Relationships between Basic Soils-Engineering Equations and Basic Ground-Water Flow Equations

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SYMBOLS USED IN THIS TEXT

Symbol	*Dimension*	*SI unit*	*Description*
B	L	m	Total thickness of layers $b_1 + b_2 + \ldots b_n$
$B.E.$	----	----	Barometric efficiency
C_c	----	----	Compression index
E_k	FL^{-2}	Pa	Constrained modulus of elasticity of structure skeleton of medium (assume inelastic range unless otherwise stated)
E_{system}	FL^{-2}	Pa	Elasticity of layered system
E_t	FL^{-2}	Pa	Total elasticity of medium
E_{t1}	FL^{-2}	Pa	Total elasticity of layer 1
E_{t2}	FL^{-2}	Pa	Total elasticity of layer 2
E_{tn}	FL^{-2}	Pa	Total elasticity of n^{th} layer
E_w	FL^{-2}	Pa	Bulk modulus of elasticity of water
H	L	m	Thickness of sample used in consolidation test
H_o	L	m	Stepwise head change at time zero
H_{dr}	L	m	Longest drainage path
K	LT^{-1}	$m \cdot s^{-1}$	Hydraulic conductivity
K_{system}	LT^{-1}	$m \cdot s^{-1}$	Hydraulic conductivity (horizontal) of layered system
K'	LT^{-1}	$m \cdot s^{-1}$	Vertical hydraulic conductivity
K'_{system}	LT^{-1}	$m \cdot s^{-1}$	Vertical hydraulic conductivity of system
M_w	M	kg	Mass of water
S	----	----	Storage coefficient of aquifer
S_k	----	----	Storage coefficient due to elasticity of aquifer skeleton
S_s	L^{-1}	m^{-1}	Specific storage
S_{sk}	L^{-1}	m^{-1}	Specific storage due to elasticity of aquifer skeleton
S_{sw}	L^{-1}	m^{-1}	Specific storage due to elasticity of water
$S_{s\ system}$	L^{-1}	m^{-1}	Specific storage of layered system
S_w	----	----	Storage coefficient due to elasticity of water
T	L^2T^{-1}	$m^2 \cdot s^{-1}$	Transmissivity
T_{system}	L^2T^{-1}	$m^2 \cdot s^{-1}$	Transmissivity of layered system
V_s	L^3	m^3	Volume of solids
V_t	L^3	m^3	Volume, total

Symbol	*Dimension*	*SI unit*	*Description*
V_v	L^3	m^3	Volume of voids
V_w	L^3	m^3	Volume of water
$T.E.$	----	----	Tidal efficiency
a_v	L^2F^{-1}	Pa^{-1}	Coefficient of compressibility
b	L	m	Thickness of aquifer
b_1	L	m	Thickness layer 1
b_2	L	m	Thickness layer 2
b_n	L	m	Thickness of n^{th} layer
c_v	L^2T^{-1}	$m^2 \cdot s^{-1}$	Coefficient of consolidation
e	----	----	Void ratio
e_o	----	----	Initial void ratio
g	LT^{-2}	$m \cdot s^{-2}$	Standard acceleration due to gravity
h	L	m	Head, total
h_s	L	m	Steady-state potentiometric head
h'	L	m	Excess head
h'_1	L	m	Excess head at time 1
m_v	L^2F^{-1}	Pa^{-1}	Coefficient of volume compressibility
n	----	----	Porosity
p	FL^{-2}	Pa	Pressure
p_a	FL^{-2}	Pa	Pressure artesian
p_l	FL^{-2}	Pa	Force resulting from loading
q	L^3T^{-1}	$m^3 \cdot s^{-1}$	Volumetric flow rate
t	T	s	Time
t_o	T	s	Time zero or initial time
t_1	T	s	Elapsed time since initial conditions
t_{50c}	T	minutes	Time, in minutes, for 50 percent of primary consolidation
t_{90c}	T	minutes	Time, in minutes, for 90 percent of primary consolidation
v_w	LT^{-1}	$M \cdot s^{-1}$	Rate of flow of water per unit area (specific discharge)
u	FL^{-2}	Pa	Excess pore pressure
x	L	m	Cartesian coordinate
y	L	m	Cartesian coordinate
z	L	m	Cartesian coordinate or elevation
γ_w	FL^{-3}	$Pa \cdot m^{-1}$ or $N \cdot m^{-3}$	Unit weight of water
Δ	----	----	Finite difference change and unit length elemental volume
ϵ_z	----	----	Confined strain in vertical direction
ρ_w	ML^{-3}	$kg \cdot m^{-3}$	Density of water
s_{z-}	FL^{-2}	Pa	Effective stress (vertical)
T	----	----	Time factor (dimensionless time)

SI UNITS AND CONVERSIONS

The International System of Units (SI units) is a modernized metric system. The unit of time is s (seconds). The unit of mass is kg (kilogram), and the unit of length is m (meter). The unit of force is N (newton) and is that force which gives a mass of 1 kilogram an acceleration of 1 meter per second per second. The unit of pressure or stress is Pa (pascal) which is N/m² (newton per square meter).

SI units may use prefixes, including the following:

tera	T	10^{12}	kilo	k	10^{3}	micro	μ	10^{-6}
giga	G	10^{9}	milli	m	10^{-3}	nano	n	10^{-9}
mega	M	10^{6}				pico	p	10^{-12}

A similar metric system is the CGS system, which is based on units of centimeters, grams, and seconds. Units found in the CGS system include erg, dyne, poise, and stokes. CGS units will not be used in this report.

The following conversions may be useful to hydrologists and soils engineers:

atmosphere (atm)
= 1.013×10^{5} Pa

centipoise
= 1×10^{-3} Pa·s

centistoke
= 1×10^{-6} $m^{2}\cdot s^{-1}$

foot (ft)
= 3.048×10^{-1} m

foot of water (4°C)
= 2.989×10^{3} Pa

cubic foot per second (ft³/s)
= 2.832×10^{-2} $m^{3}\cdot s^{-1}$

gallon (gal)
= 3.785×10^{-3} m^{3}

gallon per minute (gal/min)
= 6.309×10^{-5} $m^{3}\cdot s^{-1}$

inch (in)
= 2.540×10^{-2} m

inch of mercury (0°C)
= 3.386×10^{3} Pa

inch of water (4°C)
= 2.491×10^{2} Pa

kilogram force (kgf)
= 9.807×10^{0} N

kilogram force per centimeter squared (kgf/cm²)
= 9.807×10^{4} Pa

kilogram force per meter squared (kgf/m²)
= 9.807×10^{0} Pa

pound (lb)
= 4.536×10^{-1} kg

pound force (lbf)
= 4.448×10^{0} N

pound force per foot squared (lbf/ft²)
= 4.788×10^{1} Pa

pound force per inch squared (lbf/in²) (psi)
= 6.895×10^{3} Pa

liter fluid or gas (L)
= 1.000×10^{-3} m^{3}

density of water (ρ_w at 4°C)
= 1×10^{3} $kg\cdot m^{-3}$

free fall (g)
= 9.807×10^{0} $m\cdot s^{-2}$

unit weight of water (γ_w)
= 9.806×10^{3} $Pa\cdot s^{-1}$

K (units of ft/s)
= 3.048×10^{-1} $m\cdot s^{-1}$

K (units of m/s)
= 1×10^{0} $m\cdot s^{-1}$

c_v (units of inches²/hour)
= 1.79×10^{1} $m^{2}\cdot s^{-1}$

c_v (units of centimeters²/day)
= 1.16×10^{-3} $m^{2}\cdot s^{-1}$

c_v (units of m²/s)
= 1×10^{0} $m^{2}\cdot s^{-1}$

RELATIONSHIPS BETWEEN BASIC SOILS-ENGINEERING EQUATIONS AND BASIC GROUND-WATER FLOW EQUATIONS

By DONALD G. JORGENSEN

ABSTRACT

The many varied though related terms developed by ground-water hydrologists and by soils engineers are useful to each discipline, but their differences in terminology hinder the use of related information in interdisciplinary studies. Equations for the Terzaghi theory of consolidation and equations for ground-water flow are identical under specific conditions. A combination of the two sets of equations relates porosity to void ratio and relates the modulus of elasticity to the coefficient of compressibility, coefficient of volume compressibility, compression index, coefficient of consolidation, specific storage, and ultimate compaction. Also, transient ground-water flow is related to coefficient of consolidation, rate of soil compaction, and hydraulic conductivity. Examples show that soils-engineering data and concepts are useful to solution of problems in ground-water hydrology.

INTRODUCTION

The study of ground-water flow in porous media and certain aspects of soils-engineering studies are related, but a comparison of the terminology used in these fields does not reveal a common area of investigation. The wide disparity in the terminology for common concepts is illustrated by the use of different terms to express the volume of pore space in a porous medium. Other differences in terminology appear throughout the literature and especially in the descriptions of the compressibility of soils. Each discipline has useful concepts expressed as indices, moduli, or coefficients; but unfortunately for those involved in interdisciplinary studies, many of the terms are not interchangeable and do not allow the use of all available data. Another difficulty in using the data results from the employment of unique units of measurement in each discipline, such as gallons per day per foot or tons per square inch. In addition, the constants used in the equations are cumbersome to use in interdisciplinary studies.

The purpose of this paper is to relate some of the common or similar terms and concepts used in soils engineering to similar terms and concepts used in ground-water hydraulics and to illustrate how the equations can be used in a ground-water investigation.

The derivations of the ground-water equations presented here are similar to those presented by Jacob (1950, chap. 5). The derivations of the soils-engineering equations are similar to the presentation by Terzaghi and Peck (1948). For more complete derivations of the equations, the reader should consult these references. Scott (1963) presents an excellent derivation of equations that are useful in both fields. Definitions of many of the terms used in this paper are listed by Poland and others (1972).

The terms and symbols selected for discussion are not presumed to be an ultimate choice of terminology but were selected to familiarize the reader with related concepts of each field. For example, the terms porous medium, aquifer, and soil are used almost synonomously. The term chosen is generally the term commonly used in the discipline that most often collects that particular type of data.

BASIC PRINCIPLES AND TERMINOLOGY

Consider a small elemental volume of a porous medium (fig. 1) with flow across all faces. It would be useful to write a mass-balance equation for a small interval of time (Δt) which would specify flow across all boundaries.

The mass-balance equation is

$$[\text{mass leaving} - \text{mass entering}] = -[\text{final mass} - \text{initial mass}], \tag{1}$$

or symbolically,

$$\begin{aligned}&[(\rho q)_{x+\Delta x}\Delta t + (\rho q)_{y+\Delta y}\Delta t + (\rho q)_{z+\Delta z}\Delta t] - \\ &[(\rho q)_x \Delta t + (\rho q)_y \Delta t + (\rho q)_z \Delta t] = - \\ &[(\Delta x \Delta y \Delta z)(n\rho)_{t+\Delta t} - (\Delta x \Delta y \Delta z)(n\rho)_t],\end{aligned} \tag{1a}$$

where

ρ = density of water

q = volumetric flow rate,

$(\rho q)_x$ = mass flow rate at $\Delta y - \Delta z$ plane or at x face,

n = porosity,

$(\Delta x \Delta y \Delta z)(n\rho)_{t+\Delta t}$ = mass of fluid in element after interval Δt,

$(\Delta x \Delta y \Delta z)(n\rho)_t$ = mass of fluid in element at time t.

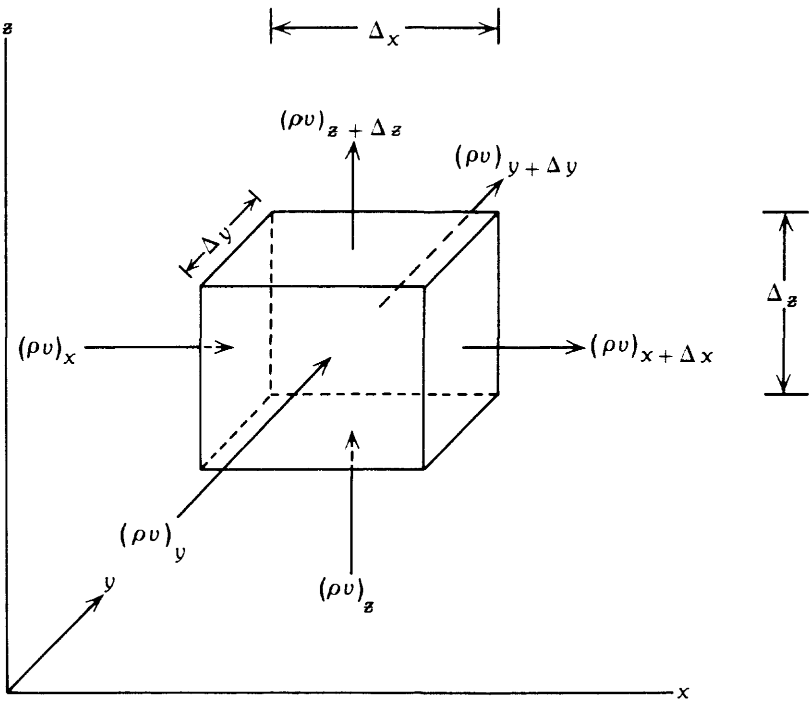

FIGURE 1.—Elemental volume of a saturated porous medium.

Rearranging the left hand term of equation 1a and dividing by $\Delta x \Delta y \Delta z \Delta t$ and noting that $(\Delta x \Delta y \Delta z)$ $(n\rho)$ equals the mass of water yields

$$-\frac{[(\rho q)_{x+\Delta x}-(\rho q)_x]}{\Delta x \Delta y \Delta z}-\frac{[(\rho q)_{y+\Delta y}-(\rho q)_y]}{\Delta x \Delta y \Delta z}-\frac{[(\rho q)_{z+\Delta z}-(\rho q)_z]}{\Delta x \Delta y \Delta z}$$
$$=\frac{(\Delta M_{w\ t+\Delta t}-\Delta M_{w\ t})}{\Delta x \Delta y \Delta z \Delta t}. \tag{1b}$$

The velocity across each face is

$$v_x=\frac{q_x}{\Delta y \Delta z},\ v_y=\frac{q_y}{\Delta x \Delta z},\ \text{and}\ v_z=\frac{q_z}{\Delta x \Delta y}; \tag{2}$$

that is, velocity is volume rate across each face divided by area of the face. Substituting equation 2 into the left side of equation 1b results in

$$-\left[\frac{(\rho v)_{x+\Delta x}-(\rho v)_x}{\Delta x}\right]-\left[\frac{(\rho v)_{y+\Delta y}-(\rho v)_y}{\Delta y}\right]-\left[\frac{(\rho v)_{z+\Delta z}-(\rho v)_z}{\Delta z}\right]$$
$$=\frac{\Delta M_{w\ t+\Delta t}-\Delta M_{w\ t}}{\Delta x \Delta y \Delta z \Delta t} \tag{3}$$

Noting the form of a derivative as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{y(x+\Delta x) - y(x)}{\Delta x},$$

and taking Δ to the limit, equation 3 becomes

$$-\frac{\delta(\rho v_x)}{\delta x} - \frac{\delta(\rho v_y)}{\delta y} - \frac{\delta(\rho v_z)}{\delta z} = \frac{1}{\Delta x \Delta y \Delta z} \frac{\delta(\Delta M_w)}{\delta t}. \tag{3a}$$

Expanding the derivatives,

$$-\frac{\rho \delta v_x}{\delta x} - \frac{\rho \delta v_y}{\delta y} - \frac{\rho \delta v_z}{\delta z} - \left[\frac{v_x \delta \rho}{\delta x} + \frac{v_y \delta \rho}{\delta y} + \frac{v_z \delta \rho}{\delta z}\right] = \frac{1}{\Delta x \Delta y \Delta z} \frac{\delta(\Delta M_w)}{\delta t}. \tag{3b}$$

The terms within the brackets in equation 3b are not considered in many ground-water problems because normally their net effect is relatively very small. Thus, equation 3b reduces to:

$$-\frac{\rho \delta v_x}{\delta x} - \frac{\rho \delta v_y}{\delta y} - \frac{\rho \delta v_z}{\delta z} = \frac{1}{\Delta x \Delta y \Delta z} \frac{\delta(\Delta M_w)}{\delta t}. \tag{3c}$$

The change of mass as expressed in equation 1 can be written as

$$\frac{\delta(\Delta M_w)}{\delta t} = \left[n \rho_w \frac{\delta(\Delta z)}{\delta t} + \rho_w \Delta z \frac{\delta n}{\delta t} + n \Delta z \frac{\delta \rho_w}{\delta t}\right] \Delta x \Delta y. \tag{4}$$

Equation 4 assumes vertical movement in the z direction and negligible movement in the x and y directions. It also assumes that the soil (porous medium) has no flexural strength and that the load is transferred downward undiminished. (The two assumptions are not completely correct in that horizontal, as well as vertical, strains have been measured near discharging wells. Also, most aquifer material, even though unconsolidated, has some flexural strength.)

Porosity (n) is the ratio of the volume of voids (V_v) to the total volume (V_t), or

$$n = \frac{V_v}{V_t}. \tag{5}$$

The constrained modulus of elasticity (E_k) is defined as the ratio of vertical effective stress (s_z), which is total stress minus artesian pressure, to laterally confined strain (ϵ_z), or

$$E_k = \frac{s_z}{\epsilon_z}. \tag{6}$$

Because of the assumptions of confined and undiminished stress the strain is also assumed to be only vertical, or the confined strain is the ratio of the change in height $\Delta(\Delta z)$ to the original height (Δz_o); thus, equation 6 can be expressed as

$$E_k = \frac{\Delta s_z}{-\Delta(\Delta z)/\Delta z_o} . \tag{6a}$$

The constrained modulus of elasticity defined in equations 6 and 6a is symbolized as D by some authors (Lambe and Whitman, 1969, p. 151–161).

Jacob (1950, p. 329) used α, which he defined as the reciprocal of the modulus of elasticity of the porous medium. Jacob also used β as the reciprocal of the bulk modulus of elasticity of water.

For liquids and gases, the bulk modulus of elasticity is the ratio of the change in pressure to the corresponding ratio change of volume to original volume. The change of volume results from change in density. For example, the bulk modulus of elasticity of water (E_w) is

$$E_w = \frac{-\Delta p}{\Delta V_w/V_w} = \frac{\Delta p}{\Delta \rho_w/\rho_w} , \tag{7}$$

where Δp = change of water pressure, and ΔV_w = change of volume of water.

The height of the elemental volume shown in figure 1 changes with the vertical effective stress (s_z) as follows:

$$d(\Delta z) = \frac{-\Delta z_o}{E_k} ds_z , \tag{8}$$

where Δz_o is the initial height of the element.

Because the compression of grains is very small in an unconsolidated medium, the change in volume must be largely due to the change in porosity. Assuming that the volume of solid material (V_s) remains constant, then

$$\Delta V_s = (1-n)(\Delta x \Delta y \Delta z) = \text{constant}, \tag{9}$$

and

$$d(\Delta V_s) = [(1-n)\, d(\Delta z) - \Delta z\, dn]\Delta x \Delta y = 0. \tag{9a}$$

Thus,

$$(\Delta z)\, dn = (1-n)\, d(\Delta z), \tag{9b}$$

or

$$d(\Delta z) = \frac{\Delta z}{1-n} dn. \tag{9c}$$

Equation 9c can be used to calculate ultimate compaction $[d(\Delta z)]$ if the original porosity (n_o), change in porosity (dn), and the original height of the element (Δz_0) are known. Combining and rearranging equations 8 and 9c yields

$$dn = \frac{-(1-n)}{E_k} ds_z. \tag{10}$$

Differentiating equation 10 with respect to time gives

$$\frac{\delta n}{\delta t} = \frac{-(1-n)}{E_k} \frac{\delta s_z}{\delta t}. \tag{10a}$$

Equation 10a states that the rate of change of porosity is a function of the elasticity of the soil and the rate of change of the stress.

Figure 2 shows the forces that act on a unit area in the plane of contact between a confining layer and an aquifer. At the contact, the downward force due to loading conditions (p_l) is equal to the sum of upward force of the artesian pressure (p_a) and the effective stress (s_z) borne by the aquifer skeleton by grain to grain contact (Poland and others, 1972, p. 6). Soils engineers commonly use the term pore pressure, which is conceptually identical with artesian pressure as used here. If the loading remains constant and if the artesian pressure is decreased, the effective stress will increase. Referring to figure 2 and the definition of effective stress, it follows that

$$p_a + s_z = p_l = \text{constant}, \tag{11}$$

from which it follows that

$$dp_a = -ds_z. \tag{12}$$

Domenico (1972, p. 213–216) gives a procedure of constructing stress diagrams to graphically show effective stress in aquifers.

Dropping the subscript a, the relationship between the change in thickness of the porous medium due to the change in pressure may be determined by substituting $-dp$ for ds_z in equation 8 and differentiating with respect to time:

$$\frac{\delta(\Delta z)}{\delta t} = \frac{\Delta z_0}{E_k} \frac{\delta p}{\delta t}. \tag{13}$$

The rate of change in porosity resulting from the change in pressure can be determined by utilizing equation 12 to modify equation 10a:

$$\frac{\delta n}{\delta t} = \frac{(1-n)}{E_k} \frac{\delta p}{\delta t}. \tag{14}$$

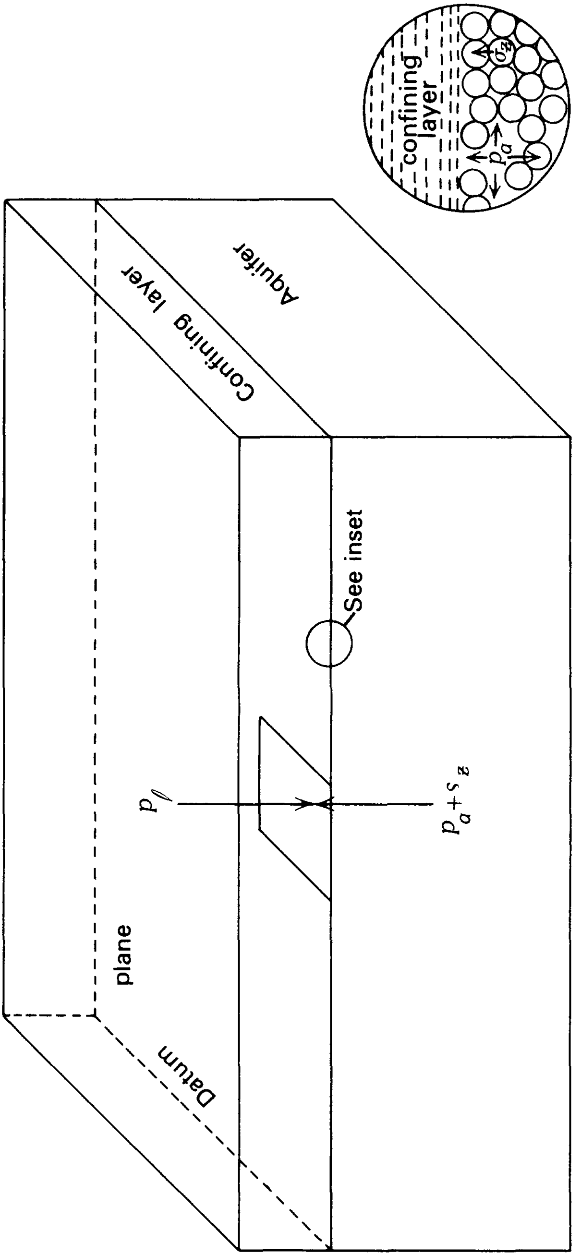

FIGURE 2.—Forces on a subsurface plane.

EXAMPLE 1

Problem:

A sand layer underlies a silt layer which is 30 m thick. The water table in the silt is at a depth of 4.0 m. A transducer is in a sand layer just below the silt layer. The transducer indicates that the pore pressure within the sand is 3.4×10^5 Pa. The dry bulk density of the silt is 1.6×10^3 kg/m³. Estimate the porosity of the silt and calculate the effective stress in the skeleton of sand layer.

Solution:

The densities of the grain materials of most clays, silts, sands, and gravels range from 2.4×10^3 kg/m³ to 2.8×10^3 kg/m³. For many estimates, a density of 2.65×10^3 kg/m³ can be assumed.

First, to determine the porosity of the silt, the mass of the silt must equal the mass of the voids plus the mass of the solids, or

$$V_t \rho_t = V_v \rho_v + V_s \rho_s. \tag{a}$$

Thus,

$$V_t(1.6 \times 10^3 \text{kg/m}^3) = V_v\,(0.00\ \text{kg/m}^3) + V_s\,(2.65 \times 10^3\ \text{kg/m}^3).$$

Assuming a unit volume,

$$V_s = (1.00\ \text{m}^3)\,\frac{(1.6 \times 10^3\ \text{kg/m}^3)}{(2.65 \times 10^3\ \text{kg/m}^3)} = 0.60\ \text{m}^3.$$

Because volume total equals the volume of voids plus the volume of the solids,

$$V_v = V_t - V_s = 1.0 - 0.60 = 0.40.$$

Porosity is equal to ratio of the volume of voids to volume of solids, or

$$n = \frac{V_v}{V_t} = \frac{0.40\ \text{m}^3}{1.00\ \text{m}^3} = 0.40.$$

Effective stress is that portion of the total pressure or load which is not borne by artesian pressure. The total pressure is the load of the overlying saturated silt below the water table and the silt above the water table to the ground level. The bulk density for the materials above the water table is given as 1.6×10^3 kg/m³. Thus, the load from the silt above the water table is calculated as

$$p_l = \rho g h, \text{ or}$$

$$p_l = (1.6 \times 10^3\ \text{kg/m}^3)\,(9.807\ \text{m/s}^2)\,(4.0\ \text{m}) = 6.3 \times 10^4\ \text{Pa}. \tag{b}$$

The calculated density of the saturated silt layer using equation (a) is

$$\rho_t = \frac{(V_v \rho_v + V_s \rho_s)}{V_t} = \frac{(0.40)(1.0\times 10^3\ \text{kg/m}^3) + (0.60)(2.65\times 10^3\ \text{kg/m}^3)}{1.00}$$

$$= 2.0\times 10^3\ \frac{\text{kg}}{\text{m}^3}\ .$$

The load caused by the saturated silt is

$$p_l = \rho_t\, gh = (2.0\times 10^3\ \frac{\text{kg}}{\text{m}^3})\ (9.807\ \frac{\text{m}}{\text{s}^2})\ (26.0\ \text{m}) = 5.1\times 10^5\ \text{Pa}.$$

Thus, total load is

$$p_l = 6.3\times 10^4\ \text{Pa} + 5.1\times 10^5\ \text{Pa} = 5.7\times 10^5\ \text{Pa}.$$

The artesian pressure at the top of the sand is given as 3.4×10^5 Pa. The effective stress from equation 11 is

$$s_z = p_l - p_a = 5.7\times 10^5\ \text{Pa} - 3.4\times 10^5\ \text{Pa} = 2.3\times 10^5\ \text{Pa}.$$

It is useful to establish the relation between the change in volume and the change in effective stress; we again make use of the assumption that the change in volume of the soil grains is small in comparison to the change in the volume of water. Therefore,

$$dV_t = dV_w\ , \tag{15}$$

or dividing by V_w gives

$$\frac{dV_w}{V_w} = \frac{dV_t}{V_w}\ . \tag{15a}$$

Recalling the definition of porosity and noting that volume of water equals volume of voids for a saturated soil,

$$V_w = nV_t\ . \tag{16}$$

Substituting equation 16 into equation 15a gives:

$$\frac{dV_w}{V_w} = \frac{dV_t}{nV_t}\ . \tag{16a}$$

Because the change in Δx and Δy is assumed to be negligible,

$$d\ V_t = \Delta x\ \Delta y\ d(\Delta z), \tag{16b}$$

and

$$\frac{d\,V_t}{V_t} = \frac{\Delta x\,\Delta y d(\Delta z)}{\Delta x\,\Delta y\,\Delta z} = \frac{d(\Delta z)}{\Delta z}. \tag{16c}$$

Equation 8 can be rewritten as

$$d\,V_t = \frac{-V_t}{E_k}\,ds_z\,. \tag{17}$$

Equation 16a can be used with equation 17 to form

$$d\,s_z = -E_k n\,\frac{dV_w}{V_w}. \tag{18}$$

GROUND-WATER FLOW EQUATIONS

Substituting equations 4, 8, 13, 14, and 7 into equation 3a, assuming $\rho_w \cong \rho_{wo}$ and $\Delta z_o \cong \Delta z$, and taking Δ to the limits, the mass-balance equation is

$$-\frac{\delta(\rho_w v_x)}{\delta x} - \frac{\delta(\rho_w v_y)}{\delta y} - \frac{\delta(\rho_w v_z)}{\delta z} = \rho_w n\left(\frac{1}{E_w} + \frac{1}{E_k n}\right)\frac{\delta p}{\delta t}. \tag{19}$$

Equation 19 may be considered as a general differential equation for ground-water flow in porous medium. The equation was first derived from purely hydraulic principles by Jacob (1950, chap. 5).

Darcy's law, as applied to isotopic porous medium (hydraulic conductivity equal and constant in all directions) is often stated as

$$v_x = -K\frac{\delta h}{\delta x},\ v_y = -K\frac{\delta h}{\delta y},\ \text{and}\ v_z = -K\frac{\delta h}{\delta z},\ \text{where}\ h = z + \frac{p}{\gamma}. \tag{20}$$

The partial derivatives are

$$\frac{\delta v_x}{\delta x} = -K\frac{\delta^2 h}{\delta x^2},\ \frac{\delta v_y}{\delta y} = -K\frac{\delta^2 h}{\delta y^2},\ \text{and}\ \frac{\delta v_z}{\delta z} = -K\frac{\delta^2 h}{\delta z^2}. \tag{20a}$$

The partial derivative, as shown by Jacob (1950, p. 332), for water density (ρ_w) in most normal conditions can be shown to be

$$\frac{\delta \rho_w}{\delta x} \approx \frac{\rho_w^2 g h}{E_w}\frac{\delta h}{\delta x},\ \frac{\delta \rho_w}{\delta y} \approx \frac{\rho_w^2 g}{E_w}\frac{\delta h}{\delta y},\ \text{and}\ \frac{\delta \rho_w}{\delta z} \approx \frac{\rho_w^2 g}{E_w}\left(\frac{\delta h}{\delta z} - 1\right). \tag{21}$$

Substitution of equations 20 and 21 into equation 19 gives

$$\left[K\left(\frac{\delta^2 h}{\delta x^2} + \frac{\delta^2 h}{\delta y^2} + \frac{\delta^2 h}{\delta z^2}\right)\right] + \frac{K\rho_w g}{E_w}\left(\frac{\delta h}{\delta x}\right)^2 + \left(\frac{\delta h}{\delta y}\right)^2 + \left(\frac{\delta h}{\delta z}\right)^2 - \left(\frac{\delta h}{\delta z}\right)\Big] = n\left(\frac{1}{E_w} + \frac{1}{E_k n}\right)\frac{\delta p}{\delta t}. \tag{22}$$

For essentially horizontal flow in saturated materials, the second term on the left side of equation 22 is often very small as compared to the first term and is neglected (Jacob, 1950). Equation 22 thus reduces to

$$K\left[\frac{\delta^2 h}{\delta x^2}+\frac{\delta^2 h}{\delta y^2}+\frac{\delta^2 h}{\delta z^2}\right]=\left(\frac{n}{E_w}+\frac{1}{E_k}\right)\frac{\delta p}{\delta t}, \tag{23}$$

which is a common form of the equation for ground-water flow. Recall that total head at a point in an aquifer is

$$h=z+\frac{p}{\rho_w g}, \tag{23a}$$

where z is elevation head, which is equal to the elevation of the point above a datum, and $\frac{p}{\rho_w g}$ is pressure head, which is the height of a column of static water that can be supported by the static pressure at the point. Then we note that

$$\frac{\delta h}{\delta t}=\frac{1}{\rho_w g}\,\frac{\delta p}{\delta t}=\frac{1}{\gamma_w}\,\frac{\delta p}{\delta t}. \tag{23b}$$

Substituting 23b into 23 and rearranging gives a common form of the flow equation:

$$\frac{\delta^2 h}{\delta x^2}+\frac{\delta^2 h}{\delta y^2}+\frac{\delta^2 h}{\delta z^2}=\frac{n\gamma_w b}{Kb}\left(\frac{1}{E_w}+\frac{1}{E_k n}\right)\frac{\delta h}{\delta t}, \tag{24}$$

where b = thickness of the aquifer or soil layer.

If steady-state conditions exist, there is no change in pressure or head with time; therefore,

$$K\left[\frac{\delta^2 h}{\delta x^2}+\frac{\delta^2 h}{\delta y^2}+\frac{\delta^2 h}{\delta z^2}\right]=0, \tag{25}$$

or

$$\frac{\delta^2 h}{\delta x^2}+\frac{\delta^2 h}{\delta y^2}+\frac{\delta^2 h}{\delta z^2}=0. \tag{25a}$$

Equation 25 is one of the common forms used to express ground-water flow for steady-state conditions.

Equation 24 can be simplified if we examine the concept of transmissivity (T) and storage coefficient (S). Transmissivity is the rate at which water of the prevailing kinematic viscosity is transmitted through a unit width of aquifer under a unit hydraulic gradient (Lohman and others, 1972, p. 13). It is equal to the hydraulic conductivity (K) across the saturated part of the aquifer or

$$T=Kb. \tag{26}$$

The storage coefficient is the volume of water that an aquifer releases or takes into storage per unit surface area per unit change in head (Lohman and others, 1972, p. 13). The storage coefficient for saturated porous media is

$$S=\rho_w g b\left(\frac{n}{E_w}+\frac{1}{E_k}\right). \tag{27}$$

(In soils engineering, the term S is used to describe the degree of saturation; the two terms are not related.)

Substitution of equation 26 and 27 into equation 24 in two-dimensional form gives

$$\frac{\delta^2 h}{\delta x^2}+\frac{\delta^2 h}{\delta y^2}=\frac{S}{T}\frac{\delta h}{\delta t}, \tag{28}$$

which is the equation commonly used to solve many ground-water flow problems. The hydraulic diffusivity (T:S) of the aquifer material is described in the ratio.

Some criticisms of the validity of equation 22 and its different formulations have been presented because on one side of the equation the net inward flux is calculated for the elemental volume without deformation to compute the rate of change of mass inside the unit volume; while on the other side, the element itself is deformed (Cooper, 1966, p. 4786).

In addition, some controversy also exists as to whether the theory of elasticity can be applied accurately to the flow of water and the deformation of porous media. These criticisms, although pertinent, are beyond the purpose and scope of this paper.

The storage coefficient as described by equation 27 applies to saturated porous medium. It is convenient to think of the storage coefficient in terms of storage related to the elasticity of the water (S_w), as well as storage related to the elasticity of the porous medium (S_k). The relation for equation 27 is

$$S=S_w+S_k=\frac{n\rho_w g b}{E_w}+\frac{\rho_w g b}{E_k}. \tag{29}$$

Specific storage (S_s) is the storage coefficient per unit thickness, in units of 1/length (Lohman and others, 1972, p. 13). It is expressed as:

$$S_s=\frac{S}{b}. \tag{30}$$

Another specific-storage relation is

$$S_s=\rho_w g\left(\frac{n}{E_w}+\frac{1}{E_k}\right). \tag{31}$$

Equation 31 can also be written as

$$S_s = S_{sw} + S_{sk} = \frac{n\rho_w g}{E_w} + \frac{\rho_w g}{E_k}. \tag{32}$$

The concept of a storage coefficient for porous medium was expanded by Jacob (1941, p. 786) to describe that porous medium that contains both water and gas. Taylor (1968, p. C194) develops the concept needed to calculate the storage coefficient of a water-saturated porous medium containing both dissolved gas and water. Scott (1963, p. 161–181) develops a similar concept by using the rate of change of weight instead of the storage coefficient.

EXAMPLE 2

Problem:

Results from an aquifer test indicate that the storage coefficient of a sand aquifer is 5×10^{-5}. The log of a test hole at the aquifer-test site recorded an aquifer thickness of 30 m. Is the value for the storage coefficient rational?

Solution:

The storage coefficient consists of at least two parts. One part results from the elasticity of the water. A second part results from the elasticity of the aquifer skeleton.

Considering the storage coefficient related to the water, values for porosity and the modulus of elasticity of water are needed. A porosity of 0.40 can generally be used for approximations for most unconsolidated silts, sands, or gravels.

The value of the bulk modulus of elasticity of water (E_w) is rather insensitive to pressure changes because density of water at a particular temperature is relatively constant. An estimate of E_w is 2.2×10^9 Pa.

Specific storage of the water as calculated by equation 32 is

$$S_{sw} = \frac{n\rho g}{E_w} = \frac{(.40)\,(1.00 \times 10^3\ \mathrm{kg \cdot m^{-3}})\,(9.81\ \mathrm{m \cdot s^{-2}})}{(2.2 \times 10^9\ \mathrm{Pa})}$$

$$= \frac{(.40)\,(9.81 \times 10^3\ \mathrm{N \cdot m^{-3}})}{2.2 \times 10^9\ \mathrm{N \cdot m^{-2}}} = 1.8 \times 10^{-6}\ \mathrm{m^{-1}}.$$

The storage coefficient is the product of specific storage and aquifer thickness, or

$$S_w = bS_{sw} = (30\ \mathrm{m})\,(1.78 \times 10^{-6}\ \mathrm{m^{-1}}) = 5.4 \times 10^{-5}.$$

The storage coefficient from the aquifer test is irrational because the storage coefficient of the aquifer as related to the compressibility of the water is as large as the total storage coefficient determined from the aquifer test. For this to be true, S_k would have to be zero.

Barometric and tidal efficiency result from the stress-strain relationship (elasticity). It has been noted that changes in atmospheric pressure or tides has caused water-level changes in artesian (confined) aquifers. It is assumed that water is not moving into or out of the porous medium (Lohman and others, 1972, p. 2). Thus, the mass of water in the elemental volume is constant; then the differential of the mass is zero. This is expressed as:

$$M_w = \rho_w V_w = \text{constant},$$

and (33)

$$d\rho_w = -\rho_w \frac{dV_w}{V_w} .$$

From equation 33 and from equation 7 and assuming $\rho_w \cong \rho_{wo}$, it follows:

$$d\rho = -E_w \frac{dV_w}{V_w} . \tag{34}$$

Finally, dividing equation 34 by equation 18 results in

$$\frac{dp}{ds_z} = \frac{E_w}{E_k n} . \tag{35}$$

The derivation of the equations for barometric and tidal efficiency will not be shown here. The relation can be found in most ground-water hydrology texts, such as DeWeist (1965, p. 184–192), and also can be obtained from equation 35 and other equations previously given.

Barometric efficiency (*B.E.*) is the ratio of the change in water level in a well to the change in atmospheric pressure (*dp*). Barometric efficiency is

$$B.E. = \frac{-1}{\frac{dp}{ds_z} + 1} . \tag{36}$$

Equation 36 indicates that an increase in atmospheric pressure will decrease the artesian pressure. Theoretically, a saturated porous medium under water-table conditions would have a barometric efficiency of zero.

The relation between barometric efficiency and atmospheric pressure results when equation 35 is substituted into equation 36:

$$B.E. = \frac{-E_k n}{E_w + E_k n} . \tag{37}$$

Tidal efficiency (*T.E.*) is the ratio of the change in artesian pressure to the change in nonbarometric uniform load. As tidal head (or pressure) increases, the artesian pressure in the aquifer increases. The relation is

$$T.E. = \frac{dp/ds_z}{1 + \frac{dp}{ds_z}} . \tag{38}$$

Substituting equation 35 into equation 38 gives

$$T.E. = \frac{E_w/nE_k}{1+E_w/E_k n} = \frac{E_w}{E_w+E_k n}\,. \tag{39}$$

The tidal efficiency of a porous medium under water-table conditions should be 1.

The relation between tidal efficiency and barometric efficiency is

$$|T.E.| + |B.E.| = 1. \tag{40}$$

EXAMPLE 3

Problem:

The depth to water in a well unaffected by other forces increased 0.11 m after a barometric change of 53 mm (millimeters) of mercury. The well is screened in an artesian aquifer that has a porosity of 0.47. Determine the constrained modulus of elasticity of the aquifer.

Solution:

By definition, *B.E.* is the ratio of water-level change to atmospheric pressure expressed in the same units. Because 1 mm head of mercury is equivalent to 133.3 Pa,

$$\Delta p_{atm.} = \left(\frac{133.3 \text{ Pa}}{\text{mm}}\right) (53 \text{ mm}) = 7{,}100 \text{ Pa}.$$

Similarily, because the head of one meter of water is equivalent to 9.8×10^3 Pa,

$$\Delta p_w = \frac{9.81 \times 10^3 \text{ Pa}}{\text{m}} (0.11 \text{ m}) = 1{,}100 \text{ Pa}.$$

Therefore,

$$B.E. = \frac{\Delta p_w}{\Delta p_{atm.}} = \frac{-1{,}100 \text{ Pa}}{7{,}100 \text{ Pa}} = -0.15.$$

Noting that the bulk modulus of elasticity of water is 2.2×10^9 Pa, the constrained modulus of elasticity can be calculated from equation 37:

$$E_k = \frac{-E_w(B.E.)}{(B.E.)\,(n)+n} = \frac{(-2.2 \times 10^9 \text{ Pa})\,(-.15)}{(-.15)\,(.47)+(.47)} = 8.3 \times 10^8 \text{ Pa}.$$

RELATIONS OF SOILS-ENGINEERING TERMS TO GROUND-WATER FLOW EQUATIONS

Application of the principles of soil mechanics to an engineering problem often requires that the elastic properties of a porous medium be determined. A common soil-mechanics test is a "consolidation" test (Hough, 1957, p. 101), sometimes termed a compression test. The procedure is to load a sample that is confined in a cylinder and to measure the resulting deformation.

Figure 3 is a graph of void ratio versus effective stress for a clay sample and a sand sample. In general, unconsolidated clay is more easily deformed than coarser-grained materials, such as silt, sand, or gravel. The elasticity of a clay is related to many factors, including past loading, rate of loading, porosity, hydraulic conductivity, and mineralogy.

VOID RATIO

Void ratio (e) is the ratio of the volume of voids to the volume of solids; while porosity is the ratio of the volume of voids to the total volume. The void-ratio equation is

$$e = V_v / V_s. \tag{41}$$

The relationships between void ratio and porosity are

$$e = n/(1-n), \tag{42}$$

and

$$n = e/(1+e). \tag{43}$$

The implicit differential of equation 43 is

$$de = (1+e)^2 dn. \tag{44}$$

Substituting equations 43 and 44 into equation 9c gives

$$d(\Delta z) = \frac{\Delta z}{1+e}\, de, \tag{45}$$

which is one equation commonly used in soils engineering to predict ultimate compaction. Correspondingly, equations 42 and/or 44 can be substituted into ground-water flow equations in which porosity is expressed. In soils engineering, the convention of initial conditions is generally used; therefore, equation 45 is written

$$d(\Delta z) = \frac{(\Delta z)_0}{1+e_0}\, de, \tag{45a}$$

where the subscript o refers to initial conditions.

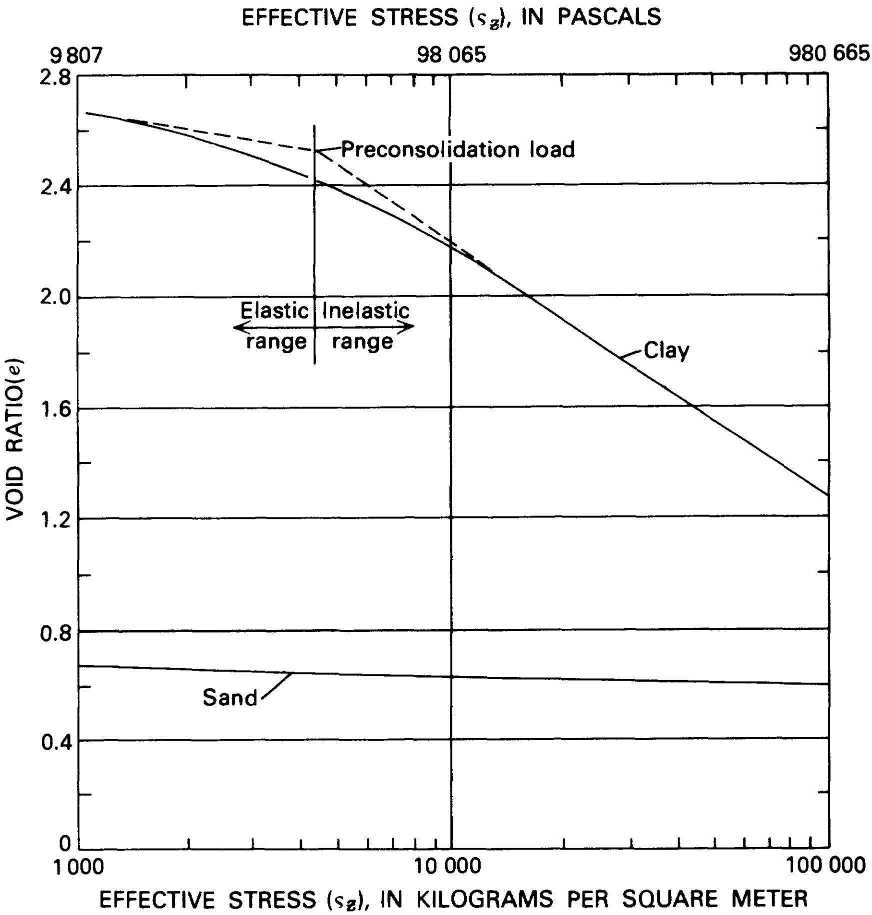

FIGURE 3.—Compression of clay and sand samples.

In using data from consolidation tests, the appropriate value is the value obtained from testing at the same effective stress that exists under natural conditions. Use of the data involves the assumption that the small sample being tested is representative of the entire body. Another assumption is that the sample is undisturbed, although there is no practical way to collect an undisturbed sample. It is also assumed that the test is conducted under the same conditions of pore pressure, water quality, rate of loading, lateral pressure, temperature, etc. that occur in nature. Also, it should be pointed out that different testing laboratories use different techniques, which affect the transfer value of the test data. The user of the data should be aware of the many assumptions; however, this does not mean that results from consolidation tests are unusable or unreliable. Valuable predictions have been made by using the results from a few consolidation tests.

COEFFICIENT OF COMPRESSIBILITY

The slope of the curve in figure 3 in the inelastic range (virgin compression) is termed the coefficient of compressibility (a_v) (Hough, 1957, p. 113). Mathematically, it is

$$a_v = \frac{-de}{ds_e} \cong \frac{-\Delta e}{\Delta s_e} \ . \tag{46}$$

The value of a_v can be determined from a semilog plot of void ratio versus effective stress such as figure 3, but it is best determined from an arithmetic plot as shown in figure 4. It is apparent that the value for a_v is not a constant and must be determined at the appropriate load.

The subscript v denotes virgin compression, or compression in the inelastic range. Another subscript such as e could be used to describe the characteristics of the medium in the elastic range. Figures 3 and 4 show the elastic and inelastic ranges. Figure 5 shows the elastic nature of the sample when the stress is removed. Notice that the "decompression" line has an average slope very nearly equal to the slope of the previous elastic curve. If the sample is stressed again, the curve would probably form a hysteresis loop as inferred by the arrow-dashed line.

To predict the behavior of an unconsolidated porous medium, it is necessary to know if the medium is being stressed in the elastic or inelastic range. The relation between a_v and E_k is found by combining equations 8, 45, and 46 to give

$$a_v = (1 + e_o)/E_k \ , \tag{47}$$

where the meaning of E_k has been generalized to include the inelastic range also.

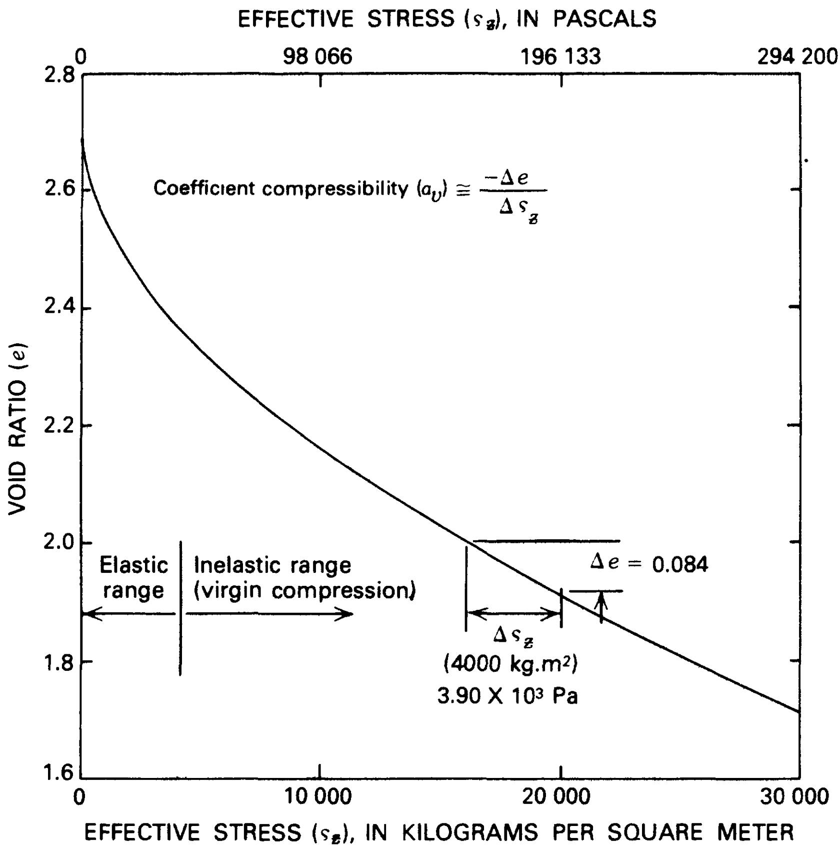

FIGURE 4.—Compression of a clay sample.

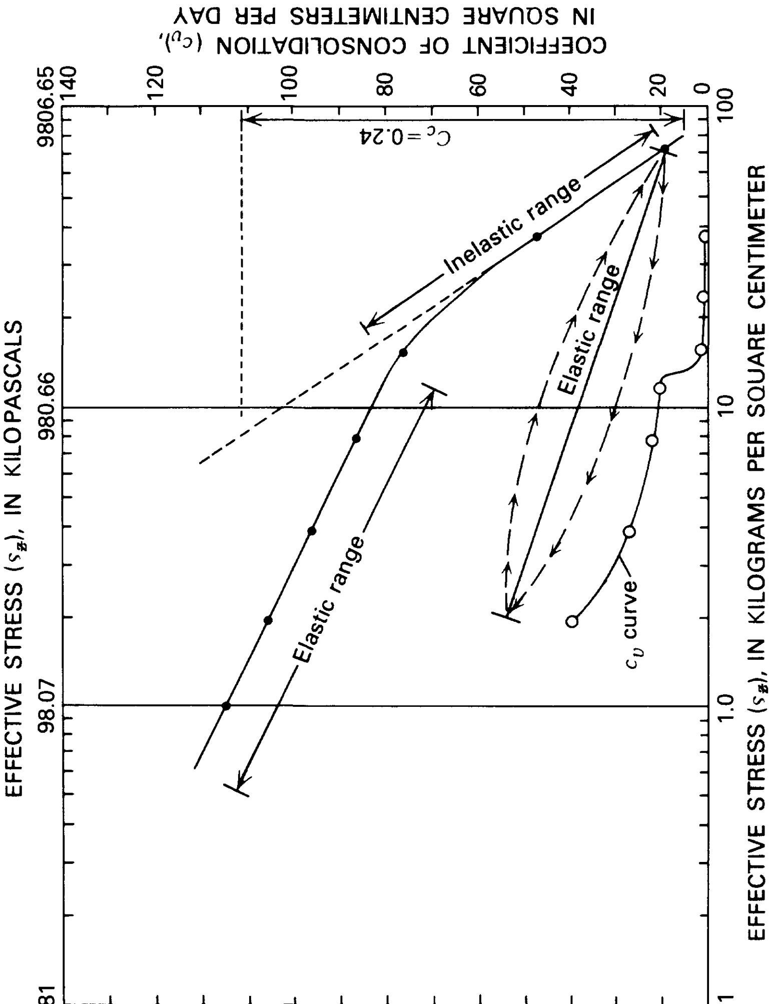

FIGURE 5.—Results of a consolidation test.

EXAMPLE 4

Problem:

An aquifer system includes several easily compressible clay layers. The total thickness of the clay layers is 50 m. Withdrawals from the aquifer have lowered the potentiometric surface 200 m. What would the cumulative decrease in thickness of the clay layer be if the consolidation test in figure 5 is representative and if initial effective stress is 2.94×10^6 Pa?

Solution:

The increase in effective stress is directly proportional to decrease in artesian pressure. The unit weight of water, γ_w, is 9.81×10^3 Pa$\cdot$m^{-1}; thus change of stress equals change of head times γ_w or

$$\Delta s_{\text{g}} = (200.\ \text{m})\,(9.81 \times 10^3\ \text{Pa}\cdot\text{m}^{-1}) = 1.96 \times 10^6\ \text{Pa}.$$

Figure 5 shows that the change of void ratio between 30 kg/cm^2 and 50 kg/cm^2 is 0.26 less 0.32 or -0.06.

The ultimate compaction of the layers can be determined from equation 45a:

$$d(\Delta z) = \frac{(\Delta z)_0}{(1+e_0)}\,de = \frac{(50\ \text{m})\,(-0.06)}{(1+.32)} = -2.3\ \text{m}.$$

EXAMPLE 5

Problem:

Consolidation-test data for an easily compressible clay sample is shown in figure 4. The range of effective vertical stress to be applied to the clay is from 1.57×10^5 Pa (16,000 $\text{kg} \cdot \text{m}^{-2}$) to 1.96×10^5 Pa (20,000 $\text{kg} \cdot \text{m}^{-2}$). Determine the coefficient of compressibility, the range of porosity, and the constrained modulus of elasticity.

Solution:

The value of a_v is the slope of the compression line (fig. 4) and is defined by equation 46;

$$a_v = \frac{-(-0.084)}{(3.9 \times 10^4 \text{ Pa})} = 2.2 \times 10^{-6} \text{ Pa}^{-1}.$$

The value of e_o (fig. 4) is 2.00. The value of E_k can be determined from equation 47;

$$E_k = \frac{(1+e_o)}{a_v} = \frac{(1+2.00)}{(2.2 \times 10^{-6} \text{ Pa}^{-1})} = 1.4 \times 10^6 \text{ Pa}.$$

The void ratio read from the curve (fig. 4) ranges from 2.00 to 1.92. Porosity values can be calculated from equation 43;

$$n = \frac{e}{1+e} = \frac{2.00}{1+2.00} = 0.67 \,,$$

and

$$n = \frac{1.92}{1+1.92} = 0.66 \,.$$

COEFFICIENT OF VOLUME COMPRESSIBILITY

The coefficient of volume compressibility (m_v) is the reciprocal of the constrained modulus of elasticity of the unconsolidated porous medium in the inelastic range, or

$$m_v = \frac{1}{E_k} \,. \tag{48}$$

The coefficient (m_v) is defined as the compression of a soil layer, per unit of original thickness per unit increase of effective stress in the load range exceeding preconsolidation stress. (Modified after Terzaghi and Peck, 1948, p. 64.) A similar term could be used to describe the compressibility in the elastic range. The relation of a_v to m_v is

$$a_v = (1 + e_o)\, m_v. \tag{49}$$

COMPRESSION INDEX

The coefficient of compressibility, as shown in figure 4, is not a constant and must be determined for each condition of loading. A similar term called the compression index (C_c) is defined by the slope of the closest straight-line fit of a curve defined by a plot of void-ratio values versus the logarithm of the load (effective stress) of soil being tested in the inelastic range (Lambe, 1962, p. 83). Typically, the compression curve for the elastic range will also define a straight line, so a coefficient could also be defined for this range. Unfortunately, the compression curves for some soils do not define a straight line in either range. Nevertheless, the index is useful in describing many soils.

The compression index in Δ form and differential form is

$$C_c = \frac{-\Delta e}{\Delta \log_{10} s_{\not{e}}} \approx \frac{-de}{d\,(\log_{10} s_{\not{e}})} \tag{50}$$

The relation between a_v and C_c for the inelastic range can be shown by combining equations 46 and 50, both of which contain de:

$$a_v = C_c \frac{d\,(\log_{10} s_{\not{e}})}{ds_{\not{e}}} \tag{51}$$

Evaluating $d\,(\log_{10} s_{\not{e}})$,

$$d(\log_{10} s_{\not{e}}) = \frac{1}{s_{\not{e}}}(\log_{10}\text{exponential})\, ds_{\not{e}} = \frac{0.434 ds_{\not{e}}}{s_{\not{e}}} \tag{51a}$$

Accordingly, equation 51 becomes

$$a_v = 0.434\, C_c / s_{\not{e}}\ . \tag{52}$$

The relation of C_c to E_k is found by combining equations 47 and 52 to give

$$C_c = (1 + e_o) s_{\not{e}} / 0.434\, E_k\ . \tag{53}$$

Several empirical methods have been developed for estimating C_c for both fine- and coarse-grained soils. Descriptions and examples of some of these methods are given in Hough's basic soils engineering text (1957, p. 114–118) and by Johnson and others (1968, p. 39).

COEFFICIENT OF CONSOLIDATION

The factors that determine the rate and amount of compaction are numerous, complex, and beyond the scope of this paper. However, a useful parameter that relates the effects of both storage and hydraulic conductivity at a given void-ratio range is the coefficient of consolidation (c_v) (Hough, 1957, p. 127). The relation is

$$c_v = \frac{KE_k}{\gamma_w} = \frac{KE_k}{\rho_w g} . \tag{54}$$

The formulation of equation 54 will be given later. As for most other soils terms, c_v is defined for stress in the inelastic range. The usual form of the equation results from the substitution of equation 47 into equation 54,

or

$$c_v = \frac{K(1+e_o)}{a_v \gamma_w} . \tag{55}$$

Seaber and Vecchioli (1966, p. 109–111) show an example of using c_v to determine hydraulic conductivity.

The values of c_v are calculated from consolidation-test data by the use of equations. The most commonly used equation is

$$c_v = \frac{\mathrm{T}\left(\frac{H}{2}\right)^2}{t_{xc}} , \tag{56}$$

where H = thickness of sample, t_{xc} = time for x percent of primary consolidation (drainage of pore water), and T is "dimensionless time factor" to be discussed in more detail in a following section.

The value of c_v is not constant and is generally calculated for each incremental load of the consolidation test. Details of the procedure are found in numerous texts and manuals such as *Soils Testing for Engineers* (Lambe, 1962, p. 74–87). The calculated values of c_v are often plotted as semilogarithmic compression curves (fig. 5).

The relationship between specific storage resulting from aquifer elasticity, hydraulic conductivity, and coefficient of consolidation is

$$c_v = \frac{K}{S_{sk}} . \tag{57}$$

The coefficient is essentially the hydraulic diffusivity ($T : S$), the inverse of which appears in equation 28. Unless an aquifer is relatively thick or rigid, the contribution of water elasticity to specific storage is relatively small and is generally ignored.

EXAMPLE 6

Problem:

Using the consolidation data for the clay sample shown in figure 5, calculate the constrained modulus of elasticity, the compression index, the porosity, and the transmissivity at an effective stress of 3.63×10^6 Pa (37 kg$\cdot$cm^{-2}). The sample is from a 10 m-thick clay layer which has an isotropic hydraulic conductivity.

Solution:

The value of compression index (C_c) is read directly from figure 5 as 0.24,

or $$C_c = 0.24.$$

The porosity can be calculated from the void ratio. The void ratio at $s_{\bar{z}} = 37$ kg$\cdot$cm^{-2} is read directly from the curve as $e = 0.29$. Porosity is related to void ratio by equation 43,

or $$n = \frac{e}{1+e} = \frac{0.29}{1+0.29} = 0.22\ .$$

The constrained modulus of elasticity can be calculated from equation 53,

or $$E_k = \frac{(1+e_0)\, s_{\bar{z}}}{0.434\ C_c} = \frac{(1+0.29)\,(3.63 \times 10^6\ \text{Pa})}{(0.434)(0.24)} = 4.5 \times 10^7\ \text{Pa}.$$

The hydraulic conductivity is calculated using equation 54 and the c_v value from figure 5.

First converting to SI metric units;

$$c_v = (2.00\ \text{cm}^2/\text{day}) \frac{(1 \times 10^{-2}\ \text{m/cm})^2}{(8.64 \times 10^4\ \text{s/day})} = 2.31 \times 10^{-9}\ \text{m}^2\cdot\text{s}^{-1}.$$

$$K = \frac{c_v \rho_w g}{E_k} = \frac{(2.31 \times 10^{-9}\ \text{m}^2\cdot\text{s}^{-1})\,(1.00 \times 10^3\ \text{kg}\cdot\text{m}^{-3})\,(9.806\ \text{m}\cdot\text{s}^{-2})}{(4.5 \times 10^7\ \text{Pa})}$$

$$= 5.03 \times 10^{-13}\ \text{m}\cdot\text{s}^{-1}\ .$$

Equation 26 relates hydraulic conductivity to transmissivity:

$$T = Kb = (5.03 \times 10^{-13}\ \text{m}\cdot\text{s}^{-1})\,(10\ \text{m}) = 5.03 \times 10^{-12}\ \text{m}^2\cdot\text{s}^{-1}\ .$$

If the convention of using the subscript o is used, the equation 57 is

$$c_v = \frac{K}{S_{sk}} \left(\frac{1+e_o}{1+e} \right). \tag{57a}$$

If the parenthetic term is very nearly 1, then little error results.

HYDRAULIC-CONDUCTIVITY TESTS

Equations 54, 55, and 57 relate hydraulic conductivity (K), the coefficient of consolidation (c_v), and other characteristics. These equations allow determination of hydraulic conductivity from compression-test data. Two common tests used to determine hydraulic conductivity are the constant head and the variable head permeameter tests. Comparisons by Domenico and Mifflin (1965, p. 566) and Wolff (1970, p. 202) indicate that values of hydraulic conductivity obtained by permeameter tests and by consolidation tests are in general agreement. A study conducted by the California Department of Water Resources (1971) indicated computed hydraulic conductivities were consistently lower than hydraulic conductivities from the permeameter test. Permeameter tests should be conducted at the same void ratio as the void ratio at which c_v was determined. Other factors, such as chemical composition of water used in the tests, must also be considered.

An aquifer test (or pumping test) is the field method most often used by ground-water hydrologists to determine hydraulic conductivity. Although the method is widely used, it is often difficult to obtain representative hydraulic-conductivity values from pumping tests because the water pumped from an aquifer comes from many sources, including clays and silts that have low hydraulic conductivities. The mathematical model used to analyze aquifer-test data frequently does not account for the complex hydrology of most ground-water systems.

Some progress has been made in obtaining better results from aquifer tests in a simple layered system consisting of layers of both high and low hydraulic conductivities; for example, Hantush (1960) developed a mathematical model to describe "leaky" aquifers. Others, such as Neuman and Witherspoon (1968, 1969), have developed mathematical models to describe the flow of water to wells in leaky aquifers.

SPECIFIC-STORAGE CURVES

For most ground-water flow problems, it is assumed that the hydraulic conductivity and specific storage of an aquifer are constant. However, if the aquifer is composed of some easily compressible material, the assumption is invalid. Specific storage due to aquifer compressibility can be calculated from consolidation-test data. The combination of equations 47, 54, and 57 results in

$$S_{sk} = \frac{a_v \gamma_w}{(1+e_o)} \ . \tag{58}$$

Similarly, the relation of specific storage to the compression index is found by substituting equation 52 into 58, which results in

$$S_{sk} = \frac{0.434\, C_c \gamma_w}{s_{\bar{z}}\ (1+e_o)} \ . \tag{59}$$

Equations 58 and 59 allow determination of specific storage even if c_v data are not available. Because c_v is not constant, equation 57 indicates that specific storage is not a constant and is similar to c_v in being a function of the loading.

Figure 6 is a semilogarithmic plot of specific storage versus effective stress. Data points were calculated by using equation 59 and figure 5. The specific storage of each of several confining layers composed of easily compressible and homogeneous clay, for which the effective stress is known, can be read directly from the plot.

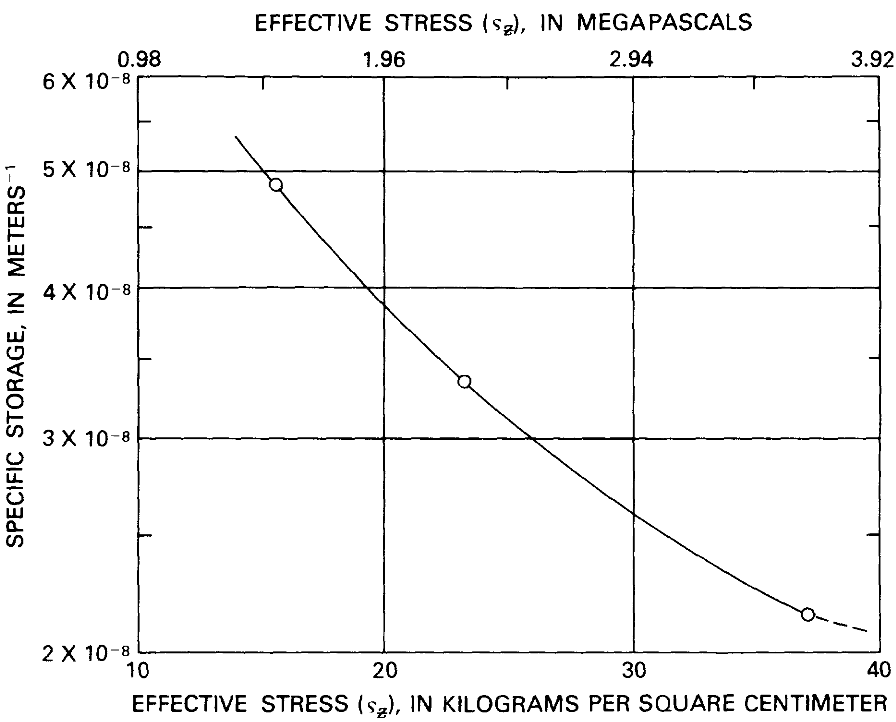

FIGURE 6.—Specific storage and effective stress.

A curve similar to the curve in figure 6 may be determined for easily compressible materials that occur at various depths (J. F. Poland, personal commun.). Each point on such a curve is the specific storage at field loading as calculated from the results of a consolidation test of a representative sample. If easily compressible materials have the same general geologic origins and elastic characteristics, the curve may be representative of an area. Under these conditions, a graph similar to figure 6 and a graph of the values of c_v could be used to approximate the hydraulic conductivity and the storage coefficient for the silts and clays.

COMPACTION AND TRANSIENT GROUND-WATER FLOW

Figure 7 shows an easily compressible semiconfining layer separating two aquifers. In situation A, water is withdrawn initially from the aquifers at t_o, some initial time prior to t_1. This withdrawal lowers the potentiometric head rapidly in both aquifers but more slowly in the semiconfining layer. The initial stepwise change in head is termed H_o. Situation B represents a transient condition, and situation C represents steady-state conditions after an infinite amount of time. The difference between the potentiometric heads in the semiconfining layer at t_1 and the steady-state potentiometric head is the excess head h'_1 at t_1.

Soils engineers use the term excess pore pressure (u) in preference to excess head. Excess pore pressure is the pore pressure at any point in a saturated porous medium in excess of the pore pressure that would exist at that point if steady flow conditions had been attained throughout the medium (modified from Poland and others, 1972, p. 4). Soils engineers generally prefer to use pressure units in formulating most relations. Equation 11 can be expanded for the transient state:

$$\begin{aligned} p_l &= s_z + p_a \\ &= s_z + (u + \gamma_w h_s), \end{aligned} \tag{60}$$

where $u = \gamma_w h'$.

Equation 28 can be written for one dimension by using excess head and by assuming that S_{sw} is negligible, as

$$\frac{\delta^2 h'}{\delta_z{}^2} = \frac{S}{T}\frac{\delta h'}{\delta t} = \frac{S_{sk} b}{K' b}\frac{\delta h'}{\delta t}, \tag{61a}$$

where K' is vertical hydraulic conductivity. Substituting equations 54 and 57 into equation 61a results in

$$\frac{\delta^2 h'}{\delta z^2} = \frac{S_{sk}}{K'}\frac{\delta h'}{\delta t} = \frac{1}{c_v}\frac{\delta h'}{\delta t} = \frac{\gamma_{w'}}{K' E_k}\frac{\delta h'}{\delta t}. \tag{61b}$$

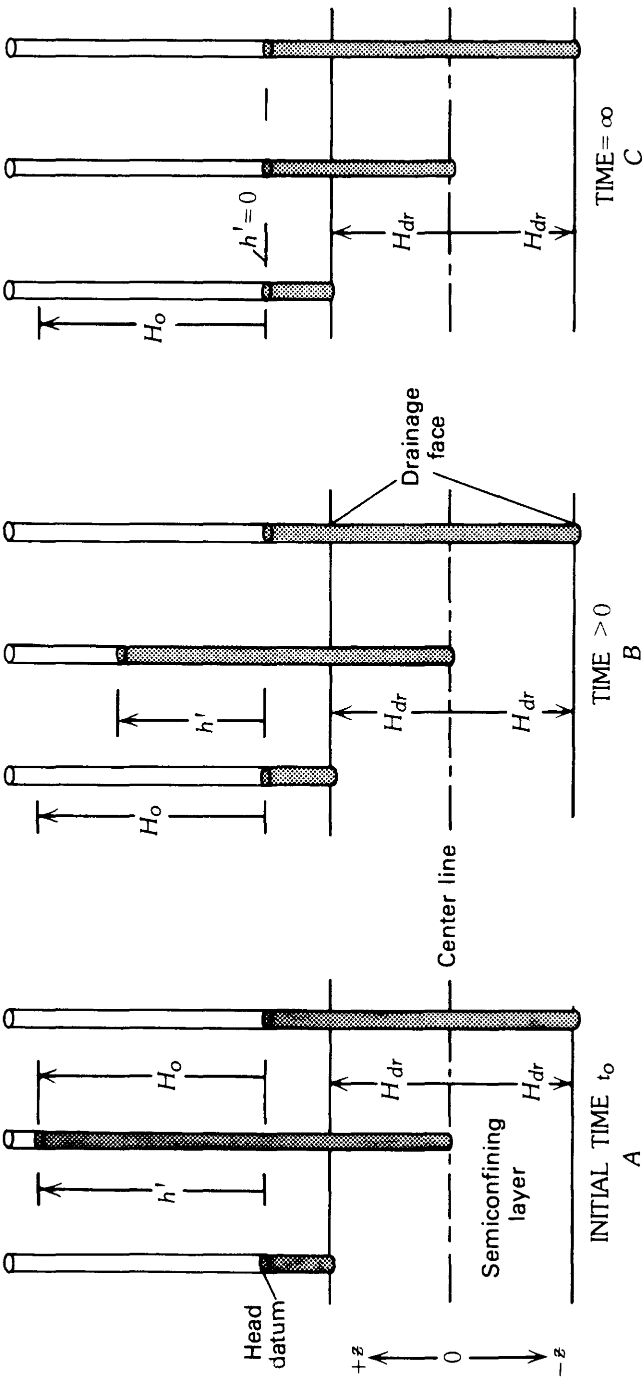

FIGURE 7.—Potentiometric head in a semiconfining layer. *A*, Initial head; *B*, Transient condition; and *C*, Steady-state condition.

Equation 61b is identical in form and meaning to

$$\frac{\delta^2 u}{\delta z^2} = \frac{1}{c_v}\frac{\delta u}{\delta t}, \tag{62}$$

which is commonly used in soils-engineering texts. Equations 61b and 62 are useful in solving many soils-engineering and ground-water flow problems for small stresses; however, these equations are unsuitable for large stresses because c_v, K, and S_s are not constant for easily compressible soils.

Soils engineers find it useful to define and use a parameter called a time factor. Hydrologists use an identical parameter which is termed dimensionless time (T). The relations are

$$\mathrm{T} = \frac{K't}{S_s H_{dr}^2} = \frac{c_v t}{H_{dr}^2}. \tag{63}$$

H_{dr} is the longest drainage path.

The analytical solution of 61a, 61b, or 62 for the boundary conditions shown in figure 7 is

$$\frac{h'}{H_0} = 1 - \sum_{n=0}^{\infty} (-1)^n \left[\operatorname{erfc} \frac{(2n+1) - z/H_{dr}}{2\mathrm{T}} + \operatorname{erfc} \frac{(2n+1) + z/H_{dr}}{2\mathrm{T}}\right]. \tag{64}$$

Equation 64 is analogous to an equation for heat flow reported by Carslaw and Jaeger (1959, p. 97). Figure 8 is a graphical solution to equation 64.

Often it is desirable to determine the average excess head, h'_{av}, at some time for the entire layer. The analytic solution is

$$\frac{h'_{av}}{H_0} = 1 - 2\sqrt{\mathrm{T}}\left[\frac{1}{\sqrt{\pi}} + 2\sum_{n=1}^{\infty} (-1)^n \operatorname{ierfc} \frac{n}{\sqrt{\mathrm{T}}}\right]. \tag{65}$$

Equation 65 is similar to that given by Hough (1957, p. 129) and analogous to an equation of heat flow by Carslaw and Jaeger (1959, p. 97). A graphical solution of equation 65 is also shown in figure 8. Figure 8 shows the ratio of excess head to initial head change. Figure 8 can also be used to determine excess pore pressure by substituting a ratio of excess pore pressure to initial change of pore pressure for the h'/H_0 ratio. Additionally, the ratio of change of thickness to total or ultimate change of thickness can also be substituted for the ratio of h'/H_0. Use of this substitute ratio and use of the curve depicting the average of the entire layer allow calculation of settlement or subsidence with time.

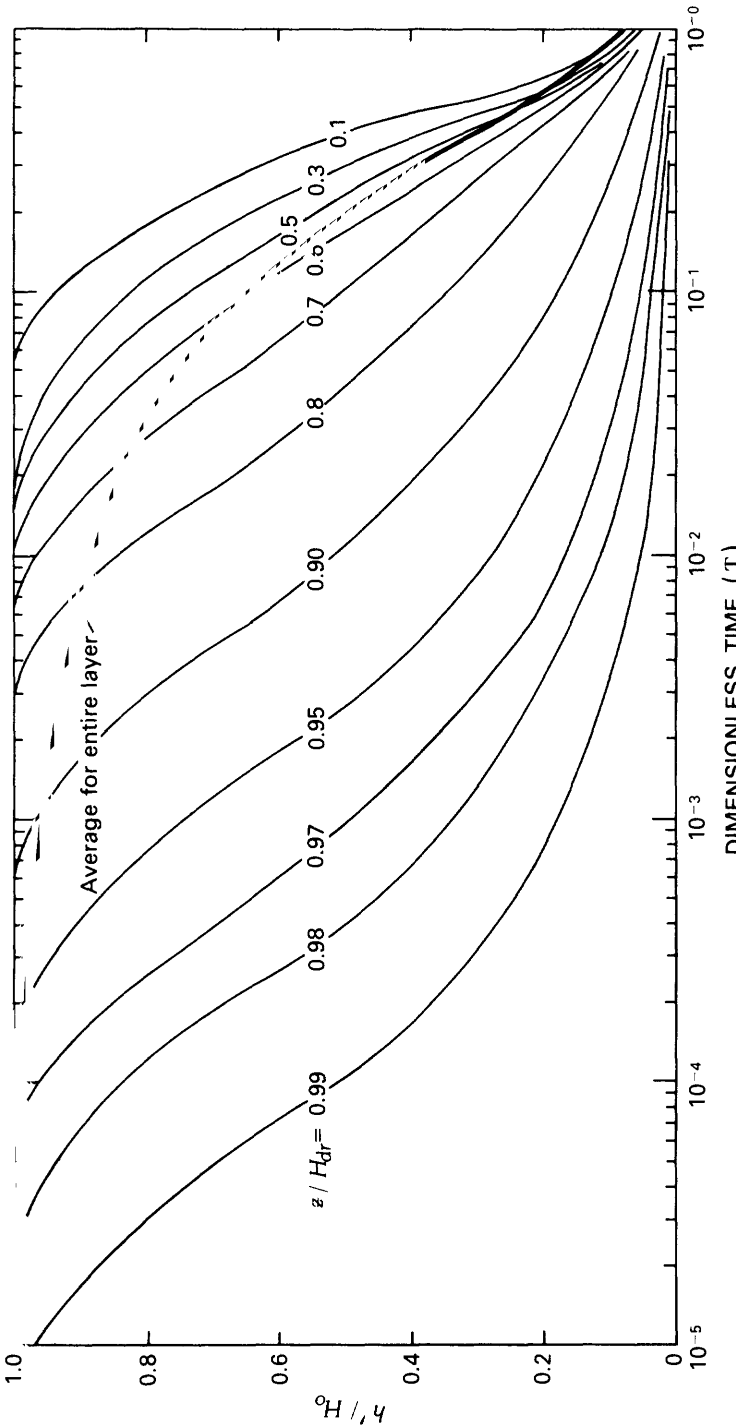

FIGURE 8.—Excess-head distribution in a layer with a stepwise head change.

EXAMPLE 7

Problem:

A cursory inspection of an electric log apparently indicated a clay layer 100-m thick was within a thick sand aquifer. The clay layer had a vertical hydraulic conductivity of 5.00×10^{-7} m/s and a specific storage of 2.40×10^{-2}/m. A closer examination indicated a thin sand layer, which would act as a drainage face, at a depth of 25 m below the top of the clay layer. If a transducer had been installed 5 m below the top of the clay layer, what would have been the difference in excess head after 1-month time for the two conditions if the potentiometric head had been lowered 70 m in the sand portions of the aquifer?

Solution:

Calculating excess head if the thickness of the included clay layer had been 100 m,

$$\frac{z}{H_{dr}} = \frac{50\text{m} - 5\text{m}}{50\text{m}} = 0.9 ,$$

and from equation 63

$$T = \frac{K't}{S_s H_{dr}^2} = \frac{(5.00 \times 10^{-7}\ \text{m} \cdot \text{s}^{-1})\ (30\ \text{day})\ (86400\text{s} \cdot \text{day}^{-1})}{(2.40 \times 10^{-2}\text{m}^{-1})\ (50\ \text{m})^2} = 2.16 \times 10^{-2}.$$

Entering figure 8 using $T = 2.16 \times 10^{-2}$ and $z/H_{dr} = 0.9$, the value of $h'/H_0 = 0.39$. Because $H_0 = 70$ m, thus

$$h' = (0.39)\ (70.\ \text{m}) = 27\ \text{m}.$$

Calculating the excess head in the 25-m thick layer,

$$\frac{z}{H_{dr}} = \frac{12.5\ \text{m} - 5.0\ \text{m}}{12.5\ \text{m}} = 0.6,$$

and from equation 63

$$T = \frac{K't}{S_s (H_{dr})^2} = \frac{(5.00 \times 10^{-7}\ \text{m} \cdot \text{s}^{-1})\ (30\ \text{day})\ (86400\ \text{s} \cdot \text{day}^{-1})}{(2.40 \times 10^{-2}\text{m}^{-1})\ (12.50\ \text{m})^2}$$

$$= 3.46 \times 10^{-1}.$$

Again using figure 8, $h'/H_0 = 0.30$. Because $H_0 = 70$ m, thus

$$h' = (0.30)\ (70.\ \text{m}) = 21\ \text{m}.$$

The difference in excess head between the two conditions would be about 6 m. This difference emphasizes the importance of locating all drainage faces when calculating excess head.

EXAMPLE 8

Problem:

The head both above and below a 20.0-m thick clay layer has been instantaneously lowered 40.0 m. From a compression test it was determined that the clay had a void ratio (e) of 1.20 and a compression index (C_c) of 0.180. Initial effective stress ($s_{\bar{z}}$) is $2.45 \times 10^{+6}$ Pa. The coefficient of consolidation (c_v) is 3.47×10^{-9} m²/s. Determine the ultimate change of thickness and the change of thickness after 1 year.

Solution:

The change of effective stress resulting from the lowering of the head is

$\Delta s_{\bar{z}} = \rho_w g\ \Delta h = (1.00 \times 10^3\ \mathrm{kg \cdot m^{-3}})\ (9.806\ \mathrm{m \cdot s^{-2}})\ (40.0\ \mathrm{m}) = 3.92 \times 10^5$ Pa.
Let $\Delta s_{\bar{z}} = ds_{\bar{z}}$.

Combining equations 50 and 51a yields

$$de = \frac{-C_c\,(0.434\ ds_{\bar{z}})}{s_{\bar{z}}} = \frac{-(0.180)\,(0.434)\,(3.92 \times 10^5\ \mathrm{Pa})}{(2.45 \times 10^6\ \mathrm{Pa})}$$

$$= -1.25 \times 10^{-2} = -0.0125.$$

Knowing de, the ultimate change of thickness can be determined using equation 45a:

$$d(\Delta \bar{z}) = \frac{(\Delta \bar{z})_0}{1 + e_0}\,de = \frac{(20.0\ \mathrm{m})\,(-0.0125)}{1 + 1.20} = -0.114\ \mathrm{m}.$$

That is, the ultimate change is -0.114 m. The negative sign indicates decrease in thickness of the layer.

The determination of thickness after one year is a transient problem. At this point it is useful to note that the change of porosity, or the change of void ratio, or the change in layer thickness is directly proportional to the change of head within a layer. In this case it will be useful to determine the average change of head for the entire layer at one year and relate this change to the steady-state (ultimate) change of thickness.

The value of dimensionless time is calculated from equation 63:

$$\mathrm{T} = \frac{c_v t}{H_{dr}^2} = \frac{(3.47 \times 10^{-9}\ \mathrm{m^2/s})\,(1.00\ \mathrm{year})\,(365.\ \mathrm{days \cdot year^{-1}})\,(86400.\ \mathrm{s \cdot day^{-1}})}{(10.0\ \mathrm{m})^2}$$

$$= 1.09 \times 10^{-3}.$$

From figure 8 using the "average for the entire layer" curve,

$$h'/H_0 = 0.96.$$

Remember that excess head and change of thickness are proportional

or
$$\frac{h'}{H_0} = \frac{\Delta z}{\Delta z_0} = 0.96 \; .$$

Thus $\Delta z = (0.96)\,(0.114 \text{ m}) = 0.109$ m. Therefore, the change of thickness after 1 year is 0.114 m minus 0.109 m, or 0.005 m.

Figure 9 shows the longest drainage path for a single drainage-face problem. Equation 64 and 65 can be used to solve a single drainage-face problem, as shown in figure 9, if the longest drainage path is correctly defined.

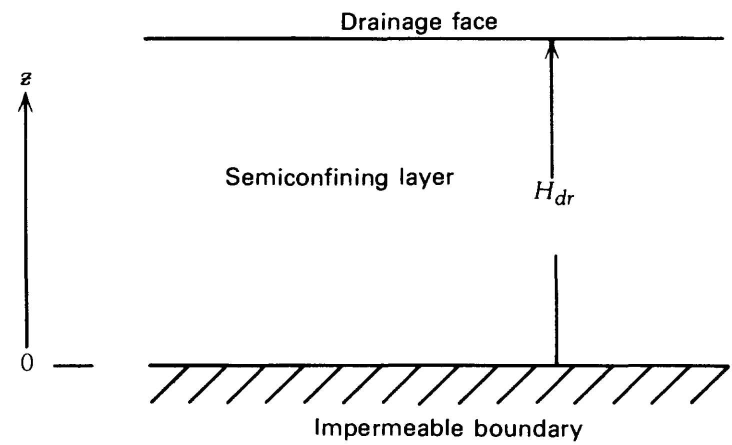

FIGURE 9.—Longest drainage path for a layer with a single drainage face.

Several other analytical solutions are available for various boundary conditions. Hanshaw and Bredehoeft (1968, p. 1109) show a graphic solution of excess head versus dimensionless time for a finite layer with a stepwise head change at one boundary.

Helm (1975, p. 465–468) describes the application of equation 61a for both virgin and elastic loading conditions. In a later paper, Helm (1976, p. 375–391) considers both hydraulic conductivity and specific storage as stress dependent.

LAYERED SYSTEMS

All of the equations developed thus far are for single layers; however, most systems studied in nature consist of many layers of material, each with its own characteristics.

For some purposes, the storage coefficient of a layered system can be considered to be equal to the sum of the storage coefficients of the individual layers. The analysis by Hantush (1960) of pumping from an aquifer, overlain and/or underlain by a semipervious layer shows that after a long enough period of pumping, the drawdowns can be computed assuming an "effective" storage coefficient equal to the sum of the storage coefficients of the aquifer and the semipervious layers. The effective storage coefficients or the storage coefficient of the system can be expressed as

$$S_{system} = S_1 + S_2 + \ldots S_n. \tag{66}$$

Substituting equation 30 into equation 66 yields:

$$S_{system} = S_{s1}b_1 + S_{s2}b_2 + \ldots S_{sn}b_n, \tag{67}$$

where S_{sn} is the specific storage of the $_n$th layer and b_n is the thickness of the $_n$th layer.

Thus the storage coefficient for the system is

$$S_{system} = S_{s\ system}B, \tag{68}$$

where $$B = b_1 + b_2 + \ldots b_n.$$

The specific storage of the system is the weighted mean of the specific storage of the individual layers:

$$S_{s\ system} = \frac{S_{s1}b_1 + S_{s2}b_2 + \ldots S_{sn}b_n}{B} \tag{69}$$

The relation between the constrained modulus of elasticity and specific storage is apparent from equations 32, 54, and 57. That is,

$$E_k = \frac{\gamma_w}{S_{sk}} \cdot \tag{70a}$$

Equation 70a is somewhat limited by the assumptions inherent in equation 54; therefore, we will use the more general equation:

$$E_k = \frac{\gamma_w}{S_s} \tag{70b}$$

Equation 70b includes the effect of the compressibility of the water.

The constrained modulus of elasticity for a layered system is found by substituting equation 69 into equation 70b:

$$E_{k\ system} = \frac{\gamma_w B}{S_{s1}b_1 + S_{s2}b_2 + \ldots S_{sn}b_n} \tag{71a}$$

Another expression is found by substituting equation 70b into equation 71, or

$$E_{k\ system} = \frac{B}{b_1/E_{k1} + b_2/E_{k2} + \ldots b_n/E_{kn}} \tag{71b}$$

The horizontal hydraulic conductivity of a layered system is the weighted mean,

or
$$K_{system}=\frac{K_1b_1+K_2b_2+\ .\ .\ .\ K_nb_n}{B}\ . \tag{72}$$

Equation 72 can be considered to represent the maximum hydraulic conductivity of a layered system. The vertical hydraulic conductivity of a layered system is:

$$K'_{system}=\frac{B}{\dfrac{b_1}{K'_1}+\dfrac{b_2}{K'_2}+\ .\ .\ .\ \dfrac{b_n}{K'_n}} \tag{73}$$

Values obtained using equation 73 can be considered the minimum vertical hydraulic conductivity of a layered system. Most aquifer systems studied have vertical-hydraulic conductivity values within the range obtained by the use of equation 73 and 74. Some layered aquifers have vertical hydraulic conductivities which would indicate that the semipervious layers are discontinuous in horizontal direction, where others have values which would indicate that the aquifer layers were discontinuous in a horizontal direction. Of course, variations between the extremes is the most common case.

The horizontal transmissivity of a layered system is the sum of the transmissivities of each layer,

or
$$T_{system}=T_1+T_2+T_n\ . \tag{74}$$

Caution should be exercised in using equations 66 to 74. These equations are useful in approximating the response of an aquifer system over a "long" period of time. The correct manner of determining the response is to analyze each layer as an independent unit within the system. This approach is difficult in the most simple system even with the use of computers.

The investigator is then forced to use approximation such as defining the aquifer system in such a way as to allow the use of approximations, such as equations 66 to 74. Often such approximation can be made, but the investigator should test to determine if the approximations are introducing excess errors.

Helm (1975) discusses the concept of "equivalent bed thickness" which is often useful in evaluating aquifer system response. Javandel and Witherspoon (1969) also investigated multilayered aquifer systems, and the problem of whether the system could be treated as an homogeneous aquifer or a multilayered system. Riley (1969) analyzed records of subsidence and water-level declines to determine the elasticity of an easily compressible system in California.

Useful estimates of layered system characteristics can sometimes be made using equations 66 to 74 along with "typical" hydraulic properties as shown in table 1.

TABLE 1.—*Typical values or typical range of values of hydraulic properties*

Properties material	E_k (Pa)	K (m/s)	n (percent)
Clay	5×10^{5}–1.5×10^{7}	3.5×10^{-10}	35–70
Silt	5×10^{7}	3.5×10^{-7}–1×10^{-5}	20–50
Sand:			
Very fine, silty	5×10^{7}–1×10^{8}	1×10^{-5}–1×10^{-4}	15–40
Fine to medium	5×10^{7}–1×10^{8}	1×10^{-4}–2×10^{-4}	15–40
Medium	5×10^{7}–1×10^{8}	2×10^{-4}–3×10^{-4}	15–40
Medium to coarse	5×10^{7}–1×10^{8}	3×10^{-4}–3.9×10^{-4}	15–40
Coarse	5×10^{7}–1×10^{8}	3.9×10^{-4}–4.2×10^{-4}	15–40
Very coarse	5×10^{7}–1×10^{8}	4.2×10^{-4}–4.6×10^{-4}	15–40
Sand and gravel	1×10^{8}–2×10^{8}	4.6×10^{-4}–1×10^{-3}	15–40

The values listed in table 1 are typical values for the constrained modulus of elasticity of different materials. Values used in the table are modification of values reported by P. A. Domenico and M. D. Mifflin (1965), P. A. Emery (1966), and S. W. Lohman (1972). Caution should be exercised in using the values at any particular site. Also note that typical ranges are given. However, this is not the complete range. For example, the maximum hydraulic conductivity ranges for the material listed would be nearly zero to greater than the values listed. The "complete" range is nearly useless for estimating typical hydraulic properties, therefore the table showing typical values was prepared to aid in preparing estimates.

It should also be noted that the values of hydraulic conductivity listed in table 1 are somewhat lower than many reported values in the literature. Generally the higher values are based on laboratory analyses of samples that have been "repacked." Specific storage values are obtained by applying equation 70a to the values of E_k. Additionally, the proportion resulting from elasticity of water should also be included. The modulus of elasticity of water (E_w) for most ground water is about 2.2×10^9 Pa. Comparing E_w with the values of E_k in table 1 and considering the reduction by porosity, it becomes evident that the contribution of specific storage from water, although small, is not always insignificant as is commonly assumed.

SUMMARY AND CONCLUSIONS

The many varied but related terms developed by ground-water hydrologists and soils engineers are useful to each discipline, and their acceptance is obviously related to their utility in their respective disciplines. However, use of these terms in interdisciplinary studies is hampered by the fact that the terms are narrowly defined to meet the objectives of each discipline.

Ground-water equations developed by Jacob (1950) are identical to the equations of the Terzaghi theory of consolidation for specific assumptions. A combination of the two sets of equations relates porosity to void ratio and relates modulus of elasticity to specific storage, storage coefficient, coefficient of compressibility, coefficient of volume compressibility, compression index, coefficient of consolidation, and ultimate compaction. In addition, transient ground-water flow is related to the coefficient of consolidation, rate of soil compaction, and hydraulic conductivity. Examples of soils-engineering data and concepts applied to ground-water problems demonstrate the usefulness of the interdisciplinary approach.

Most terms relating to compressibility, elasticity, storage, and specific storage could be eliminated by the use of the constrained modulus of elasticity. Some equations that include the change in porosity or the change in void ratios could be simplified by using the most appropriate term. Terms relating to diffusivity, excess pressure or head, and stress could be standardized. Such standardization is unlikely, but an effort by investigators in each discipline to use the terminology of basic physics would facilitate the transfer of data and techniques.

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