A Galerkin Finite-Element Flow Model to Predict the Transient Response of a Radially Symmetric Aquifer

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Prepared in Cooperation with the Nassau County Department of Public Works, Suffolk County Department of Health Services, Suffolk County Water Authority, and New York State Department of Environmental Conservation

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CONTENTS

FIGURES

GLOSSARY

A	Cross sectional area of an element (L^2).
[A]	Coefficient matrix.
[B]	Time-dependent coefficient matrix.
I	General surface integral for a linear triangular element.
{F}	Vector containing the flux terms.
K_r	Hydraulic conductivity in the radial (r) direction (LT^{-1}).
K_z	Hydraulic conductivity in the vertical (z) direction (LT^{-1}).
[LHS]	Coefficient matrix containing all terms on the left hand side of an equation.
M	Number of nodes.
N	Number of elements.
NE	Number of elements with boundaries at the well.
Q	Total well discharge (L^3T^{-1}).
Q_e	Well discharge through element e (L^3T^{-1}).
R	Radius of basin (L).
R	Residual.
{RHS}	Vector comprising all terms on the right hand side of an equation.
S	Storage coefficient.
S_s	Specific storage coefficient (L^{-1}).
S_y	Specific yield.
T	Transmissivity (L^2T^{-1}).
W	Recharge rate (LT^{-1}).
$W(u)$	Well function of u, an exponential integral.
a_i	A constant in the basis function.
a_{ij}	A member of matrix [A].
b	Thickness of aquifer (L).
b_i	A coefficient in the basis function.
b_{ij}	A member of matrix [B].
c_i	A coefficient in the basis function.
e	Element number.
f_i	Flux term.
i	Node number.
j	Node number.
k	Node number.
k	Time level.
l	Length long the side of an element (L).
m	Number of nodes.
n_r	Outward pointing directional cosine.
n_z	Outward pointing directional cosine.
q	Specific discharge (LT^{-1}).
q_r	Flux through an element in the radial direction (LT^{-1}).
q_z	Flux through an element in the vertcal direction (LT^{-1}).
r	Radial direction or distance from the center line of the well (L).
$\bar{r}$	Average radius of an element (L).
r°	Radial distance of a free surface node (L).
r_w	radius of the well (L).
s	drawdown (L)
$s_j(t)$	time dependent coefficient associated with node j.
$\hat{s}$	approximation of s (L).
t	time (T).
u	$\frac{r^2S}{4Tt}$
z	vertical direction or distance (L).
Σ	summation.
α°	coefficient associated with the unconfined boundary.
Δ	area of triangular element.
∂	partial differential.
$\int_{\Gamma'} ds$	surface integral over the outer boundary.
μ	a general dependent variable used as an example.
π	3.1416.
$\phi_j^e(r, z)$	linear basis function associated with node j in element e.
Ψ	$\frac{r}{b}\sqrt{\frac{K_z}{K_r}}$.

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Abstract

A computer program developed to evaluate radial flow of ground water, such as at a pumping well, recharge basin, or injection well, is capable of simulating anisotropic, inhomogenous, confined, or pseudo-unconfined (constant saturated thickness) conditions. Results compare well with those calculated from published analytical and model solutions. The program is based on the Galerkin finite-element technique.

A sample model run is presented to illustrate the use of the program; supplementary material provides the program listing as well as a sample problem data set and output. From the text and other material presented, one can use the program to predict drawdowns from pumping and ground-water buildups from recharge in a radially symmetric ground-water system.

INTRODUCTION

Several aspects of ground-water hydrology that concern radial flow address such questions as the distribution of drawdowns near a pumping well or ground-water buildup beneath a circular recharge basin or at a recharge well. To study and evaluate the performance of wells or basins and determine short- and long-term effects of their operation on the ground-water system, it is necessary to represent the system's physical properties mathematically and to calculate the response to given rates of pumping or withdrawal.

Questions concerning simple hydrologic situations can be solved through published analytical solutions, which generally provide accurate predictions. However, problems concerning more complex flow systems such as multi-aquifer systems require analysis by numerical methods. The purpose of this report is to introduce a computer program capable of solving many of the more complex radial-flow problems. The program was developed in cooperation with the Nassau County Department of Public Works, the Suffolk County Department of Health Services, the Suffolk County Water Authority, and the New York State Department of Environmental Conservation.

MODEL ASSUMPTIONS AND CAPABILITIES

The numerical model of ground-water flow described in this report simulates transient radial flow of ground water in which the flow field is two dimensional and symmetric around a central axis. Figure 1 illustrates the conceptual flow system to be simulated by the model. The numerical technique used to solve the transient radial flow equations is called the Galerkin finite-element method.

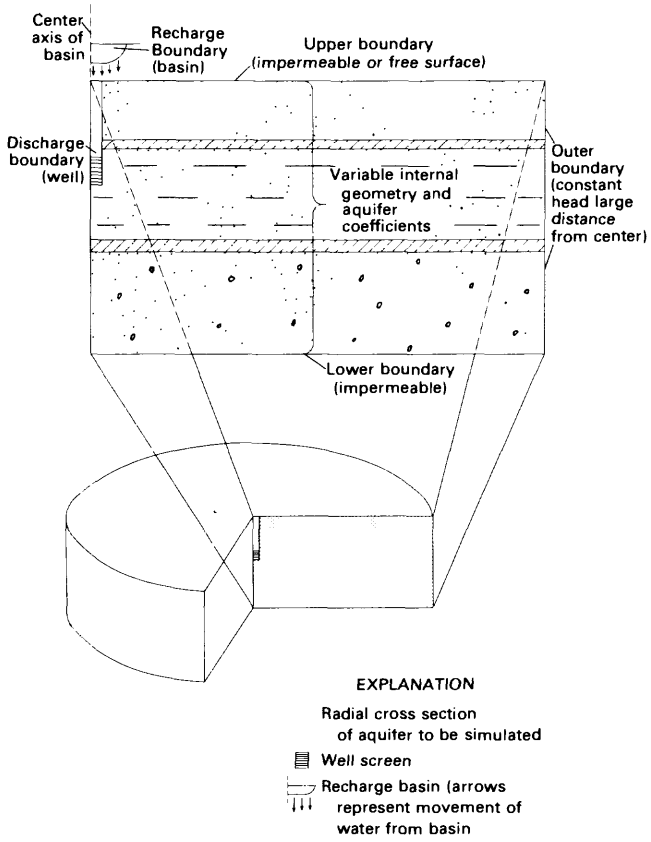

Figure 1. Conceptual model of aquifer flow system.

Use of the model requires certain assumptions, or simplifications, which must be evaluated before the program can be used successfully. These assumptions are

1. The flow field is radially symmetric.
2. No seepage face occurs in the well.
3. The saturated thickness of an unconfined aquifer does not change significantly during pumping or recharge.
4. The aquifer is finite in extent. (A constant-potential boundary far from the well or basin is generally established in the model.)

5. The well is pumped at either a constant rate or at stepped rates.
6. The specific yield of the aquifer is a constant.
7. The coefficient of specific storage (S_s) is a constant over the entire model grid.

Even with these assumptions, few of which are entirely true in the real system, the computer program can be used to predict drawdowns or buildups resulting from a wide range of conditions. The program is capable of simulating anisotropic, inhomogeneous, confined, and pseudo-unconfined (constant saturated thickness) conditions. (The pseudo-unconfined aquifer is so called because the element configuration in the model is constant, which means that, even though the model nodes representing the free surface have a storage coefficient representing unconfined or water-table conditions, the saturated thickness does not decrease in response to drawdowns. Thus, caution or "engineering judgment" must be used when the predicted drawdown at the free surface represents a significant percentage of the aquifer thickness.) In addition, the well-bore geometry can be simulated, and the well screen can be partially penetrating and screened in zones of differing hydraulic conductivity.

Linear triangular elements are used to represent the flow field. The radial section to be simulated is represented as a net, or grid, of connected elements, as depicted in figure 2. The model program allows each element to be assigned a value of the hydraulic conductivity in the radial (K_r) and vertical (K_z) directions. In contrast, the coefficient of specific storage (S_s) and the specific yield (S_y) are treated as constants throughout the grid. The treatment of the storage coefficients as constants is not a restriction of the solution technique but only the manner in which it was programmed; this is discussed later in the section "Possible Program Modifications."

THEORY

Solution of Radial Flow Equation by Galerkin Finite-Element Method

The equation that describes two-dimensional radial flow of ground water in cross section (Cooley, 1974, p. 20) is

$$\frac{\partial}{\partial r}\left(K_r r \frac{\partial s}{\partial r}\right) + \frac{\partial}{\partial z}\left(K_z r \frac{\partial s}{\partial z}\right) = S_s r \frac{\partial s}{\partial t}, \qquad (1)$$

where

s = drawdown (L),
r = radial distance (L),
z = vertical distance (L),
t = time (T),
K_r= radial hydraulic conductivity (LT^{-1}),
K_z= vertical hydraulic conductivity (LT^{-1}),
S_s= coefficient of specific storage (L^{-1}).

To simulate the continuous system, equation 1 is approximated by a series of linear algebraic equations. These linear equations are derived from the finite-element method by the Galerkin method of weighted residuals (Pinder and Gray, 1977, p. 54).

The linear equations are generated by discretizing, or dividing, the entire flow field, or aquifer cross section (fig. 1), into separate elements (linear triangular elements in this model). The drawdown in the flow field can be approximated through a linear basis function ϕ (r, z), which is defined separately for each element. The drawdown is approximated as

$$\hat{s} = \sum_{e=1}^{N} \sum_{j=1}^{M} s_j(t)\, \phi_j^e(r,z) \qquad (2)$$

where

$\hat{s}$ = approximation of s,
N= number of elements,
M= number of nodes,
ϕ_j^e (r, z) = linear basis function associated with node j in element e,
$s_j(t)$ = time dependent coefficient associated with node j,

Substitution of $\hat{s}$ for s into equation 1 and rearrangement of terms gives

$$\frac{\partial}{\partial r}\left(K_r r \frac{\partial \hat{s}}{\partial r}\right) + \frac{\partial}{\partial z}\left(K_z r \frac{\partial \hat{s}}{\partial z}\right) - S_s r \frac{\partial \hat{s}}{\partial t} \approx 0. \qquad (3)$$

The residual is the amount by which the equation with $\hat{s}$ varies from the actual solution of zero. Therefore, the residual R can be defined as:

$$R = \frac{\partial}{\partial r}\left(K_r r \frac{\partial \hat{s}}{\partial r}\right) + \frac{\partial}{\partial z}\left(K_z r \frac{\partial \hat{s}}{\partial z}\right) - S_s r \frac{\partial \hat{s}}{\partial t}. \qquad (4)$$

If $\hat{s}$ were exact, the residual would be zero. Galerkin's

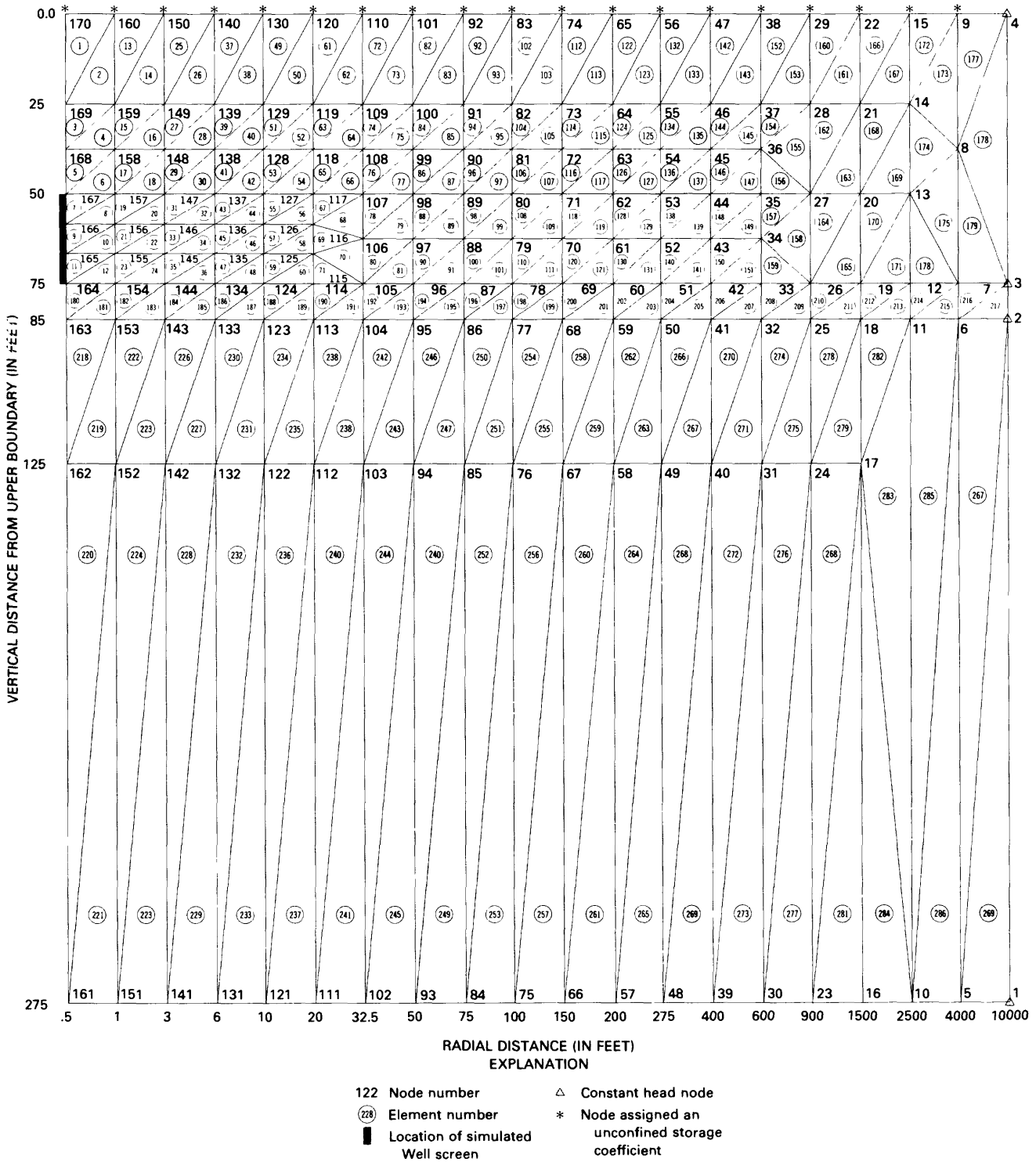

Figure 2. Modeled aquifer section showing configuration of triangular elements.

method attempts to force the residual toward zero over the entire domain by weighting the residual by the basis function. The weighted residual equation is

$$\int_r \int_z R\phi_i dz\, dr = 0 \qquad (i = 1,2,3,\ldots.m). \qquad (5)$$

Each node has an approximate solution $\hat{s}$ and therefore a weighted residual equation. Because the number of nodes equals the number of equations, with use of appropriate boundary and initial conditions, this set of simultaneous ordinary differential equations in time can be solved for $\hat{s}$.

The set of equations is derived by substituting for R into our weighted residual equation (eq. 5), which gives

$$\int_r \int_z \left\{ \frac{\partial}{\partial r}\left[K_r r \frac{\partial}{\partial r}\left(\sum_{e=1}^{N}\sum_{j=1}^{M} s_j(t)\phi_j^e(r,z)\right)\right]\phi_i^e + \frac{\partial}{\partial z}\left[K_z r \frac{\partial}{\partial z}\left(\sum_{e=1}^{N}\sum_{j=1}^{M} s_j(t)\phi_j^e(r,z)\right)\right]\phi_i^e - S_s r \frac{\partial}{\partial t}\left(\sum_{e=1}^{N}\sum_{j=1}^{M} s_j(t)\phi_j^e(r,z)\right)\phi_i^e \right\} dz\, dr = 0 \qquad (6)$$

$$(\text{for } i = 1,2,3,\ldots.m).$$

Before equation 6 can be simplified, the basis function must be defined. For this model, the linear triangular element was selected. The basis function (ϕ) is in the form of a plane and is expressed as

$$\phi_i(r,z) = a_i + b_i r + c_i z \qquad (7)$$

where the coefficients are

$a_i = (r_j z_k - r_k z_j)/2\Delta$,
$b_i = (z_j - z_k)/2\Delta$,
$c_i = (r_k - r_j)/2\Delta$,
Δ = area of triangular element,

and

i,j,k = element nodes, in counterclockwise order.

Thus, a basis function is defined for each node in each element. A property of the basis function defined is

$$\phi_i^e + \phi_j^e + \phi_k^e = 1 \qquad (8)$$

where

i, j, and k are the three nodes in element e.

The basis function has a value only over the element for which it is defined and is zero over all other elements. Its value is unity at the node for which it is defined and decreases linearly to zero at the other two nodes associated with the element. Figure 3 depicts the basis function ϕ_i^e.

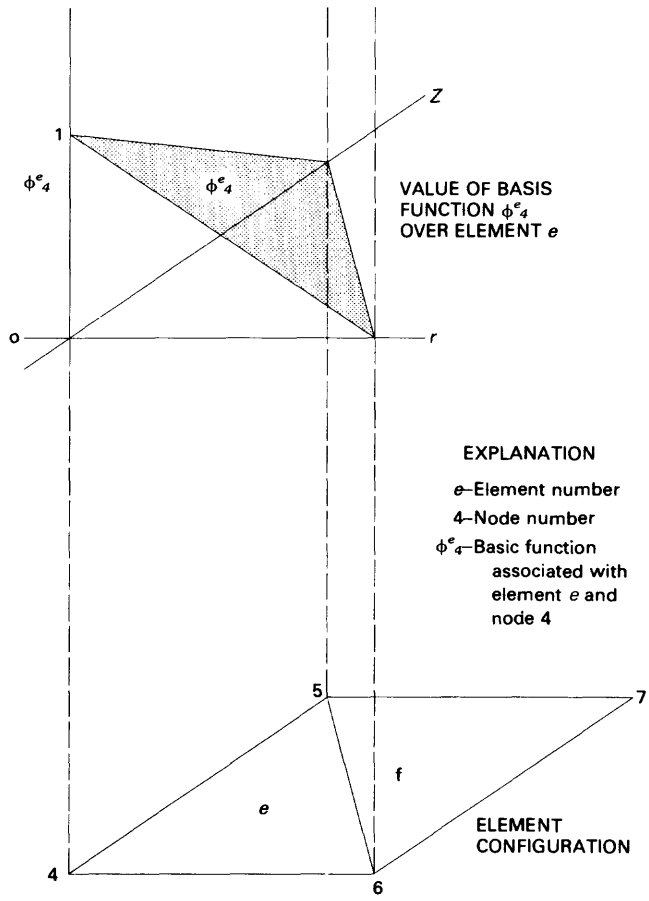

Figure 3. Typical linear triangular basis function.

Two additional assumptions that will allow for simplification of equation 6 are (a) both radial and vertical hydraulic conductivity are defined as a constant in each element, and (b) an average radius ($\bar{r}$) is defined for each element. This average radius ($\bar{r}$) is defined as

$$\bar{r} = (r_i + r_j + r_k)/3 \qquad (9)$$

This average radius ($\bar{r}$) was defined similarly by Pinder and Gray (1977, p. 139).

Substituting and using the notation for the inner product as

$$\int_r \int_z \phi_j^e\, \phi_i^e\, dz\, dr = \left\langle \phi_j^e, \phi_i^e \right\rangle \qquad (10)$$

equation 6 can be written as

$$\sum_{j=1}^{M}\sum_{e=1}^{N}\left[s_j(t) K_r^e \bar{r}^e \left\langle \frac{\partial^2 \phi_j^e}{\partial r^2}, \phi_i^e \right\rangle + s_j(t) K_z^e \bar{r}^e \left\langle \frac{\partial^2 \phi_j^e}{\partial z^2}, \phi_i^e \right\rangle - S_s r_j \left\langle \phi_j^e, \phi_i^e \right\rangle \frac{\partial s_j(t)}{\partial t}\right] = 0 \qquad (11)$$

$$(i = 1,2,3,\ldots.m)$$

Because the basis function (ϕ) is a linear function, the second derivative is trivial (zero on the interior of an element). To circumvent this problem, Green's Theorem or Integration by Parts (Pinder and Gray, 1977, p. 83) is used. This changes the inner products to

$$\left\langle \frac{\partial^2 \phi_j^e}{\partial r^2}, \phi_i^e \right\rangle = - \left\langle \frac{\partial \phi_j^e}{\partial r}, \frac{\partial \phi_i^e}{\partial r} \right\rangle + \int_\Gamma \phi_i^e \frac{\partial \phi_j^e}{\partial r} n_r \, ds \quad (12a)$$

and

$$\left\langle \frac{\partial^2 \phi_j^e}{\partial z^2}, \phi_i^e \right\rangle = - \left\langle \frac{\partial \phi_j^e}{\partial z}, \frac{\partial \phi_i^e}{\partial z} \right\rangle + \int_\Gamma \phi_i^e \frac{\partial \phi_j^e}{\partial z} n_z ds \quad (12b)$$

where

$\int_\Gamma ds$ represents a surface integral over the outer boundary, and n_r and n_z are outward-pointing directional cosines. These surface integrals (flux terms) become the forcing functions of the problem for all flux boundaries. The surface integrals become zero for impermeable boundaries and are readily defined for discharge and recharge boundaries.

Substituting the result of Green's Theorem into equation 11 results in the final form, which is a set of ordinary differential equations:

$$\sum_{j=1}^{M} \sum_{e=1}^{N} \Big[s_j(t) K_r^e \bar{r}^e \left\langle \frac{\partial \phi_j^e}{\partial r}, \frac{\partial \phi_i^e}{\partial r} \right\rangle + s_j(t) K_z^e \bar{r}^e \left\langle \frac{\partial \phi_j^e}{\partial z}, \frac{\partial \phi_i^e}{\partial z} \right\rangle + S_s r_j \left\langle \phi_j^e, \phi_i^e \right\rangle \frac{\partial s_j(t)}{\partial t} - s_j(t) K_r^e \bar{r}^e \int_\Gamma \phi_i^e \frac{\partial \phi_j^e}{\partial r} n_r ds - s_j(t) K_z^e \bar{r}^e \int_\Gamma \phi_i^e \frac{\partial \phi_j^e}{\partial z} n_z ds \Big] = 0 \quad (13)$$

$$(i = 1,2,3, \ldots . m)$$

The inner products are non-zero only when the node indices i and j are part of the same element. Thus, the inner products can be defined over an element. The integration formulae for a linear triangular element are

$$\left\langle \frac{\partial \phi_j}{\partial r}, \frac{\partial \phi_i}{\partial r} \right\rangle = b_i b_j \triangle \quad (14a)$$

$$\left\langle \frac{\partial \phi_j}{\partial z}, \frac{\partial \phi_i}{\partial z} \right\rangle = c_i c_j \triangle \quad (14b)$$

$$\left\langle \phi_j, \phi_i \right\rangle = \triangle/6 \qquad (j = i) \quad (14c)$$

$$\left\langle \phi_j, \phi_i \right\rangle = \triangle/12 \qquad (j \neq i) \quad (14d)$$

When the integration formulae are substituted into equation 13 as indicated by the summation indices, the result is a set of M (the number of unknowns) simultaneous ordinary differential equations. Thus, application of the Galerkin finite-element technique has changed the continuous partial differential equation into a set of simultaneous ordinary differential equations.

Boundary Conditions

Constant-head (zero drawdown) boundaries are readily handled in the finite-element method. The nodes that are set at a constant zero drawdown are not unknowns, and the equations associated with those nodes drop out of the set of simultaneous equations.

Flux boundaries (Neumann boundary conditions) enter into the finite element representation of the radial-flow equation by means of the surface integrals generated by using Green's Theorem (eq. 12). For linear triangular elements, the surface integral, which can be represented in general terms as

$$I = \int_\Gamma \left(\frac{\partial \mu}{\partial n} \right) \phi_i \, ds, \quad (15)$$

can be integrated to become the following (Pinder and Gray, 1977, p. 124):

$$I = \frac{q \, l}{2} \quad (16)$$

where

q = average value of $\frac{\partial \mu}{\partial n}$ applied along the side, and

l = length along the side.

In examining the flux boundaries that occur in the radial-flow problem, the evaluation of the surface integrals is identical to equation 16.

Ground-water discharge to the well is treated as a flux across the boundary of the elements at the well-screen location. The surface integral is reduced as

$$- \int_\Gamma s_j(t) K_r^e \bar{r}^e \phi_i^e \frac{\partial \phi_j^e}{\partial r} n_r ds = - \bar{r} \int_\Gamma \phi_i K_r \frac{\partial \hat{s}}{\partial r} \, ds, \quad (17)$$

from Darcy's law, $K_r \frac{\partial \hat{s}}{\partial r} = q_r$, which is the flux across the well boundary.
Then

$$- \bar{r} \int_\Gamma \phi_i q_r^e ds = - \bar{r} \, q_r^e \, \frac{(z_k - z_i)}{2} \quad (18)$$

where

$z_k - z_i$ = length of the element along the well screen, and

q_r^e = flux through element e.

At the well, the boundary of the element is at r_w (well radius), and the sum of all flows from the elements must be equal to total discharge (Q).

Examining q_r^e more closely and again using Darcy's law, it can be shown that

$$q_r^e = \frac{Q}{A} = \frac{Q_e}{2\pi r_w (z_k^e - z_i^e)}, \quad (19)$$

where

Q_e = flow through element e.

Substituting this result into the right-hand side of equation 18 gives

$$-\bar{r}q_r^e\frac{(z_k - z_i)}{2}=\frac{\bar{r}\,Q_e(z_k - z_i)}{4\pi r_w(z_k - z_i)}. \quad (20)$$

If $\bar{r}\approx r_w$, then

$$-\bar{r}\int_\Gamma \phi_i q_r^e\, ds = Q_e/4\pi . \quad (21)$$

Therefore, the surface integral along the well simplifies down to equation 21.

The discharge from an individual element (Q_e) is calculated in a manner similar to that used for a model prepared for the U.S. Geological Survey by Intercomp Resource Development and Engineering, Inc. (1976, p. B.2)[1]. The discharge of an element (Q_e) in relation to the entire discharge (Q) is proportional to the transmissivity of the element divided by the transmissivity of the entire screen length. Thus,

$$Q_e = \frac{Q\,K_{r_e}(z_k - z_i)}{\sum_{n=1}^{NE} K_{r_n}(z_k - z_i)}, \quad (22)$$

where

Q_e is the discharge from the eth element,

NE is the number of elements with boundaries at the well,

Q is the total discharge, and

$(z_k - z_i)$ is the length of the element boundary at the well.

Therefore, the model allocates discharge rates along the well bore according to equation 22 for wells screened in aquifers of varying hydraulic conductivity. This boundary condition is not exact; it only approximates the theoretical withdrawal distribution of the well. The correct boundary condition is an equal drawdown along the well bore. Therefore, the model results will be somewhat in error if this condition is not realized.

Recharge is treated as a flux across the top surface of the aquifer. The surface integral in this case reduces as

$$-\int_\Gamma K_z\bar{r}\frac{\partial\hat{s}}{\partial z}\phi_i n_z ds=\frac{\bar{r}q_z(r_j - r_k)}{2}=-2\pi\bar{r}(r_j - r_k)q_z/4\pi, \quad (23)$$

where

q_z = recharge in flux units (ft/day).

The term $2\pi\bar{r}(r_j - r_k)$ represents the surficial area of the top element, where the average circumference is $2\pi\bar{r}$, and $(r_j - r_k)$ is the width of the element at the surface. Therefore, $2\pi\bar{r}(r_j - r_k)$ is simply an approximation of $\pi(r_j^2 - r_k^2)$, which is the actual surface area. The recharge at the surface then simplifies as

$$-\int_\Gamma K_z\bar{r}\frac{\partial\hat{s}}{\partial z}\phi_i n_z ds = -(r_j^2 - r_k^2)\,q_z/4. \quad (24)$$

This is actually the same expression as the well discharge (or recharge) because $\pi(r_j^2 - r_k^2)q_z$ is the amount of water being added into the element.

The water in storage released by the movement of the free surface (water table) can be simulated, although the actual movement of the free surface is not simulated. Thus, the solution for unconfined aquifers is valid only when the drawdown equals a small percentage of the total aquifer thickness.

For the unconfined boundary condition, the surface integral for the top boundary is reduced by a few assumptions. These were first described by Boulton (1954) and later by Stallman (1963), who show that

$$S_y\frac{\partial s}{\partial t}=K_r\left(\frac{\partial s}{\partial r}\right)^2+K_z\left[\left(\frac{\partial s}{\partial z}\right)^2-\frac{\partial s}{\partial z}\right], \quad (25)$$

where

S_y is called the specific yield and represents the volume of water which the rock or soil, after being saturated, will yield by gravity divided by the volume of the rock or soil (Lohman, 1972). This assumes that once a particle of water is on the free surface, it never leaves that surface. The additional assumption that the squared derivatives are much smaller than the first-order derivatives gives

$$S_y\frac{\partial s}{\partial t}\approx -K_z\frac{\partial s}{\partial z}. \quad (26)$$

The surface integral for the top horizontal boundary can therefore be evaluated as

$$-\int_\Gamma K_z\bar{r}\frac{\partial\hat{s}}{\partial z}\phi_i n_z ds\approx\int_\Gamma S_y\bar{r}\frac{\partial\hat{s}}{\partial t}\phi_i n_z ds\approx\frac{\bar{r}\,(r_j^o - r_k^o)\,S_y}{2}\frac{\partial\hat{s}}{\partial t}, \quad (27)$$

where

r^o indicates the radial distance of a free surface node.

By the same analysis for the surface area as in the recharge term, the boundary can be approximately evaluated as

$$\approx\frac{(r_j^{o2} - r_k^{o2})\,S_y}{4}\frac{\partial\hat{s}}{\partial t} \quad (28)$$

Matrix Equations

The preceding section describes the finite-element equations that are used to generate a system of simultaneous ordinary differential equations from the general equation for radial flow of ground water in cross section. The equations are produced by calculating an element matrix (a set of coef-

[1] Use of trade names is for identification purposes only and does not imply endorsement by the U.S. Geological Survey.

ficients for that element). Because all elements contain three nodes, each element is associated with a three-by-three matrix. The coefficients for each element are then assembled into one "global" coefficient matrix, which represents the entire radial section of aquifer.

The global matrix equation to be solved (Pinder and Gray, 1977, p. 81) can be represented by

$$[A]\{s(t)\} + [B]\left\{\frac{ds(t)}{dt}\right\} = \{F\} \qquad (29)$$

where

[A] = a_{ij}, which is made up of terms of the form

$$\sum_{e=1}^{N}\left[\left\langle\frac{\partial\phi_i}{\partial r},\frac{\partial\phi_j}{\partial r}\right\rangle+\left\langle\frac{\partial\phi_i}{\partial z},\frac{\partial\phi_j}{\partial z}\right\rangle\right],$$

[B] = b_{ij}, which is made up of terms of the form

$$\sum_{e=1}^{N}\left[\langle\phi_i,\phi_j\rangle+\alpha^o S_y\right],$$

{F} = f_i, which is made up of terms of the form

$$\sum_{e=1}^{N}\left[\int_\Gamma f_i\, ds\right],$$

for $i = 1, 2, \ldots . M$
and $j = 1, 2, \ldots . M$

α^o = coefficient associated with the unconfined boundary condition,
f_i = flux term.

The time derivative is then approximated using a backward difference formulation. When the finite difference approximation is used for the time derivative, the set of ordinary differential equations becomes a set of simultaneous algebraic equations. The backward difference approximation is unconditionally stable and is represented simply as

$$\frac{ds_j(t)}{dt} = \frac{s_j^{k+1} - s_j^k}{t_{k+1} - t_k} \qquad (30)$$

where

k is the time level.

The matrix equation can then be written:

$$[A]\left\{s^{k+1}\right\}+[B]\left\{\frac{s^{k+1} - s^k}{\Delta t}\right\} = \{F\} \qquad (31)$$

or simply,

$$\left([A]+\frac{1}{\Delta t}[B]\right)\left\{s^{k+1}\right\} = \{F\}+\frac{1}{\Delta t}[B]\{s^k\}. \qquad (32)$$

Let:

$$\left([A]+\frac{1}{\Delta t}[B]\right)=[LHS] \qquad (33a)$$

$$\left(\{F\}+\frac{1}{\Delta t}[B]\left\{s^k\right\}\right)=\{RHS\}. \qquad (33b)$$

The final matrix equation is

$$[LHS]\left\{s^{k+1}\right\} = \{RHS\} \qquad (34)$$

from which the drawdown at the new time step can then be solved. The technique used for this model is a direct solution technique.

The method that is used to generate the global matrices is actually less complicated than in equation 13. Because the inner products are zero when i and (or) j are not nodes of element e, the global matrix can be calculated more easily by using a modified form:

$$\sum_{e=1}^{N}\sum_{j=1}^{3}\Big[s_j(t)K_r^e\bar{r}^e\left\langle\frac{\partial\phi_j^e}{\partial r},\frac{\partial\phi_i^e}{\partial r}\right\rangle+s_j(t)K_z^e\bar{r}^e\left\langle\frac{\partial\phi_j^e}{\partial z},\frac{\partial\phi_i^e}{\partial z}\right\rangle$$

$$+S_s r_j\left\langle\phi_j^e\cdot\phi_i^e\right\rangle\frac{\partial s_j(t)}{\partial t}-s_j(t)K_r^e\bar{r}^e\int_\Gamma\phi_i^e\frac{\partial\phi_j^e}{\partial r}n_r ds$$

$$-s_j(t)K_z^e\bar{r}^e\int_\Gamma\phi_i^e\frac{\partial\phi_j^e}{\partial z}n_z ds\Big]=0$$

$$(i=1,2,3) \qquad (13 \text{ modified})$$

where

i =1, 2, 3 and j=1, 2, 3 are the three nodes in element e, and

N = number of elements.

Thus, for each element, a three-by-three element matrix is defined. These matrices are then summed over all elements to obtain the global matrix to be used in the final matrix equations.

COMPUTER PROGRAM

The input and output of the program data are not complex, and changes in the basic computer code to improve input-output or to add additional information should be straightforward to implement. The computer code consists of four routines—the main, subroutine CHECK, subroutine MLTBM, and subroutine SOLVE, which are listed in supplemental data I. A brief summary of each follows.

Main

The main program runs the input, output, and equation setup. The equation setup is the calculation of the coefficients for the boundary conditions and the global coefficient matrices.

To save computer storage, the program uses banded matrix storage. The global coefficient matrix has a half band width equal to the largest difference between two adjacent node numbers. To run the program, the global array must be dimensioned properly (dimensions are given in sup-

plemental data IV), but the program calculates the band width, and the operator should note the size required.

Subroutine CHECK

This subroutine checks the input data to see if they meet the requirements as stated in supplemental data III (Data Input Formats) and in the section "Design Considerations for the Finite-Element Grid." Data are checked for three criteria: (1) consistent constant head information, (2) proper vertical coordinate system, and (3) proper ordering of nodes in an element. If the input data fail to meet one of these criteria the reason is printed, and the program is terminated.

Subroutine MLTBM

This small subroutine multiplies an array stored in a transformed banded manner with a vector.

Subroutine SOLVE

This subroutine solves the final matrix equation and returns the drawdown solution. This routine was originally programmed by James O. Duguid of the Oak Ridge National Laboratory, (R. B. Wells, J. W. Mercer, and C. R. Faust, written commun., 1976). SOLVE accepts a fully banded nonsymmetric matrix. During the solution process, the routine is called twice. The first call upper triangularizes the matrix using the Gauss-Doolittle method, and the second call solves the triangularized form by back substitution.

EVALUATION OF THE MODEL

To insure that the model theory and programming are correct, comparisons were made between results from this model and those obtained from published problem solutions. Although the comparisons with analytical solutions by Theis (Lohman, 1972), and Hantush (Walton, 1970, p. 370) were done for simple examples, the close fit indicates that the model is sound. The comparison of model results to the simple analytical solutions is also intended to give some insight into the discretization errors of the finite-element model.

In a more complex test, results of Stallman's (Lohman, 1972) analog-model type curves were compared with results obtained by the model described in this report, and again, the similarity of results indicates that the model is valid. The comparison with Stallman's model results is primarily intended to check the trends in the response of the finite-element model to a complex problem. Although there may be better solutions available to compare the results of the finite-element model against, the purpose of this comparison is only to further substantiate that the finite-element model can simulate complex two-dimensional radial solutions. A discussion of both tests follows.

Comparison with Analytical Solutions

In comparing the model results to Theis' analytical solution, the radial cross section was set to resemble as closely as possible the conditions of the analytical solution. The only condition that could not be met was that of an infinite aquifer, because the model must have a radial boundary at a finite distance. For comparison with the analytical solutions, a zero-drawdown boundary 10,000 ft from the center of the radial section was simulated.

To verify the model response for a well pumping under transient artesian conditions, model results were compared with the Theis type curve (from Lohman, 1972). The Theis solution is for a fully penetrating well in an infinite confined aquifer with no vertical movement of water. The model results were transformed into Theis' dimensionless coefficients u and $W(u)$ from their relationship to drawdown and time:

$$u = \frac{r^2 S}{4T} \frac{1}{t} \tag{35a}$$

$$W(u) = \frac{4\pi T}{Q}(s) \tag{35b}$$

where

r = radial distance from well,
S = storage coefficient,
T = transmissivity,
Q = well discharge,
t = time,
s = drawdown.

Transformed model results for two different simulations are plotted against the Theis type curve in figure 4. The first simulation is plotted with circles, and the first point plotted is the worst comparison point. The second simulation used the same aquifer conditions as the first, except that a smaller initial time step (DELT) was used. This second simulation is plotted with squares and indicates that part of the error in the first simulation was probably due to the truncation error associated with the time derivative. Thus, the smaller initial time step gives a better match with the Theis type curve.

The next check on the model response was to evaluate its accuracy in predicting water-table mounding beneath a circular recharge basin. In this check, the assumption of no recharge, which was used in formulating the upper unconfined boundary, was violated in that both the recharge flux and free-surface boundary conditions were used simultaneously.

The simplified form of Boulton's free-surface equation (eq. 26) is

$$S_y \frac{\partial s}{\partial t} \approx -K_z \frac{\partial s}{\partial z},$$

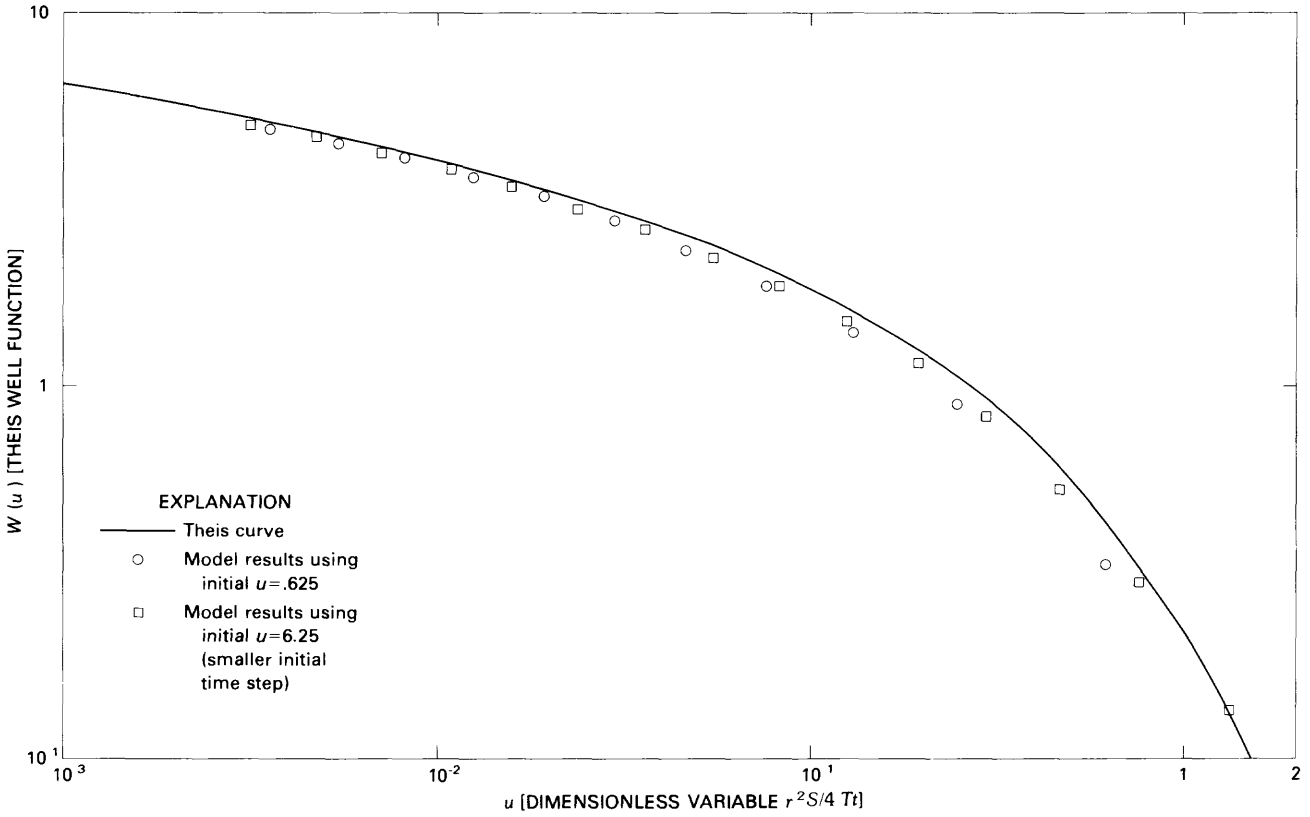

Figure 4. Comparison of model results with Theis' dimensionless analytical solution.

which relates two expressions for the rate of downward movement at the surface (Bennett and others, 1967). Boulton evaluated the substantial time derivative at the free surface, which made it necessary to assume that a particle of water always stays on the free surface. However, an equation analogous to the simplified equation (eq. 26) can be derived using a mass balance on an elemental volume at the free surface. This analogous equation allows recharge at the free surface and relates the boundary flux to the storage term and the recharge rate, which can be written:

$$-K_z\frac{\partial s}{\partial z} = S_y\frac{\partial s}{\partial t} - W(r) \qquad (36)$$

where

$W(r)$ is the recharge rate.

This equation relates three expressions for the rate of downward movement at the free surface. Thus, comparison of the model results to the analytical solution for a recharge mound derived by Hantush (Walton, 1970) provides a check on the appropriateness of the boundary condition as expressed in equation 36.

To reproduce Hantush's solution, a simulation was made for a one-dimensional homogenous aquifer with the following properties:

b = thickness of aquifer = 700 ft,
R = radius of basin = 100 ft,
W = recharge rate = 10 ft/d,
K_r = radial hydraulic conductivity = 100 ft/d,
K_z = vertical hydraulic conductivity = 1,000,000 ft/d (The reason for the high vertical hydraulic conductivity is to approximate no vertical head gradient—the Dupuit assumption [Lohman, 1972, p. 11]),
S_s = specific storage = 0,
S_y = specific yield = 0.3.

Model results are compared with Hantush's solution in figure 5. The results show a reasonably close match between the analytical solution and the model results.

Comparison with Model Solutions

To fully evaluate the accuracy of the model results, it was necessary to simulate a pumping well in an unconfined, anisotropic aquifer. Stallman (Lohman, pl. 6, 1972) developed dimensionless type curves for such a case, with five families of curves, each for a different vertical screen setting.

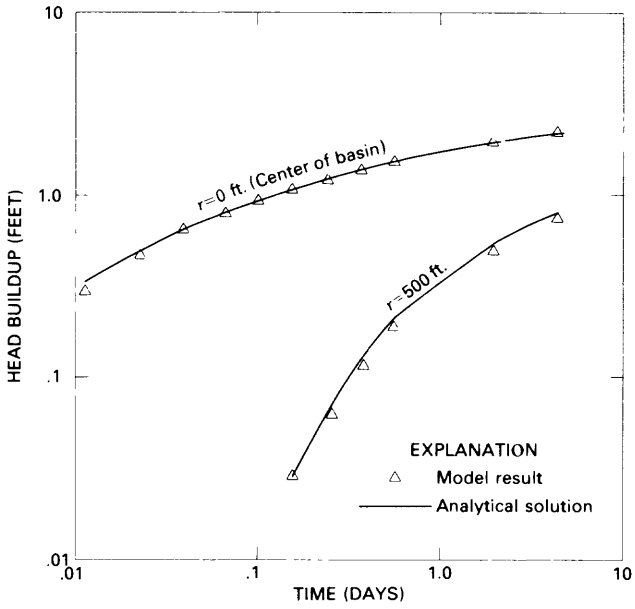

Figure 5. Comparison of model results with Hantush's analytical solution for head buildup in aquifer directly beneath center of circular recharge basin and 500 ft from basin center.

Figures 6 and 7 compare results from the model described in this paper to results generated by Stallman's electric-analog model. Figure 6 indicates drawdowns at the water table, and figure 7 indicates drawdowns at the bottom of the aquifer. Model results are for various psi (Ψ) values:

$$\Psi = \frac{r}{b}\sqrt{\frac{K_z}{K_r}}, \tag{37}$$

where

b = thickness of aquifer,
r = radial distance of observation point from center,
K_r = radial hydraulic conductivity,
K_z = vertical hydraulic conductivity.

As can be seen in figures 6 and 7, the predicted response of the aquifer has the same characteristics for both solutions. The response at the water table is very different from the predicted response at the base of the aquifer and both solutions produce the same trends. Since Stallman's analog model was subject to errors due to space discretization and inaccuracies associated with the electrical components (resistors and capacitors usually have accuracies in the range of ± 10 percent) and the finite-element method presented is subject to errors due to space and time discretization, the results were expected to be slightly different. Taking this into account, the results appear to compare well even though some differences exist.

Observations on Model Behavior

Although the model worked well in all tests, some noteworthy features emerged during some of the tests.

1. The average radius in the integration formula (eq. 9) is an approximation. Zienkiewicz (1971, p. 79–80) observed that because a fairly fine subdivision is required with linear triangular elements, this "one-point" integration is satisfactory. With a coarse element mesh, however, errors about the true solution become apparent, and in some cases the predicted drawdown will be slightly smaller or larger than in the actual system, depending on the three radius values of each element. In the tests involving comparison with the analytical solutions, most errors were insignificant compared to the magnitude of the drawdown.
2. The FORTRAN computer code was written for the CDC 7600 computer[2] at Brookhaven National Laboratories. In most test cases, it ran for the minimum charge. The computer code should be capable of running on most computers with no necessary modifications, but the cost may vary considerably among computer centers.

Possible Program Modifications

The computer program presented contains assumptions that satisfied the author's needs for local application but may not satisfy the user's needs; for example, the assumption of a constant coefficient of specific storage (S_s) over the entire grid. However, this constant in the program could easily be changed into an array that could assign a different specific storage to each element, and this can be accomplished by simply adding a read statement to read a specific storage for each element and changing the specific-storage constant in the calculation of the element matrices to an array associated with each element.

Two major modifications are being evaluated for future implementation. Although they would require extensive reprogramming, they could offer major advantages. The first would involve use of a logarithmic transformation on the radial coordinate system so that the flow system could be evaluated without use of the average-radius (eq. 9) approximation. As discussed in the previous section, this approximation introduces some error, and the logarithmic coordinate transformation should eliminate it. The second modification would simulate the vertical movement of the free surface (the

[2]Use of brand names is for identification purposes only and does not imply endorsement by the U.S. Geological Survey.

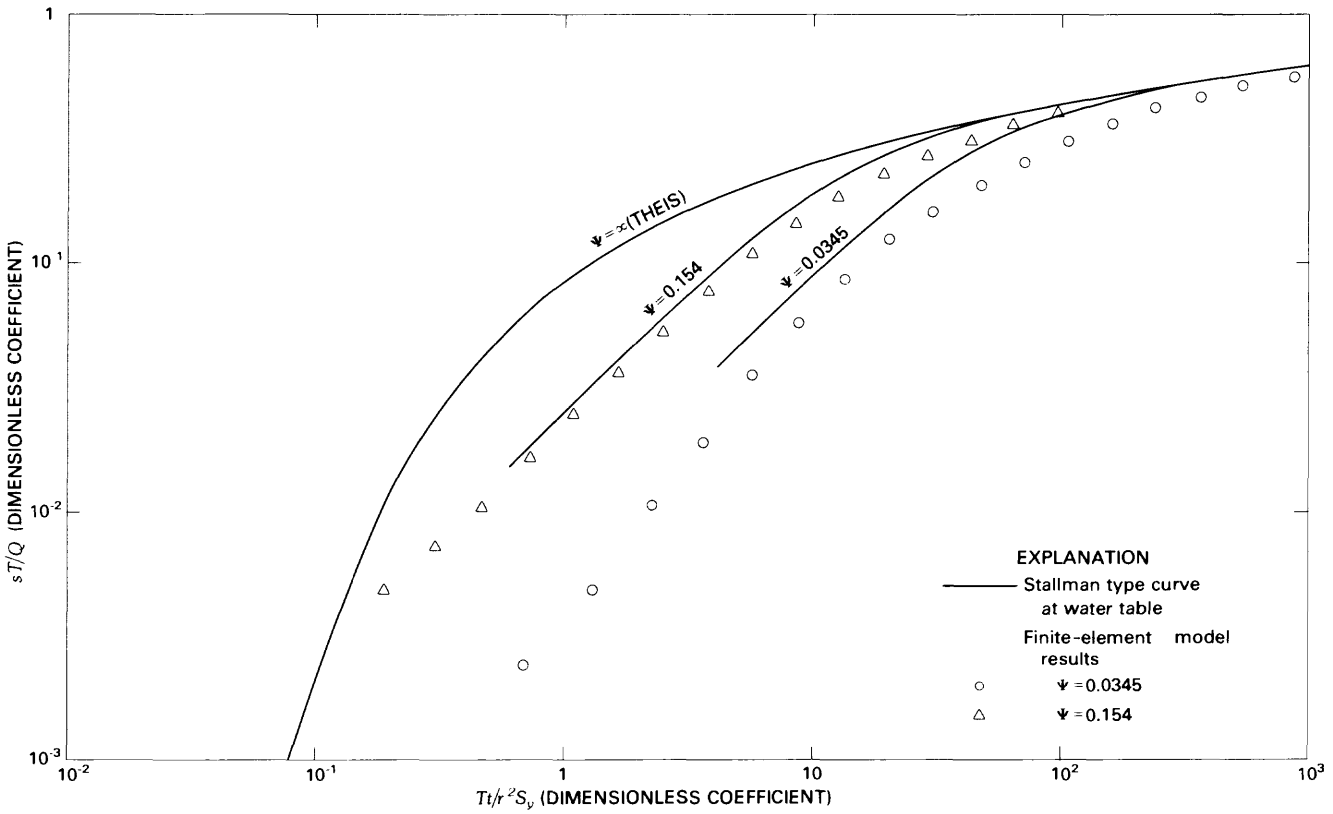

Figure 6. Comparison of water-table drawdowns as predicted by Stallman's analog model and the Galerkin finite-element model.

solution of the nonlinear problem), which may probably be accomplished by an iterative technique in which the element configuration heaves (or distorts) in the vertical dimension.

DESIGN CONSIDERATIONS FOR THE FINITE-ELEMENT GRID

In designing a finite-element grid, two types of design constraints must be met—those imposed by the mathematical procedures and those imposed by the computer-programming procedure. Both should be understood before the user attempts to make use of the program.

Three design considerations are associated with the mathematical methods used. First, because the solution presented involves linear triangular elements and requires use of an average radius for each element, the flow field should be represented in as much detail as possible (a fine grid). Second, the solution technique assumes that the nodal order for each element is specified in counterclockwise direction (also noted in supplemental data III). Third, as mentioned previously, the global coefficient matrices are sparse-banded, and the smaller the band width, the more efficient the solution technique for both time and storage requirements. The band width is determined by the largest difference between two node numbers in an element. Thus, an efficient nodal order improves the efficiency of the solution. The program also outputs the band width for the user to check the array dimensions.

Two additional design considerations are necessary because of the way in which the mathematical procedure was programmed, and both must be met for the program to work. The first is that the constant zero-drawdown nodes must be numbered first, which means that if constant drawdown is to be represented by three nodes, they must be nodes 1, 2 and 3. The second consideration is that the vertical coordinate system must start with zero at the free surface and increase in a positive manner to indicate the depth of the node below the top boundary. This second restriction is due solely to the manner in which the surface area of the top boundary is calculated.

SAMPLE SIMULATION

To illustrate the use of the model, the radial cross section shown in figure 8 was simulated. The cross section shown is typical of Long Island, New York, and is composed

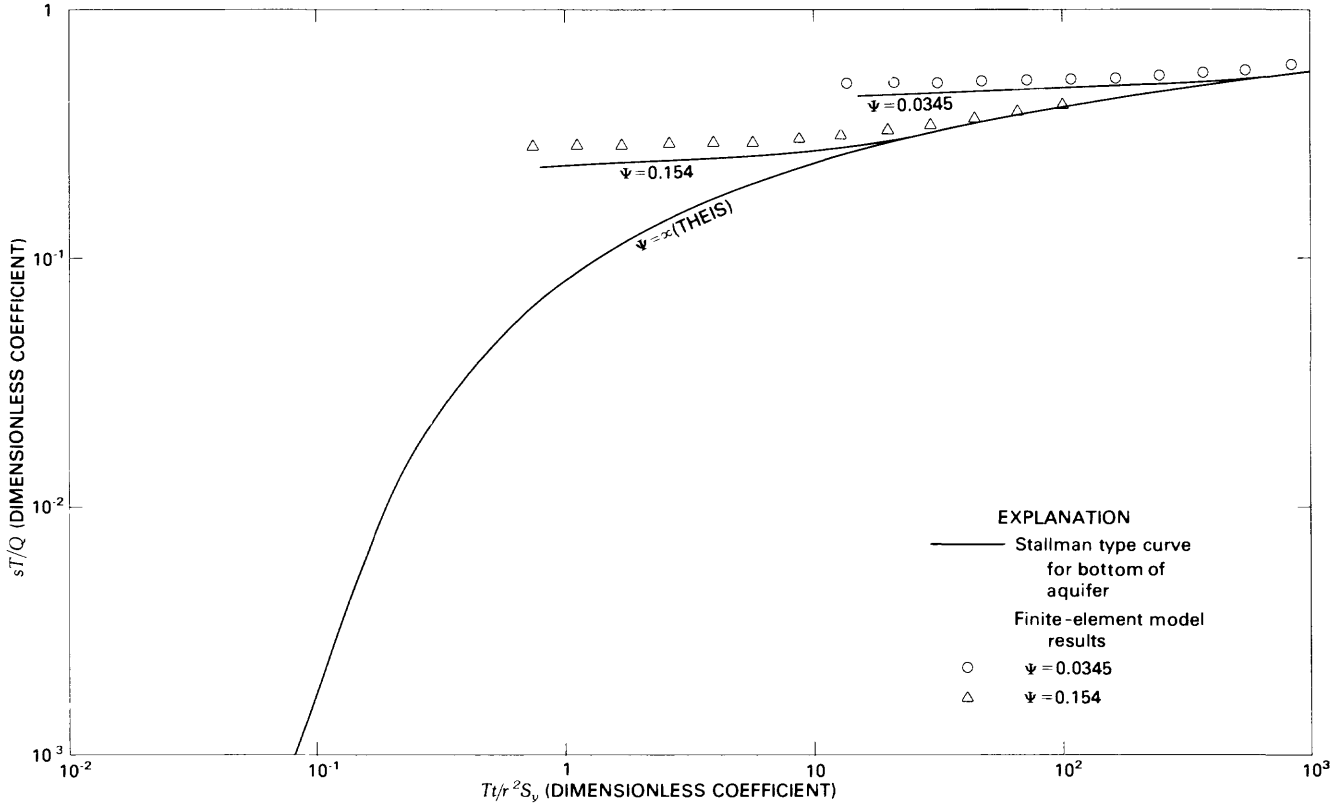

Figure 7. Comparison of drawdowns at the base of an aquifer as predicted by Stallman's analog model and the Galerkin finite-element model.

of two aquifers separated by a confining unit. The element configuration used to simulate this section is shown in figure 2.

Supplemental data III and V give the input formats and data used for this model run. The well screen is in the bottom 25 feet of the upper aquifer. The coefficients used for the simulation are

K_z (vertical hydraulic conductivity of upper glacial aquifer) = 27.0 ft/d,

K_r (horizontal hydraulic conductivity of upper aquifer) = 270.0 ft/d

K_z (vertical hydraulic conductivity of confining unit) = 0.001 ft/d,

K_r (horizontal hydraulic conductivity of confining unit) = 0.01 ft/d,
K_z (vertical hydraulic conductivity of lower aquifer) = 1.4 ft/d,
K_r (horizontal hydraulic conductivity of lower aquifer) = 50.0 ft/d,
Q (pumping rate) = 1.0 ft^3/s,
S_y (specific yield) = 0.25,
S_s (specific storage) = 1.0×10^{-6} ft^{-1}.

The results for one time step are presented in supplemental data VI. The results simply present the total time into the simulation and the drawdown for each node in the finite-element configuration.

CONCLUSIONS

The Galerkin finite-element radial-flow ground-water model described can simulate anisotropic, inhomogeneous, confined, and pseudo-unconfined aquifer conditions. Comparisons between radial-flow model results and published solutions are satisfactory. The program described allows more freedom in representing the field conditions than previously published methods.

The program, names of variables, data input formats, and array dimensioning are described in the supplemental data. From the text and supplemental data, one can use the program and modify the input and output of data as necessary to obtain accurate predictions.

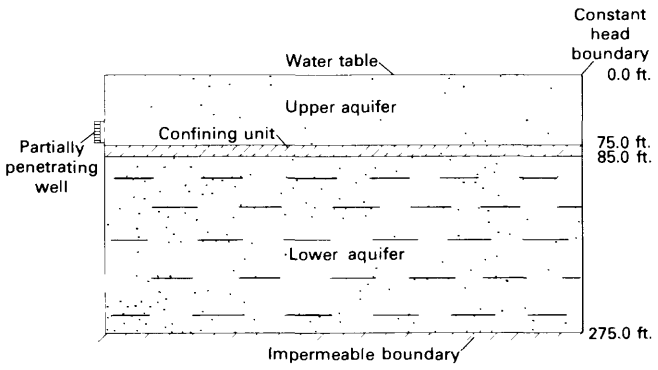

Figure 8. Radial cross section of aquifer for sample simulation.

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SUPPLEMENTAL DATA

SUPPLEMENTAL DATA I

FORTRAN IV Program Listings

A. Main Program

```
      PROGRAM RAD(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT,TAPE7=OUTPUT)    MAN  10
      DIMENSION NG(288,3), PR(288), PZ(288), RE(170), ZE(170), IFLUX(170MAN  20
     1), IQ(170), IRCH(170), NSCON(170), F(170), S1(170), S3(166), S5(16MAN  30
     26), F2(166), WG1(166,21), WTRA1(166,21), R(3), Z(3), ZI(3), RI(3),MAN  40
     3 L(3), W(3,3), WR(3,3), WZ(3,3), WW(3,3), WST(3,3), NND(50), TITLEMAN  50
     4(20)                                                               MAN  60
C*********************************************************************  MAN  70
C                                                                        MAN  80
C     A GALERKIN FINITE-ELEMENT FLOW MODEL FOR THE TRANSIENT             MAN  90
C     RESPONSE OF A RADIALLY SYMMETRIC AQUIFER                           MAN 100
C                                                                        MAN 110
C       PR, PZ = FT/DAY  ; ZE, RE=FT  ; Q=CFS ; TIME =DAYS              MAN 120
C     QRCH=FT/DAY (+ MEANS RECHARGE ; - MEANS DISCHARGE)                 MAN 130
C                                                                        MAN 140
C     CONSTANT DRAWDOWN NODES MUST BE NUMBERED FIRST                     MAN 150
C            (I.E.   1,2,3,.....)                                        MAN 160
C     PPR=PRIMARY RADIAL HYDRAULIC CONDUCTIVITY                          MAN 170
C     PPZ=PRIMARY VERTICAL HYDRAULIC CONDUCTIVITY                        MAN 180
C     NDIF=# OF DIFFERING HYDRAULIC CONDUCTIVITY ELEMENTS                MAN 190
C     NE=# OF ELEMENTS; NN=# OF NODES; NCH=# OF CONSTANT DRAWDOWN NODES  MAN 200
C     NQ=# OF NODES DISCHARGING; NND=NODES DISCH.                        MAN 210
C     IF NODE HAS AN 'UNCONFINED' BOUNDARY PUT A '1' IN COL. 35          MAN 220
C     IF NODE HAS A RECHARGE TOP BOUNDARY PUT A '1' IN COL. 40           MAN 230
C     IF NODE HAS A CONSTANT ZERO DRAWDOWN PUT A '1' IN COL. 45          MAN 240
C                                                                        MAN 250
C*********************************************************************  MAN 260
C                                                                        MAN 270
C     CALCULATE CONSTANTS                                                MAN 280
C                                                                        MAN 290
      TPI=3.1416*2.                                                      MAN 300
      CONV1=1440.*60.                                                    MAN 310
C                                                                        MAN 320
      READ (5,11) TITLE                                                  MAN 330
   11 FORMAT (20A4)                                                      MAN 340
      WRITE (6,12) TITLE                                                 MAN 350
   12 FORMAT (1H1,10X,20A4)                                              MAN 360
      READ (5,13) PPR,PPZ,IPP                                            MAN 370
   13 FORMAT (2F10.0,I10)                                                MAN 380
      READ (5,14) S,SY                                                   MAN 390
   14 FORMAT (2F10.0)                                                    MAN 400
      WRITE (6,15) S                                                     MAN 410
   15 FORMAT (1HO,36H COEFFICIENT OF   SPECIFIC  STORAGE=,E12.5,5H 1/FT)MAN 420
      WRITE (6,16) SY                                                    MAN 430
   16 FORMAT (1HO,17H SPECIFIC YIELD =,F10.4)                            MAN 440
      WRITE (6,17) PPR,PPZ                                               MAN 450
   17 FORMAT (1HO,38H THE PRIMARY HYDRAULIC CONDUCTIVITY = ,F10.3,12H  RMAN 460
     1ADIALLY &,F10.3,21H  VERTICALLY (FT/DAY))                          MAN 470
      READ (5,18) NE,NN,NCH,NQ,NDIF                                     MAN 480
   18 FORMAT (5I5)                                                       MAN 490
      WRITE (6,19) NE,NN,NCH,NQ,NDIF                                     MAN 500
   19 FORMAT (1HO,I5,13H   ELEMENTS  ,I5,9H  NODES  ,I5,22H  CONSTANT VAMAN 510
     1LUE NODES,I5,18H NODES DISCHARGING,I5,23H ELEMENTS OF DIFF. H.C.) MAN 520
      IF (NQ.EQ.0) GO TO 24                                              MAN 530
      WRITE (6,20)                                                       MAN 540
   20 FORMAT (1HO,17HDISCHARGING NODES)                                  MAN 550
      DO 23 I=1,NQ                                                       MAN 560
      READ (5,21) NND(I)                                                 MAN 570
```

FORTRAN IV Program Listings—Continued.

A. Main Program—Continued.

```
   21 FORMAT (I5)                                                          MAN 580
      WRITE (6,22) NND(I)                                                  MAN 590
   22 FORMAT (1X,I9)                                                       MAN 600
   23 CONTINUE                                                             MAN 610
   24 DO 25 I=1,NE                                                         MAN 620
      PR(I)=PPR                                                            MAN 630
      PZ(I)=PPZ                                                            MAN 640
   25 CONTINUE                                                             MAN 650
C                                                                          MAN 660
C     DEFINE ELEMENTS OF DIFFERENT CONDUCTIVITIES                          MAN 670
C                                                                          MAN 680
      IF (NDIF.EQ.0) GO TO 31                                              MAN 690
      WRITE (6,26)                                                         MAN 700
   26 FORMAT (1HO,1X,39H ELEMENTS WITH DIFFERENT CONDUCTIVITIES)           MAN 710
      WRITE (6,27)                                                         MAN 720
   27 FORMAT (1HO,1X,7HELEMENT,5X,11HRADIAL H.C.,5X,13HVERTICAL H.C.)      MAN 730
      DO 30 K=1,NDIF                                                       MAN 740
      READ (5,28) IE,PR(IE),PZ(IE)                                         MAN 750
   28 FORMAT (I10,2F10.0)                                                  MAN 760
      WRITE (6,29) IE,PR(IE),PZ(IE)                                        MAN 770
   29 FORMAT (1X,I5,10X,F10.3,10X,F10.3)                                   MAN 780
   30 CONTINUE                                                             MAN 790
C                                                                          MAN 800
C     READ NODAL ORDER OF EACH ELEMENT                                     MAN 810
C                                                                          MAN 820
   31 WRITE (6,32)                                                         MAN 830
   32 FORMAT (1HO,7HELEMENT,10X,11HNODAL ORDER)                            MAN 840
      DO 35 I=1,NE                                                         MAN 850
      READ (5,33) IE,NG(IE,1),NG(IE,2),NG(IE,3)                            MAN 860
   33 FORMAT (4I5)                                                         MAN 870
      WRITE (6,34) IE,NG(IE,1),NG(IE,2),NG(IE,3)                           MAN 880
   34 FORMAT (5X,I5,5X,3(2X,I5))                                           MAN 890
   35 CONTINUE                                                             MAN 900
C                                                                          MAN 910
C     READ NODAL COORDINATES FOR EACH NODE                                 MAN 920
C                                                                          MAN 930
      WRITE (6,36)                                                         MAN 940
   36 FORMAT (1HO,17HNODAL INFORMATION)                                    MAN 950
      WRITE (6,37)                                                         MAN 960
   37 FORMAT (1X,4HNODE,10X,1HR,10X,1HZ,10X,10HUNCONFINED,10X,16HSURFACEMAN 970
     1 RECHARGE,10X,13HCONSTANT HEAD)                                      MAN 980
      DO 40 J=1,NN                                                         MAN 990
      IQ(J)=0                                                              MAN1000
      READ (5,38) IND,RE(IND),ZE(IND),IFLUX(IND),IRCH(IND),NSCON(IND)      MAN1010
   38 FORMAT (I10,2F10.0,3I5)                                              MAN1020
      WRITE (6,39) IND,RE(IND),ZE(IND),IFLUX(IND),IRCH(IND),NSCON(IND)     MAN1030
   39 FORMAT (1X,I5,2F10.2,3(15X,I5))                                      MAN1040
   40 CONTINUE                                                             MAN1050
C                                                                          MAN1060
C     CHECK INPUT DATA FOR CONSISTANCY                                     MAN1070
C                                                                          MAN1080
      CALL CHECK(NE,NN,NCH,NG,RE,ZE,NSCON,IERR)                            MAN1090
      IF (IERR.EQ.1) GO TO 91                                              MAN1100
C                                                                          MAN1110
C     DEFINE DISCHARGING NODES                                             MAN1120
C                                                                          MAN1130
      IF (NQ.EQ.0) GO TO 46                                                MAN1140
      DO 41 K=1,NQ                                                         MAN1150
      IQ(NND(K))=1                                                         MAN1160
```

FORTRAN IV Program Listings—Continued.

A. Main Program—Continued.

```
   41 CONTINUE                                                          MAN1170
C                                                                       MAN1180
C     DEFINE TOTAL TRANSMISSIVITY OF SCREEN LENGTH                      MAN1190
C                                                                       MAN1200
      TRTOT=0.                                                          MAN1210
      DO 45 J=1,NE                                                      MAN1220
      DO 44 I=1,3                                                       MAN1230
      IF (IQ(NG(J,I)).NE.1) GO TO 44                                    MAN1240
      IF (I.EQ.1) GO TO 42                                              MAN1250
      IF (IQ(NG(J,I-1)).NE.1) GO TO 44                                  MAN1260
      TI=PR(J)*ABS(ZE(NG(J,I))-ZE(NG(J,I-1)))                          MAN1270
      GO TO 43                                                          MAN1280
   42 IF (IQ(NG(J,3)).NE.1) GO TO 44                                    MAN1290
      TI=PR(J)*ABS(ZE(NG(J,1))-ZE(NG(J,3)))                            MAN1300
   43 TRTOT=TRTOT+TI                                                    MAN1310
   44 CONTINUE                                                          MAN1320
   45 CONTINUE                                                          MAN1330
C                                                                       MAN1340
C     DETERMINE HALF BAND WIDTH AND NEEDED MATRIX WIDTH                 MAN1350
C                                                                       MAN1360
   46 IHBW1=0                                                           MAN1370
      DO 50 I=1,NE                                                      MAN1380
      DO 49 J=1,3                                                       MAN1390
      IF (J.EQ.1) GO TO 47                                              MAN1400
      IF (NSCON(NG(I,J-1)).EQ.1) GO TO 49                               MAN1410
      IF (NSCON(NG(I,J)).EQ.1) GO TO 49                                 MAN1420
      IHBW=IABS(NG(I,J)-NG(I,J-1))                                      MAN1430
      GO TO 48                                                          MAN1440
   47 IF (NSCON(NG(I,1)).EQ.1) GO TO 49                                 MAN1450
      IF (NSCON(NG(I,3)).EQ.1) GO TO 49                                 MAN1460
      IHBW=IABS(NG(I,1)-NG(I,3))                                        MAN1470
   48 IF (IHBW.LT.IHBW1) GO TO 49                                       MAN1480
      IHBW1=IHBW                                                        MAN1490
   49 CONTINUE                                                          MAN1500
   50 CONTINUE                                                          MAN1510
      M=2*IHBW1+1                                                       MAN1520
      WRITE (6,51) IHBW1,M                                              MAN1530
   51 FORMAT (1HO,22HTHE HALF BAND WIDTH IS,I5,32H  AND THE WIDTH OF THEMAN1540
     1 MATRIX IS,I5)                                                    MAN1550
      DO 52 II=1,NN                                                     MAN1560
      S1(II)=0.0                                                        MAN1570
   52 CONTINUE                                                          MAN1580
      MM=NN-NCH                                                         MAN1590
C                                                                       MAN1600
C     SET UP PUMPING PERIOD                                             MAN1610
C                                                                       MAN1620
      TITO=0.0                                                          MAN1630
      DO 90 NPP=1,IPP                                                   MAN1640
      DO 53 II=1,NN                                                     MAN1650
      F(II)=0.0                                                         MAN1660
```

FORTRAN IV Program Listings—Continued.

A. Main Program—Continued.

```
   53 CONTINUE                                                                    MAN1670
      READ (5,54) Q,QRCH,DELT,TMAXF,TSM,NTS                                       MAN1680
   54 FORMAT (5F10.0,I10)                                                         MAN1690
      DT=DELT                                                                     MAN1700
      TM=0.0                                                                      MAN1710
      DO 55 I=1,NTS                                                               MAN1720
      TM=TM+DT                                                                    MAN1730
      IF (TM.GE.TMAXF) GO TO 56                                                   MAN1740
      DT=TSM*DT                                                                   MAN1750
   55 CONTINUE                                                                    MAN1760
      GO TO 57                                                                    MAN1770
   56 DELT=TMAXF/TM*DELT                                                          MAN1780
      NTS=I                                                                       MAN1790
   57 WRITE (6,58) NPP,DELT,NTS                                                   MAN1800
   58 FORMAT (1H1,14HPUMPING PERIOD,I10,/,20H INITIAL TIME STEP =,F10.5,         MAN1810
     15H DAYS,/,46H NUMBER OF TIME STEPS IN THIS PUMPING PERIOD =,I10)           MAN1820
      WRITE (6,59) Q                                                              MAN1830
   59 FORMAT (1HO,11HDISCHARGE= ,E12.5,6H   CFS)                                  MAN1840
      WRITE (6,60) QRCH                                                           MAN1850
   60 FORMAT (1HO,10HRECHARGE= ,F7.2,8H  FT/DAY)                                  MAN1860
      Q=Q*CONV1                                                                   MAN1870
C                                                                                 MAN1880
C     VALUE OF SINK MATRIX                                                        MAN1890
C                                                                                 MAN1900
      QTOT=0.0                                                                    MAN1910
      DO 65 K=1,NE                                                                MAN1920
      DO 61 J=1,3                                                                 MAN1930
      L(J)=NG(K,J)                                                                MAN1940
      R(J)=RE(L(J))                                                               MAN1950
      Z(J)=ZE(L(J))                                                               MAN1960
   61 CONTINUE                                                                    MAN1970
      DO 64 I=1,3                                                                 MAN1980
      DO 63 J=1,3                                                                 MAN1990
      IF (IQ(L(J)).NE.1) GO TO 62                                                 MAN2000
      IF (IQ(L(I)).NE.1) GO TO 62                                                 MAN2010
      BLEN=SQRT((ABS(R(I)-R(J)))**2+(ABS(Z(I)-Z(J)))**2)                          MAN2020
      F(L(I))=Q*PR(K)*BLEN/(2.*TRTOT)+F(L(I))                                     MAN2030
      QTOT=Q*PR(K)*BLEN/(2.*TRTOT)+QTOT                                           MAN2040
   62 IF (IRCH(L(J)).NE.1) GO TO 63                                               MAN2050
      IF (IRCH(L(I)).NE.1) GO TO 63                                               MAN2060
      AREA=3.1416*ABS(R(J)**2-R(I)**2)                                            MAN2070
      F(L(I))=QRCH*AREA/2.+F(L(I))                                                MAN2080
   63 CONTINUE                                                                    MAN2090
   64 CONTINUE                                                                    MAN2100
   65 CONTINUE                                                                    MAN2110
      DO 66 I=1,MM                                                                MAN2120
      F2(I)=F(I+NCH)                                                              MAN2130
   66 CONTINUE                                                                    MAN2140
C                                                                                 MAN2150
C     CHECK DISCHARGE                                                             MAN2160
```

FORTRAN IV Program Listings—Continued.

A. Main Program—Continued.

```
C                                                                       MAN2170
      QTOT=QTOT/CONV1                                                   MAN2180
      WRITE (6,67) QTOT                                                 MAN2190
   67 FORMAT (1HO,28H CALCULATED WELL DISCHARGE =,F10.2,4H CFS)         MAN2200
C                                                                       MAN2210
C     START TIME LOOP                                                   MAN2220
C                                                                       MAN2230
      TIME=0.0                                                          MAN2240
      DO 89 IT=1,NTS                                                    MAN2250
C                                                                       MAN2260
C     INITIALIZE VARIABLES                                              MAN2270
C                                                                       MAN2280
      DO 69 I=1,MM                                                      MAN2290
      DO 68 J=1,M                                                       MAN2300
      WTRA1(I,J)=0.0                                                    MAN2310
      WG1(I,J)=0.0                                                      MAN2320
   68 CONTINUE                                                          MAN2330
   69 CONTINUE                                                          MAN2340
      DO 79 K=1,NE                                                      MAN2350
      DO 70 J=1,3                                                       MAN2360
      L(J)=NG(K,J)                                                      MAN2370
      R(J)=RE(L(J))                                                     MAN2380
      Z(J)=ZE(L(J))                                                     MAN2390
   70 CONTINUE                                                          MAN2400
C                                                                       MAN2410
C     AVERAGE DISTANCE R OF ELEMENT                                     MAN2420
C                                                                       MAN2430
      RBAR=(R(1)+R(2)+R(3))/3.                                          MAN2440
C                                                                       MAN2450
      DO 72 I=1,3                                                       MAN2460
      DO 71 J=1,3                                                       MAN2470
      W(I,J)=0.0                                                        MAN2480
   71 CONTINUE                                                          MAN2490
   72 CONTINUE                                                          MAN2500
      ZI(1)=Z(2)-Z(3)                                                   MAN2510
      ZI(2)=Z(3)-Z(1)                                                   MAN2520
      ZI(3)=Z(1)-Z(2)                                                   MAN2530
      RI(1)=R(3)-R(2)                                                   MAN2540
      RI(2)=R(1)-R(3)                                                   MAN2550
      RI(3)=R(2)-R(1)                                                   MAN2560
      DEL=(ZI(1)*R(1)+ZI(2)*R(2)+ZI(3)*R(3))/2.                        MAN2570
C                                                                       MAN2580
C     CALCULATE ELEMENT MATRICIES                                       MAN2590
C                                                                       MAN2600
      DO 78 I=1,3                                                       MAN2610
      DO 77 J=1,3                                                       MAN2620
      WR(I,J)=TPI*RBAR*PR(K)*ZI(I)*ZI(J)/(4.*DEL)                       MAN2630
      WZ(I,J)=TPI*RBAR*PZ(K)*RI(I)*RI(J)/(4.*DEL)                       MAN2640
      IF (I.EQ.J) GO TO 73                                              MAN2650
      WST(I,J)=S*TPI*R(I)*DEL/(DELT*12.)                                MAN2660
      GO TO 74                                                          MAN2670
   73 WST(I,J)=S*TPI*R(I)*DEL/(DELT*6.)                                 MAN2680
   74 WW(I,J)=WR(I,J)+WZ(I,J)                                           MAN2690
      IF (I.NE.J) GO TO 76                                              MAN2700
      IF (IFLUX(L(I)).NE.1) GO TO 76                                    MAN2710
      AREA=0.0                                                          MAN2720
      DO 75 KK=1,3                                                      MAN2730
      IF (KK.EQ.I) GO TO 75                                             MAN2740
      IF (IFLUX(L(KK)).NE.1) GO TO 75                                   MAN2750
      AREA=3.1416*ABS(R(I)*R(I)-R(KK)*R(KK))                            MAN2760
```

FORTRAN IV Program Listings—Continued.

A. Main Program—Continued.

```
   75 CONTINUE                                                          MAN2770
      W(I,J)=-SY*AREA/(2.*DELT)                                         MAN2780
C                                                                       MAN2790
C       ASSEMBLE ELEMENT MATRICIES INTO GLOBAL                          MAN2800
C                                                                       MAN2810
   76 IF (NSCON(L(J)).EQ.1) GO TO 77                                    MAN2820
      IF (NSCON(L(I)).EQ.1) GO TO 77                                    MAN2830
      II=NG(K,I)-NCH                                                    MAN2840
      JJ=NG(K,J)-NCH                                                    MAN2850
      MTRAN=JJ-II+(M+1)/2                                               MAN2860
C                                                                       MAN2870
C       GLOBAL TRANSIENT MATRIX                                         MAN2880
C                                                                       MAN2890
      WTRA1(II,MTRAN)=WST(I,J)+W(I,J)+WTRA1(II,MTRAN)                   MAN2900
C                                                                       MAN2910
C       GLOBAL MATRIX                                                   MAN2920
C                                                                       MAN2930
      WG1(II,MTRAN)=WW(I,J)+WG1(II,MTRAN)                               MAN2940
   77 CONTINUE                                                          MAN2950
   78 CONTINUE                                                          MAN2960
   79 CONTINUE                                                          MAN2970
C                                                                       MAN2980
C                                                                       MAN2990
C       SUM OF GLOBAL AND TRANSIENT COEFFICIENT MATRICIES               MAN3000
C                 ON LEFT HAND SIDE                                     MAN3010
C                                                                       MAN3020
      DO 81 I=1,MM                                                      MAN3030
      DO 80 J=1,M                                                       MAN3040
      WG1(I,J)=WTRA1(I,J)+WG1(I,J)                                      MAN3050
   80 CONTINUE                                                          MAN3060
   81 CONTINUE                                                          MAN3070
C                                                                       MAN3080
C       MULT. OF MATRICIES ON RIGHT HAND SIDE                           MAN3090
C                                                                       MAN3100
      DO 82 I=1,MM                                                      MAN3110
      S3(I)=S1(I+NCH)                                                   MAN3120
      S5(I)=0.0                                                         MAN3130
   82 CONTINUE                                                          MAN3140
      CALL MLTBM(WTRA1,S3,S5,MM,M)                                      MAN3150
      DO 83 I=1,MM                                                      MAN3160
      S5(I)=S5(I)+F2(I)                                                 MAN3170
   83 CONTINUE                                                          MAN3180
C                                                                       MAN3190
C       SOLUTION                                                        MAN3200
C                                                                       MAN3210
      CALL SOLVE(1,WG1,S5,MM,IHBW1,MM,M)                                MAN3220
      CALL SOLVE(2,WG1,S5,MM,IHBW1,MM,M)                                MAN3230
      NNN=NCH+1                                                         MAN3240
      DO 84 I=NNN,NN                                                    MAN3250
      S1(I)=S5(I-NCH)                                                   MAN3260
   84 CONTINUE                                                          MAN3270
      TIME=TIME+DELT                                                    MAN3280
      TIMM=TIME*1440.                                                   MAN3290
      ACTT=TIME+TITO                                                    MAN3300
C                                                                       MAN3310
C       OUTPUT                                                          MAN3320
C                                                                       MAN3330
      WRITE (6,85) ACTT                                                 MAN3340
   85 FORMAT (1H1,30HTOTAL TIME IN THE SIMULATION =,F10.3,5H DAYS)      MAN3350
      WRITE (6,86) TIME,TIMM                                            MAN3360
```

FORTRAN IV Program Listings—Continued.

A. Main Program—Continued.

```
   86 FORMAT (1HO, 18HTHE DRAWDOWN AFTER, E10.3, 9H  DAYS OR, E10.3, 26HMIN.  MAN3370
     1IN THE PUMPING PERIOD)                                                  MAN3380
      WRITE (6,87)                                                            MAN3390
   87 FORMAT (1HO, 10X, 12H NODE NUMBER, 5X, 8HDRAWDOWN, 10X, 12H NODE NUMBER, MAN3400
     15X, 8HDRAWDOWN, 10X, 12H NODE NUMBER, 5X, 8HDRAWDOWN)                   MAN3410
      WRITE (6,88) ((I,S1(I)), I=1,NN)                                        MAN3420
   88 FORMAT (10X, I5, 10X, F10.3, 10X, I5, 10X, F10.3, 10X, I5, 10X, F10.3)  MAN3430
      DELT=DELT*TSM                                                           MAN3440
   89 CONTINUE                                                                MAN3450
      TITO=TIME+TITO                                                          MAN3460
   90 CONTINUE                                                                MAN3470
      GO TO 93                                                                MAN3480
   91 WRITE (6,92)                                                            MAN3490
   92 FORMAT (1H1, 49H**TERMINATION OF PROGRAM DUE TO INPUT DATA ERRORS)      MAN3500
   93 STOP                                                                    MAN3510
      END                                                                     MAN3520-
```

B. Subroutine CHECK

```
      SUBROUTINE CHECK(NE, NN, NCH, NG, RE, ZE, NSCON, IERR)                  CHK 010
      DIMENSION NG(NE,3), RE(NN), ZE(NN), NSCON(NN)                           CHK 020
C                                                                             CHK 030
C     THIS SUBROUTINE CHECKS THE ELEMENT INPUT DATA FOR CONSISTANCY           CHK 040
C                                                                             CHK 050
      IERR=0                                                                  CHK 060
      NCK=0                                                                   CHK 070
C                                                                             CHK 080
C     CHECK NUMBER AND ORDER OF CONSTANT HEAD NODES                           CHK 090
C                                                                             CHK 100
C     FIRST CHECK IF CONSTANT HEAD NODES ARE THE FIRST NODES NUMBERED         CHK 110
C                                                                             CHK 120
      DO 10 I=1,NCH                                                           CHK 130
      NCK=NSCON(I)+NCK                                                        CHK 140
   10 CONTINUE                                                                CHK 150
      IF(NCK.EQ.NCH) GO TO 30                                                 CHK 160
      IERR=1                                                                  CHK 170
      WRITE(6,20) NCH,NCK                                                     CHK 180
   20 FORMAT(1H1,20H*** PROGRAM EXPECTED, I5, 39H CONSTANT HEAD NODES BUT     CHK 190
     1ONLY THE FIRST, I5, 28H  NODES WERE FLAGGED AS SUCH)                    CHK 200
C                                                                             CHK 210
C     THEN CHECK THE TOTAL NUMBER OF CONSTANT HEAD FLAGS                      CHK 220
C                                                                             CHK 230
   30 NCK=0                                                                   CHK 240
      DO 40 I=1,NN                                                            CHK 250
      NCK=NSCON(I)+NCK                                                        CHK 260
   40 CONTINUE                                                                CHK 270
      IF(NCK.EQ.NCH) GO TO 60                                                 CHK 280
      IERR=1                                                                  CHK 290
      WRITE(6,50)                                                             CHK 300
   50 FORMAT(1HO,71H***TOTAL NUMBER OF CONSTANT HEAD FLAGS DOES NOT AGRECHK 310
     1E WITH NCH(# CODED))                                                    CHK 320
C                                                                             CHK 330
C     NEXT CHECK IS TO INSURE THAT ALL ELEMENTS ARE NUMBERED                  CHK 340
C     COUNTERCLOCKWISE                                                        CHK 350
C                                                                             CHK 360
   60 DO 100 K=1,NE                                                           CHK 370
      L=NG(K,1)                                                               CHK 380
      M=NG(K,2)                                                               CHK 390
      N=NG(K,3)                                                               CHK 400
```

FORTRAN IV Program Listings—Continued.

B. Subroutine CHECK—Continued.

```
      A=(RE(L)-RE(N))*ZE(M)+(RE(M)-RE(L))*ZE(N)+(RE(N)-RE(M))*ZE(L)      CHK 410
      IF(A.LT.0.) GO TO 100                                             CHK 420
      IERR=1                                                            CHK 430
      WRITE(6,70) K                                                     CHK 440
   70 FORMAT(1HO,17H***EITHER ELEMENT,I5,23H  IS NUMBERED CLOCKWISE,/,11CHK 450
     1X,92HOR THE VERTICAL COORDINATES ARE NOT POSITIVE DOWNWARD STARTINCHK 460
     2G WITH ZERO AT THE TOP BOUNDARY)                                  CHK 470
  100 CONTINUE                                                          CHK 480
      RETURN                                                            CHK 490
      END                                                               CHK 500-
```

C. Subroutine MLTBM

```
      SUBROUTINE MLTBM(A,B,R,MM,M)                                      MLT 010
      DIMENSION A(MM,M),B(MM),R(MM)                                     MLT 020
C                                                                       MLT 030
C     MULT. OF A BANDED MATRIX(ORIGINALLY MM*MM)                        MLT 040
C     WITH A VECTOR MATRIX                                              MLT 050
C     ( COMPACTED BANDED MATRIX OF M*MM AND VECTOR OF MM*1 )            MLT 060
C                                                                       MLT 070
      DO 1 I=1,MM                                                       MLT 080
      R(I)=0.                                                           MLT 090
      DO 2 J=1,M                                                        MLT 100
      K=J+I-(M+1)/2                                                     MLT 110
      IF(K.LT.1) GO TO 2                                                MLT 120
      IF(K.GT.MM) GO TO 2                                               MLT 130
      R(I)=A(I,J)*B(K)+R(I)                                             MLT 140
    2 CONTINUE                                                          MLT 150
    1 CONTINUE                                                          MLT 160
      RETURN                                                            MLT 170
      END                                                               MLT 180-
```

D. Subroutine SOLVE

```
      SUBROUTINE SOLVE(KKK,B,R,NEQ,IHALFB,NDIM,MDIM)                    SOL  10
C     ************************************************                  SOL  20
C                                                                       SOL  30
C   ASYMMETRIC BAND MATRIX EQUATION SOLVER                              SOL  40
C     ORIGINALLY PROGRAMED BY JAMES O. DUGUID                           SOL  50
C                                                                       SOL  60
C   KKK=1 TRIANGULARIZES THE BAND MATRIX B                              SOL  70
C   KKK=2 SOLVES FOR RIGHT SIDE R, SOLUTION RETURNS IN R                SOL  80
C                                                                       SOL  90
      DIMENSION B(NDIM,MDIM), R(NDIM)                                   SOL 100
      NRS=NEQ-1                                                         SOL 110
      IHBP=IHALFB+1                                                     SOL 120
      IF (KKK.EQ.2) GO TO 30                                            SOL 130
C                                                                       SOL 140
C  TRIANGULARIZE MATRIX A USING DOOLITTLE METHOD                        SOL 150
C                                                                       SOL 160
      DO 20 K=1,NRS                                                     SOL 170
      PIVOT=B(K,IHBP)                                                   SOL 180
      KK=K+1                                                            SOL 190
      KC=IHBP                                                           SOL 200
      DO 10 I=KK,NEQ                                                    SOL 210
      KC=KC-1                                                           SOL 220
      IF (KC.LE.0) GO TO 20                                             SOL 230
      C=-B(I,KC)/PIVOT                                                  SOL 240
      B(I,KC)=C                                                         SOL 250
      KI=KC+1                                                           SOL 260
      LIM=KC+IHALFB                                                     SOL 270
      DO 10 J=KI,LIM                                                    SOL 280
      JC=IHBP+J-KC                                                      SOL 290
```

FORTRAN IV Program Listings—Continued.

D. Subroutine SOLVE—Continued.

```
   10 B(I,J)=B(I,J)+C*B(K,JC)                                          SOL 300
   20 CONTINUE                                                         SOL 310
      GO TO 100                                                        SOL 320
C                                                                      SOL 330
C  MODIFY LOAD VECTOR R                                                SOL 340
C                                                                      SOL 350
   30 NN=NEQ+1                                                         SOL 360
      IBAND=2*IHALFB+1                                                 SOL 370
      DO 70 I=2,NEQ                                                    SOL 380
      JC=IHBP-I+1                                                      SOL 390
      JI=1                                                             SOL 400
      IF (JC.LE.0) GO TO 40                                            SOL 410
      GO TO 50                                                         SOL 420
   40 JC=1                                                             SOL 430
      JI=I-IHBP+1                                                      SOL 440
   50 SUM=0.0                                                          SOL 450
      DO 60 J=JC,IHALFB                                                SOL 460
      SUM=SUM+B(I,J)*R(JI)                                             SOL 470
   60 JI=JI+1                                                          SOL 480
   70 R(I)=R(I)+SUM                                                    SOL 490
C                                                                      SOL 500
C   BACK SOLUTION                                                      SOL 510
C                                                                      SOL 520
      R(NEQ)=R(NEQ)/B(NEQ,IHBP)                                        SOL 530
      DO 90 IBACK=2,NEQ                                                SOL 540
      I=NN-IBACK                                                       SOL 550
      JP=I                                                             SOL 560
      KR=IHBP+1                                                        SOL 570
      MR=MINO(IBAND,IHALFB+IBACK)                                      SOL 580
      SUM=0.0                                                          SOL 590
      DO 80 J=KR,MR                                                    SOL 600
      JP=JP+1                                                          SOL. 610
   80 SUM=SUM+B(I,J)*R(JP)                                             SOL 620
   90 R(I)=(R(I)-SUM)/B(I,IHBP)                                        SOL 630
  100 RETURN                                                           SOL 640
      END                                                              SOL 650-
```

SUPPLEMENTAL DATA II

Definition of Selected Program Variables

A - area of element used to check nodal order

ACTT - total time in simulation

AREA - surface area of an element along the top boundary

BLEN - length of screen between two nodes

CONV1 - conversion factor

DEL - area of triangular element

DELT - time step (days)

DT - dummy time step used in setting up pumping period

F - vector containing the flux terms in all nodes on the right hand side of equation

F2 - vector containing the flux terms for all unknown nodes on the right hand side of equation

EI - element number

IERR - error flag for input data

IFLUX - unconfined boundary flag

IHBW1 - half band width

IND - node number

IPP - number of pumping periods

IQ - discharge boundary flag

IRCH - recharge boundary flag

L - global node number in an element

M - band width of global matricies

MM - number of unknowns

MTRAN - transformed column location for the compacted banded matrix

NCH - number of constant head nodes

NDIF - number of elements with a different hydraulic conductivity than the primary hydraulic conductivity

NE - number of elements

NG - array storing nodal order for each element

NN - number of nodes

NND - nodes along the well screen

NPP - pumping period number

NQ - number of nodes along the well screen

NSCON - flag for constant head nodes

NTS - number of time steps in pumping period

PPR - the primary hydraulic conductivity in the radial direction

PPZ - the primary hydraulic conductivity in the vertical direction

PR - the hydraulic conductivity in the radial direction

PZ - the hydraulic conductivity in the vertical direction

Q - discharge of the well (cfs)

QRCH - recharge rate of the basin (ft/d)

QTOT - calculated discharge of well after distribution along the well bore

R - radius of the nodes for an element

RBAR - average radius for an element

RE - radial distance of node

RI - coefficient for calculation of element arrays

S - Specific storage S_s (ft^{-1})

SY - specific yield S_y (unitless)

S1 - drawdowns

S3 - drawdowns at unknown nodes at previous time step

S5 - vector representing right hand side of final matrix equation

T1 - increment of transmissivity along well screen for an element

TIME - total time of pumping period in days

TIMM - total time of pumping period in minutes

TITLE - title to be printed at start of computer output

TMAXF - maximum time of simulation (days)

TPI - constant

TRTOT - total transmissivity along the well screen

TSM - time-step multiplier

W - element matrix for unconfined storage coefficients

WG1 - global matrix for non-time dependent coefficients and also total matrix on the left hand side of final matrix equation

WR - element matrix for radial direction coefficients

WST - element matrix for storage coefficients

WTRA1 - global matrix for time dependent coefficients

WW - sum of WR and WZ

WZ - element matrix for vertical direction coefficients

Z - vertical location of the nodes for an element

ZE - vertical location of a node

ZI - coefficient for calculation of element arrays

SUPPLEMENTAL DATA III

Data Input Formats

Group 1: Title and problem setup

Card	Columns	Format	Variable	Definition
1	1-80	20A4	TITLE	Any title the user wishes to print on one line at start of output.
2	1-10	F10.0	PPR	Primary radial hydraulic conductivity (ft/d)
	11-20	F10.0	PPZ	Primary vertical hydraulic conductivity (ft/d)
	21-30	I10	IPP	number of pumping periods

Note: PPR and PPZ are the hydraulic conductivities assigned to all elements unless redefined in data Group 3.

3	1-10	F10.0	S	Coefficient of compressive storage (S_s) in ft^{-1}
	11-20	F10.0	SY	Specific yield (unitless)
4	1-5	I5	NE	Number of elements in finite element mesh
	6-10	I5	NN	Number of nodes in finite element mesh
	11-15	I5	NCH	Number of constant-head nodes in finite element mesh
	16-20	I5	NQ	Number of nodes associated with well screen
	21-25	I5	NDIF	Number of elements having different hydraulic conductivities than the primary ones on card 2

Data Input Formats—Continued.

Group 2: Nodes along the well screen

NQ number of cards

Card	Columns	Format	Variable	Definition
--	1-5	I5	NND(I)	Node that is along well screen. All nodes along screen must be identified because water is withdrawn only from elements with two nodes along screen.

Group 3: Elements of different hydraulic conductivities

DIF number of cards

Card	Columns	Format	Variable	Definition
--	1-10	I10	IE	Element number
	11-20	F10.0	PR(IE)	Radial hydraulic conductivity of element (ft/d)
	21-30	F10.0	PZ(IE)	Vertical hydraulic conductivity of element (ft/d)

Group 4: Nodal configuration of each element

NE number of cards

Card	Columns	Format	Variable	Definition
--	1-5	I5	IE	Element number
	6-10	I5	NG(IE,1)	First node of triangular element.
	11-15	I5	NG(IE,2)	Second node in counter-clockwise direction of triangular element.
	16-20	I5	NG(IE,3)	Third node in counter-clockwise direction of triangular element.

See note on node numbers in Data Group 5.

Group 5: Location of each node and flags for constant head, unconfined boundary and recharge boundary.

NN number of cards

Card	Columns	Format	Variable	Definition
--	1-10	I10	IND	Node number
	11-20	F10.0	RE(IND)	Radial location of node (ft)
	21-30	F10.0	ZE(IND)	Vertical location of node (ft)
	35	I5	IFLUX(IND)	If set to 1, node is treated as unconfined and specific-yield boundary condition is applied.
	40	I5	IRCH(IND)	If set to 1, node is part of a recharge boundary. This is for ground-water mounding problems and is only the top boundary. Recharge occurs only to elements with two nodes on top boundary.
	45	I5	NSCON(IND)	If set to 1, node is considered to be a constant zero drawdown (constant head).

Important Note: For this program, constant-drawdown nodes (constant head) must be the first nodes numbered and must be sequential. The program assumes that if there are four constant-head nodes (NCH = 4), they are nodes 1, 2, 3, and 4.

Data Input Formats—Continued.

Group 6: Pumping period information

IPP number of cards

Card	Columns	Format	Variable	Definition
--	1-10	F10.0	Q	Pumping rate (ft^3/s)
	11-20	F10.0	QRCH	Recharge rate (ft/d)
	21-30	F10.0	DELT	Initial time step (days)
	31-40	F10.0	TMAXF	Maximum length of pumping period (days)
	41-50	F10.0	TSM	Time-step multiplier (each time step after DELT is multiplied by TSM).
	51-60	I10	NTS	Number of time steps in pumping period.

The program has two options for the pumping period:

1. To simulate a given number of time steps, set TMAXF to a value larger than the expected simulation period. The program will use NTS, TSM, and DELT as coded.
2. To simulate a given pumping period, set NTS larger than the number required for the simulation period (for example, 100). The program will compute the exact DELT (which will be $\leq$ DELT coded) and NTS to arrive exactly at TMAXF on the last time step.

SUPPLEMENTAL DATA IV

Array Dimensioning

Several of the arrays must be dimensioned for each specific grid setup.

The arrays can be divided into six different groups:

(1) Arrays associated with the number of elements (NE)
ARRAY NAME (Size)
a) NG (NE, 3)
b) PR (NE)
c) PZ (NE)

(2) Arrays associated with the total number of nodes (NN).
ARRAY NAME (Size)
a) RE (NN)
b) ZE (NN)
c) IFLUX (NN)
d) IQ (NN)
e) IRCH (NN)
f) NSCON (NN)
g) F (NN)
h) S1 (NN)

(3) Arrays associated with the number of unknown nodes (MM). The number of unknown nodes (MM) equals the number of nodes (NN) minus the number of constant head nodes (NCH). Therefore, MM=NN-NCH.
ARRAY NAME (Size)
a) S3 (MM)
b) S5 (MM)
c) F2 (MM)

(4) The compacted global arrays associated with the number of unknown nodes (MM) and the band width of the global coefficient matrix (M). The band width (M) is equal to twice the largest difference between two adjacent node numbers plus one.
ARRAY NAME (Size)
a) WG1 (MM,M)
b) WTRA1 (MM,M)

Array Dimensioning—Continued.

(5) Array associated with the number of nodes along the well screen (NQ). This array is dimensioned at 50 in the program, which should be sufficient for most problems.
ARRAY NAME (Size)
a) NND (NQ)

(6) Arrays that are a constant size regardless of problem.
ARRAY NAME (Size)
a) R (3)
b) Z (3)
c) ZI (3)
d) RI (3)
e) L (3)
f) W (3,3)
g) WR (3,3)
h) WZ (3,3)
i) WW (3,3)
j) WST (3,3)
k) TITLE (20)

SUPPLEMENTAL DATA V

Input Data for Test Problem

```
SAMPLE MODEL SIMULATION OF A LONG ISLAND, NEW YORK PUMP TEST
  270         27                  3
  000001        25
  288  170     4     4  169
  167
  166
  165
  164
        180   01          001
        181   01          001
        182   01          001
        183   01          001
        184   01          001
        185   01          001
        186   01          001
        187   01          001
        188   01          001
        189   01          001
        190   01          001
        191   01          001
        192   01          001
        193   01          001
        194   01          001
        195   01          001
        196   01          001
        197   01          001
        198   01          001
        199   01          001
        200   01          001
        201   01          001
        202   01          001
        203   01          001
        204   01          001
        205   01          001
        206   01          001
        207   01          001
        208   01          001
        209   01          001
        210   01          001
        211   01          001
        212   01          001
        213   01          001
        214   01          001
        215   01          001
        216   01          001
        217   01          001
        218   50        1  4
        219   50        1  4
        220   50        1  4
        221   50        1  4
        222   50        1  4
        223   50        1  4
        224   50        1  4
        225   50        1  4
        226   50        1  4
        227   50        1  4
        228   50        1  4
        229   50        1  4
        230   50        1  4
        231   50        1  4
        232   50        1  4
        233   50        1  4
        234   50        1  4
        235   50        1  4
        236   50.       1  4
        237   50        1  4
        238   50        1  4
        239   50        1  4
        240   50        1  4
        241   50        1  4
        242   50        1  4
        243   50        1  4
        244   50        1  4
        245   50        1  4
        246   50        1  4
        247   50        1  4
        248   50.       1  4
        249   50        1  4
        250   50        1  4
        251   50        1  4
        252   50        1  4
        253   50        1  4
        254   50        1. 4
        255   50        1  4
        256   50        1  4
        257   50        1. 4
        258   50.       1. 4
        259   50.       1  4
        260   50        1  4
        261   50        1  4
        262   50        1  4
        263   50        1  4
        264   50        1  4
        265   50        1  4
        266   50        1  4
        267   50        1  4
        268   50        1  4
        269   50        1  4
        270   50        1  4
        271   50        1  4
        272   50        1  4
        273   50        1  4
        274   50        1  4
        275   50        1  4
        276   50        1. 4
        277   50        1  4
        278   50        1  4
        279   50        1  4
        280   50        1  4
        281   50        1  4
        282   50        1  4
        283   50        1  4
        284   50        1  4
        285   50        1  4
        286   50        1  4
        287   50        1  4
        288   50        1  4
    1   170   169   160
    2   160   169   159
    3   159   169   168
```

Input Data for Test Problem—Continued.

4	168	158	159
5	158	168	167
6	167	157	158
7	167	166	157
8	157	166	156
9	156	166	165
10	165	155	156
11	155	165	164
12	164	154	155
13	160	159	150
14	150	159	149
15	149	159	158
16	158	148	149
17	158	157	148
18	148	157	147
19	147	157	156
20	156	146	147
21	156	155	146
22	146	155	145
23	145	155	154
24	154	144	145
25	140	150	149
26	149	139	140
27	139	149	148
28	148	138	139
29	138	148	147
30	147	137	138
31	137	147	146
32	146	136	137
33	136	146	145
34	145	135	136
35	145	144	135
36	135	144	134
37	140	139	130
38	130	139	129
39	139	138	129
40	129	138	128
41	128	138	137
42	137	127	128
43	137	136	127
44	127	136	126
45	126	136	135
46	135	125	126
47	125	135	134
48	134	124	125
49	130	129	120
50	120	129	119
51	119	129	128
52	128	118	119
53	128	127	118
54	118	127	117
55	117	127	126
56	126	116	117
57	126	125	116
58	116	125	115
59	115	125	124
60	124	114	115
61	120	119	110
62	110	119	109
63	109	119	118
64	118	108	109
65	118	117	108
66	108	117	107
67	107	117	116
68	116	106	107
69	116	115	106
70	106	115	105
71	105	115	114
72	110	109	101
73	101	109	100
74	100	109	108
75	108	99	100
76	108	107	99
77	99	107	98
78	98	107	106
79	106	97	98
80	106	105	97
81	97	105	96
82	101	100	92
83	92	100	91
84	91	100	99
85	99	90	91
86	99	98	90
87	90	98	89
88	89	98	97
89	97	88	89
90	97	96	88
91	88	96	87
92	92	91	83
93	83	91	82
94	82	91	90
95	90	81	82
96	90	89	81
97	81	89	80
98	80	89	88
99	88	79	80
100	88	87	79
101	79	87	78
102	83	82	74
103	74	82	73
104	73	82	81
105	81	72	73
106	81	80	72
107	72	80	71
108	71	80	79
109	79	70	71
110	79	78	70
111	70	78	69
112	74	73	65
113	65	73	64
114	64	73	72
115	72	63	64
116	72	71	63
117	63	71	62
118	62	71	70
119	70	61	62
120	70	69	61
121	61	69	60
122	65	64	56
123	56	64	55
124	55	64	63
125	63	54	55
126	63	62	54
127	54	62	53
128	53	62	61
129	61	52	53
130	61	60	52
131	52	60	51
132	56	55	47
133	47	55	46
134	46	55	54
135	54	45	46
136	54	53	45
137	45	53	44
138	44	53	52
139	52	43	44
140	52	51	43
141	43	51	42
142	47	46	38
143	38	46	37
144	37	46	45
145	45	36	37
146	45	44	36
147	36	44	35
148	35	44	43
149	43	34	35
150	43	42	34
151	34	42	33
152	38	37	29
153	29	37	28
154	28	37	36
155	36	27	28
156	36	35	27
157	27	35	34
158	34	26	27
159	34	33	26
160	29	28	22
161	22	28	21
162	21	28	27
163	27	20	21
164	27	26	20
165	20	26	19
166	22	21	15
167	15	21	14
168	14	21	20
169	20	13	14
170	20	19	13
171	13	19	12
172	15	14	9
173	9	14	8
174	8	14	13
175	13	7	8
176	13	12	7
177	9	8	4
178	4	8	3
179	3	8	7
180	164	163	154
181	154	163	153
182	154	153	144
183	144	153	143
184	144	143	134
185	134	143	133
186	134	133	124
187	124	133	123
188	124	123	114

Input Data for Test Problem—Continued.

```
189  114  123  113
190  114  113  105
191  105  113  104
192  105  104   96
193   96  104   95
194   96   95   87
195   87   95   86
196   87   86   78
197   78   86   77
198   78   77   69
199   69   77   68
200   69   68   60
201   60   68   59
202   60   59   51
203   51   59   50
204   51   50   42
205   42   50   41
206   42   41   33
207   33   41   32
208   33   32   26
209   26   32   25
210   26   25   19
211   19   25   18
212   19   18   12
213   12   18   11
214   12   11    7
215    7   11    6
216    7    6    3
217    3    6    2
218  163  162  153
219  153  162  152
220  152  162  161
221  161  151  152
222  153  152  143
223  143  152  142
224  142  152  151
225  151  141  142
226  143  142  133
227  133  142  132
228  132  142  141
229  141  131  132
230  133  132  123
231  123  132  122
232  122  132  131
233  131  121  122
234  123  122  113
235  113  122  112
236  112  122  121
237  121  111  112
238  113  112  104
239  104  112  103
240  103  112  111
241  111  102  103
242  104  103   95
243   95  103   94
244   94  103  102
245  102   93   94
246   95   94   86
247   86   94   85
248   85   94   93
249   93   84   85
250   86   85   77
251   77   85   76
252   76   85   84
253   84   75   76
254   77   76   68
255   68   76   67
256   67   76   75
257   75   66   67
258   68   67   59
259   59   67   58
260   58   67   66
261   66   57   58
262   59   58   50
263   50   58   49
264   49   58   57
265   57   48   49
266   50   49   41
267   41   49   40
268   40   49   48
269   48   39   40
270   41   40   32
271   32   40   31
272   31   40   39
273   39   30   31
274   32   31   25
275   25   31   24
276   24   31   30
277   30   23   24
278   25   24   18
279   18   24   17
280   17   24   23
281   23   16   17
282   18   17   11
283   11   17   10
284   10   17   16
285   11   10    6
286    6   10    5
287    6    5    2
288    2    5    1
```

1	10000	275		1
2	10000	85		1
3	10000	75		1
4	10000	0	1	1
5	4000	275		
6	4000	85		
7	4000	75		
8	4000	37 5		
9	4000	0		
10	2500	275		
11	2500	85		
12	2500	75		
13	2500	50		
14	2500	25		
15	2500	0	1	
16	1500	275		
17	1500	125		
18	1500	85		
19	1500	75		
20	1500	50		
21	1500	25		
22	1500	0	1	
23	900	275		
24	900	125		
25	900	85		
26	900	75		
27	900	50		
28	900	25.		
29	900	0	1	
30	600	275		
31	600	125		
32	600	85		
33	600	75		
34	600	62 5		
35	600	50		
36	600	37 5		
37	600	25		
38	600	0	1	
39	400	275		
40	400	125		
41	400.	85		
42	400	75		
43	400	62 5		
44	400	50		
45	400	37 5		
46	400	25		
47	400	0	1	
48	275	275		
49	275.	125.		
50	275	85		
51	275	75		
52	275	62. 5		
53	275	50.		
54	275	37 5		
55	275	25		
56	275	0	1	
57	200	275		
58	200	125		
59	200	85		
60	200	75		
61	200	62 5		
62	200	50		
63	200	37. 5		
64	200	25		
65	200	0	1	
66	150	275.		
67	150	125		
68	150	85		
69	150	75		
70	150	62. 5		
71	150	50		
72	150	37 5		
73	150	25		
74	150	0	1	
75	100	275		
76	100	125		
77	100	85		
78	100	75		
79	100	62 5		
80	100	50		
81	100	37 5		
82	100	25		
83	100	0	1	
84	75	275		
85	75	125		

Input Data for Test Problem—Continued.

```
 86   75      85
 87   75      75
 88   75      62.5
 89   75      50
 90   75      37.5
 91   75      25
 92   75       0          1
 93   50     275
 94   50     125
 95   50      85
 96   50      75
 97   50      62.5
 98   50      50
 99   50      37.5
100   50      25
101   50       0          1
102   32.5   275
103   32.5   125
104   32.5    85
105   32.5    75
106   32.5    62.5
107   32.5    50
108   32.5    37.5
109   32.5    25
110   32.5     0          1
111   20     275
112   20     125
113   20      85.
114   20      75
115   20      66.67
116   20      58.33
117   20      50
118   20      37.5
119   20      25
120   20       0          1
121   10     275
122   10     125
123   10      85.
124   10      75.
125   10      66.67
126   10      58.33
127   10      50
128   10      37.5
129   10      25
130   10       0          1
131    6     275
132    6     125
133    6      85
134    6      75
135    6      66.67
136    6      58.33
137    6      50
138    6      37.5
139    6      25
140    6       0          1
141    3     275
142    3     125
143    3      85
144    3      75
145    3      66.67
146    3      58.33
147    3      50
148    3      37.5
149    3      25
150    3       0          1
151    1     275
152    1     125
153    1      85
154    1      75
155    1      66.67
156    1      58.33
157    1      50
158    1      37.5
159    1      25
160    1       0          1
161    5     275
162    5     125
163    5      85
164    5      75
165    5      66.67
166    5      58.33
167    5      50
168    5      37.5
169    5      25
170    5       0          1
-1            01         2          1.5          50.
 0            01         2          1.5          50.
-1            01         2          1.5          50.
```

SUPPLEMENTAL DATA VI

Selected Output for Test Problem

```
          SAMPLE MODEL SIMULATION OF A LONG ISLAND, NEW YORK PUMP TEST

 COEFFICIENT OF   SPECIFIC  STORAGE=   10000E-05 1/FT

 SPECIFIC YIELD =      2500

 THE PRIMARY HYDRAULIC CONDUCTIVITY =    270 000  RADIALLY &    27 000  VERTICALLY (FT/DAY)

  288   ELEMENTS    170  NODES      4  CONSTANT VALUE NODES    4 NODES DISCHARGING  109 ELEMENTS OF DIFF  H C

DISCHARGING NODES
      167
      166
      165
      164

  ELEMENTS WITH DIFFERENT CONDUCTIVITIES

 ELEMENT     RADIAL H C      VERTICAL H.C
  180                010                001
  181                010                001
  182                010                001
  183                010                001
  184                010                001
  185                010                001
  186                010                001
  187                010                001
  188                010                001
  189                010                001
  190                010                001
  191                010                001
  192                010                001
  193                010                001
  194                010                001
  195                010                001
  196                010                001
  197                010                001
  198                010                001
  199                010                001
  200                010                001
  201                010                001
  202                010                001
  203                010                001
  204                010                001
  205                010                001
  206                010                001
  207                010                001
  208                010                001
  209                010                001
  210                010                001
  211                010                001
  212                010                001
  213                010                001
  214                010                001
  215                010                001
  216                010                001
  217                010                001
  218             50 000             1 400
  219             50 000             1 400
  220             50 000             1 400
  221             50 000             1 400
  222             50 000             1 400
  223             50 000             1 400
  224             50 000             1 400



 THE HALF BAND WIDTH IS   10  AND THE WIDTH OF THE MATRIX IS   21

 PUMPING PERIOD         1
 INITIAL TIME STEP =     00777 DAYS
 NUMBER OF TIME STEPS IN THIS PUMPING PERIOD =        12

 DISCHARGE=  - 10000E+01   CFS

 RECHARGE=   -0 00  FT/DAY

  CALCULATED WELL DISCHARGE =      -1 00 CFS
```

Selected Output for Test Program—Continued.

TOTAL TIME IN THE SIMULATION = 037 DAYS

THE DRAWDOWN AFTER .369E-01 DAYS OR 531E+02MIN IN THE PUMPING PERIOD

NODE NUMBER	DRAWDOWN	NODE NUMBER	DRAWDOWN	NODE NUMBER	DRAWDOWN
1	0.000	2	0.000	3	0 000
4	0.000	5	.000	6	000
7	.000	8	.000	9	000
10	000	11	000	12	000
13	.000	14	000	15	000
16	000	17	000	18	000
19	000	20	000	21	000
22	000	23	000	24	000
25	000	26	002	27	002
28	.001	29	000	30	000
31	.000	32	000	33	016
34	015	35	014	36	011
37	.008	38	001	39	000
40	001	41	001	42	067
43	065	44	059	45	049
46	.036	47	005	48	000
49	.001	50	001	51	183
52	176	53	159	54	131
55	096	56	013	57	000
58	.001	59	001	60	354
61	340	62	300	63	243
64	175	65	024	66	000
67	001	68	002	69	575
70	546	71	472	72	370
73	257	74	035	75	000
76	001	77	002	78	981
79	923	80	768	81	565
82	370	83	050	84	000
85	.001	86	002	87	1 359
88	1 265	89	1.014	90	700
91	434	92	059	93	000
94	001	95	002	96	1 962
97	1 820	98	1 388	99	866
100	499	101	.066	102	000
103	001	104	.002	105	2 680
106	2.496	107	1.811	108	995
109	538	110	070	111	000
112	.001	113	.003	114	3 654
115	3 456	116	3.049	117	2 231
118	1.084	119	.559	120	073
121	.000	122	.001	123	003
124	4 863	125	4.765	126	4 271
127	2.806	128	1 146	129	569
130	.073	131	.000	132	001
133	003	134	5.804	135	5 770
136	5.247	137	3.222	138	1.164
139	572	140	.073	141	.000
142	001	143	.003	144	7.029
145	7.121	146	6 580	147	3 763
148	1.173	149	.573	150	.073
151	.000	152	.001	153	.003
154	8.777	155	9.156	156	8 607
157	4 555	158	1.177	159	573
160	073	161	.000	162	001
163	003	164	10.000	165	10 513
166	9 965	167	5.087	168	1 177
169	573	170	.073		

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