\pi^2 Z}\sin\frac{\pi x}{l}$ $\pi x/l=18°$ sin 18°=0.309	$\frac{1}{3}e^{-9\pi^2 Z}$	$\frac{1}{3}e^{-9\pi^2 Z}\sin\frac{3\pi x}{l}$ $3\pi x/l=54°$ sin 54°=0.809	$\frac{1}{5}e^{-25\pi^2 Z}$	$\frac{1}{5}e^{-25\pi^2 Z}\sin\frac{5\pi x}{l}$ $5\pi x/l=90°$ sin 90°=1	$\frac{1}{7}e^{-49\pi^2 Z}$	$\frac{1}{7}e^{-49\pi^2 Z}\sin\frac{7\pi x}{l}$ $7\pi x/l=126°$ sin 126°=0.809	Sum of columns 1, 2, 3, and 4	h/h_0 Product of column 5 and $(4/\pi)$
			1		2		3		4	5	
0.01	0.0987	0.906	0.280	0.137	0.111	0.0170	0.017	0.001	0.001	0.409	0.520
.02	.1974	.821	.254	.0564	.045	.0014	.001	-----	-----	.300	.382
.03	.2961	.744	.230	.0232	.019	-----	-----	-----	-----	.249	.317
.04	.3948	.674	.208	.0095	.008	-----	-----	-----	-----	.216	.275
.05	.4935	.610	.183	.0039	.003	-----	-----	-----	-----	.186	.237
.06	.5922	.553	.171	.0016	.002	-----	-----	-----	-----	.173	.220
.08	.7896	.454	.140	-----	-----	-----	-----	-----	-----	.140	.178
.10	.9870	.373	.115	-----	-----	-----	-----	-----	-----	.115	.146
.20	1.974	.139	.043	-----	-----	-----	-----	-----	-----	.043	.055
.30	2.961	.0518	.016	-----	-----	-----	-----	-----	-----	.016	.020
.40	3.948	.0193	.0060	-----	-----	-----	-----	-----	-----	.0060	.0076
.50	4.935	.00719	.0022	-----	-----	-----	-----	-----	-----	.0022	.0028
.60	5.922	.00267	.00083	-----	-----	-----	-----	-----	-----	.00083	.00106
.80	7.896	.00037	-----	-----	-----	-----	-----	-----	-----	-----	-----
1.00	9.870	.00005	-----	-----	-----	-----	-----	-----	-----	-----	-----

TABLE 15.—*Values of* h/h₀ *for the indicated arguments* Z *or* Z′ *and* x/l *or* y/w

Z or *Z′*	*x/l* or *y/w*					
	0.05	0.1	0.2	0.3	0.4	0.5
0.01	0.275	0.520	0.843	0.965	0.997	0.999
.02	.197	.382	.682	.866	.952	.975
.03	.162	.317	.584	.775	.884	.918
.04	.139	.275	.516	.698	.808	.845
.05	.122	.237	.462	.629	.736	.772
.06	.111	.220	.416	.569	.668	.702
.08	.090	.178	.340	.467	.550	.578
.10	.074	.146	.279	.384	.452	.474
.20	.028	.055	.104	.143	.168	.177
.30	.010	.020	.039	.053	.063	.066
.40	.0038	.0076	.014	.020	.023	.025
.50	.0014	.0028	.0053	.0074	.0087	.0092
.60	.0005	.0011	.0020	.0027	.0032	.0034
.80	----------	----------	----------	.00038	.00045	.00047
1.00	----------	----------	----------	----------	----------	----------

5 reveals why it was expedient to prepare figure 23 as a semilog plot. If common logarithms (logs to the base 10) are taken of both sides of equation 5, Z is seen to vary inversely with $\log_{10} h/h_0$. Therefore, in the region of $Z>0.08$, a semilog plot of arithmetic values of Z versus log values of h/h_0 produces a straight line. Hence, to plot part of any desired curve for intermediate values of x/l (or y/l) not shown in figure 23: assume a convenient value of Z; use equation 5 to compute the corresponding value of h/h_0; plot the result; and draw a straight line through the plotted point parallel to the straight-line parts of the family of curves. Computations for values needed to extend any curve into the region $Z<0.08$ can be made by the method shown in table 14.

THE SOLUTION OF HYDRAULIC PROBLEMS FOR A RECTANGULAR AQUIFER

Procedures for the solution of hydraulic problems for a rectangular aquifer are best explained by the use of illustrative examples. For an aquifer of known dimensions, the ratio T/S can be determined if the water-level record for a specific well is available and the water-table recession that will occur after a sudden rise of the water level can be predicted if the aquifer constants are known.

First, consider the problem of determining the ratio T/S from the water-level record of a given well. The areal extent of a highly permeable aquifer, surrounded by four canals, is reasonably approximated as a rectangle 10 miles long and 3 miles wide. A cloudburst caused an abrupt water-table rise of 2.60 feet throughout the aquifer. The rise and subsequent decline of the water level in a well at the coordinate position $x=3$ miles and $y=0.6$ mile were recorded on the chart of a water-stage recording gage, from which chart were derived the data that are given in table 16 and that are plotted in figure 24. The ratio T/S is to be computed.

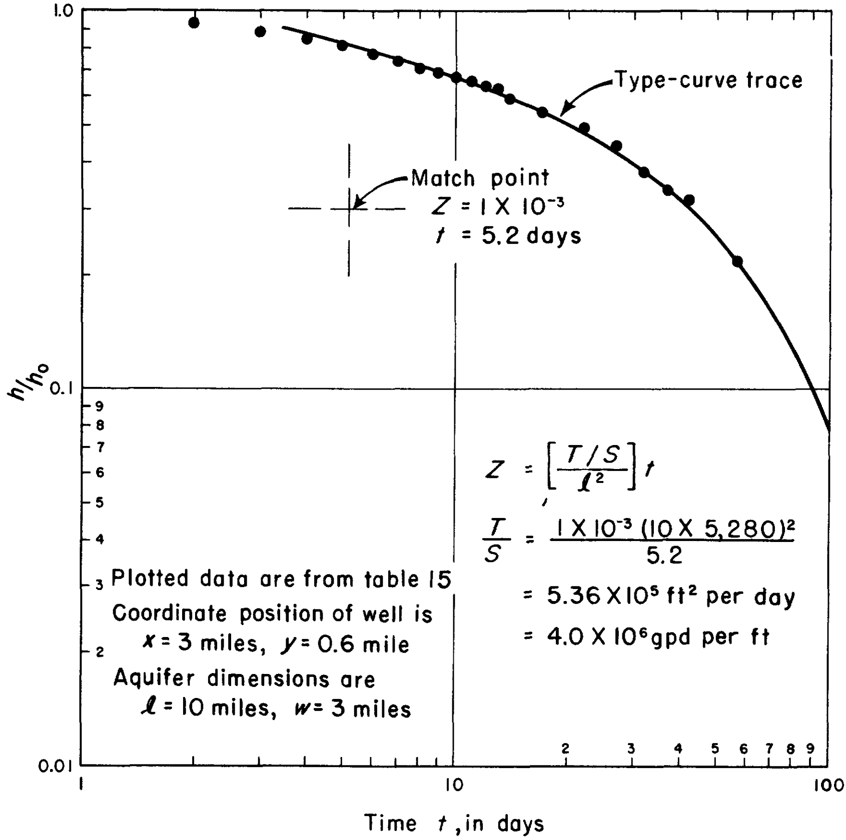

Figure 24.—Curve of recession data developed from the recording-gage chart of a specific well that taps a rectangular aquifer.

Table 16.—*Water-table recession data for an observation well at coordinate position* x=*3 miles and* y=*0.6 mile in a rectangular aquifer where* h_0=*2.60* feet at t=0

[Data in this table are plotted in fig. 24]

t (days)	h (feet)	h/h_0	t (days)	h (feet)	h/h_0	t (days)	h (feet)	h/h_0
0	2.60	1.000	8	1.85	0.712	17	1.40	0.538
1	2.51	.965	9	1.79	.688	22	1.27	.488
2	2.42	.932	10	1.75	.673	27	1.14	.438
3	2.28	.877	11	1.70	.654	32	.97	.373
4	2.19	.842	12	1.65	.635	37	.87	.334
5	2.10	.808	13	1.62	.623	42	.82	.315
6	2.00	.769	14	1.54	.593	57	.57	.219
7	1.92	.738						

Equation 4 shows that, for any specified point in the aquifer, h/h_0 varies as the product of the exponential functions of Z and Z'. Inasmuch as t is the only variable in either Z or Z', log plots of h/h_0 versus t, Z, or Z' are curves of the same shape. Thus, the log plot of corresponding values of Z (or Z') and h/h_0 for a specific point in the aquifer produces a curve against which can be matched the log data plot of corresponding values of t and h/h_0 for that point. Solution for the ratio T/S then involves selecting a convenient match point and substituting its coordinates in equation 2a if the plot is of Z versus h/h_0, or in equation 2b if the plot is of Z' versus h/h_0.

For the given aquifer and observation point, the following relationships exist:

$$\frac{x}{l}=0.3;\ \frac{y}{w}=0.2;\ \frac{l}{w}=\frac{10}{3};$$

and by substitution in equation 3

$$Z'=\frac{100Z}{9}.$$

Preparation of the curve to be matched to the data plot (fig. 24) requires that values of Z (or Z') be assumed and that equation 4 be solved for the corresponding values of h/h_0. The computations are greatly simplified by use of values shown in figure 23 and table 15. Results of the computations appear in columns 1 through 5 of table 17 and the data in columns 1 and 5 are plotted in figure 25.

The data curve in figure 24 is superposed on the curve in figure 25, keeping the coordinate axes parallel and the h/h_0 scales alined; the data plot is translated in the t-coordinate direction until the best fit is found. From a convenient match point, corresponding values of Z and t are determined (fig. 24). Substitution of these values in equation 2a yields the following computation:

$$\frac{T}{S}=\frac{Zl^2}{t}=\frac{1\times10^{-3}(10\times5{,}280)^2}{5.2}$$
$$=5.36\times10^5\ (\text{ft})^2\text{ per day}$$
$$=5.36\times10^5\ (\text{ft})^3\text{ per day per ft}$$
$$=4.0\times10^6\text{ gpd per ft.}$$

From the above ratio, T is found easily if S is known or can be estimated with reasonable reliability.

In the foregoing problem, the water-table recession at a specific point, after a sudden and areawide rise in water level, can be predicted if the aquifer constants are known. For convenience, assume that the ratio $T/S=4.0\times10^6$ gpd per ft and that the water table rose abruptly 5.0 feet. The recession of the water table at the well in the preceding problem is depicted best by a recession curve plotted in terms of residual head versus time. If none of the previous computations were available, the analytical procedure would involve as-

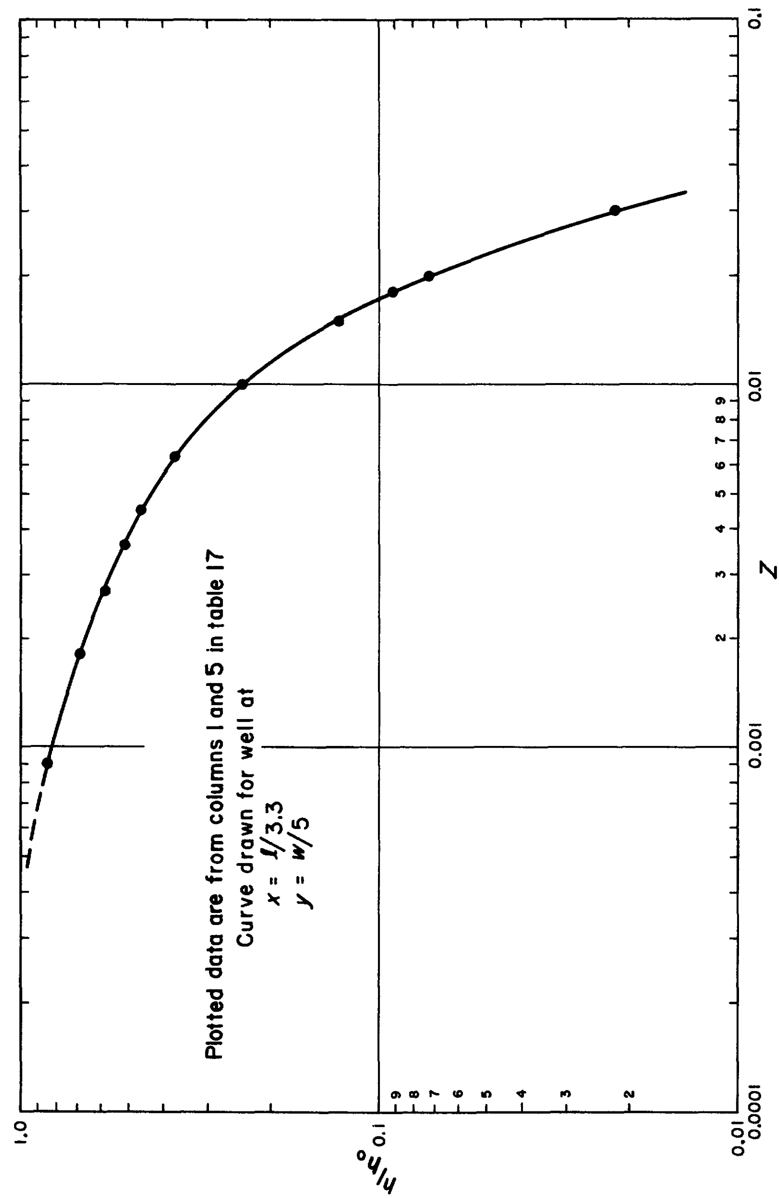

FIGURE 25.—Z-Curve for a specific well that taps a rectangular aquifer.

TABLE 17.—*Computed water-table recession data for an observation well at coordinate position* x=*3 miles and* y=*0.6 mile in a rectangular aquifer*

[Data in columns 1 and 3 were computed by use of equation 4 or were assumed; those in column 2 were obtained from either fig. 23 or table 15 for $x/l=0.3$ and those in column 4 from either fig. 23 or table 15 for $y/w=0.2$; those in column 5 are the products of columns 2 and 4; those in column 6 were computed on the assumption that $h_0=5.00$ ft and those in column 7 by use of the ratio $T/S=4.0\times10^6$ gpd per ft, which produces the relation $t_{days}=5.2\times10^3\ Z$ or $t_{days}=4.7\times10^2\ Z'$. Data in columns 1 and 5 are plotted in fig. 25 and those in columns 6 and 7 in figs. 26 and 27]

Z	$(h/h_0)_l$	Z'	$(h/h_0)_w$	$(h/h_0)_{lw}$	h (feet)	t (days)
1	2	3	4	5	6	7
0.0000	1.00	0.00	1.000	1.00	5.00	0
.0009	1.00	.01	.843	.84	4.20	4.7
.0018	1.00	.02	.682	.68	3.40	9.4
.0027	.99	.03	.584	.58	2.90	14
.0036	.99	.04	.516	.51	2.55	19
.0045	.99	.05	.462	.46	2.30	24
.0063	.98	.07	.375	.37	1.85	33
.01	.965	.111	.25	.24	1.20	52
.015	.92	.17	.14	.13	.65	78
.018	.88	.20	.104	.092	.46	94
.02	.866	.222	.084	.073	.36	104
.03	.775	.333	.028	.022	.11	156
.04	.698	.444	.0094	.0066	.033	208
.05	.629	.555	.0033	.0021	.010	260

suming a value of Z (or Z') and computing the corresponding values of h/h_0 and t, repeating the process until the plotted results adequately define the desired recession curve. In this problem, however, the computations in columns 1 through 5 of table 17 are used. The h values in columns 6 and 7 of table 17 are obtained by multiplying the corresponding h/h_0 values by h_0, or 5.0; the t values are obtained by using equation 2a or 2b. The log plot of corresponding values of t and h gives the recession curve in figure 26, whereas the arithmetic plot in figure 27 is the more familiar concave-upward recession curve, which is similar to the record that would appear on the chart of a water-stage recording gage.

The first plotted point on the recession curve (fig. 27 and table 17) represents a time when nearly 5 days of the recession period had elapsed and for which 16 percent of the total possible head decline had occurred; the lowest available value of Z' was used to determine the coordinates of this point. If points representing earlier times and less decline of head are desired, each of the family of curves (fig. 23) must be expanded by replotting and by computing additional points in the region $0<Z<0.1$. The straight-line segments of the family of curves are for times that exceed 1 month.

The two illustrated applications of equation 4 may be repeated readily for any desired point in the rectangular aquifer. Owing to symmetry, the solution for any specific point in one quadrant of the rectangle applies also to three other points similarly placed in the remaining three quadrants. Thus, the foregoing analysis for the

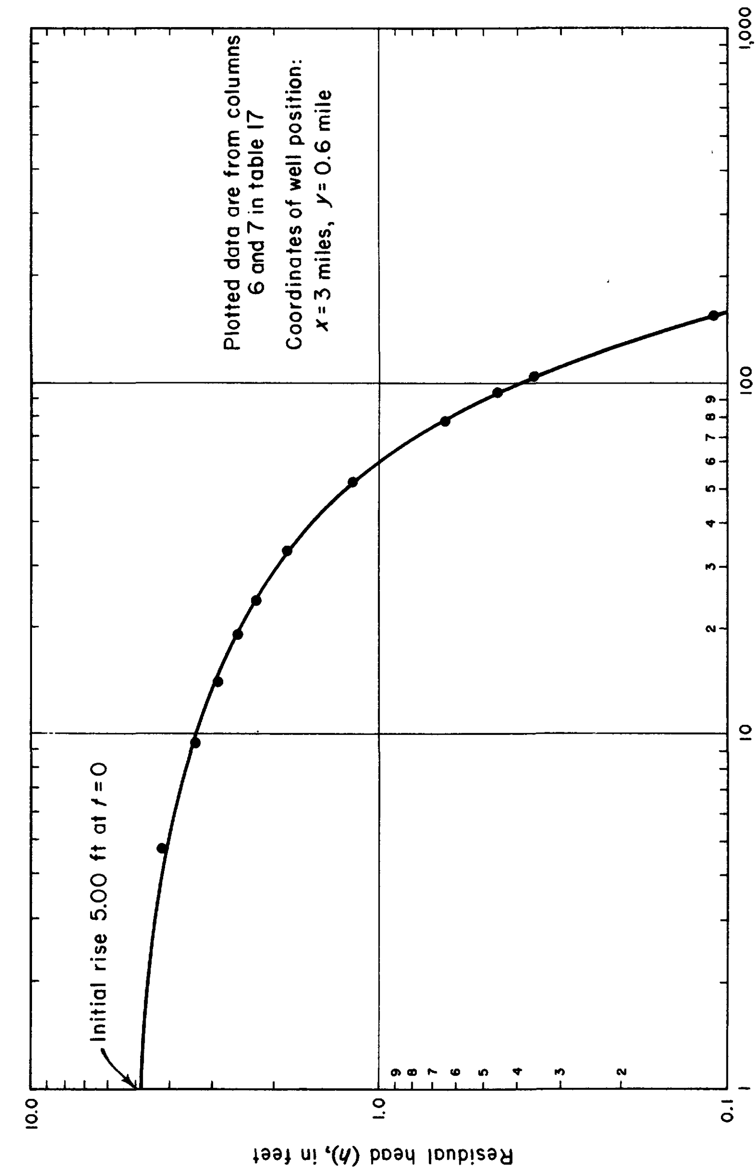

FIGURE 26.—Log plot of the computed water-level recession curve for a specific well that taps a rectangular aquifer.

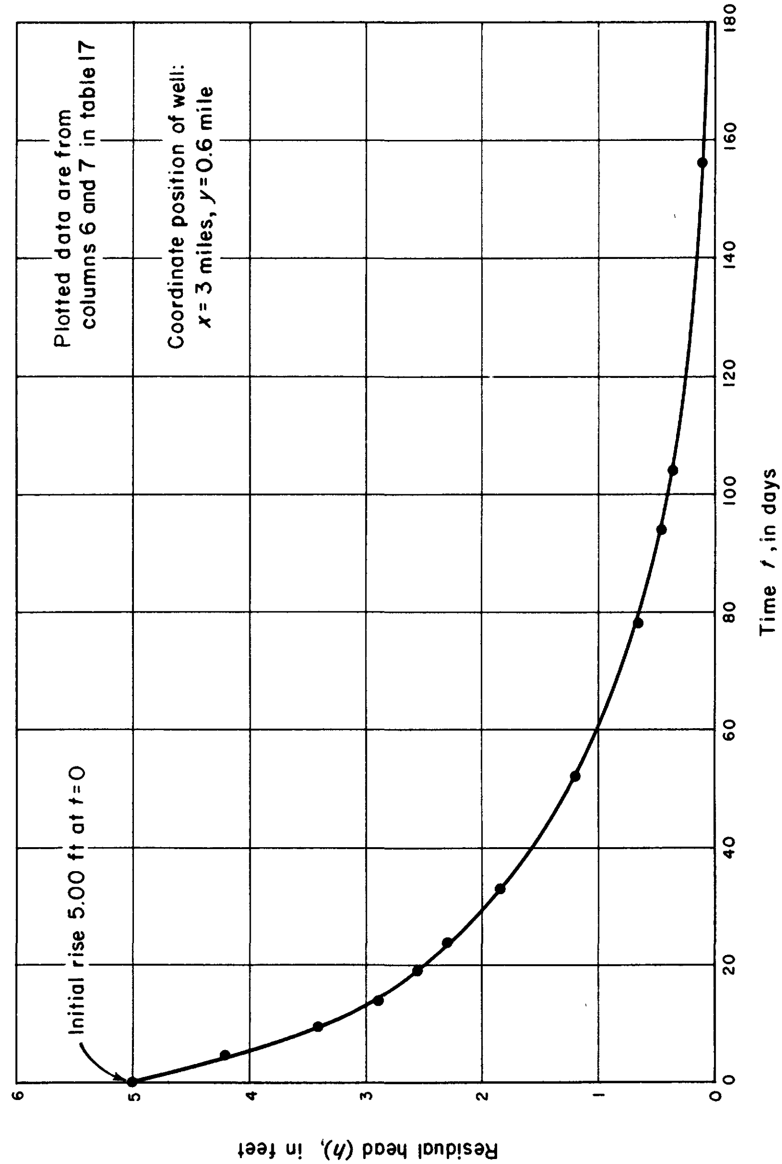

FIGURE 27.—Arithmetic plot of the computed water-level recession curve for a specific well that taps a rectangular aquifer.

well at the coordinates $x=3$ miles and $y=0.6$ mile would apply equally to the following three additional coordinate locations: $x=3$ miles, $y=2.4$ miles; $x=7$ miles, $y=0.6$ mile; and $x=7$ miles, $y=2.4$ miles.

CONCLUSION

In the foregoing analysis of ground-water movement in a rectangular aquifer, it has been postulated that recharge from a cloudburst caused a sudden rise of the water table throughout an aquifer in which the water level initially was the same as that in the bordering ditches or canals. The analogous heat-flow theory is not so restrictive. The basic theory, translated into ground-water terms, requires only that the water-level position initially be the same everywhere throughout the aquifer and the bordering ditches and that the initial situation be followed by an instantaneous change of the water level throughout either the aquifer or the peripheral surface-water body. Thus, in addition to the cloudburst effect already described, other circumstances that could produce the sudden water-level change include: lowering the water level in the bordering ditches or canals by operating large-capacity drainage pumps or by opening the gates on check dams (the lowering occurs in a relatively short time and the water level is then held at the new position); or raising the water level in bordering canals by introducing water for irrigation (the water level is held at the new raised position).

The analytical procedures described in this paper afford a means either for determining the hydraulic characteristics of a rectangular aquifer by the use of water-level data from recording-gage charts or, conversely, if the hydraulic characteristics are known, for predicting water-table recession curves for any point in the aquifer. Such analyses afford solutions to such practical problems as the prediction of water-table rises near irrigation canals and the determination of the adequacy of the spacing of drainage ditches around nursery plots or other areas in which control of the water-table position is critical.

DRAWDOWNS CAUSED BY A WELL DISCHARGING UNDER EQUILIBRIUM CONDITIONS FROM AN AQUIFER BOUNDED BY A FINITE STRAIGHT-LINE SOURCE

By CHARLES V. THEIS

ABSTRACT

The method for determining the drawdowns caused by a well discharging, under equilibrium conditions, from an aquifer bounded by a finite straight-line source is analogous to that used by Muskat for a similar problem. The formula for the drawdown is virtually a special case of the Muskat formula; it can be reduced to the Thiem formula if the line source is of infinite extent. Drawdowns in an aquifer that has a finite line-source boundary are greater than those in an aquifer that has an infinite line-source boundary but are smaller than those in an aquifer that is without boundaries.

THE PROBLEM AND A FORMULA FOR ITS SOLUTION

The drawdowns caused by a well discharging, under equilibrium conditions, from an aquifer that has a finite line-source boundary can be determined from a formula analogous to one developed by Muskat (1937, p. 186–192) for a similar problem. The general method for finding the drawdown at any point in the vicinity of the well is given below.

The relative positions of the finite line source, the discharging well, and the point for which the drawdown is to be determined can be shown graphically. Let the line source of length $2c$ lie along the X-axis of a Cartesian coordinate system in such a way that it is bisected by the Y-axis; hence, the source extends from $-c$ to $+c$ on the X-axis. Let the well be located at the point (x_0,y_0) and let the point at which the drawdown is desired be (x,y) (fig. 28). If c is then considered to be the basic unit of reference, the coordinates of these points become $(x_0/c,y_0/c)$ and $(x/c,y/c)$, respectively.

If the Cartesian system of coordinates is transformed by the equations

$$x/c=\cosh \xi \cos \eta$$

and

$$y/c=\sinh \xi \sin \eta,$$

the resulting system of orthogonal coordinates represents the stream lines and the equipotential lines from the line source. The chart in figure 29 gives the values of ξ and η corresponding to values of x/c

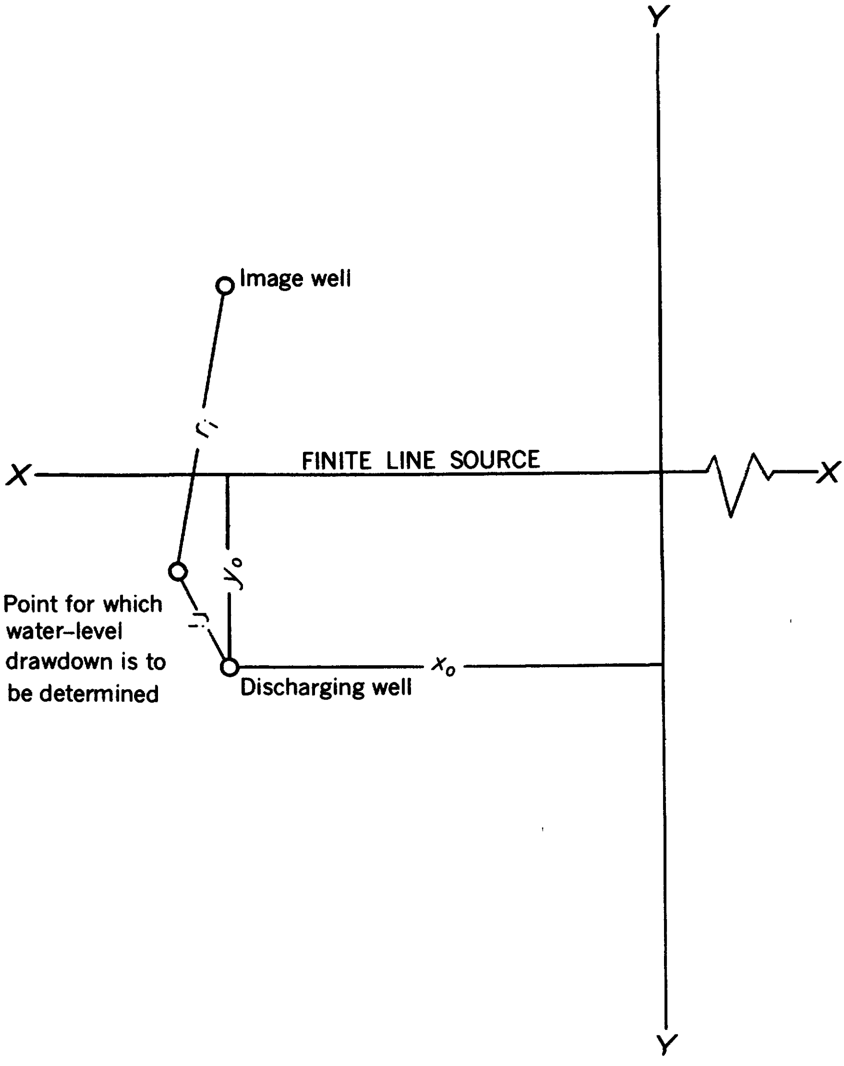

FIGURE 28.—Sketch illustrating the relative positions of a finite line source, discharging well, image well, and point for which water-level drawdown is to be determined.

and y/c. To use this chart, find the point $(x/c, y/c)$ using the Cartesian coordinates and read the values, at that point, of ξ and η given by the two sets of hyperbolic curves; ξ and η are, of course, in natural, or radian, measure. Tables of the hyperbolic cosines (cosh ξ), hyperbolic sines (sinh ξ), cosines (cos η), and sines (sin η) can be found, among other places, in the Handbook of Chemistry and Physics (Hodgman, 1953–54, p. 136–139, 180–186).

According to Muskat (1937), the equation for the solution of a

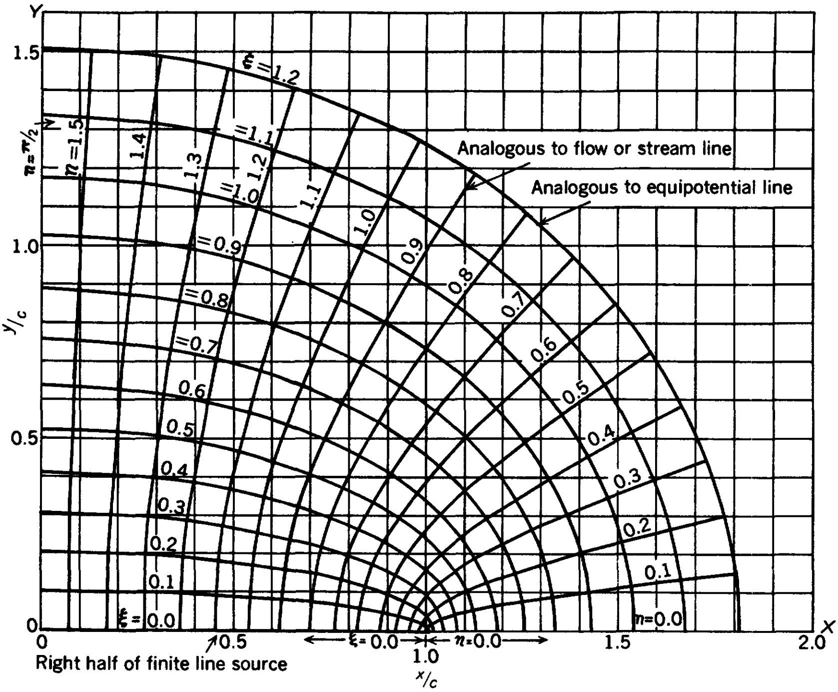

FIGURE 29.—Diagram for effecting the transformation from rectangular coordinates to a form of hyperbolic coordinates.

problem analogous to finding the drawdown, under conditions of steady-state flow, caused by a well discharging water from an aquifer bounded by a finite line source is

$$p=p_e+\frac{q}{2}\log_e\left[\frac{\cosh(\xi+\xi_0)-\cos(\eta-\eta_0)}{\cosh(\xi-\xi_0)-\cos(\eta-\eta_0)}\times\frac{\cosh(\xi+\xi_0)-\cos(\eta+\eta_0)}{\cosh(\xi-\xi_0)-\cos(\eta+\eta_0)}\right]$$

in which p is the pressure at the point (ξ,η), p_e is the pressure at the source, and q is what Muskat calls the "strength" of the well. The drawdown, s, at the point (x,y), which is the point (ξ,η), can be considered analogous to the pressure at (ξ,η). As no drawdown occurs along the line source, p_e can be disregarded. In terms of Q and T, q is equivalent to $229.2Q/T$; hence, $q/2$ equals $114.6Q/T$; and $2.303q/2$, where 2.303 is the conversion factor needed to change natural logarithms to logarithms that have the base 10, is equal to $264Q/T$. Consequently, the equation for the required drawdown is

$$s=\frac{264Q}{T}\log_{10}\left[\frac{\cosh(\xi+\xi_0)-\cos(\eta-\eta_0)}{\cosh(\xi-\xi_0)-\cos(\eta-\eta_0)}\times\frac{\cosh(\xi+\xi_0)-\cos(\eta+\eta_0)}{\cosh(\xi-\xi_0)-\cos(\eta+\eta_0)}\right]. \tag{1}$$

In using this equation, it should be remembered that the hyperbolic cosine is always positive [cosh x=cosh $(-x)$] and that the natural cosine is positive for any angle in the first and fourth quadrants and negative for an angle in the second and third quadrants

$$[\cos x = \cos(-x) \text{ and } \cos(\pi - x) = -\cos x].$$

Where the line source is infinite in length ($c=\infty$), equation 1 can be shown to reduce to

$$s = (264Q/T \log_{10}(r_i^2/r_r^2) = 528 \log_{10}(r_i/r_r), \tag{2}$$

in which

r_i = the distance from the image well to the point for which the drawdown is desired,

and

r_r = the distance from the real well to the point for which the drawdown is desired.

The drawdown, s, caused by a well discharging from an aquifer that has an infinite line-source boundary, is the difference between the drawdown, s_r, caused by the discharging well and the buildup, s_i, caused by the theoretical recharging image well; hence

$$s = s_r - s_i,$$

and equation 2 is readily recognized as the Thiem equation for steady-state conditions.

A PRACTICAL APPLICATION OF THE FORMULA

In the Lufkin area in Texas, the southward-dipping Carrizo sand is the principal aquifer. The outcrop of the Carrizo, north and northeast of Lufkin, is the intake area and can be regarded as a finite line source. The pumping at Lufkin is assumed to have reached equilibrium conditions, and the drawdown at Nacogdoches is desired. Figure 28 shows the relative position of all the elements of the problem: the outcrop of the Carrizo sand extends from $-c=4$ to $+c=4$ on the X-axis; the coordinate location of the pumping at Lufkin is $x_0=-3$ and $y_0=-1$, and the coordinate location of Nacogdoches is $x=-3.25$ and $y=-0.5$. Then $x_0/c=0.75$, $y_0/c=0.25$, $x/c=0.812$, and $y/c=0.125$, and corresponding values from figure 29 are $\xi_0=0.35$, $\eta_0=0.78$, $\xi=0.20$, and $\eta=0.65$. Substitution of these values in equarion 1 gives

$$s=\frac{264Q}{T}\log_{10}\left[\frac{\cosh(0.55)-\cos(-0.13)}{\cosh(-0.15)-\cos(-0.13)}\cdot\frac{\cosh(0.55)-\cos(1.43)}{\cosh(-0.15)-\cos(1.43)}\right]$$

$$=\frac{264Q}{T}\log_{10}\left[\frac{1.1551-0.9915}{1.0113-0.9915}\cdot\frac{1.1551-0.1399}{1.0113-0.1399}\right]$$

$$=\frac{264Q}{T}\log_{10} 9.64$$

$$=\frac{264Q}{T}\,0.985.$$

If the boundary of the aquifer had been assumed to be an infinite straight-line source, the drawdown would have been

$$s=\frac{264Q}{T}\log_{10}\frac{r_i^2}{r_r^2}$$

$$=\frac{264Q}{T}\log_{10}\frac{(1.5)^2+(0.25)^2}{(0.5)^2+(0.25)^2}$$

$$=\frac{265Q}{T}\log_{10} 7.4$$

$$=\frac{264Q}{T}0.87.$$

If the other factors are equal, the ratio of the required drawdown for steady-state flow in an aquifer that has finite line-source boundary to the drawdown in an aquifer that has an infinite line-source boundary is 0.985/0.87, or 1.13. Hence, as a result of pumping at Lufkin, the drawdown at Nacogdoches had the value computed by Guyton (1942) sometime during the early stages of pumping and gradually approached a value that is 13 percent greater than Guyton's computation as equilibrium was approached.

CONCLUSION

If a discharging well taps an aquifer that has a finite line-source boundary, the drawdown, after equilibrium has been reached, for any point in the vicinity can be found from equation 1. The drawdown is greater than if the aquifer had had an infinite line-source boundary and less than if it had been unbounded.

CHART FOR DETERMINATION OF THE PERCENTAGE OF PUMPED WATER BEING DIVERTED FROM A STREAM OR DRAIN

By Charles V. Theis and Clyde S. Conover

ABSTRACT

If an aquifer is connected hydraulically with a stream or drain, the discharge of a well near the stream or drain consists partly of water diverted from that surface source. Provided the hydraulic constants of the aquifer are known, that fractional part of the discharge which consists of water diverted from the stream or drain can be determined for any given time from a chart that is based on a formula derived by Theis. Certain idealizations of the aquifer system set forth by Theis apply equally to the chart and formula. This paper includes not only instructions for use of the chart but also an illustrative example.

DESCRIPTION OF THE CHART

If a water-table aquifer and a stream (or drain) are hydraulically connected, part of the discharge of a well near the stream consists of water diverted from that surface source of supply. The diversion from the stream can be either a diminution of ground-water accretion to the stream or a movement of water from the stream into the aquifer and toward the well. The theoretical percentage of the pumped water that is being diverted from the stream at any given time after pumping begins can be determined from a chart (fig. 30) if the hydraulic constants (T and S) of the aquifer and other factors are known. Unlike the chart presented by Conover (1954, fig. 15), which could be used for only one combination of hydraulic constants, this chart can be used for any combination of constants. To the extent that the lowering of the water table in areas of withdrawal permits additional recharge to the aquifer or reduces the losses due to evapotranspiration, the actual percentage of the pumped water that is from the stream will be smaller than the percentage determined from the chart. These factors, of course, require individual evaluation in specific areas of withdrawal from wells.

The chart is based on the following equation developed by Theis (1941, p. 736)

$$P=\frac{2}{\pi}\int_0^{\frac{\pi}{2}} e^{-k \sec^2 u} du,$$

in which

P=the percentage of the pumped water being diverted from the stream or drain,

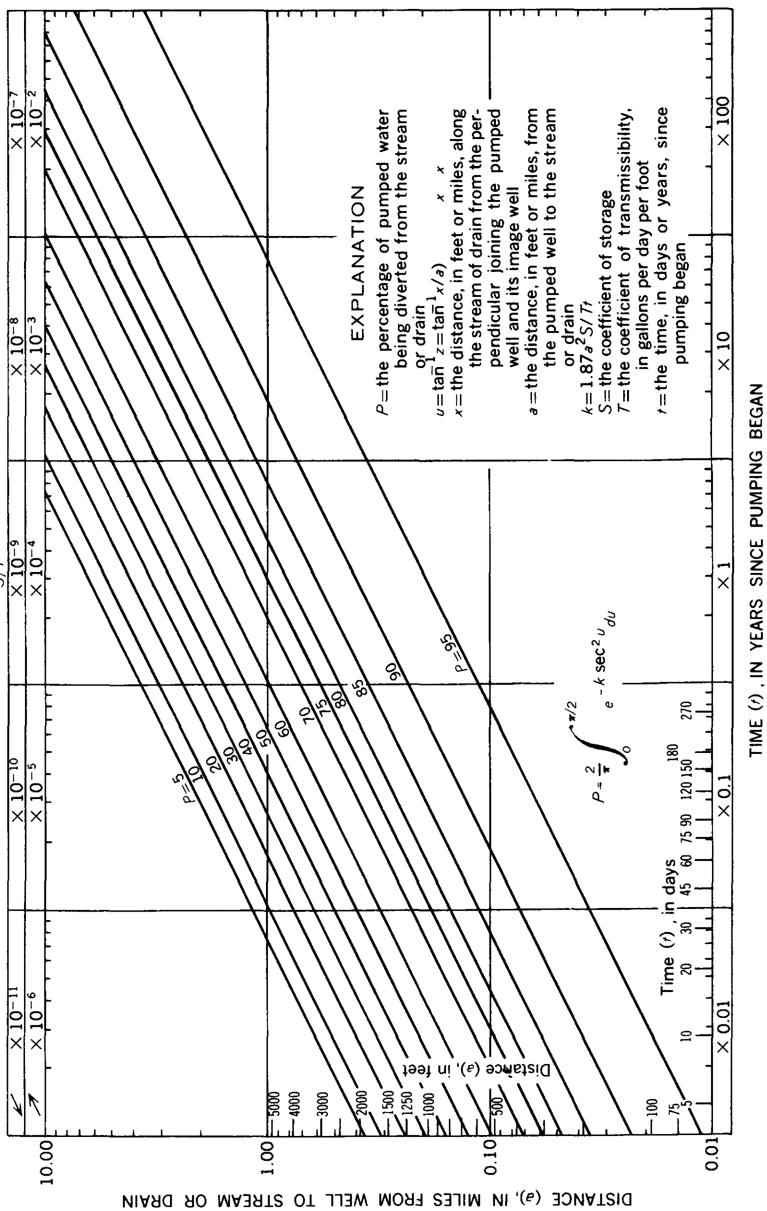

FIGURE 30.—Chart for determination of the percentage of pumped water being diverted from a stream or drain.

$u = \tan^{-1} z = \tan^{-1}(x/a)$,

x = the distance, in feet or miles, along the stream or drain from the perpendicular joining the pumped well and its image well,

a = the distance, in feet or miles, from the pumped well to the stream or drain,

$k = 1.87a^2S/Tt$,

t = the time, in days or years, since pumping began,

and T and S are as previously defined.

DIRECTIONS FOR USE OF THE CHART

The chart is used in the following manner:

1. Compute S/T and locate the value on one of the two horizontal scales at the top of the chart.

2. If the S/T value is found on the lower of the two scales, locate the value of a on the vertical scale at the left side of the chart; on the other hand, if the S/T value is found on the upper scale, locate the value of a on the vertical scale at the right side of the chart.

3. From this a value, proceed in a direction parallel to the family of diagonal lines to the vertical line through the S/T value.

4. From this point of intersection, move horizontally to the vertical line through the desired time (t) value.

5. If this latter intersection falls on one of the family of diagonal lines, the percentage of the pumped water being diverted from the surface source can be read directly; if the point falls between two lines, the percentage can be determined by simple interpolation.

The following example illustrates the above procedure. Assume a water-table aquifer that is hydraulically connected with a river and for which T=100,000 gpd per ft and S=0.20. This aquifer is tapped by a well, located 1 mile from the river, that has been pumped continuously for 1 year. To determine from the chart the percentage of the discharge being diverted from the river, the step-by-step procedure would be as described below:

1. $S/T = \frac{0.20}{100{,}000} = 2\times10^{-6}$.

2. Because this S/T value is on the lower of the two horizontal scales at the top of the chart, locate the value $a=1$ on the left-hand vertical scale.

3. From this $a=1$ value, proceed in a direction parallel to the family of diagonal lines to the vertical line through $S/T=2\times10^{-6}$.

4. From this point of intersection, move horizontally to the vertical line through the time $t=1$.

5. As this latter intersection lies about midway between the diagonal

lines labeled 40 percent and 50 percent, the percentage of water being diverted from the river is about 45 percent.

CONCLUSION

Pumping water from a well that taps an aquifer which is hydraulically connected with a stream or drain reduces the flow of surface water either by reducing the flow of ground water to the stream or drain or by causing the surface water to infiltrate the aquifer and percolate toward the well. If the hydraulic constants of the aquifer are known, the theoretical percentage of surface water in the discharge of the well at a given distance from the stream or drain and at a given time since pumping began can be determined readily through use of the chart described in this paper.

LOCUS CIRCLES AS AN AID IN THE LOCATION OF A HYDROGEOLOGIC BOUNDARY

By Edward A. Moulder

ABSTRACT

If the position of the straight line that represents the boundary of an aquifer is unknown and cannot be determined from aquifer-test data, the general area of its location can be found by determining all the possible boundary positions and by using the available geologic information. The locus of the possible locations of the image well reflected over the boundary from the discharging well is a circle that has for its center an observation well that had been used in making the aquifer test. The locus of the possible boundary reflection points (a boundary reflection point is the point where the boundary intersects the line joining the discharging well and the image well) is a circle that has a radius that is half of that of the image-well circle and a center that is midway on the line between the discharging well and the image well. The boundary through any reflection point is a line perpendicular to the line through that point and the discharging well, and the corresponding image well lies on the line through the discharging well and the boundary reflection point at the point where it intersects the image-well locus.

UNDETERMINED BOUNDARIES

Frequently the number or distribution of observation wells used in an aquifer test is not adequate for identification of the specific location of the hydrogeologic boundaries. If a boundary can be represented by a straight line of infinite extent, the general location of the boundary generally can be determined by correlating the available geologic information with possible boundary positions. The possible boundary positions can be determined by means of two circles; one circle represents the locus of all possible locations of the image well that is reflected over the boundary from the discharging well, and the other circle represents the locus of all possible boundary reflection points (a boundary reflection point is the point of intersection of the line that represents the boundary and the line that joins the discharging well and the image well).

CONSTRUCTION OF THE IMAGE-WELL AND BOUNDARY-REFLECTION-POINT LOCI

To find the locus of all possible locations of the image well, let point *P* denote the position of the discharging well and point *O* the position of an observation well, and draw the line *PO* (fig. 31). The dis-

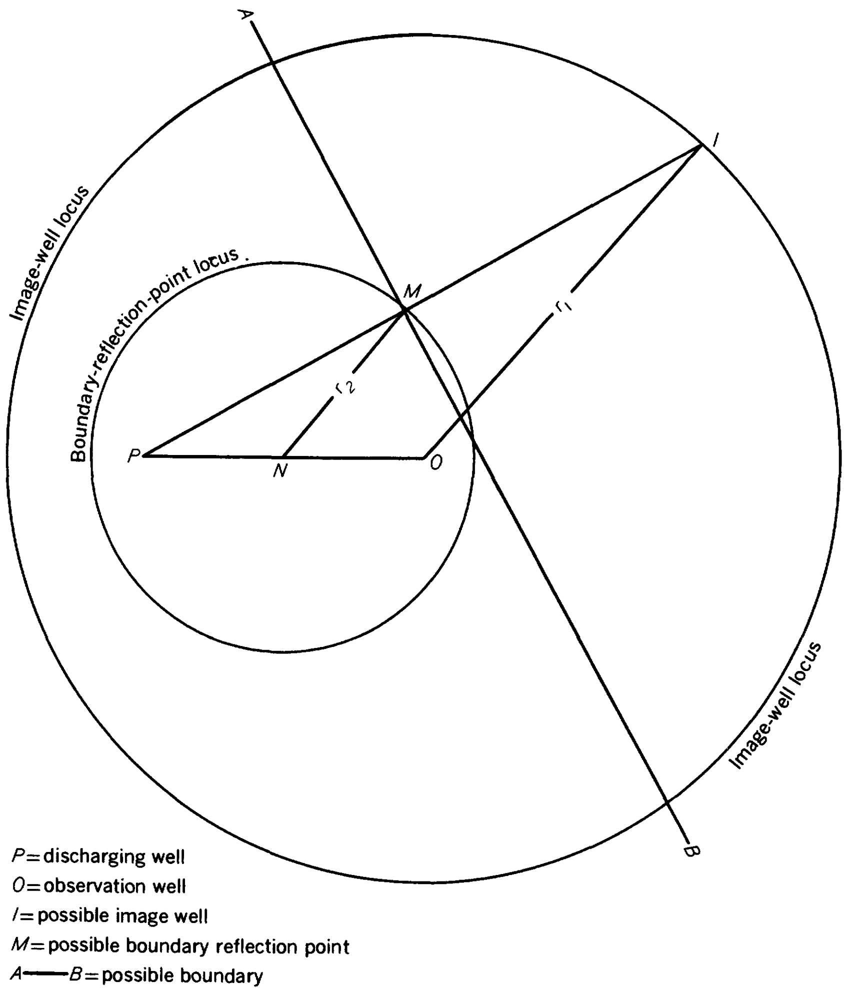

FIGURE 31.—Sketch showing the loci of the possible locations of a boundary reflection point and the corresponding image well.

tance, r_1, from the observation well O to the image well can be found from the aquifer-test data. Using the distance r_1 as the radius, draw a circle with O as the center—this circle represents the locus of all the possible locations of the image well.

To locate one possible boundary, assume that the image well is located at any point I on the circle and draw the line, PI; then, the boundary is represented by an infinite straight line, AB, that is perpendicular to *PI* at its midpoint, *M*. Hence, if *I* represents the position of the image well corresponding to the discharging well, then *M* is the required boundary reflection point and AB represents the boundary of the aquifer.

To find the locus of all possible boundary-reflection points, locate a point *N* on the line *PO* midway between *P* and *O*. With N as the center, draw a circle of radius *MN*, or r_2—this is the required locus. The boundary through any boundary reflection point, *M*, is perpendicular to *PM* at *M*, and the corresponding image well lies on the line through *PM* at the point where the line intersects the circle representing the locus of all locations of the image well.

In figure 31, line *PI* equals 2 *PM* and *PO* equals 2 *PN*, by construction; the included angle at *P* is common to triangles *PMN* and *PIO*. As the two triangles have a common angle and the sides adjacent to the angle are proportional, they are similar. From the properties of similar triangles, radius r_2 must equal one-half of radius r_1. Hence, the locus circle of all the potential boundary reflection points has a radius that is half of that of the locus circle for the corresponding locations of the image well and a center that is midway on the line between the discharging well and the observation well.

CONCLUSION

The reflection point on the straight line that represents the hydrogeologic boundary of an aquifer can be determined from the intersection of two or more boundary-reflection-point loci, drawn for two or more observation wells. The boundary passes through this point and lies along the line that is perpendicular to the line that joins this point and the discharging well.

SPACING OF WELLS

By Charles V. Theis

ABSTRACT

If a thick, areally extensive aquifer is to be tapped by several wells, consideration should be given to the most advantageous spacing of the wells. The problem is primarily economic because wells that are farther apart are more efficient, owing to less interference between the wells, but the cost of the connecting pipeline and electrical installations is less when the wells are closer together. The distance between wells that will result in the minimum cost of both installation and operation can be determined mathematically and is found to depend on the pumping rate of the wells and the transmissibility of the aquifer. However, latitude should be allowed for convenience in spacing the wells and for other factors that cannot be evaluated quantitatively.

THE ECONOMIC PROBLEM

In a thick aquifer, the problem of properly spacing wells generally is one of economics; less interference occurs if the wells are farther apart, but the cost of installation and operation of the wells is much greater. The cost, insofar as it is affected by the spacing of the wells, may be reduced to an annual unit charge consisting of (a) the cost of lifting the water against the additional drawdown caused by the interference in each well from the other discharging wells, and (b) the cost of the pipeline connecting the wells and of the electrical installations, including the maintenance, depreciation, and capital charges on the cost of installation.

COST ANALYSIS

The determination of the best spacing for two wells that are to be pumped at the same rate from an areally extensive aquifer is not difficult. To find the spacing necessary for operation at minimum cost, consider the average cost values over a year's time. Let

s' = the drawdown, in feet, in one discharging well caused by the pumping of the other well;

Q' = the pumping rate of each well, in gallons per minute;

r' = the distance, in feet, between the two discharging wells;

c = the cost, in dollars, to raise a gallon of water 1 foot, consisting largely of power charges but also including some additional charges on the equipment;

k = the capitalized cost, in dollars per year per foot, for the maintenance, depreciation, original cost of the pipeline, etc.;

C = the total yearly cost, in dollars.

Then, the general expression for the total yearly cost for the operation of the two wells can be written in the form

$$C = 2 \times 365 \times 1{,}440 \; cQ's' + kr'$$
$$= 1.05 \times 10^6 \; cQ's' + kr'. \qquad (1)$$

In the vicinity of the discharging wells, steady-state conditions can be assumed to have prevailed throughout most of the 1-year pumping period. Thus, the drawdown in either discharging well that results from the pumping of the other well is given by the following approximate expression (Cooper and Jacob, 1946):

$$s' = \frac{114.6Q'}{T}\left(-0.577 - \log_e \frac{1.87r'^2S}{Tt}\right)$$
$$= \frac{114.6Q'}{T}\left(-0.577 - \log_e \frac{1.87S}{Tt} - \log_e r'^2\right). \qquad (2)$$

Substituting this value for s' into equation 1 gives

$$C = 1.05 \times 10^6 \left[\frac{114.6Q'^2c}{T}\left(-0.577 - \log_e \frac{1.87S}{Tt} - 2\log_e r'\right)\right] + kr'$$
$$= \frac{1.20 \times 10^8 Q'^2c}{T}\left(-0.577 - \log_e \frac{1.87S}{Tt} - 2\log_e r'\right) + kr'. \qquad (3)$$

The minimum cost will occur for that distance at which the first derivative of C with respect to r' equals zero. Differentiating equation 3 and equating the resulting expression to zero yields

$$\frac{dC}{dr'} = \frac{1.2 \times 10^8 Q'^2c}{T}\left(\frac{-2}{r'}\right) + k = 0,$$

from which the optimum well spacing, r_o', is found to be

$$r_o' = \frac{2.4 \times 10^8 Q'^2c}{kT}. \qquad (4)$$

As a final test of the propriety of using the approximate relation given by equation 2, the value of u, as computed from the relation

$$u = \frac{1.87 r_o'^2 S}{Tt},$$

should prove to be less than 0.02.

To illustrate the use of equation 4, assume the availability of electrical power to be 1½ cents per kilowatt-hour and an overall efficiency of 50 percent. The power charge, c, for lifting water will then be about 1×10^{-7} dollars per gallon per foot. Assuming the cost of pipeline and electric wiring to be \$10 per foot and capitalizing this at 10 percent gives a cost, k, of \$1 per foot per year for capital charges, depreciation, and maintenance. If the coefficient of transmissibility, T, for the aquifer is assumed to be 50,000 gpd per ft and the discharge, Q', to be 500 gpm from each well, then

$$r_o' = \frac{2.4 \times 10^8 \times 10^{-7} \times 25 \times 10^4}{1 \times 5 \times 10^4} = 120 \text{ feet.}$$

Inspection of equation 4 shows that, if the cost factors, c and k, are constant, the optimum spacing, r_o', of two discharging wells varies directly with the square of the pumping rate, Q', and inversely with the coefficient of transmissibility, T. Thus, r_o' will be greater for an aquifer of low transmissibility than for an aquifer of high transmissibility; or, within the same aquifer, r_o' will be greater for a high pumping rate than for a low rate. The only aquifer property involved in determining r_o' is seen to be the transmissibility. Hence, for equal pumping rates, the values of r_o' will be identical for water-table and artesian aquifers of the same transmissibility; however, as equation 3 shows, the total annual cost for pumping will be less for a water-table aquifer, which has a comparatively large S, than for an artesian aquifer, which has a small S.

The graph of corresponding values of the total annual cost and the distance between the two discharging wells is a curve that is relatively flat in the neighborhood of the minimum value; placing wells at distances that are somewhat more or less than that found by equation 4 would not appreciably increase the cost.

CONCLUSION

The spacing of wells in extensive aquifers generally is governed by the convenience of operation as well as by the hydrogeologic conditions in the area. An analysis, in which appropriate hydraulic constants are used, should be made whenever comparable field problems require investigation of the criteria for spacing wells, but considerable latitude should be allowed for convenience, security, and other factors that cannot be evaluated quantitatively. Obviously, if aquifer boundaries are present or if more than two wells are to be pumped, the analysis will have to be modified to take into account the added conditions.

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